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AN EXTERNAL CHARACTERIZATION OF THE PRIME COMPUTABILITY *

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Иван Н. Сосков. ВНЕШНЯЯ ХАРАКТЕРИЗАЦИЯ ПРОСТОЙ ВЫЧИСЛИМОСТИ
В настоящей работе продолжены исследования автора по максимальной вычислимости в классах алгебраических систем.

Основной результат работы состоит в том, что в любом достаточно богатом классе алгебраических систем простая вычислимость Московакиса является самой сильной среди вычислимостей, которые являются инвариантными, последовательными и обладающими подструктурным свойством.

Ivan N. Soskov. AN EXTERNAL CHARACTERIZATION OF THE PRIME COMPUTABILITY
In the paper we continue the study of the maximal concepts of computability on classes of first order structures.

The main result is that on each rich enough class of denumerable structures the Prime computability of Moschovakis is the strongest among all computabilities which are sequential, invariant and have the substructure property.

In [1] we begin the study of the so called maximal concepts of computability. With these investigations we aim at obtaining a classification of the concepts of "effective" computability on first order structures. The main idea, on which this classification is based, is to consider the behavior of a computability not only on a single structure but on a class of structures. Then one can formulate some properties and in some cases to prove that there exists a strongest computability among the computabilities which have this properties. Such computabilities are called maximal. The results in [1] are connected with the characterization of some maximal non-deterministic computabilities. It is proved there that on each rich

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enough class $\mathcal{A}$ of denumerable structures Search computability of Moschovakis [2] is the strongest among the effective and invariant on $\mathcal{A}$ computabilities and the Computability by means of recursively enumerable definitional schemes (REDS-computability) [3], [4] is the strongest among the effective and invariant on $\mathcal{A}$ computabilities which have the substructure property on $\mathcal{A}$.

The exact definitions of the notions of effectiveness and invariance of a computability and of the substructure property are given in section 1.3 of the present paper.

Here we begin the study of the maximal sequential computabilities. The main result in the paper is that on each rich enough class $\mathcal{A}$ of denumerable structures the Prime computability of Moschovakis [2] is the strongest computability among the sequential and invariant on $\mathcal{A}$ computabilities which have the substructure property on $\mathcal{A}$.

This result follows from an appropriate external characterization of the Prime computability, obtained in section 2.2.

As in [1], all results in the paper are formulated and proved only for (classes of) denumerable partial structures. The problem of generalizing the present results for classes of arbitrary structures is still open.

1. PRELIMINARIES

1.1. Notation and basic definitions.

Let $\mathcal{L} = \{f_1, \ldots, f_n; T_1, \ldots, T_k\}$ be a first order language, here $f_1, \ldots, f_n$ are functional symbols and $T_1, \ldots, T_k$ are predicate symbols. Let each f_i be a_i-ary and let each T_j be b_j-ary.

In what follows we shall consider only partial structures of the language $\mathcal{L}$ with denumerable domains, i.e. structures $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$, where A — the domain of $\mathfrak{A}$ — is a denumerable set of objects, each θ_i is an a_i-ary partial function on A and each Σ_j is a b_j-ary partial predicate on A. The structure $\mathfrak{A}$ will be called total if all initial functions $\theta_1, \ldots, \theta_n$ and all initial predicates $\Sigma_1, \ldots, \Sigma_k$ are totally defined on A.

By $|\mathfrak{A}|$ we shall denote the domain of the structure $\mathfrak{A}$.

Throughout the paper by a structure we shall mean a denumerable partial structure of the language $\mathcal{L}$ and by a total structure we shall mean a total denumerable structure of the language $\mathcal{L}$.

The partial predicates on the domain A of a structure will be identified with the partial mappings which obtain values in $\{0,1\}$, taking 0 for true and 1 for false.

Let $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ and $\mathfrak{B} = (B; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k)$ be structures.

The surjective mapping κ of A onto B is called a *strong homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff the following conditions are fulfilled:

(i) $\kappa(\theta_i(s_1, \ldots, s_{a_i})) \cong \varphi_i(\kappa(s_1), \ldots, \kappa(s_{a_i}))$ for all $s_1, \ldots, s_{a_i}$ of A.

(ii) $\Sigma_j(s_1, \ldots, s_{b_j}) \cong \sigma_j(\kappa(s_1), \ldots, \kappa(s_{b_j}))$ for all $s_1, \ldots, s_{b_j}$ of A.

Obviously if κ is an injective strong homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then κ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

The structure $\mathfrak{A}$ is called a *substructure* of $\mathfrak{B}$ and $\mathfrak{B}$ is called an *extension* of $\mathfrak{A}$ if the following conditions are satisfied:

(i) $A \subseteq B$;

(ii) $\theta_i(s_1,\ldots,s_{a_i})) \cong \varphi_i(s_1,\ldots,s_{a_i})$ for all $s_1, \ldots, s_{a_i}$ of A.

(iii) $\Sigma_j(s_1,\ldots,s_{b_j})) \cong \sigma_j(s_1,\ldots,s_{b_j})$ for all $s_1, \ldots, s_{b_j}$ of A.

By $\mathfrak{A} \subseteq \mathfrak{B}$ we shall denote that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

The structure $\mathfrak{B}$ is said to be a *total extension* of $\mathfrak{A}$, in symbols $\mathfrak{A} \subseteq_t \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and the following two conditions hold:

(i) if $1 \leq i \leq n$, then φ_i is totally defined on $B^{a_i} \setminus A^{a_i}$,

(ii) if $1 \leq j \leq n$, then σ_j is totally defined on $B^{b_j} \setminus A^{b_j}$.

Notice that if $\mathfrak{A}$ is a total structure and $\mathfrak{A} \subseteq_t \mathfrak{B}$, then $\mathfrak{B}$ is also total.

1.2. The μ-recursive operators and the μ-recursive functions.

Let $\mathbb{N}$ be the set of all natural numbers. By $\mathfrak{F}_n$, $n \geq 1$, we shall denote the set of all partial functions of n arguments on $\mathbb{N}$.

Let $n_1, \ldots, n_k$ and m be positive natural numbers. A *μ-recursive operator* of type $(n_1, \ldots, n_k \Rightarrow m)$ is called each total mapping Γ of $\mathfrak{F}_{n_1} \times \mathfrak{F}_{n_2} \times \cdots \times \mathfrak{F}_{n_k}$ into $\mathfrak{F}_m$ such that whenever $\theta_1, \ldots, \theta_k$ are elements of $\mathfrak{F}_{n_1}, \mathfrak{F}_{n_2}, \ldots, \mathfrak{F}_{n_k}$, respectively, then $\Gamma(\theta_1, \ldots, \theta_k)$ is defined uniformly with respect to $\theta_1, \ldots, \theta_k$ trough an explicit expression build up from $\theta_1, \ldots, \theta_k$ and the initial primitive recursive functions by means of the operations substitution, primitive recursion and minimization.

The μ-recursive operators are studied by Skordev in [5], [6], by Sasso in [7] and recently by Cooper [8]. In particular in [6] it is shown that the μ-recursive operators coincide with the Turing computable ones, where an operator Γ is Turing computable if there exists a Turing machine which computes the value of $\Gamma(\theta_1, \ldots, \theta_k)(x_1, \ldots, x_m)$ using oracles for $\theta_1, \ldots, \theta_k$ in a sequential way.

