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RAMSEY MULTIPLICITY $M(3,6)$ IS 2

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Николай Г. Хаджииванов, Иван Пашов. РАМСЕЕВСКАЯ КРАТНОСТЬ $M(3,6)$ ЕСТЬ 2

Обозначим через $c_p(G)$ число p-клик графа G, а через $\overline{G}$ — его дополнение. Кери сконструировал 17-вершинный граф G, для которого $c_3(G) + c_6(\overline{G}) = 0$ и доказал, что неравенство $c_3(G) + c_6(\overline{G}) \geqq 1$ имеет место для любого 18-вершинного графа G. Ив. Пашов указал 18-вершинный граф G, для которого $c_3(G) + c_6(\overline{G}) = 2$. В настоящей статье доказано, что $c_3(G) + c_6(\overline{G}) \geqq 2$ для любого 18-вершинного графа G.

Nikolay G. Khadzhiivanov, Ivan Zh. Pashov. RAMSEY MULTIPLICITY $M(3,6)$ IS 2

Let $c_p(G)$ be the number of the p-cliques of a graph G and $\overline{G}$ be the complement of G. Kéry showed a 17-vertex graph G with $c_3(G) + c_6(\overline{G}) = 0$ and proved for every 18-vertex graph G the inequality $c_3(G)+c_6(\overline{G}) \geqq 1$. I. Pashov constructed an 18-vertex graph G with $c_3(G)+c_6(\overline{G}) = 2$. In this paper is proved that $c_3(G) + c_6(\overline{G}) \geqq 2$ for every 18-vertex graph G.

1. INTRODUCTION

A set of p vertices of a graph will be called p-clique (p-anticlique) if every pair of vertices in the set are (are not) adjacent.

F. Ramsey [1] proved that for every two natural numbers p, q there exists a natural number n such that each n-vertex graph has a p-clique or a q-anticlique. The minimum n with this property is denoted by $R(p,q)$ and is called Ramsey number. Obviously we have $R(p,q) = R(q,p)$, $R(1,q) = 1$, $R(2,q) = q$. Thus, $R(p,q)$ is of interest when $3 \leqq p \leqq q$. There are known only seven such Ramsey numbers: $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(4,4) = 18$ (R. Greenwood and

A. Gleason [2]); $R(3,6) = 18$ (G. Kéry [3]); $R(3,7) = 23$ (J. Graver and J. Yackel [4]); $R(3,9) = 36$ (C. Grinstead and S. Roberts [5]).

We will denote with $c_p(G)$ the number of p-cliques of a graph G and with $\bar{c}_q(G)$ — the number of its q-anticliques; it is clear that $\bar{c}_q(G) = c_q(\overline{G})$.

The number $c_p(G) + \bar{c}_q(G)$ we will call (p,q)-multiplicity of the graph G. The minimum (p,q)-multiplicity in the set of the n-vertex graphs we will denote by $M(n;p,q)$.

It is clear that $M(n;p,q) \geqq 1$ when $n \geqq R(p,q)$ and $M(n;p,q) = 0$ when $n < R(p,q)$.

The number $M(n;3,3)$ has been determined for every n by A. Goodman [6].

The number $M(R(p,q);p,q)$ is called Ramsey multiplicity and is denoted by $M(p,q)$. Of course, this number is of interest only when $3 \leqq p \leqq q$ because $M(p,q) = M(q,p)$ and $M(1,q) = M(2,q) = 1$.

There are known only three Ramsey multiplicities: $M(3,3) = 2$ (A. Goodman [6]); $M(3,4) = 1$ (N. Khadzhiivanov and N. Nenov [7]); $M(3,5) = 4$ (I. Pashov [8]).

It is known that every 18-vertex graph has a 3-clique or a 6-anticlique (G. Kéry [3]). In this paper we will prove that there is not a graph without 3-cliques with just one 6-anticlique (see Theorem 1) and, similarly — there is not a graph without 6-anticliques with just one 3-clique (see Theorem 2). Hence $M(3,6) \geqq 2$.

On the other hand, the second author showed in [9] a 18-vertex graph without 6-anticliques and with exactly two 3-cliques. Thus $M(3,6) = 2$.

2. NOTATIONS AND PRELIMINARY RESULTS

We will write $G \in (n; \alpha \times p, \beta \times \bar{q})$ for an n-vertex graph G with $c_p(G) = \alpha$ and $\bar{c}_q(G) = \beta$.

The set of all neighbours of the vertex v in the considered graph will be denoted by $A(v)$; then $|A(v)| = d(v)$ is the degree of v. The same notation $A(v)$ will be used for the subgraph of G generated by the set $A(v)$. With $N(v)$ we will denote the set of the not adjacent to v (except v) vertices of the graph (and the correspondent generated subgraph). For the set of vertices not adjacent to either u and v (except u and v) and for the correspondent generated subgraph we will used the notation $N(u,v)$.

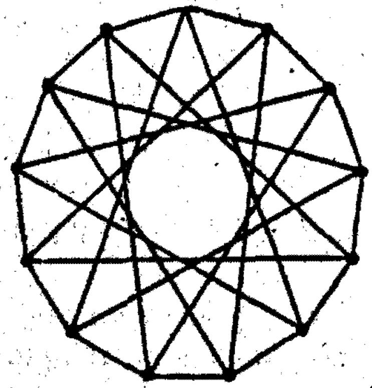

Fig. 1. The unique graph from $(13;0 \times 3, 0 \times \bar{5})$.

In the proofs of the theorems we will also use except a part of the above mentioned results the following propositions.

Proposition 1 (N. Nenov, I. Pashov and N. Khadzhiivanov [10]). *There is no graph* $G \in (13; 0 \times 3, 1 \times \bar{5})$.

Proposition 2 (G.Kéry [3]). *There is a unique graph* $G \in (13; 0 \times 3, 0 \times \bar{5})$; *it is presented in* Fig. 1.

Proposition 3 (N. Khadzhiivanov, N. Nenov and I. Pashov [11]). *There is a unique graph* $G \in (13; 1 \times 3, 0 \times \bar{5})$; *this graph is shown in* Fig. 2.

Proposition 4 (N. Nenov and N. Khadzhiivanov [12]). *There is a unique graph* $G \in (12; 0 \times 3,$

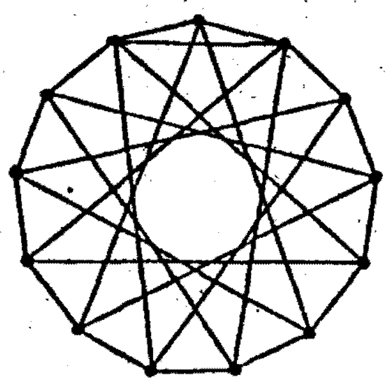

Fig. 2. The unique graph from $(13;1\times 3,0\times\overline{5})$.

