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# FACTORIZATIONS OF THE GROUPS $PSp_6(q)$ *

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*Цанко Генчев, Еленка Генчева.* ФАКТОРИЗАЦИИ ГРУПП $PSp_6(q)$

Доказан следующий результат:

Пусть $G = PSp_6(q)$ и $G = AB$, где $A$, $B$ — собственные неабелевые простые подгруппы $G$. Тогда имеет место одно из следующих:

(1) $q = 2$ и $A \cong U_3(3)$, $B \cong U_4(2)$;
(2) $q = 4$ и $A \cong J_2$, $B \cong U_4(4)$;
(3) $q = 2^n$ и $A \cong L_2(q^3)$, $B \cong L_4(q)$ или $U_4(q)$;
(4) $q = 2^n > 2$ и $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ или $U_4(q)$.

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The following result is proved.

Let $G = PSp_6(q)$ and $G = AB$, where $A$, $B$ are proper non-Abelian simple subgroups of $G$ Then one of the following holds:

(1) $q = 2$ and $A \cong U_3(3)$, $B \cong U_4(2)$;
(2) $q = 4$ and $A \cong J_2$, $B \cong U_4(4)$;
(3) $q = 2^n$ and $A \cong L_2(q^3)$, $B \cong L_4(q)$ or $U_4(q)$;
(4) $q = 2^n > 2$ and $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ or $U_4(q)$.

## INTRODUCTION

In [5, 6] we determined all the factorizations with two proper simple subgroups of some groups of Lie type of Lie rank 3. In the present work we extend this

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investigation to the simple groups $PSp_6(q)$ of Lie type $(C_3)$ over the finite field $GF(q)$. We prove the following

**Theorem.** *Let $G = PSp_6(q)$ and $G = AB$, where $A, B$ are proper non-Abelian simple subgroups of $G$. Then one of the following holds:*

(1) $q = 2$ *and* $A \cong U_3(3)$, $B \cong U_4(2)$;

(2) $q = 4$ *and* $A \cong J_2$, $B \cong U_4(4)$;

(3) $q = 2^n$ *and* $A \cong L_2(q^3)$, $B \cong L_4(q)$ *or* $U_4(q)$;

(4) $q = 2^n > 2$ *and* $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ *or* $U_4(q)$.

The factorizations of $PSp_6(q)$ into the product of two maximal subgroups have been determined in [8]. We make use of this result here.

Our notation is standard. Basic information on the known simple groups can be found in [2, 3].

In the proof below we shall freely use the following directly verified properties of the group $G = PSp_6(q)$, $q = 2^n$. Using the symplectic realization

$$G = \{X \in GL_6(q) \mid X^tTX = T\}, \quad \text{where } T = \begin{pmatrix} O & E \\ E & O \end{pmatrix}$$

($E$ is the indentity matrix), $G$ has four conjugacy classes of involutions denoted here $(2_1)$, $(2_2)$, $(2_3)$, $(2_4)$ with representatives

$$i_1 = \begin{pmatrix} E & P \\ O & E \end{pmatrix}, \quad i_2 = \begin{pmatrix} E & Q \\ O & E \end{pmatrix}, \quad i_3 = \begin{pmatrix} E & E \\ O & E \end{pmatrix}, \quad i_4 = \begin{pmatrix} E & R \\ O & E \end{pmatrix},$$

respectively, where

$$P = \operatorname{diag}(1,0,0), \quad Q = \operatorname{diag}(1,1,0), \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The only involutions of $G$ which are squares (of elements of order 4) are those in the classes $(2_2)$ and $(2_4)$. Further, $|C_G(i_2)| = q^9(q^2-1)$ and $|C_G(i_4)| = q^9(q^2-1)^2$. Lastly, $G$ has no elementary Abelian subgroup of order $q^4$ all of whose involutions are in the class $(2_4)$.

## PROOF OF THE THEOREM

Let $G = PSp_6(q)$ and $G = AB$, where $A$, $B$ are proper non-Abelian simple subgroups of $G$. The factorizations of $PSp_6(2)$ and $PSp_6(3)$ are determined in [1, 7]; this gives (1) and (3) (with $n = 1$) of the theorem. Thus we can assume that $q \geqq 4$. The list of maximal factorizations of $G$ is given in [8]. This leads, by order considerations, to the following possibilities, where $q = 2^n$:

1) $A \cong L_4(q)$, $B \cong U_3(q)$;

2) $A \cong U_4(q)$, $B \cong L_3(q)$;

3) $A \cong U_4(q)$, $B \cong G_2(\sqrt{q})$, $n$ even $> 2$;

4) $A \cong L_2(q^3)$, $B \cong L_4(q)$ or $U_4(q)$;

5) $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ or $U_4(q)$;

6) $A \cong J_2$, $B \cong U_4(4)$, $n = 2$.

Note that $PSp_6(2^n)$ contains subgroups isomorphic to $U_3(2^n)$ or $L_3(2^n)$ if and only if $n$ is even or odd, respectively.

We consider these possibilities case by case.