The close connections between the μ-recursive operators and some sequential concepts of computability on first order structures are established in [4].

Here we shall use a definition of the μ-recursive operators which is a reformulation of that one given in [4].

Let us fix the positive natural numbers $n_1, \ldots, n_k$.

Let $R^1, \ldots, R^k$ be new predicate symbols and let each R^i be $n_i + 1$-ary.

The *number theoretic predicates* (n.t. predicates) of type $(n_1, \ldots, n_k)$ are defined by means of the following inductive clauses:

(i) The empty expression Λ is a n.t. predicate;

(ii) Each expression of the form $R^i(z_1, \ldots, z_{n_i}, y)$, where $1 \leq i \leq k$ and $z_1, \ldots, z_{n_i}, y$ are arbitrary natural numbers is a n.t. predicate;

(iii) If E^1 and E^2 are n.t. predicates then so is and $(E^1 \& E^2)$.

We shall assume that $(E^1 \& \Lambda) = E^1$, and $(\Lambda \& E^2) = E^2$.

Let $\theta_1, \ldots, \theta_k$ be elements of $\mathfrak{F}_{n_1}, \mathfrak{F}_{n_2}, \ldots, \mathfrak{F}_{n_k}$, respectively. Then the k-tuple $\theta^* = (\theta_1, \ldots, \theta_k)$ will be called a *functional system* of type $(n_1, \ldots, n_k)$.

Let E be a n.t. predicate of type $(n_1, \ldots, n_k)$, and $\theta^* = (\theta_1, \ldots, \theta_k)$ be a functional system of the same type. Then the value E_{θ^*} of E over θ^* is defined by means of the following inductive clauses:

(i) If $E = \Lambda$, then $E_{\theta^*} \cong 0$;

(ii) If $E = R^i(z_1, \ldots, z_{n_i}, y)$, $1 \leq i \leq k$, then

$$E_{\theta^*} \cong \begin{cases} 0, & \text{if } \theta_i(z_1, \ldots, z_{n_i}) \cong y, \\ 1, & \text{if } \theta_i(z_1, \ldots, z_{n_i}) \text{ is defined and } \theta_i(z_1, \ldots, z_{n_i}) \neq y, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(iii) If $E = (E^1 \& E^2)$, then

$$E_{\theta^*} \cong \begin{cases} 1, & \text{if } E^1_{\theta^*} \cong 1, \\ E^2_{\theta^*}, & \text{if } E^1_{\theta^*} \cong 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Notice that it may happen that $(E^1 \& E^2)_{\theta^*}$ is defined though $E^2_{\theta^*}$ is not defined. On the other hand, $(E^1 \& E^2)_{\theta^*} \cong 0$ iff $E^1_{\theta^*} \cong 0$ and $E^2_{\theta^*} \cong 0$.

Obviously the operation "&" is associative though not commutative. In what follows we shall write the n.t. predicates in the form $E^1 \& E^2 \& \dots \& E^l$, $l \geq 0$, omitting the brackets.

Let E^1 and E^2 be n.t. predicates of type $(n_1, \dots, n_k)$. Then E^1 and E^2 are said to be contrary if

$$E^1 = E^0 \& R^i(z_1, \dots, z_{n_i}, y) \& Q^1 \text{ and } E^2 = E^0 \& R^i(z_1, \dots, z_{n_i}, x) \& Q^2,$$

where $x \neq y$ and E^0, Q^1, Q^2 are n.t. predicates.

Clearly if E^1 and E^2 are contrary, then for each functional system θ^* of type $(n_1, \dots, n_k)$, if $E^1_{\theta^*} \cong 0$, then $E^2_{\theta^*}$ is defined and $E^2_{\theta^*} \cong 1$.

If E is a n.t. predicate of type $(n_1, \dots, n_k)$ and $y \in \mathbb{N}$, then $E \supset y$ will be called a *number theoretic expression* of type $(n_1, \dots, n_k)$.

Assume that an effective coding of the n.t. expressions is fixed. We shall use $(E^v \supset y^v)$ to denote the n.t. expression with code v.

A *treelike number theoretic scheme* of type $(n_1, \dots, n_k)$ will be called each r.e. set $\{E^v \supset y^v\}_{v \in V}$ of n.t. expressions of type $(n_1, \dots, n_k)$ such that if v_1 and v_2 are distinct elements of V, then E^{v_1} and E^{v_2} are contrary.

Given a treelike n.t. scheme $S = \{E^v \supset y^v\}_{v \in V}$ and a functional system θ^* of the same type as S, define the value S_{θ^*} of S over θ^* by the equivalence $S_{\theta^*} \cong y \iff \exists v(v \in V \ \& \ E^v_{\theta^*} \cong 0 \ \& \ y = y^v)$.

Notice that if $S = \{E^v \supset y^v\}_{v \in V}$ is a treelike n.t. scheme, θ^* is a functional system of the same type as S and $S_{\theta^*} \cong y$, then for all $v \in V$, $E^v_{\theta^*}$ is defined.

The following proposition is a reformulation of Theorem 7.11 of [4].

Proposition 1. *The total mapping Γ of $\mathfrak{F}_{n_1} \times \mathfrak{F}_{n_2} \times \dots \times \mathfrak{F}_{n_k}$ into $\mathfrak{F}_m$ is a μ-recursive operator if and only if there exists a recursive function $\gamma(n, x_1, \dots, x_m)$ such that the following is true for all $x_1, \dots, x_m$ of $\mathbb{N}$:*

(i) *For each n, $\gamma(n, x_1, \dots, x_m)$ yields a code of a n.t. expression $E^{\gamma(n,x_1,\dots,x_m)} \supset y^{\gamma(n,x_1,\dots,x_m)}$ of type $(n_1, \dots, n_k)$.*

(ii) *The set $S^{x_1,\dots,x_m} = \left\{E^{\gamma(n,x_1,\dots,x_m)} \supset y^{\gamma(n,x_1,\dots,x_m)}\right\}_{n \in \mathbb{N}}$ is a treelike scheme.*

(iii) *If $\theta^* = (\theta_1, \dots, \theta_k)$ is a functional system of type $(n_1, \dots, n_k)$, then $\Gamma(\theta_1, \dots, \theta_k)(x_1, \dots, x_m) \cong y \iff S^{x_1,\dots,x_m}_{\theta^*} \cong y$.*

For our purposes the following evident consequence of Proposition 1 is sufficient.

Proposition 2. *Let Γ be a μ-recursive operator of type $(n_1, \dots, n_k \Rightarrow m)$. Suppose that $x_1, \dots, x_m$ are fixed elements of $\mathbb{N}$. Then there exists a treelike n.t. scheme S of type $(n_1, \dots, n_k)$ such that whenever $\theta^* = (\theta_1, \dots, \theta_k)$ is a functional system of type $(n_1, \dots, n_k)$, then $\Gamma(\theta_1, \dots, \theta_k)(x_1, \dots, x_m) \cong y \iff S_{\theta^*} \cong y$.*

Let $\mathfrak{B} = (\mathbb{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k)$ be a structure over the natural numbers. We shall use $\mathfrak{B}$ to denote and the functional system $(\varphi_1, \dots, \varphi_n, \sigma_1, \dots, \sigma_k)$ of type $(a_1, \dots, a_n, b_1, \dots, b_k)$.