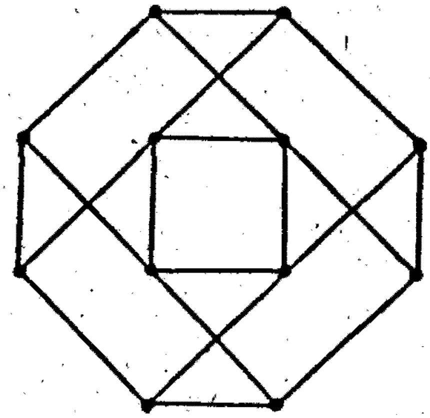

Fig. 3 The unique 20-edge graph from $(12;0\times 3,0\times\overline{5})$.

$0\times\overline{5})$ *with* 20 *edges*; *it is presented in* Fig. 3.

3. EVERY 18-VERTEX GRAPH WITHOUT 3-CLIQUES HAS AT LEAST TWO 6-ANTICLIQUES

Lemma 1. *Let* $G\in(18;0\times 3,1\times\overline{6})$ *and let* A *be the unique 6-anticlique in* G. *If* u *and* v *are not adjacent vertices of* G *and* $\{u,v\}\not\subset A$, *then* $|A(u)\cap A(v)|\leqq d(u)+d(v)-8$.

Proof. Clearly, $N(u,v)$ does not contain 3-cliques and 4-anticliques because if M is a 4-anticlique in $N(u,v)$, then $M\cup\{u,v\}$ is a 6-anticlique in G and is different from A. From $R(3,4)=9$, we have $|N(u,v)|\leqq 8$. But $|N(u,v)|=16-|A(u)\cup A(v)|=16-d(u)-d(v)+|A(u)\cap A(v)|$. Hence $|A(u)\cap A(v)|\leqq d(u)+d(v)-8$.

Lemma 2. *If* $G\in(18;0\times 3,1\times\overline{6})$, *then* $d(v)\leqq 6$ *for every vertex* v *and the equality is achieved for at most one vertex.*

Proof. For every vertex v the set $A(v)$ is an anticlique because G has no 3-cliques. Since G has only one 6-anticlique — A, we have $d(v)=|A(v)|\leqq 6$ and $A(v)=A$ when $d(v)=6$.

If $d(u)=d(v)=6$, then $A(u)=A(v)=A$. Hence u and v are not adjacent (G has no 3-cliques) and $|A(u)\cap A(v)|=6$. This contradicts Lemma 1).

Lemma 3. *If* $G\in(18;0\times 3,1\times\overline{6})$, *then* $d(v)\geqq 4$ *for every vertex* v.

Proof. Assume $d(v)\leqq 3$, i.e. $|N(v)|\geqq 14$. Remember $M(3,5)>1$, but $c_3(N(v))=0$, so $N(v)$ has at least two 5-anticliques K_1 and K_2. Then $K_1\cup\{v\}$ and $K_2\cup\{v\}$ are two different 6-anticliques in G. This contradiction completes the proof of the lemma.

Lemma 4. *If* $G\in(18;0\times 3,1\times\overline{6})$ *and* v *is a vertex of* G *with* $d(v)=4$, *then* $N(v)\in(13;0\times 3,0\times\overline{5})$.

Proof. Obviously $|N(v)|=13$, $c_3(N(v))=0$ and any 5-anticlique in $N(v)$ forms with v a 6-anticlique in G. Hence $\overline{c}_5(N(v))\leqq 1$.

The assumption $\overline{c}_5(N(v))=1$ implies $N(v)\in(13;0\times 3,1\times\overline{5})$ which contradicts Proposition 1. Hence $\overline{c}_5(N(v))=0$, i.e. $N(v)\in(13;0\times 3,0\times\overline{5})$.

Lemma 5. *If* $G\in(18;0\times 3,1\times\overline{6})$ *and* A *is the unique 6-anticlique in* G, *then* $d(v)\geqq 5$ *for every* $v\in A$.

P r o o f. Assume $v \in A$ and $d(v) = 4$ (see Lemma 3). According to Lemma 4, $N(v) \in (13; 0\times 3, 0\times \overline{5})$, which is impossible because $N(v)$ contains the 5-anticlique $A\backslash\{v\}$.

Lemma 6. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A is the unique 6-anticlique in G, then $d(v) \geqq 5$ for each $v \notin A$.*

P r o o f. Let us assume $v_0 \notin A$ and $d(v_0) = 4$ (see lemma 3). By Lemma 4, $N(v_0) \in (13; 0 \times 3, 0 \times \overline{5})$. Hence (see Proposition 2) $N(v_0)$ is isomorphic to the 4-regular graph in Fig. 1.

Denote by m the number of edges $[u, v]$ with $u \in A(v_0)$, $v \in N(v_0)$. Obviously, $m = \sum_{v\in N(v_0)} (d(v) - 4)$. This equality and Lemma 2 yields $m \leqq 12.1 + 1.2 = 14$.

On the other hand, it is clear that $m = \sum_{u\in A(v_0)} (d(v) - 1)$. Hence

$$\sum_{u\in A(v_0)} d(v) \leqq 18. \tag{1}$$

The set $A(v_0)$ contains at least two vertices from A because otherwise $N(v_0)$ will have a 5-anticlique and v_0 will be in a 6-anticlique, different from A. Let $\{u_1, u_2\} \subset A(v_0) \cap A$ and u_3, u_4 be the other two vertices in $A(v_0)$. By Lemma 5, $\sum_{u\in A(v_0)} d(u) \geqq 10 + d(u_3) + d(u_4)$. This inequality and (1) yield $d(u_3) + d(u_4) \leqq 8$. Then Lemma 3 gives $d(u_3) = d(u_4) = 4$.

Apply Lemma 1 for vertices u_3 and u_4 ($u_3 \notin A$, by Lemma 5). Thus we conclude $|A(u_3) \cap A(u_4)| \leqq 0$, which is absurdity because $v_0 \in A(u_3) \cap A(u_4)$. The received contradiction completes the proof of the Lemma.

Lemma 7. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$, then G is 5-regular.*

P r o o f. By Lemmas 5 and 6 we know that $d(v) \geqq 5$ for every vertex v of G and by Lemma 2 that $d(v) \leqq 6$ with equality at most for one vertex v. Hence G has at least 17 vertices of degree 5. If the rest vertex has degree 6, then the number of the vertices of an odd degree will be odd, which is impossible. Consequently, all vertices of G have degree 5.