**Cases 1), 2), 3).** Here $|A \cap B| = q^2-1$, $q^2-1$, and $q-1$, respectively. Each of the groups $U_3(q)$, $L_3(q)$, and $G_2(\sqrt{q})$ has elements of order 4 and hence $B$ contains involutions from at least one of the classes $(2_2)$ and $(2_4)$ of $G$. Let $L_4^\varepsilon(q)$ denote $L_4(q)$ if $\varepsilon = +$ and $U_4(q)$ if $\varepsilon = -$. Then any subgroup $A \cong L_4^\varepsilon(q)$ of $G$ contains involutions from each of the classes $(2_2)$ and $(2_4)$. Indeed, $A$ has two conjugacy classes of involutions, central and noncentral, which are squares and so they belong to $(2_2)$ or $(2_4)$. The central involutions in $A$ have centralizers of order divisible by $(q-\varepsilon 1)^2$, whence they are from the class $(2_4)$. But $L_4^\varepsilon(q)$ has an elementary Abelian subgroup of order $q^4$. As $G$ has no elementary Abelian subgroup of order $q^4$ with involutions only from $(2_4)$, it follows that $A$ contains involutions also from $(2_2)$. Thus $A$ and $B$ necessarily have a common involution, which implies $2 \mid |A \cap B|$, a contradiction.

Now we prove the existence of the factorizations in the remaining three cases 4), 5), 6).

**Case 4).** Here we use the following symplectic realization of the group $L_2(q^3)$ in $PSp_6(q)$, $q = 2^n$. Let $K$ be a field extension of $k = GF(q)$ of degree 3. There is an element $\omega$ of $K$ such that $1$, $\omega$, $\omega^2$ form a basis of $K$ over $k$, and set $\omega^3 = p_0 + p_1\omega + p_2\omega^2$, where $p_i \in k$ $(i = 0, 1, 2)$. Further, let

$$S = \begin{pmatrix} a_0 + a_1\omega + a_2\omega^2 & b_0 + b_1\omega + b_2\omega^2 \\ c_0 + c_1\omega + c_2\omega^2 & d_0 + d_1\omega + d_2\omega^2 \end{pmatrix}, \quad a_i, b_i, c_i, d_i \in k \ \ (i = 0, 1, 2)$$

be any matrix with $\det S = 1$. Then the following matrices form a subgroup $A$ of $G$ isomorphic to $L_2(q^3)$:

$$W = \begin{pmatrix} a_0 & p_0a_2 & w_{13} & b_0 & b_1 & b_2 \\ a_1 & a_0 + p_1a_2 & w_{23} & b_1 & b_2 + p_0^{-1}(p_2b_0 + p_1b_1) & p_0^{-1}(b_0 + p_1b_2) \\ a_2 & a_1 + p_2a_2 & w_{33} & b_2 & p_0^{-1}(b_0 + p_1b_2) & p_0^{-1}(b_1 + p_2b_2) \\ c_0 & p_0c_2 & w_{61} & d_0 & d_1 & d_2 \\ p_0c_2 & w_{61} & w_{62} & p_0d_2 & d_0 + p_1d_2 & d_1 + p_2d_2 \\ w_{61} & w_{62} & w_{63} & p_0d_1 + p_0p_2d_2 & w_{65} & w_{66} \end{pmatrix},$$

where

$$w_{13} = p_0a_1 + p_0p_2a_2, \quad w_{23} = p_1a_1 + (p_1p_2 + p_0)a_2, \quad w_{33} = a_0 + p_2a_1 + (p_2^2 + p_1)a_2,$$

$$w_{61} = p_0c_1 + p_0p_2c_2, \quad w_{62} = p_0c_0 + p_0p_2c_1 + p_0c_2(p_2^2 + p_1),$$

$$w_{63} = p_0p_2c_0 + p_0c_1(p_2^2 + p_1) + p_0c_2(p_2^3 + p_0), \quad w_{65} = p_1d_1 + d_2(p_0 + p_1p_2),$$

$$w_{66} = d_0 + p_2d_1 + d_2(p_2^2 + p_1).$$

The isomorphism is given by the map $S \mapsto W$.

Now let $B \cong L_4^\varepsilon(q)$ be a subgroup of $G$. Then $|A \cap B| \geq q^3 - \varepsilon 1$ and, using the subgroup list of $A$, we see that $|A \cap B| \leq 2(q^3 - \varepsilon 1)$. The above realization of $A$ shows that the involutions in $A$ are from the class $(2_3)$ of $G$. However (as shown above), any $L_4^\varepsilon(q)$ subgroup of $G$ contains involutions only from the classes $(2_2)$ and $(2_4)$. Thus $2 \nmid |A \cap B|$ and then $|A \cap B| = q^3 - \varepsilon 1$, which implies $G = AB$. This is (3) of the theorem.

**Case 5).** We use the following two realizations of the group $G = PSp_6(q)$, $q = 2^n$:

(i) $$PSp_6(q) = \{X \in GL_6(q) \mid X^t H X = H\},$$

where $H = \mathrm{diag}(J, J, J)$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(ii) $$PSp_6(q) = \{Y \in GL_6(q) \mid Y^t I Y = I\},$$

where $I = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$.