A partial a-ary function φ on $\mathbb{N}$ is said to be *μ-recursive* in $\mathfrak{B}$ if there exists a μ-recursive operator Γ of type $(a_1, \dots, a_n, b_1, \dots, b_k \Rightarrow a)$ such that $\Gamma(\mathfrak{B}) = \varphi$.

Clearly the μ-recursive in $\mathfrak{B}$ functions are exactly those which can be obtained from the initial primitive recursive functions and from the basic functions and predicates of $\mathfrak{B}$ by means of substitution, primitive recursion and minimization.

We are finishing this section with some words concerning the relationships between the relative μ-recursiveness and the relative partial recursiveness.

Let $\mathfrak{B}$ be a structure and let $|\mathfrak{B}| = \mathbb{N}$. A partial function φ on $\mathbb{N}$ is said to be partial recursive in $\mathfrak{B}$ if $\varphi = \Delta(\mathfrak{B})$, where Δ is a partial recursive operator. For the definitions of the recursive, of the partial recursive operators and of the enumeration operators the reader may consult [9].

Since each μ-recursive operator is partial recursive and even recursive, each μ-recursive in $\mathfrak{B}$ function is partial recursive in $\mathfrak{B}$.

If the structure $\mathfrak{B}$ is total then the μ-recursive in $\mathfrak{B}$ functions coincide with the partial recursive in $\mathfrak{B}$ functions. But there are examples of partial structures in which the μ-recursive functions are a proper subclass of the partial recursive functions. Such examples are given in [10] and [5].

Intuitively, the μ-recursive in $\mathfrak{B}$ functions are those which are computable by means of sequential procedures using the basic functions and predicates of $\mathfrak{B}$ while the partial recursive in $\mathfrak{B}$ functions are those which are computable by means of arbitrary non-deterministic (parallel) procedures using the basic functions and predicates of $\mathfrak{B}$.

Speaking about non-deterministic procedures it seems natural to admit not only single-valued but also partial multiple-valued (p.m.v.) functions to be computable by means of such procedures.

A p.m.v. function φ on $\mathbb{N}$ is said to be partial recursive in the structure $\mathfrak{B}$ iff there exists an enumeration operator Γ such that $\Gamma(\mathfrak{B})$ is the graph of φ.

1.3. Computability on a class of structures.

Let $\mathcal{A}$ be a class of structures. A *computability* on $\mathcal{A}$ is called every mapping C of $\mathcal{A}$ such that if $\mathfrak{A} \in \mathcal{A}$, then $C(\mathfrak{A})$ is a set of p.m.v. functions on $\mathfrak{A}$.

The computability C is said to be *effective* on $\mathcal{A}$ if for each element $\mathfrak{B}$ of $\mathcal{A}$, if $|\mathfrak{B}| = \mathbb{N}$, then all elements of $C(\mathfrak{B})$ are partial recursive in $\mathfrak{B}$ p.m.v. functions.

A computability C is said to be *invariant* on $\mathcal{A}$ iff whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, κ is a strong homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ and $\theta \in C(\mathfrak{A})$, there exists a $\varphi \in C(\mathfrak{B})$ of the same arity as θ such that $\kappa(\varphi(s_1, \dots, s_a)) \cong \theta(\kappa(s_1), \dots, \kappa(s_a))$ for all elements $s_1, \dots, s_a$ of $|\mathfrak{B}|$.

A computability C on $\mathcal{A}$ has the *substructure property* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, $\mathfrak{A} \subseteq \mathfrak{B}$ and $\theta \in C(\mathfrak{A})$, there exists a φ in $C(\mathfrak{B})$ of the same arity as θ and such that $\theta(s_1, \dots, s_a) \cong \varphi(s_1, \dots, s_a)$ for all elements $s_1, \dots, s_a$ of $|\mathfrak{A}|$.

To explain the substructure property we have to think that there exists a computational process which computes the value of θ over the arguments $s_1, \dots, s_a$. Now, the substructure property follows from the assumption that in the course of the computation no additional information but the arguments is needed.

Let C be a computability on $\mathcal{A}$. Then C is said to be *sequential* on $\mathcal{A}$ iff whenever $\mathfrak{B} \in \mathcal{A}$ and $|\mathfrak{B}| = \mathbb{N}$, then all elements of $C(\mathfrak{B})$ are μ-recursive in $\mathfrak{B}$.

Clearly if C is a sequential computability on a class $\mathcal{A}$ of structures, then C is also effective on $\mathcal{A}$.

Let C_1 and C_2 be two computabilities on A. The computability C_1 is said to be *weaker* than C_2 on A, in symbols $C_1 \subseteq_A C_2$, iff for all $\mathfrak{A}$ in A, $C_1(\mathfrak{A}) \subseteq C_2(\mathfrak{A})$.

As we shall see in the next section, if A is a rich enough class of structures, then each sequential and invariant on A computability which has the substructure property on A is weaker than Prime computability on A.

2. PRIME COMPUTABILITY

2.1. Definition and some properties of the Prime computability.

Let $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ be a structure.

Let $\{X_1, X_2, \ldots\}$ be a denumerable set of variables. We shall use the capital letters X, Y, Z to denote variables.

If τ is a term in the language $\mathcal{L}$, then we shall write $\tau(X_1, X_2, \ldots, X_a)$ to denote that all of the variables in τ are among $X_1, X_2, \ldots, X_a$.

If $\tau(X_1, X_2, \ldots, X_a)$ is a term, $s_1, \ldots, s_a$ are arbitrary elements of A, then with $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ we shall denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements $s_1, \ldots, s_a$.

Let T_0 be a new unary predicate symbol and Σ_0 be the total predicate $\lambda s.0$ on A.

Termal predicates in the language $\mathcal{L}$ are defined by the inductive clauses:

If $T \in \{T_0, \ldots, T_k\}$, T is b-ary and $\tau^1, \ldots, \tau^b$ are terms, then any of $T(\tau^1, \ldots, \tau^b)$ and $\neg T(\tau^1, \ldots, \tau^b)$ is a termal predicate;

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(X_1, X_2, \ldots, X_a)$ be a termal predicate whose variables are among X_1, $X_2, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of A. The value $\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Π over $s_1, \ldots, s_a$ in $\mathfrak{A}$ is defined by the inductive clauses:

If $\Pi = T_j(\tau^1, \ldots, \tau^{b_j})$, $0 \leq j \leq k$, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \Sigma_j(\tau^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), \ldots, \tau^{b_j}_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a));$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \begin{cases} 0, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 1, \\ 1, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0, \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \begin{cases} 1, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 1, \\ \Pi^2_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

If Π is a termal predicate and τ is a term, then $Q = (\Pi \supset \tau)$ is called a *conditional term*.

Let $Q(X_1, \ldots, X_a)$ be a conditional term with variables among $X_1, \ldots, X_a$ and let $s_1, \ldots, s_a$ be elements of A. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Q over $s_1, \ldots, s_a$ in $\mathfrak{A}$ is defined by

$$Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong t$$
$$\iff \Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0 \;\&\; \tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong t.$$

Assume that an effective coding of the expressions of the language $\mathcal{L}$ is fixed. By $Q^v = (\Pi^v \supset \tau^v)$ we shall denote the conditional term with code v.