Lemma 8. *Let $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A be the unique 6-anticlique in G. If $u \notin A$, then $|A(u) \cap A| \geqq 2$. If $u \notin A$, $v \notin A$ and $A(u) \cup A(v) \supset A$, then u and v are not adjacent.*

P r o o f. Any k vertices from A which are not adjacent to $u \notin A$ form with u a $(k + 1)$-anticlique different from A. Hence, $k \leqq 4$, i.e. $|A(u) \cap A| \geqq 2$ for every $u \notin A$.

Now let $u \notin A$, $v \notin A$ and $A(u) \cup A(v) \supset A$. Assume that u and v are adjacent. We may regard that $|A \cap A(v)| \leqq |A \cap A(u)|$. Because of $c_3(G) = 0$, $A(u) \cap A(v) = \varnothing$. Hence $|A \cap A(v)| \leqq 3$, i.e. v has at least two neighbours out of A (see Lemma 7). Consequently, there is a neighbour w of v, $w \neq u$, $w \notin A$. The adjacent vertices w, v cannot have common neighbours. From the relations $|A(w) \cap A| \geqq 2$ and $A(u) \cup A(v) \supset A$ follows $|A(w) \cap A \cap A(u)| \geqq 2$. The vertices w and u have also the vertex $v \notin A$ as a common neighbour. Thus $|A(u) \cap A(w)| \geqq 3$.

On the other hand, applying Lemma 1 for the vertices u and w we receive $|A(u) \cap A(w)| \leqq d(u) + d(w) - 8 = 5 + 5 - 8 = 2$.

The obtained contradiction completes the proof of the Lemma.

Lemma 9. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A is the unique 6-anticlique in G, then $|A(v) \cap A| \leqq 3$ for every vertex $v \notin A$.*

P r o o f. Assume $|A(v) \cap A| \geq 4$.

It is easy to see that $|A(v) \cap A| = 4$. Really, otherwise all five neighbours of v are in A and if w is the rest vertex of A, then $A(w) \cup v$ is a 6-anticlique, different from A, which is impossible.

Thus, $|A(v) \cap A| = 4$ and there is a vertex $u \notin A$ which is adjacent to v. By Lemma 8, $|A(u) \cap A| \geq 2$ and because of $A(u) \cap A(v) = \emptyset$ we have $A(u) \cup A(v) \supset A$. Again Lemma 8 yields that u and v are not adjacent. This contradiction completes the proof of the Lemma.

Theorem 1. *Every 18-vertex graph without 3-cliques contains at least two 6-anticliques.*

P r o o f. Let G be an 18-vertex graph and $c_3(G) = 0$. Assume $\bar{c}_6(G) \leq 1$. By $R(3,6) = 18$ we have $\bar{c}_6(G) \neq 0$. Hence $\bar{c}_6(G) = 1$, i.e. $G \in (18; 0 \times 3, 1 \times \bar{6})$. Denote by A the unique 6-anticlique in G.

Let B be the set of vertices v for which $|A(v) \cap A| = 3$ and $|B| = n$. Let V be the set of the vertices of G and $C = V \setminus (A \cup B)$. Obviously $B \cap A = \emptyset$ and $|C| = 12 - n$. By Lemmas 8 and 9 follows $|A(v) \cap A| = 2$ for each $v \in C$.

Let l be the number of the edges $[u, v]$ with $u \in A$ and $v \notin A$. Every vertex in A is an end point for five such edges; hence $l = 6.5 = 30$.

On the other hand, clearly, $l = \sum_{v \in B} |A(v) \cap A| + \sum_{v \in C} |A(v) \cap A| = n.3 + (12 - n).2 = 24 + n$. Hence $24 + n = 30$ and $n = 6$.

We will prove that B is an anticlique. Assume the opposite and let u, v be adjacent vertices from B. Since $A(u) \cap A(v) = \emptyset$, $|A(u) \cap A| = 3$ and $|A(v) \cap A| = 3$ we have $A(u) \cup A(v) \supset A$. From Lemma 8 follows that u and v are not adjacent.

The received contradiction shows that B is an 6-anticlique. But $B \cap A = \emptyset$, thus we obtain a contradiction with equality $\bar{c}_6(G) = 1$.

The Theorem is proved.

4. EVERY 18-VERTEX GRAPH WITHOUT 6-ANTICLIQUES HAS AT LEAST TWO 3-CLIQUES

Lemma 10. *If $G \in (18; 1 \times 3, 0 \times \bar{6})$ and T is the unique 3-clique in G, then $d(v) \leq 6$ for every vertex v. The equality may be attained only for a vertex from T.*

P r o o f. If $v \notin T$, then $A(v)$ is an anticlique; hence $d(v) = |A(v)| \leq 5$.

When $v \in T$, the subgraph $A(v)$ has only one edge; thus $d(v) = |A(v)| \leq 6$.

Lemma 11. *If $G \in (18; 1 \times 3, 0 \times \bar{6})$, then $d(v) \geq 4$ for every vertex v.*

P r o o f. If $d(v) \leq 3$, then $|N(v)| \geq 14$. Since $\bar{c}_5(N(v)) = 0$ and $M(3,5) > 1$ (see the Introduction), the subgraph $N(v)$ will have at least two 3-cliques, which is impossible.

Lemma 12. *Let $G \in (18; 1 \times 3, 0 \times \bar{6})$ and T is the 3-clique of G. If the vertex v_0 be adjacent to at least one vertex from T and $d(v_0) = 4$, then every vertex from $N(v_0) \setminus T$ is adjacent to at most one vertex from $A(v_0)$.*

P r o o f. Obviously $|N(v_0)| = 13$ and $\bar{c}_5(N(v_0)) = 0$. Since v_0 is adjacent to a vertex from T, we have $c_3(N(v_0)) = 0$.

Proposition 2 implies that the graph $N(v_0)$ is isomorphic to the graph in Fig. 1. We will use only the fact that $N(v_0)$ must be a 4-regular graph (see Fig. 1).

If $v \notin T$, by Lemma 10 we will have $d(v) \leqq 5$. Then every $v \in N(v_0)\backslash T$, having 4 neighbours in $N(v_0)$, is adjacent to at most one vertex out of $N(v_0)$, i.e. to at most one vertex from $A(v_0)$.

Lemma 13. *If $G \in (18; 1 \times 3, 0 \times \overline{6})$ and $T = \{v_1, v_2, v_3\}$ is the 3-clique of G, then $d(v_i) \geqq 5$, $i = 1, 2, 3$.*

Proof. Assume $d(v_1) < 5$. Then (see Lemma 11) $d(v_1) = 4$. Let $A(v_1)\backslash T = \{v_4, v_5\}$.