Let $X, Y \in GL_6(q)$ and $Y = T_0^{-1} X T_0$, where

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = T_0^{-1}.$$

Then $Y^t I Y = I$ if and only if $X^t H X = H$.

Now, with respect to (i), we have

$$\left\{ \left( \begin{array}{c|cc} * & 0 & \\ \hline 0 & 1 & \\ & & 1 \end{array} \right) \in PSp_6(q) \right\} \cong PSp_4(q).$$

On the other hand (see [4]), with respect to (ii), a $G_2(q)$ subgroup of $PSp_6(q)$ is generated by the matrices $X_{\pm r}(t)$, $r \in \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$, $t \in GF(q)$, where

$$X_a(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_b(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{2a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{3a+2b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix $X_{-r}(t)$ is the transpose of $X_r(t)$. Now a direct computation shows that the common elements of the above $PSp_4(q)$ and $G_2(q)$ subgroups are exactly as follows:

$$T_0 \begin{pmatrix} w & 0 & 0 & u^2+vw & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & u+vw \\ u^{-1} & 0 & 0 & u^{-1}v & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & u^{-2} & 0 & 0 & u^{-2}v \end{pmatrix} T_0 \quad (u \in GF(q)^*, \ v, w \in GF(q)),$$

$$T_0 \begin{pmatrix} u^2 & 0 & 0 & v & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & v \\ 0 & 0 & 0 & u^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u^{-2} \end{pmatrix} T_0 \quad (u \in GF(q)^*, \ v \in GF(q)).$$

Hence $|PSp_4(q) \cap G_2(q)| = q(q^2-1)$ (in fact,

$$PSp_4(q) \cap G_2(q) = T_0 \left\langle X_{3a+b}(t), X_{-(3a+b)}(t) \right\rangle T_0 \cong L_2(q)).$$

Now order consideration imply

$$PSp_6(q) = G_2(q).PSp_4(q).$$

This is the first factorization in (4) of the theorem.

Now let $A \cong G_2(q)$ be the subgroup of $G$ described in the above paragraph, and $B_1 \cong O_6^\epsilon(q) \cong L_4^\epsilon(q).2$ be a subgroup of $G$. Then $G = AB_1$ and $A \cap B_1 \cong SL_3^\epsilon(q).2$ (see [8]). Let $B$ be the $L_4^\epsilon(q)$ subgroup in $B_1$. Now $A$ has two conjugacy classes of involutions — central and non-central. It is not difficult to see that these involutions are from the classes $(2_4)$ and $(2_3)$ of $G$, respectively. Further, $A$ has a single class of $SL_3^\epsilon(q).2$ subgroups (cf. [4]) and every such subgroup contains non-central involutions. Consequently, every $SL_3^\epsilon(q).2$ subgroup of $A$ contains involutions from $(2_3)$. But (as we have seen) $B$ has no involutions from $(2_3)$. Thus $A \cap B$ is a proper subgroup of $SL_3^\epsilon(q).2$. Then (by order considerations) $G = AB$; in particular, $A \cap B \cong SL_3^\epsilon(q)$. This gives the remaining two factorizations in (4) of the theorem.

**Case 6)**. Now $q = 4$. In case 5) we proved that $G = AB$, where $A \cong G_2(4)$, $B \cong U_4(4)$, and $D = A \cap B \cong U_3(4)$. Take a subgroup $C \cong J_2$ of $A$. Then (as shown in [9]) $A = CD$. It follows that $|B \cap C| = |D \cap C| = 150$. This implies $G = BC$, the factorization in (2) of the theorem.

This completes the proof of the theorem.

## REFERENCES


1. Alnader, N., K. Tchakerian. Factorizations of finite simple groups. — Ann. Univ. Sofia, 79, 1985, 357–364.
2. Carter, R. W. Simple groups of Lie type. London, 1972.
3. Conway, J., R. Curtis, S. Norton, R. Parker, R. Wilson. Atlas of finite groups. Oxford, 1985.
4. Cooperstein, B. Maximal subgroups of $G_2(2^n)$. — J. Algebra, 70, 1981, 23–36.
5. Gentchev, Ts., E. Gentcheva. Factorizations of some groups of Lie type of Lie rank 3. — Ann. Univ., Appl. Math., 1990, (to appear).
6. Gentchev, Ts., E. Gentcheva. Factorizations of the groups $PSU_6(q)$. — Ann. Univ. Sofia, 86, 1992, 79–85.
7. Gentchev, Ts., K. Tchakerian. Factorizations of the groups of Lie type of Lie rank three over fields of 2 or 3 elements. — Ann. Univ. Sofia, 85, 1991, 83–88.
8. Liebeck, M., C. Praeger, J. Saxl. The maximal factorizations of the finite simple groups and their automorphism groups. — Memoirs AMS, 86, 1990, 1–151.
9. Tchakerian, K., Ts. Gentchev. Factorizations of the groups $G_2(q)$. — Arch. Math., 44, 1985, 230–232.


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