An a-ary partial function θ on A is called *prime computable* on $\mathfrak{A}$ iff for some recursive sequence $\{\Pi^{\rho(n)} \supset \tau^{\rho(n)}\}_{n\in\mathbb{N}}$ of conditional expressions with variables among $Z_1,\ldots,Z_r$, X_1, ..., X_a and for some fixed elements t_1, ..., t_r of A the following equivalence is true for all s_1, ..., s_a and t in A:

$$\theta(s_1,\ldots,s_a) \cong t$$

$$\iff \exists n(n \in \mathbb{N} \;\&\; \Pi_{\mathfrak{A}}^{\rho(n)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong 0$$

$$\& \; \tau_{\mathfrak{A}}^{\rho(n)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong t$$

$$\& \; \forall m(m < n \Rightarrow \Pi_{\mathfrak{A}}^{\rho(m)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong 1)).$$

Notice that each prime computable on $\mathfrak{A}$ function is single-valued.

The original definition of the prime computable functions in [2] looks out somewhat different. There the prime computable functions are defined as a subclass of the p.m.v. functions on the set A^*, where A^* is an appropriate extension of A. However for partial functions on A both definitions are equivalent, see [11] or [12].

Given a structure $\mathfrak{A}$, denote by PC($\mathfrak{A}$) the class of all prime computable on $\mathfrak{A}$ functions.

Proposition 3. *The computability* PC *is sequential, invariant and has the substructure property on each class A of structures.*

Let $Q = (\Pi \supset \tau)$ be a conditional term with variables among X_1, ..., X_a. Then Q is said to be *definite* over the elements s_1, ..., s_a, in symbols $!!Q_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$, iff $\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong 1$ or $(\Pi(X_1/s_1,\ldots,X_a/s_a) \cong 0$ and $\tau_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$ is defined).

An a-ary single-valued partial function θ is said to be *definable* on $\mathfrak{A}$ iff for some r.e. set $\{Q^v\}_{v\in V}$ of conditional terms with variables among Z_1, ..., Z_r; X_1, ..., X_a and some fixed elements t_1, ..., t_r of A the following conditions hold for all elements s_1, ..., s_a and t of A:

(i) If $\theta(s_1,\ldots,s_a)$ is defined then for each v in V,

$$!!Q^v_{\mathfrak{A}}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a).$$

(ii) $\theta(s_1,\ldots,s_a) \cong t$

$$\iff \exists v(v \in V \& Q^v(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong t).$$

Proposition 4. *Each definable on $\mathfrak{A}$ single-valued function is prime computable on $\mathfrak{A}$.*

As we shall see later, the definable on $\mathfrak{A}$ single-valued functions coincide with the prime computable ones.

2.2. External characterisation of the Prime computability.

Let $\mathfrak{A} = (A; \theta_1,\ldots,\theta_n; \Sigma_1,\ldots,\Sigma_k)$ be a structure.

An ordered pair $(\alpha, \mathfrak{B})$ is called *enumeration* of $\mathfrak{A}$ iff α is a partial surjective mapping of $\mathbb{N}$ onto A, $\mathfrak{B} = (\mathbb{N}; \varphi_1,\ldots,\varphi_n; \sigma_1,\ldots,\sigma_k)$ is a structure and the following conditions are fulfilled:

(i) The domain of α (dom(α)) is closed with respect to the partial functions $\varphi_1,\ldots,\varphi_n$;

(ii) $\alpha(\varphi_i(x_1,\ldots,x_{a_i})) \cong \theta_i(\alpha(x_1),\ldots,\alpha(x_{a_i}))$ for all $x_1,\ldots,x_{a_i}$ of dom(α), $1 \leq i \leq n$;

(iii) $\sigma_j(x_1,\ldots,x_{b_j}) \cong \Sigma_j(\alpha(x_1),\ldots,\alpha(x_{b_j}))$ for all $x_1,\ldots,x_{b_j}$ of $\mathrm{dom}(\alpha)$, $1 \leq j \leq k$;

(iv) Each φ_i is totally defined on $\mathbb{N}^{a_i} \setminus (\mathrm{dom}(\alpha))^{a_i}$ and each σ_j is totally defined on $\mathbb{N}^{b_j} \setminus (\mathrm{dom}(\alpha))^{b_j}$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of $\mathfrak{A}$. Denote by $\mathfrak{B}^*$ the structure $(\mathrm{dom}(\alpha); \varphi_1^*,\ldots,\varphi_n^*; \sigma_1^*,\ldots,\sigma_k^*)$, where each φ_i^* is the restriction of φ_i on $\mathrm{dom}(\alpha)$ and each σ_j^* is the restriction of σ_j on $\mathrm{dom}(\alpha)$. It follows from the definition that α is a strong homomorphism from $\mathfrak{B}^*$ to $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq_s \mathfrak{B}$.

A partial a-ary function θ on A is called *μ-admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff for some μ-recursive in $\mathfrak{B}$ partial function φ of a arguments on $\mathbb{N}$ and for all $x_1,\ldots,x_a$ in $\mathrm{dom}(\alpha)$, the following is true:

(i) If $\varphi(x_1,\ldots,x_a) \cong y$, then $y \in \mathrm{dom}(\alpha)$;

(ii) $\theta(\alpha(x_1),\ldots,\alpha(x_a)) \cong \alpha(\varphi(x_1,\ldots,x_a))$.

Here we shall prove the following theorem which gives an external characterization of the prime computability.

Theorem 1. *A partial function θ on A is prime computable on $\mathfrak{A}$ iff it is μ-admissible in all enumerations of $\mathfrak{A}$.*

Similar external characterisations of Search computability and of Computability by means of effectively definable schemes are obtained in [13], [14], [15] and [1].

Let $\langle .,. \rangle$ be an effective coding of the ordered pairs of natural numbers and let $\lambda x.(x)_0$ and $\lambda x.(x)_1$ be recursive functions such that $(\langle x_0, x_1\rangle)_0 = x_0$ and $(\langle x_0, x_1\rangle)_1 = x_1$. If $n \geq 2$, then by $\langle x_0,\ldots,x_n\rangle$ we shall denote $\langle x_0, \langle x_1,\ldots, \langle x_{n-1}, x_n\rangle \ldots\rangle\rangle$.

We shall assume that the coding $\langle .,. \rangle$ is chosen so that $\langle x,y\rangle > x$ and $\langle x,y\rangle > y$. For example, let $\langle x,y\rangle = 2^x 3^y$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of $\mathfrak{A}$. Suppose that $\mathfrak{B} = (\mathbb{N}; \varphi_1,\ldots,\varphi_n; \sigma_1,\ldots,\sigma_k)$. The enumeration $\langle \alpha, \mathfrak{B} \rangle$ is called *special* if whenever $1 \leq i \leq n$ and $\varphi_i(x_1,\ldots,x_{a_i}) \cong y$, then $y = \langle i, j, x_1,\ldots,x_{a_i}\rangle$ for some natural j.