We will prove that $d(v_4) = d(v_5) = 4$.

Assume $d(v_4) \neq 4$. By Lemma 11, $d(v_4) \geqq 5$. Since $A(v_4) \cap A(v_1) = \emptyset$, we have $|A(v_4) \cap N(v_1)| \geqq 4$. The set $A(v_4)$ is an anticlique because $v_4 \notin T$. Hence $A(v_4) \cap N(v_1)$ is an anticlique too.

Applying Lemma 12 for the vertex v_1 we have that every vertex v from $N(v_1)\backslash T = N(v_1)$ is adjacent to at most one vertex from $A(v_1)$; in particular, the same is valid for each vertex $v \in A(v_4) \cap N(v_1)$. But such a v is adjacent to $v_4 \in A(v_1)$; hence v is not adjacent to the vertices v_2 and v_5 from $A(v_1)$.

Thus we receive that the set $(A(v_4) \cap N(v_1)) \cup \{v_2, v_5\}$ is an anticlique, which is a contradiction because this set has at least 6 vertices.

In this way we have that $d(v_4) = d(v_5) = 4$. The non-adjacent vertices v_4 and v_5 have a common neighbour (the vertex v_1). Hence $|N(v_4, v_5)| \geqq 9$. Moreover, $c_3(N(v_4, v_5)) = 0$, because $v_1 \notin N(v_4, v_5)$. Note also that $\overline{c}_4(N(v_4, v_5)) = 0$. The obtained results for the graph $N(v_4, v_5)$ contradict the equality $R(3,4) = 9$.

Lemma 14. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and $T = \{v_1, v_2, v_3\}$ be the 3-clique of G. If $v_0 \notin T$ and v_0 is adjacent to a vertex from T, then $d(v_0) = 5$.*

Proof. Assume $d(v_0) < 5$. Then (see Lemma 11) $d(v_0) = 4$. Let v_0 be adjacent to v_1 from T and the rest three neighbours of v_0 be v_4, v_5, and v_6.

We will prove that $d(v_4) = 4$ and $A(v_4) \cap T \neq \emptyset$.

Let $A = A(v_4) \cap (N(v_0)\backslash T)$. Since the set $A(v_4)$ is an anticlique, then A is an anticlique, too. By Lemma 12 we know that every vertex from $N(v_0)\backslash T$ is adjacent to at most one vertex from $A(v_0)$. The vertices of A are adjacent to $v_4 \in A(v_0)$ and belong to $N(v_0)\backslash T$. Hence each of these vertices is not adjacent to the other vertices v_1, v_5, v_6 from $A(v_0)$.

Consequently, $A \cup \{v_1, v_5, v_6\}$ is an anticlique, so that $|A| \leqq 2$. Therefore v_4 is adjacent to at most two vertices from $N(v_0)\backslash T$. Besides v_4 is adjacent to v_0 and not adjacent to v_1, v_5, v_6. According to Lemma 11, $d(v_4) \geqq 4$. Hence v_4 must be adjacent to at least one of the vertices v_2, v_3. On the other hand, v_4 can not be adjacent to both v_2, v_3.

Consequently, $d(v_4) = 4$ and $A(v_4) \cap T \neq \emptyset$.

Similarly, $d(v_5) = 4$. The vertices v_4 and v_5 have a common neighbour (the vertex v_0); hence $|N(v_4, v_5)| \geqq 9$. It is clear that $\overline{c}_4(N(v_4, v_5)) = 0$ because v_4 and v_5 are not adjacent and G has no 6-anticliques. Then from $R(3,4) = 9$ follows that $N(v_4, v_5)$ has a 3-clique. But this is impossible because $A(v_4) \cap T \neq \emptyset$.

The obtained contradiction shows that $d(v_0) \geqq 5$. From Lemma 10 it follows $d(v_0) = 5$.

Lemma 15. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and T be the 3-clique in G. If v_0 is not adjacent to vertices from T, then $d(v_0) = 5$.*

Proof. By Lemma 10 we know that $d(v_0) \leqq 5$. Assume $d(v_0) < 5$. Then Lemma 11 yields $d(v_0) = 4$.

Obviously, $|N(v_0)| = 13$, $c_3(N(v_0)) = 1$ and $\overline{c}_5(N(v_0)) = 0$. According to Proposition 3, the graph $N(v_0)$ is isomorphic to the graph in Fig. 2. Every vertex

v from $N(v_0)\backslash T$ has degree 4 in $N(v_0)$ (see Fig. 2). Consequently, v is adjacent to at most one vertex out of $N(v_0)$ (because $d(v) \leq 5$, see Lemma 10). We will essentially use the just proved proposition in the end of the proof.

Every vertex u from $A(v_0)$ has at most one neighbour in T; hence it has at least two neighbours in $N(v_0)\backslash T$ (according to Lemma 11, $d(u) \geq 4$). Moreover, if u is not adjacent to the vertices from T or $d(u) \geq 5$, then u will have at least three neighbours in $N(v_0)\backslash T$.

We will prove that $A(v_0)$ contains at least three vertices, each with at least three neighbours in $N(v_0)\backslash T$.

This is clear, if three vertices from $A(v_0)$ have degree ≥ 5. Thus we will assume that there are $u_1, u_2 \in A(v_0)$ both of degree 4 (see Lemma 11). Then $|N(u_1, u_2)| \geq 9$ (u_1 and u_2 have a common neighbour). The vertices u_1 and u_2 are not adjacent ($v_0 \notin T$), so that $\overline{c}_4(N(u_1, u_2)) = 0$. The equality $R(3,4) = 9$ implies that $N(u_1, u_2)$ has a 3-clique, i.e. $N(u_1, u_2) \supset T$. So u_2 is not adjacent to the vertices from T. Then, as we now, u_2 has three neighbours in $N(v_0)\backslash T$.

Thus $A(v_0)$ contains at least three vertices, each with at least three neighbours in $N(v_0)\backslash T$. The fourth vertex from $A(v_0)$ has at least two neighbours in $N(v_0)\backslash T$. But $|N(v_0)\backslash T| = 10$. Consequently, there is a vertex from $N(v_0)\backslash T$ with at least two neighbours in $A(v_0)$, which is impossible (see the beginning of the proof).