A $n+k+2$-tuple $(H_1, \alpha_1, \varphi_1^1,\ldots,\varphi_n^1, \sigma_1^1,\ldots,\sigma_k^1)$ is called *finite part* (of a special enumeration) iff the following conditions are satisfied:

(i) H_1 is a finite subset of N;

(ii) α_1 is a partial mapping with a finite domain of $\mathbb{N}$ into A, $\mathrm{dom}(\alpha_1) \cap H_1 = \emptyset$;

(iii) Each φ_i^1 is an a_i-ary partial function on $H_1 \cup \mathrm{dom}(\alpha_1)$ and:

(a) $\mathrm{dom}(\alpha_1)$ is closed with respect to $\varphi_1^1,\ldots,\varphi_n^1$;

(b) if $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$, then $y = \langle i, j, x_1,\ldots,x_{a_i}\rangle$ for some natural j;

(c) if $x_1,\ldots,x_{a_i}$ are elements of $\mathrm{dom}(\alpha_1)$ and $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$, then $\theta_i(\alpha_1(x_1),\ldots,\alpha_1(x_{a_i})) \cong \alpha_1(y)$;

(iv) Each σ_j^1 is a partial predicate on $(H_1 \cup \mathrm{dom}(\alpha_1))^{b_j} \setminus (\mathrm{dom}(\alpha_1))^{b_j}$.

Let $\Delta_1 = (H_1, \alpha_1, \varphi_1^1,\ldots,\varphi_n^1, \sigma_1^1,\ldots,\sigma_k^1)$ and $\Delta_2 = (H_2, \alpha_2, \varphi_1^2,\ldots,\varphi_n^2, \sigma_1^2,\ldots,\sigma_k^2)$ be finite parts. Then $\Delta_1 \subseteq \Delta_2$ iff the following conditions are true:

(i) $H_1 \subseteq H_2$ and $\alpha_1 \subseteq \alpha_2$;

(ii) $\varphi_i^1 \subseteq \varphi_i^2$, $i = 1,\ldots,n$, and $\sigma_j^1 \subseteq \sigma_j^2$, $j = 1,\ldots,k$;

(iii) If $\varphi_i^2(x_1,\ldots,x_{a_i}) \cong y$ and $y \in \mathrm{dom}(\alpha_1)$, then $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$.

If $\Delta = (H_1, \alpha_1, \varphi_1^1, \dots, \varphi_n^1, \sigma_1^1, \dots, \sigma_k^1)$ is a finite part and $\langle \alpha, \mathfrak{B} = (\mathsf{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k) \rangle$ is a special enumeration, then $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ iff the following is true:

(i) $H_1 \cap \operatorname{dom}(\alpha) = \varnothing$ and $\alpha_1 \subseteq \alpha$;

(ii) $\varphi_i^1 \subseteq \varphi_i$, $i = 1, \dots, n$, and $\sigma_j^1 \subseteq \sigma_j$, $j = 1, \dots, k$;

(iii) If $\varphi_i(x_1, \dots, x_{a_i}) \cong y$ and $y \in \operatorname{dom}(\alpha_1)$, then $\varphi_i^1(x_1, \dots, x_{a_i}) \cong y$.

The proofs of the following two propositions can be found in [1].

Proposition 5. *Let Δ_1, Δ_2 and Δ_3 be finite parts and let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration. Then $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \Delta_3$ implies $\Delta_1 \subseteq \Delta_3$, and $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$ implies $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$.*

Proposition 6. *Let $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_q \subseteq \cdots$ be a sequence of finite parts. Let $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \dots, \varphi_n^q, \sigma_1^q, \dots \sigma_k^q)$ and suppose that the following is true:*

(i) *for each $s \in A$ there exists a q, such that $s \in \operatorname{range}(\alpha_q)$;*

(ii) *if $1 \leqq i \leqq n$, $x_1, \dots, x_{a_i}$ are elements of $\operatorname{dom}(\alpha_q)$ and $\theta_i(\alpha_q(x_1), \dots, \alpha_q(x_{a_i}))$ is defined, then for some $p \geqq q$, $\varphi_i^p(x_1, \dots, x_{a_i})$ is defined;*

(iii) *if $(x_1, \dots, x_{a_i})$ belongs to $(H_q \cup \operatorname{dom}(\alpha_q))^{a_i} \setminus (\operatorname{dom}(\alpha_q))^{a_i}$, then for some $p \geqq q$, $\varphi_i^p(x_1, \dots, x_{a_i})$ is defined;*

(iv) *for each natural number x there exists a q such that $x \in H_q$ or $x \in \operatorname{dom}(\alpha_q)$.*

Then there exists a special enumeration $\langle \alpha, \mathfrak{B} \rangle$ such that for all natural q, $\Delta_q \subseteq \langle \alpha, \mathfrak{B} \rangle$.

Let val be an effective one to one mapping of the set of all natural numbers onto the set of all variables.

Let N_0 be the set of those natural numbers which are not of the form $\langle i, j, x_1, \dots, x_{a_i} \rangle$, where $1 \leqq i \leqq n$.

Suppose that $\Delta_1 = (H_1, \alpha_1, \varphi_1^1, \varphi_2^1, \dots, \varphi_n^1, \sigma_1^1, \sigma_2^1, \dots, \sigma_k^1)$ is a finite part. Let $\operatorname{dom}(\alpha_1) = \{w_1, \dots, w_r\}$, $\alpha_1(w_i) \cong t_i$ and $\operatorname{val}(w_i) = W_i$, $i = 1, \dots, r$. Let $x_1, \dots, x_a$ be distinct elements of $\mathsf{N}_0 \setminus (H_1 \cup \operatorname{dom}(\alpha_1))$ and let $\operatorname{val}(x_i) = X_i$, $i = 1, \dots, a$.

Proposition 7. *There exists an effective way to define for each n.t. expression $E \supset y$ of type $(a_1, \dots, a_n, b_1, \dots, b_k)$ a conditional term Q with variables among $W_1, \dots, W_r, X_1, \dots, X_a$ such that for all elements $s_1, \dots, s_a$ and t of A, the following conditions are satisfied:*

(1) *If $Q\mathfrak{A}(W_1/t_1, \dots, W_r/t_r, X_1/s_1, \dots, X_a/s_a) \cong t$, then there exists a finite part $\Delta_2 \supseteq \Delta_1$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\alpha(x_1) \cong s_1, \dots, \alpha(x_a) \cong s_a$, $\alpha(y) \cong t$ and $E_{\mathfrak{B}} \cong 0$;*

(2) *If $Q\mathfrak{A}(W_1/t_1, \dots, W_r/t_r, X_1/s_1, \dots, X_a/s_a) \not\cong t$, then at least one of the following is true:*

(2.1) *There exists a finite part $\Delta_2 \supseteq \Delta_1$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $x_1, \dots, x_a$ belong to $\operatorname{dom}(\alpha)$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \operatorname{dom}(\alpha)$;*

(2.2) *For each special enumeration $\langle \alpha, \mathfrak{B} \rangle$, if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_1$, $\alpha(x_1) \cong s_1, \dots, \alpha(x_a) \cong s_a$ and $E_{\mathfrak{B}} \cong 0$, then $\alpha(y) \not\cong t$;*

(3) *If $Q\mathfrak{A}(W_1/t_1, \dots, W_r/t_r, X_1/s_1, \dots, X_a/s_a)$ is not definite, then there exists a finite part $\Delta_2 \supseteq \Delta_1$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$, and $E_{\mathfrak{B}}$ is not defined.*

A detailed proof of this proposition is given in the Appendix.