Lemma 16. *Let G be not regular graph and $G \in (18; 1 \times 3, 0 \times \overline{6})$. Then two vertices, say v_2 and v_3, from the unique 3-clique $T = \{v_1, v_2, v_3\}$ have degree 6 and all the rest vertices of G have degree 5. Moreover $N(v_1)$ contains a simple 8-cycle B which is an induced subgraph of G and has the properties:*

1. *Every vertex from $A = N(v_1)\backslash B$ is adjacent exactly to one vertex from $A(v_1)$.*
2. *Every vertex from B is adjacent exactly to two vertices from $A(v_1)$.*

P r o o f. By Lemmas 14 and 15 we have $d(v) = 5$ for every $v \notin T$ and Lemmas 10 and 13 give $5 \leq d(v_i) \leq 6$, $i = 1, 2, 3$. Since the number of the vertices of odd degree must be even, T contains one or three vertices of degree 5. But G is not regular. Hence T contains only one vertex of degree 5 — assume $d(v_1) = 5$ and $d(v_2) = d(v_3) = 6$.

Let e be the number of the edges of the 12-vertex graph $N(v_1)$ and m be the number of the edges of the type $[u, v]$ with $u \in N(v_1)$, $v \in A(v_1)$. Clearly

$$12.5 = \sum_{u \in N(v_1)} d(u) = 2e + m. \qquad (2)$$

Since every $v \in A(v_1)$ has 4 neighbours in $N(v_1)$, we have $m = 5.4 = 20$ and $e = 20$.

Thus $N(v_1)$ has 20 edges. Obviously $N(v_1) \in (12; 0 \times 3, 0 \times \overline{5})$. According to Proposition 4, $N(v_1)$ is isomorphic to the graph in Fig.3. Let B be the set of the vertices of $N(v_1)$ of degree 3 in $N(v_1)$. Clearly B is a simple 8-cycle which is an induced subgraph of G. Every vertex from the set $A = N(v_1)\backslash B$ has degree 4 in $N(v_1)$. The vertices of $N(v_1)$ have degree 5 in G. Therefore the properties 1 and 2 are obviously fulfilled and the lemma is proved.

The next lemma will be proved in assumptions and notations from Lemma 16. We know that $d(v) = 5$ when $v \notin T$. The neighbours of the vertex v_1 (which also has degree 5) we will denote by v_2, v_3, v_4, v_5, v_6. By Lemma 16, $N(v_1) = A \cup B$, where $A \cap B = \emptyset$, $|A| = 4$, B is a simple 8-cycle and A and B have the properties 1 and 2.

We will denote by B_k, $2 \leq k \leq 6$, the set of the vertices from B which are adjacent to v_k. Clearly B_k are anticliques and $|B_k| \leq 4$.

Lemma 17. *The assumptions of Lemma 16 and the last notations imply the properties:*

1. $B_2 \cap B_3 = \varnothing$.
2. $|B_i| \geqq 3$ *when* $i \in \{2,3\}$.
3. $B_i \cap B_j \neq \varnothing$ *when* $i \in \{2,3\}$ *and* $j \in \{4,5,6\}$.
4. $B \neq B_2 \cup B_3$.
5. $|B_i| = 3$ *when* $i \in \{2,3\}$.
6. $|B_i \cap B_j| = 1$ *when* $i \in \{2,3\}$ *and* $j \in \{4,5,6\}$.
7. *There is* $j_0 \in \{4,5,6\}$ *with* $|B_{j_0}| = 4$.
8. *Let* j_0 *(see the previous property) be, for instance,* 6. *Then* $B \backslash (B_2 \cup B_3) \subset B_6$ *and one of the vertices from* $B \backslash (B_2 \cup B_3)$ *is in* $B_4 \backslash B_5$ *and the other in* $B_5 \backslash B_4$.

Proof. 1. If $v \in B_2 \cap B_3$, then $\{v, v_2, v_3\}$ is a 3-clique different from T, which is impossible.

2. The set $A(v_2) \cap N(v_1)$ is a 4-anticlique, since $d(v_2) = 6$ and G has no other 3-clique except T.

Assume $|B_2| \leqq 2$. Then $|A(v_2) \cap A| \geqq 2$. Let $a, b \in A(v_2) \cap A$. According to the property 1 in Lemma 16, a and b are not adjacent to the vertices from $A(v_1) \backslash \{v_2\}$. The set $A(v_1) \backslash \{v_2\}$ is a 4-anticlique because $d(v_1) = 5$ and G has not a 3-clique, different from T. Since a and b are not adjacent, $\{a, b\} \cup (A(v_1) \backslash \{v_2\})$ is a 6-anticlique, which is a contradiction.

3. It is sufficient to prove $A(v_2) \cap A(v_j) \cap N(v_1) \neq \varnothing$ because if $v \in A(v_2) \cap A(v_j) \cap N(v_1)$, then property 1 from Lemma 16 implies $v \in B$.

Assume $A(v_2) \cap A(v_j) \cap N(v_1) = \varnothing$. The set $A(v_2) \backslash \{v_1\}$ is a 5-anticlique because $d(v_2) = 6$ and G has no other 3-clique except T. We have $A(v_2) \backslash \{v_1\} = \{v_3\} \cup (A(v_2) \cap N(v_1))$. Our assumption yields that v_j is not adjacent to the vertices from $A(v_2) \cap N(v_1)$. On the other hand, v_j is obviously not adjacent to v_3. Hence $\{v_j\} \cup (A(v_2) \backslash \{v_1\})$ is a 6-anticlique, which is a contradiction.

4. Assume the opposite, i.e. $B = B_2 \cup B_3$. Then $|B_2| = |B_3| = 4$, since $|B_k| \leqq 4$. Hence v_2 and v_3 have no neighbours in A. Then every vertex from A is adjacent exactly to one vertex from $A(v_1) \backslash \{v_2, v_3\} = \{v_4, v_5, v_6\}$. Therefore, there are vertices a and b from A with a common neighbour from $\{v_4, v_5, v_6\}$; we may regard that a and b are adjacent to v_4.

According to property 3, v_4 and v_i, $i = 2, 3$, have a common neighbour in B; we will denote it by b_i. Obviously $b_2 \neq b_3$ because otherwise $B_2 \cap B_3 \neq \varnothing$. The set $\{a, b, b_2, b_3\}$ is a 4-anticlique because $A(v_4)$ contains this set and $v_4 \notin T$. From 1 in Lemma 16 it follows that a and b are not adjacent to v_5 and v_6 and from 2 in the same lemma — b_2 and b_3 are not adjacent to v_5 and v_6.

We come to the contradiction that $\{a, b, b_2, b_3, v_5, v_6\}$ is a 6-anticlique.

5. Assume the opposite. Then we may regard (see property 2) that $|B_2| = 4$. Let $c \in B \backslash (B_2 \cup B_3)$ (see property 4). Since B_2 is a 4-anticlique in the simple 8-cycle B and $c \notin B_2$, then c is adjacent to two vertices u_1 and u_2 from B_2. According to property 2 in lemma 16, the vertex c is adjacent exactly to two vertices from $A(v_1) \backslash \{v_2, v_3\} = \{v_4, v_5, v_6\}$; we may assume that c is adjacent to v_4 and v_5. The vertices u_1 and u_2 are not adjacent to v_4 and v_5 because otherwise will be a 3-clique which contains the vertex c. The vertices u_1 and u_2 are in B_2 and $B_2 \cap B_3 = \varnothing$. Hence u_1 and u_2 are not adjacent to v_3. From property 2 in Lemma 16 it follows that u_1 has another (different from v_2) neighbour from $A(v_1)$ and we have the same for the vertex u_2. But u_1 and u_2 are not adjacent to v_3, v_4 and v_5. Hence u_1 and u_2 are adjacent to v_6.

Thus $A(u_1) \cap A(u_2) \supset \{v_2, v_6, c\}$. Moreover $d(u_1) = d(u_2) = 5$. Hence $|N(u_1, u_2)| \geq 9$. The vertices u_1 and u_2 are not adjacent (otherwise $\{c, u_1, u_2\}$ will be a 3-clique). Hence $\overline{c}_4(N(u_1, u_2)) = 0$. Then from $R(3,4) = 9$ we have $c_3(N(u_1, u_2) > 0$, i.e. $N(u_1, u_2) \supset T$. But this is impossible because $v_2 \notin N(u_1, u_2)$.

6. The sets $B_2 \cap B_4$, $B_2 \cap B_5$ and $B_2 \cap B_6$ are disjoint (see property 2 in Lemma 16) and not empty (see property 3), and $|B_2| = 3$ (see property 5). Hence $|B_2 \cap B_j| = 1$, $j = 4, 5, 6$.

7. Since $|A(v_i) \cap N(v_1)| = 4$ and $|A(v_i) \cap B| = 3$ (see property 5) we have $|A(v_i) \cap A| = 1$, $i = 2, 3$. Let $a_i = A(v_i) \cap A$. The vertices a_i have no neighbours in $\{v_4, v_5, v_6\}$ (see property 1 in Lemma 16). Each of the rest two vertices from A has one neighbour in $A(v_1)$. Hence there is a certain vertex from $\{v_4, v_5, v_6\}$ which has no neighbours in A, so all the four neighbours from $N(v_1)$ of this vertex are in B.

8. From properties 1 and 5 it follows $|B \setminus (B_2 \cup B_3)| = 2$. Let $B \setminus (B_2 \cup B_3) = \{a, b\}$. Every set B_4, B_5, B_6 has exactly one element in B_2 and B_3 (see property 6). According to the condition $|B_6| = 4$. Hence $\{a, b\} \subset B_6$.

The vertex a has another (different from v_6) neighbour in $A(v_1)$. Since a is not adjacent to v_2 and v_3, then a is adjacent to v_4 or v_5; we will assume that a is adjacent to v_4. The vertex b is also adjacent to a certain vertex from $\{v_4, v_5\}$. But b can not be adjacent to v_4. Indeed, otherwise $A(v_4) \cap A(v_6) \supset \{a, b, v_1\}$ and since $d(v_4) = d(v_6) = 5$, we obtain $|N(v_4, v_6)| \geq 9$. This is impossible because $R(3,4) = 9$, $\overline{c}_4(N(v_4, v_6)) = 0$ and $c_3(N(v_4, v_6)) = 0$ ($T \not\subset N(v_4, v_6)$, because $v_1 \notin N(v_4, v_6)$).

Thus, b is not adjacent to v_4. Hence b is adjacent to v_5, i.e. $b \in B_5 \setminus B_4$. Similarly, $a \in B_4 \setminus B_5$.

The lemma is proved.

Let us summarize the results from Lemma 17. Replace $B_i \cap B_j = b_{i,j}$ when $i \in \{2,3\}$, $j \in \{4,5,6\}$ (see property 6 in Lemma 17). Since $|B_2| = |B_3| = 3$ (see property 5 in Lemma 17) and every vertex from B belongs exactly to two B_k (see property 2 in Lemma 16), then $B_2 = \{b_{2,4}, b_{2,5}, b_{2,6}\}$ and $B_3 = \{b_{3,4}, b_{3,5}, b_{3,6}\}$ where all these $b_{i,j}$ are different.

According to properties 7 and 8 in Lemma 17, the 4-element set B_6 contains (except the vertices $b_{2,6}$ and $b_{3,6}$) the two vertices of $B \setminus (B_2 \cup B_3)$. One of these two vertices belongs to $B_4 \setminus B_5$ — we will denote it by $b_{4,6}$. The other belongs to $B_5 \setminus B_4$ — denote it by $b_{5,6}$. So $B_6 = \{b_{2,6}, b_{3,6}, b_{4,6}, b_{5,6}\}$.

Thus $B = \{b_{2,4}, b_{2,5}, b_{2,6}, b_{3,4}, b_{3,5}, b_{3,6}, b_{4,6}, b_{5,6}\}$ and $B_4 = \{b_{2,4}, b_{3,4}, b_{4,6}\}$, $B_5 = \{b_{2,5}, b_{3,5}, b_{5,6}\}$.

Lemma 18. *If $G \in (18; 1 \times 3, 0 \times \overline{6})$, then G is 5-regular.*

P r o o f. Assume the opposite. We will use Lemmas 16 and 17 and the introduced notations. So the simple 8-cycle B contains the anticliques B_k, $k = 2, 3, 4, 5, 6$. Hence the two neighbours from B of $b_{i,6}$ are $b_{m,n}$ where $m \in \{2,3,4,5\} \setminus \{i\}$ and $n \in \{4,5\} \setminus \{i\}$. Thus B is the simple 8-cycle $(b_{2,4}, b_{3,6}, b_{2,5}, b_{4,6}, b_{3,5}, b_{2,6}, b_{3,4}, b_{5,6})$. Denote by a_i, $i = 2, 3$, the unique neighbour of v_i in A. The vertex a_i have no other (different from v_i) neighbour in $A(v_1)$ (see property 1 in Lemma 16).

We will prove that a_i, $i = 2, 3$, is adjacent to $b_{4,6}$. First of all, the set $M = \{a_2, b_{2,4}, b_{2,6}, v_3, v_5\}$ is a 5-anticlique. Indeed, the set $\{a_2, b_{2,4}, b_{2,6}, v_3\}$ contains in the anticlique $A(v_2) \setminus \{v_1\}$ and v_5 is not adjacent to all vertices from this set.