Now we are ready for the proof of Theorem 1.

Proof of Theorem 1. Let θ be an a-ary partial function on A. If θ is prime computable on $\mathfrak{A}$, then clearly θ is μ-admissible in each enumeration of $\mathfrak{A}$.

Suppose now that θ is not prime computable on $\mathfrak{A}$. We can assume that θ is single-valued. Obviously if θ is not single-valued, then θ is not μ-admissible in any enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$. By Proposition 4, θ is not definable on $\mathfrak{A}$.

Let us fix an enumeration of the μ-recursive operators of type $(a_1, \dots, a_n, b_1, \dots, b_k \Rightarrow a)$. By Γ_n we shall denote the μ-recursive operator with number n.

We shall construct a special enumeration $\langle \alpha^0, \mathfrak{B}^0 \rangle$ such that θ is not admissible in it. The definition of $\langle \alpha^0, \mathfrak{B}^0 \rangle$ will be performed by steps. In each step q we shall define a finite part $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \dots, \varphi_n^q, \sigma_1^q, \dots, \sigma_k^q)$ so that $\Delta_q \subseteq \Delta_{q+1}$. After that we shall define $\langle \alpha^0, \mathfrak{B}^0 \rangle$ as a special enumeration such that $\Delta_q \subseteq \langle \alpha^0, \mathfrak{B}^0 \rangle$ for all q.

We shall consider four kinds of steps. With the first three kinds we shall ensure that the conditions (i) — (iv) of the hypothesis of Proposition 6 are true. With the steps $q = \langle 4m+3, n \rangle$ we shall ensure that if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$ is a special enumeration and $\zeta = \Gamma_n(\mathfrak{B})$, then for some $x_1, \dots, x_a$ of $\mathrm{dom}(\alpha)$, at least one of the following two conditions is not true:

(4) $\quad \zeta(x_1, \dots, x_a) \cong y \Rightarrow y \in \mathrm{dom}(\alpha)$,

(5) $\quad \theta(\alpha(x_1), \dots, \alpha(x_a)) \cong \alpha(\zeta(x_1, \dots, x_a))$.

Let $s_0, s_1, \dots$ be an arbitrary enumeration of A.

Let $\Delta_0 = (\emptyset, \alpha_0, \varphi_1^0, \dots, \varphi_n^0, \sigma_1^0, \dots, \sigma_k^0)$, where $\alpha_0, \varphi_1^0, \dots, \varphi_n^0, \sigma_1^0, \dots, \sigma_k^0$ are totally undefined.

Suppose that $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \dots, \varphi_n^q, \sigma_1^q, \dots, \sigma_k^q)$ is defined. We shall consider the following cases:

1. $(q)_0 = 4m$, for some $m \in \mathbb{N}$. Let x be the first natural number, which does not belong to $\mathrm{dom}(\alpha_q) \cup H_q$ and let s be the first element of the sequence $s_0, s_1, \dots$ which does not belong to $\mathrm{range}(\alpha_q)$. If such s does not exist then let s be an arbitrary element of A. Define $\alpha_{q+1}(x) \cong s$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$ for $z \neq x$ and let $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi_1^q, \dots, \varphi_n^q, \sigma_1^q, \dots, \sigma_k^q)$.

2. $q = \langle 4m+1, \langle i, x_1, \dots, x_{a_i} \rangle \rangle$, where $m \in \mathbb{N}$, $1 \leq i \leq n$, $x_1, \dots, x_{a_i}$ are elements of $\mathrm{dom}(\alpha_q)$, $\varphi_i^q(x_1, \dots, x_{a_i})$ is not defined and $\theta_i(\alpha_q(x_1), \dots, \alpha_q(x_{a_i}))$ is defined. Let $y = \langle i, j, x_1, \dots, x_{a_i} \rangle$ be an element of $\mathbb{N} \setminus (\mathrm{dom}(\alpha_q) \cup H_q)$. Let $\alpha_{q+1}(y) \cong \theta_i(\alpha_q(x_1), \dots, \alpha(x_{a_i}))$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$ for $z \neq x$. Let $\varphi_i^{q+1}(x_1, \dots, x_{a_i}) \cong y$ and $\varphi_i^{q+1}(z_1, \dots, z_{a_i}) \cong \varphi_i^q(z_1, \dots, z_{a_i})$ for $(z_1, \dots, z_{a_i}) \neq (x_1, \dots, x_{a_i})$. Define $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi_1^q, \dots, \varphi_{i-1}^q, \varphi_i^{q+1}, \varphi_{i+1}^q, \dots, \varphi_n^q, \sigma_1^q, \dots, \sigma_k^q)$.

3. $q = \langle 4m+2, \langle i, x_1, \dots, x_{a_i} \rangle \rangle$, where $m \in \mathbb{N}$, $(x_1, \dots, x_{a_i})$ belongs to $(H_q \cup \mathrm{dom}(\alpha_q))^{a_i} \setminus (\mathrm{dom}(\alpha_q))^{a_i}$ and $\varphi_i^q(x_1, \dots, x_{a_i})$ is not defined. Let $y = \langle i, j, x_1, \dots, x_{a_i} \rangle$ be an element of $\mathbb{N} \setminus (\mathrm{dom}(\alpha_q) \cup H_q)$ and let $H_{q+1} = H_q \cup \{y\}$. Let $\varphi_i^{q+1}(x_1, \dots, x_{a_i}) \cong y$ and $\varphi_i^{q+1}(z_1, \dots, z_{a_i}) \cong \varphi_i^q(z_1, \dots, z_{a_i})$ for $(z_1, \dots, z_{a_i}) \neq (x_1, \dots, x_{a_i})$. Let $\Delta_{q+1} = (H_{q+1}, \alpha_q, \varphi_1^q, \dots, \varphi_{i-1}^q, \varphi_i^{q+1}, \varphi_{i+1}^q, \dots, \varphi_n^q, \sigma_1^q, \dots, \sigma_k^q)$.

4. $q = \langle 4m+3, n \rangle$, for some $m, n \in \mathbb{N}$. Let $\mathrm{dom}(\alpha_q) = \{w_1, \dots, w_r\}$, and let us fix some distinct elements $x_1, \dots, x_a$ of $\mathbb{N}_0 \setminus (H_q \cup \mathrm{dom}(\alpha_q))$. Let $\mathrm{val}(w_i) = W_i$, $i = 1, \dots, r$, and $\mathrm{val}(x_i) = X_i$, $i = 1, \dots, a$.

By Proposition 2, there exists a treelike n.t. scheme $S = \{E^v \supset y^v\}_{v \in V}$ such that for each structure $\mathfrak{B}$ over the natural numbers

(6) $$\Gamma_n(\mathfrak{B})(x_1, \dots, x_a) \cong y \Longleftrightarrow \exists v (v \in V \,\&\, E^v_{\mathfrak{B}} \cong 0 \,\&\, y = y^v).$$

By Proposition 7, there exists an effective way to define for each $v \in V$ a conditional term $Q^v(W_1, \ldots, W_r, X_1, \ldots, X_a)$ satisfying the conditions (1), (2) and (3) with respect to $E^v \supset y^v$, $x_1, \ldots, x_a$ and Δ_q. So we obtain the r.e. set $\{Q^v\}_{v \in V}$ of conditional terms.