The set $\{b_{4,6}, b_{2,4}, b_{2,6}, v_3, v_5\}$ is a 5-anticlique, too. Indeed, as we already know, $\{b_{2,4}, b_{2,6}, v_3, v_5\}$ is an anticlique. On the other hand, $b_{4,6}$ is not adjacent to $b_{2,4}$ (otherwise $\{b_{4,6}, b_{2,4}, v_4\}$ will be a 3-clique); $b_{4,6}$ is not adjacent to $b_{2,6}$ (by virtue of similar reasons) and $b_{4,6}$ is not adjacent to v_3 and v_5.

If we assume that $b_{4,6}$ is not adjacent to a_2, then $\{b_{4,6}\} \cup M$ will be a 6-anticlique, which is impossible.

We proved that a_2 is adjacent to $b_{4,6}$. In the same way we can prove that a_3 is adjacent to $b_{4,6}$. For this purpose it is sufficient to note that $\{a_3, b_{3,4}, b_{3,6}, v_2, v_5\}$ and $\{b_{4,6}, b_{3,4}, b_{3,6}, v_2, v_5\}$ are anticliques.

Thereby we arrive at the conclusion that $A(b_{4,6}) \supset \{a_2, a_3, b_{2,5}, b_{3,5}, v_4, v_6\}$, which is a contradiction because $d(b_{4,6}) = 5$ (see Lemma 16).

The obtained contradiction completes the proof of the lemma.

Lemma 19. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and T be the unique 3-clique in G. As we know by Lemma 18, G is a 5-regular graph. Denote by v_0 a vertex of G, $v_0 \notin T$ and let v_0 be adjacent to a vertex from T. Then $N(v_0)$ has a simple 8-cycle B which is an induced subgraph in G and has the properties:*

1. *Every vertex from $A = N(v_0)\backslash B$ is adjacent exactly to one vertex from $A(v_0)$.*
2. *Every vertex from B is adjacent exactly to two vertices from $A(v_0)$.*

The proof of this lemma is similar to the proof of Lemma 16 and we will not present it.

In relation with Lemma 19 we will introduce some notations.

Denote by v_i, $1 \leq i \leq 5$, the neighbours of v_0. Let $v_1 \in T$ and $\Delta_i = A(v_i) \cap N(v_0)$. Clearly, $|\Delta_i| = 4$, Δ_1 contains a unique edge and the other Δ_i are anticliques.

Lemma 20. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$. In the notations from (and after) Lemma 19 we may affirm that:*

1. *$|\Delta_i \cap \Delta_j| \leq 1$ when $i < j$.*
2. *$\Delta_k \subset B$ for a certain k.*
3. *$|\Delta_i \cap B| = 3$ when $i \neq k$.*
4. *$k = 1$.*

P r o o f. 1. Assume the opposite and let $\{a, b\} \subset \Delta_i \cap \Delta_j$. First we will prove that $i > 1$. Assume $i = 1$. Then $T \not\subset N(v_1, v_j)$; hence $c_3(N(v_1, v_j)) = 0$. The vertices v_1 and v_j are not adjacent and therefore $\bar{c}_4(N(v_1, v_j)) = 0$. We have $A(v_1) \cap A(v_j) \supset \{v_0, a, b\}$ and $d(v_1) = d(v_j) = 5$, so that $|N(v_1, v_j)| \geq 9$. The obtained result contradicts the equality $R(3, 4) = 9$.

Thus, $i > 1$. Hence Δ_i and Δ_j are 4-anticliques.

It is easy to see that $\Delta_i \cup \Delta_j \not\subset B$. Indeed, if $\Delta_i \cup \Delta_j \subset B$, then since B is a simple 8-cycle and the 4-anticliques Δ_i, Δ_j have a common vertex, certainly $\Delta_i = \Delta_j$. Thus, the vertices of Δ_i have no other (different from v_i and v_j) neighbours in $A(v_0)$ (see Lemma 19), thereby $\Delta_i \cup (A(v_0)\backslash\{v_i, v_j\})$ is a 7-anticlique in G, which is an absurdity.

Therefore we may assume that $\Delta_i \not\subset B$. Then $\Delta_i \cap A \neq \emptyset$ and let $c \in \Delta_i \cap A$.

We will prove that $M = \{a, b, c\} \cup (A(v_0)\backslash\{v_i, v_j\})$ is an anticlique. Indeed, $\{a, b, c\} \subset A(v_i)$; hence $\{a, b, c\}$ is an anticlique. On the other hand, c is not adjacent to the vertices from $A(v_0)\backslash\{v_i\}$ (see property 1 in Lemma 19), and a and b are not adjacent to the vertices from $A(v_0)\backslash\{v_i, v_j\}$ (see property 2 in Lemma 19). Since $A(v_0)$ is an anticlique, then M is really a 6-anticlique.

The obtained contradiction shows that $|\Delta_i \cap \Delta_j| \leq 1$.

2. Remind that every vertex from A is adjacent to exactly one vertex from $A(v_0)$. Since $|A| = 4$ and $|A(v_0)| = 5$, certainly there is a vertex v_k which has no neighbours in A. Hence $\Delta_k \subset B$.

3. First we will prove that if $i \neq k$, then $\Delta_i \not\subset B$. Assume the opposite, i.e. that there is an $i \neq k$ with $\Delta_i \subset B$. Since $\Delta_k \cup \Delta_i \subset B$, then the vertices from A are not adjacent to the vertices v_k and v_i, so that every vertex from A is adjacent to exactly one vertex from $A(v_0)\backslash\{v_k, v_i\}$. We have $|A| = 4$ and $|A(v_0)\backslash\{v_k, v_i\}| = 3$. Hence, there are two vertices a and b in A with a common neighbour $v_n \in A(v_0)\backslash\{v_k, v_i\}$.

The vertices a and b must be adjacent because otherwise $\{a, b\} \cup (A(v_0)\backslash\{v_n\})$ will be a 6-anticlique (see property 1 in Lemma 19). Thus $\{a, b, v_n\}$ is a 3-clique. But v_1 is the unique vertex from $A(v_0)$ which is a vertex from a 3-clique, so $n = 1$.

Consequently $k \neq 1$ and $i \neq 1$. Then Δ_k and Δ_i are 4-anticliques. From $|\Delta_k \cap \Delta_i| \leq 1$ (see property 1) it follows that Δ_k and Δ_i are different 4-anticliques in the simple 8-cycle B, so that $\Delta_k \cap \Delta_i = \varnothing$.

This shows that $B = \Delta_k \cup \Delta_i$. From property 1 it follows that Δ_m $(m \neq k, i)$ has at most two elements in B, so that $|\Delta_m \cap A| \geq 2$. The subscript m takes three values and $|A| = 4 < 6$. Consequently, there are such m_1, m_2 that $\Delta_{m_1} \cap \Delta_{m_2} \cap A \neq \varnothing$. This contradicts to property 1 in Lemma 19.

Thus $\Delta_i \not\subset B$ when $i \neq k$, i.e. $\Delta_i \cap A \neq \varnothing$. Since $\Delta_{i_1} \cap \Delta_{i_2} \cap A = \varnothing$ when $i_1 \neq i_2$, then the four sets $\Delta_i \cap A$, $i \neq k$, are 1-element sets, i.e. $|\Delta_i \cap B| = 3$ when $i \neq k$.

4. Assume the opposite. We may suppose that $k = 2$ (see property 2). Then Δ_2 is a 4-anticlique in B. Number the vertices of B consecutively: $B = (c_1, c_2, \ldots, c_7, c_8)$. By property 1, $|\Delta_i \cap \Delta_j| \leq 1$, $i \neq j$; say $i < j$. If the set $B \cap \Delta_i \cap \Delta_j$ is not empty, then we denote by $b_{i,j}$ the unique element of this set. In this way (see property 2 in Lemma 19) every vertex of B has a single notation.

Without loss of generality we may assume that the 4-anticlique Δ_2 consists of vertices $c_1 = b_{2,3}$, $c_3 = b_{2,4}$, $c_5 = b_{2,5}$ and $c_7 = b_{1,2}$. Since no one of the vertices v_2, v_3, v_4, v_5 is a vertex from a 3-clique, then $c_2 = b_{1,5}$ and $c_4 = b_{1,3}$.

According to property 3, $\Delta_1 \cap B = \{c_2, c_4, c_7\}$. Then c_6, $c_8 \notin \Delta_1$; hence $c_6 = b_{3,4}$ and $c_8 = b_{4,5}$.

From property 3 it follows $|\Delta_i \cap A| = 1$ when $i \neq 2$. Let $a_i = \Delta_i \cap A$. It is clear that the vertices a_i are different (see property 1 in Lemma 19).

Obviously, the sets $\{b_{2,4}, b_{3,4}, b_{2,3}, v_1, v_5\}$ and $M = \{a_3, b_{3,4}, b_{2,3}, v_1, v_5\}$ are 5-anticliques (v_0 and v_3 do not belong to 3-cliques). The vertex a_3 is adjacent to $b_{2,4}$ because otherwise $M \cup \{b_{2,4}\}$ will be a 6-anticlique. The vertex a_5 is adjacent to $b_{2,4}$, too. Indeed, the sets $\{b_{2,4}, b_{2,5}, b_{4,5}, v_1, v_3\}$ and $\{a_5, b_{2,5}, b_{4,5}, v_1, v_3\}$ are anticliques; so if a_5 is not adjacent to $b_{2,4}$, we have a 6-anticlique in G.

Thus $A(c_3) \supset \{c_2, c_4, v_2, v_4, a_3, a_5\}$, which contradicts the equality $d(c_3) = 5$.

The proof of the lemma is completed.

Theorem 2. *Every 18-vertex graph without 6-anticliques contains at least two 3-cliques.*

P r o o f. Let G be an 18-vertex graph and $\bar{c}_6(G) = 0$. Assume $c_3(G) \leq 1$. From the equality $R(3,6) = 18$ it follows $c_3(G) = 1$; hence $G \in (18; 1 \times 3, 0 \times \bar{6})$.

We will use the notations introduced in Lemmas 19 and 20.

We know that $\Delta_1 \subset B$ (see properties 2 and 4 in Lemma 20) and Δ_1 contains a unique edge. We may suppose that this edge is $[c_1, c_2]$ (the set B induces a simple 8-cycle in G). Then $T = \{v_1, c_1, c_2\}$ is the unique 3-clique of G and $c_3, c_8 \notin \Delta_1$. At least one of the vertices c_4, c_7 is in Δ_1 (because $|\Delta_1| = 4$ and T is the unique 3-clique of G). By the symmetry we may suppose that $c_4 \in \Delta_1$. Then $c_6 \in \Delta_1$ or $c_7 \in \Delta_1$.

We will prove that $c_7 \in \Delta_1$. Assume the opposite, i.e. $c_6 \in \Delta_1$.

Without loss of generality we assume $c_1 = b_{1,2}$, $c_2 = b_{1,3}$, $c_4 = b_{1,4}$, $c_6 = b_{1,5}$. It is easy to see that $c_3 = b_{2,5}$ and $c_5 = b_{2,3}$. Then $\Delta_2 \cap B = \{c_1, c_3, c_5\}$ (see property 3 in Lemma 20) and $c_7 \notin \Delta_2$. Moreover $c_7 \notin \Delta_1$, $c_7 \notin \Delta_5$ (since $c_6 \in \Delta_5$). This shows that $c_7 = b_{3,4}$. It is easy to see that $c_8 \notin \Delta_i$, $i = 1, 2, 3, 4$ and hence, c_8 can not have two neighbours in $A(v_0)$.

The obtained contradiction shows that $c_7 \in \Delta_1$. Therefore $\Delta_1 = \{c_1, c_2, c_4, c_7\}$. We may suppose that $c_1 = b_{1,2}$, $c_2 = b_{1,3}$, $c_4 = b_{1,4}$, $c_7 = b_{1,5}$. Then $c_3 = b_{2,5}$ and $c_8 = b_{3,4}$. Consequently, the 3-anticlique $\Delta_5 \cap B = \{c_3, c_7, c_5\}$. On the other hand $c_5 \notin (\Delta_1 \cup \Delta_4)$; hence $c_5 = b_{2,5}$ or $c_5 = b_{3,5}$. But we have already $c_3 = b_{2,5}$; hence $c_5 = b_{3,5}$. Finally, we have $c_6 \notin (\Delta_1 \cup \Delta_3 \cup \Delta_5)$ and hence $c_6 = b_{2,4}$.

Let $a_i = \Delta_i \cap A$, $i = 2, 3, 4, 5$ (see property 3 in Lemma 20). The vertices a_i are different (see property 1 in Lemma 19).

It is clear that $M = \{c_4, c_2, c_8, v_2, v_5\}$ and $\{a_3, c_2, c_8, v_2, v_5\}$ are anticliques. If we assume that a_3 and c_4 are not adjacent, then $\{a_3\} \cup M$ will be a 6-anticlique.

Consequently, a_3 is adjacent to c_4. In order to prove in this way that a_2 is adjacent to c_4 , it is sufficient to keep in mind that $\{c_4, c_6, c_1, v_3, v_5\}$ and $\{a_2, c_6, c_1, v_3, v_5\}$ are anticliques. Finally, $A(c_4) \supset \{c_3, c_5, a_2, a_3, v_1, v_4\}$, which contradicts the equality $d(c_4) = 5$.

The theorem is proved.

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