Let $\alpha(w_i) \cong t_i$, $i = 1, \ldots, r$. Define the a-ary p.m.v. function ζ on A by the equivalence: $t \in \zeta(s_1, \ldots, s_a) \iff \exists v(v \in V \,\&\, Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong t)$.

We shall consider the following cases:

I. For some $s_1, \ldots, s_a$, t of A, $t \in \zeta(s_1, \ldots, s_a)$ but $\theta(s_1, \ldots, s_a) \not\cong t$. Then for some $v \in V$, $Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong t$. By (1), there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$, $\alpha(y^v) \cong t$ and $E_{\mathfrak{B}} \cong 0$.

Let $\Delta_{q+1} = \Delta$. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$ and let $\zeta = \Gamma_n(\mathfrak{B})$. By (6), $\alpha(\zeta(x_1, \ldots, x_a)) \cong \alpha(y^v) \cong t$. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a) \not\cong t$. So in this case (5) is not true.

II. Let for all $s_1, \ldots, s_a$, t of A, if $\zeta(s_1, \ldots, s_a) \cong t$, then $\theta(s_1, \ldots, s_a) \cong t$. Hence ζ is single-valued. We have the following subcases:

a) For some $s_1, \ldots, s_a$ of A, $\zeta(s_1, \ldots, s_a)$ is defined but there exists a $v \in V$ such that Q^v is not definite over $t_1, \ldots, t_r$, $s_1, \ldots, s_a$.

Clearly in this case $\theta(s_1, \ldots, s_a)$ is also defined.

Since Q^v satisfies (3) with respect to $x_1, \ldots, x_a$, $E^v \supset y^v$ and Δ_q, there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $\alpha(x_i) \cong s_i$, $i = 1, \ldots, n$ and $E^v_{\mathfrak{B}}$ is not defined. Let $\Delta_{q+1} = \Delta$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\Gamma_n(\mathfrak{B}) = \zeta$. Using (6) and the fact that S is a treelike scheme, we obtain that $\zeta(x_1, \ldots, x_a)$ is not defined and, hence, $\alpha(\zeta(x_1, \ldots, x_a))$ is not defined. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a)$ and, hence, $\theta(\alpha(x_1), \ldots, \alpha(x_a))$ is defined.

So we receive that the condition (5) is not true.

b) If $\zeta(s_1, \ldots, s_a)$ is defined then for all $v \in V$, $!!Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$. It follows from here that ζ is definable on $\mathfrak{A}$. Since θ is not definable on $\mathfrak{A}$, $\zeta \not\cong \theta$. Hence there exist $s_1, \ldots, s_a$ and t in A such that $\theta(s_1, \ldots, s_a) \cong t$ and $\zeta(s_1, \ldots, s_a) \not\cong t$. By the definition of ζ, for all $v \in V$, $Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \not\cong t$. Therefore for all Q^v, at least one of the conditions (2.1) and (2.2) is true with respect to $x_1, \ldots, x_a$, $E^v \supset y^v$ and Δ_q.

Suppose that there exists a $v \in V$ such that (2.1) is true for Q^v. Then there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $x_1, \ldots, x_a$ belong to $\mathrm{dom}(\alpha)$, $E^v_{\mathfrak{B}} \cong 0$ and $y^v \notin \mathrm{dom}(\alpha)$. Let $\Delta_{q+1} = \Delta$. Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\zeta = \Gamma_n(\mathfrak{B})$. By (6), $\zeta(x_1, \ldots, x_a) \cong y^v$ but $y^v \notin \mathrm{dom}(\alpha)$. So the condition (4) fails.

Let us suppose that for all Q^v, (2.2) is true.

Let $\alpha_{q+1}(x_1) \cong s_1, \ldots, \alpha_{q+1}(x_a) \cong s_a$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$, for $z \notin \{x_1, \ldots, x_a\}$. Let $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi_1^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots, \sigma_k^q)$. Clearly Δ_{q+1} is a finite part and $\Delta_{q+1} \supseteq \Delta_q$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\Gamma_n(\mathfrak{B}) = \zeta$. Since $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$, $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$ and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_q$. Then, by (2.2), if $v \in V$ and $E^v_{\mathfrak{B}} \cong 0$, then $\alpha(y^v) \not\cong t$. Therefore $\alpha(\zeta(x_1, \ldots, x_a)) \not\cong t$. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a) \cong t$. From here it follows that the condition (5) fails.

Define $\Delta_{q+1} = \Delta_q$ in the other cases.

Let $\langle\alpha^0, \mathfrak{B}^0\rangle$ be a special enumeration such that $\langle\alpha^0, \mathfrak{B}^0\rangle \supseteq \Delta_q$ for all q. Let us suppose that θ is μ-admissible in $\langle\alpha^0, \mathfrak{B}^0\rangle$. Then for some μ-recursive in $\mathfrak{B}$ function ζ (4) and (5) are true for all $x_1, \ldots, x_a \in \mathrm{dom}(\alpha^0)$. Let Γ_n be the μ-recursive operator of type $(a_1, \ldots, a_n; b_1, \ldots, b_k \Rightarrow a)$ such that $\Gamma_n(\mathfrak{B}) = \zeta$.

Since $\langle\alpha^0, \mathfrak{B}^0\rangle \supseteq \Delta_{\langle 3, n\rangle+1}$, at least one of (4) and (5) is not true for some $x_1, \ldots, x_a$ of $\mathrm{dom}(\alpha^0)$. A contradiction.

2.3. The main result

A class $\mathcal{A}$ of structures is *closed under homomorphic counter-images* iff whenever $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}$ is a structure and there exists a strong homomorphism κ from $\mathfrak{B}$ to $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{A}$.

A class $\mathcal{A}$ of structures is *closed under total extensions* iff whenever $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}$ is a structure and $\mathfrak{A} \subseteq_t \mathfrak{B}$, then $\mathfrak{B} \in \mathcal{A}$.

Two natural examples of closed under homomorphic counter-images and under total extensions classes of structures are the class $\mathcal{A}_0$ of all structures and the class $\mathcal{A}_t$ of all total structures.

Theorem 2. *Let $\mathcal{A}$ be a class of structures and let $\mathcal{A}$ be closed with respect to homomorphic counter-images and total extensions. Let C be a computability on $\mathcal{A}$ which is sequential, invariant and let C have the substructure property on $\mathcal{A}$. Then $C \subseteq_{\mathcal{A}} \mathrm{PC}$.*

Proof. Let $\mathfrak{A}$ be an element of $\mathcal{A}$ and let $\theta \in C(\mathfrak{A})$. By Theorem 1, to prove that $\theta \in \mathrm{PC}(\mathfrak{A})$ it is sufficient to show that θ is μ-admissible in all enumerations of $\mathfrak{A}$. Let $\langle\alpha, \mathfrak{B} = (\mathbb{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k)\rangle$ be an enumeration of $\mathfrak{A}$. Denote by φ_i^* the restriction of φ_i on $\mathrm{dom}(\alpha)$, $i = 1, \ldots, n$ and by σ_j^* the restriction of σ_j on $\mathrm{dom}(\alpha)$. Let $\mathfrak{B}^* = (\mathrm{dom}(\alpha); \varphi_1^*, \ldots, \varphi_n^*; \sigma_1^*, \ldots, \sigma_k^*)$. From the definition of the notion of an enumeration of $\mathfrak{A}$ it follows that α is a strong homomorphism from $\mathfrak{B}^*$ to $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq_t \mathfrak{B}$. Then $\mathfrak{B}^*$ and $\mathfrak{B}$ are elements of $\mathcal{A}$. By the invariance of C, there exists a φ^* in $C(\mathfrak{B}^*)$ such that for all $x_1, \ldots x_a$ of $\mathrm{dom}(\alpha)$, $\alpha(\varphi^*(x_1, \ldots, x_a)) \cong \theta(\alpha(x_1), \ldots, \alpha(x_a))$. From the substructure property of C it follows that there exists an a-ary function φ in $C(\mathfrak{B})$ such that for all $x_1, \ldots, x_a$ of $\mathrm{dom}(\alpha)$, $\varphi(x_1, \ldots, x_a) \cong \varphi^*(x_1, \ldots, x_a)$. Finally, since C is sequential, φ is μ-recursive in $\mathfrak{B}$. So, we obtain that θ is μ-admissible in $\langle\alpha, \mathfrak{B}\rangle$.

This result should be compared with Theorem 4 in [1], which states that if $\mathcal{A}$ is a closed under homomorphic counter-images and total extensions class of structures, then each effective and invariant on $\mathcal{A}$ computability C which has the substructure property on $\mathcal{A}$ is weaker than REDS-computability.

So we obtain that in some sense Prime computability is the sequential counterpart of REDS-computability. Namely both computabilities have the same model-theoretic properties but the first is sequential while the second is non-deterministic.

From the above results it follows that on each total structure the prime computable functions coincide with the single-valued REDS-computable functions. An example of a partial structure on which the prime computable functions are a proper subclass of the single-valued REDS-computable functions is given in [16].

APPENDIX

Here we shall give a detailed proof of Proposition 7. The enumeration of the propositions in the Appendix will be independent of that one used in the main text.

We shall suppose fixed a structure $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n, \Sigma_1, \ldots, \Sigma_k)$ and a finite part $\Delta = (H_1, \alpha_1, \varphi_1^1, \ldots, \varphi_n^1, \sigma_1^1, \ldots, \sigma_k^1)$.

Let $\mathrm{dom}(\alpha_1) = \{w_1, \ldots, w_r\}$ and let $x_1, \ldots, x_a$ be distinct elements of $\mathbb{N}_0 \setminus (H_1 \cup \mathrm{dom}(\alpha_1))$.

Recall that $\mathfrak{A}$ is a structure of the first order language $\mathcal{L} = (f_1, \ldots, f_n; T_1, \ldots, T_k)$, where each f_i is a_i-ary and each T_j is b_j-ary.

A finite subset L of $\mathbb{N}^3$ will be called a *termal tree* if the following conditions are true:

(i) If $(i, z, y) \in L$, then $1 \leq i \leq n$, $z = \langle z_1, \ldots, z_{a_i} \rangle$ for some elements $z_1, \ldots, z_{a_i}$ of $\mathbb{N}$ and $y = \langle i, j, z \rangle$ for some $j \in \mathbb{N}$.

(ii) If $(i, z, y_1) \in L$ and $(i, z, y_2) \in L$, then $y_1 = y_2$.

(iii) If $(i, z, y) \in L$, then $y \notin \mathrm{dom}(\alpha_1)$.

(iv) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, and $\varphi_i^1(z_1, \ldots, z_{a_i})$ is defined, then $\varphi_i^1(z_1, \ldots, z_{a_i}) \cong y$.

Let us fix a termal tree L.

Immediate consequences of the condition (i) of the definition and of the choice of the coding function $\langle.,.\rangle$ are:

(T1) If (i_1, z_1, y) and (i_2, z_2, y) are elements of L, then $i_1 = i_2$ and $z_1 = z_2$.

(T2) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $z_j < y$, $j = 1, \ldots, a_i$.

If $(i, z, y) \in L$, then z is called a *L-predecessor* of y. Those natural numbers which have not L-predecessors will be called *L-prime.*

Notice that by (iii) of the definition and by the choice of $x_1, \ldots x_a$, all $w_1, \ldots, w_r, x_1, \ldots, x_a$ are L-prime.

For each natural number y, let $|y|_L = 0$, if y is L-prime, and let $|y|_L = |z_1|_L + \cdots + |z_{a_i}|_L + 1$, if $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$.

Let $y \in \mathbb{N}$. Define the finite subset $[y]$ of $\mathbb{N}$ by means of the following inductive clauses:

If $|y|_L = 0$, then $[y] = \{y\}$.

If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $[y] = \{y\} \cup [z_1] \cup \ldots \cup [z_{a_i}]$.

Define the binary relation "$\leq_L$" on $\mathbb{N}$ by the equivalence $z \leq_L y \iff z \in [y]$.

Now we shall list some properties of the relation "$\leq_L$" which follow easily from the definition.

(R0) If $|y|_L = 0$ and $z \leq_L y$, then $z = y$.

(R1) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $z_j \leq_L y$, $j = 1, \ldots, a_i$.

(R2) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$ and $u \leq_L y$, then $u = y$ or $\exists j(1 \leq j \leq a_i \;\&\; u \leq_L z_j)$.

(R3) $y \leq_L y$.

(R4) $z \leq_L y \;\&\; y \leq_L u \Rightarrow z \leq_L u$.

(R5) $z \leq_L y \;\&\; y \leq_L z \Rightarrow y = z$.

Let $y \in \mathbb{N}$. The subset $(L)_y$ of L is defined by the equivalence

$$(i, z, u) \in (L)_y \iff (i, z, u) \in L \;\&\; u \leq_L y.$$

We have the following properties of $(L)_y$.

(P1) $(L)_y = \varnothing \iff |y|_L = 0$.

(P2) If $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$, then $(L)_y = \{(i, \langle z_1, \ldots, z_{a_i}\rangle, y)\} \cup (L)_{z_1} \cup \ldots \cup (L)_{z_{a_i}}$.

If $u = \langle z_1, \ldots, z_{a_i}\rangle$ and $1 \leqq j \leqq a_i$, then we shall use $(u)_j$ to denote z_j.

(P3) $z \leq_L y \iff z = y$ or $\exists i \exists j \exists u \exists v (1 \leqq i \leqq n \;\&\; 1 \leqq j \leqq a_i \;\&\; z = (u)_j \;\&\; (i, u, v) \in (L)_y)$.

P r o o f. In the right to left direction (P3) follows from (R1), (R3), (R4) and from the definition of $(L)_y$.

The proof in the other direction is by induction on $|y|_L$.

If $|y|_L = 0$, then (P3) follows from (R0).

Let $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$. Suppose that $z \leq_L y$ and $z \neq y$, then, by (R2), $z \leq_L z_j$ for some j, $1 \leqq j \leqq a_i$. If $z = z_j$, then let $u = \langle z_1, \ldots, z_{a_i}\rangle$ and $y = v$. Clearly $z = (u)_j$. Otherwise, by the induction hypothesis, there exist i_0, j_0, u and v such that $1 \leqq i_0 \leqq n$, $1 \leqq j_0 \leqq a_{i_0}$, $z = (u)_{j_0}$ and $(i_0, u, v) \in (L)_{z_j}$. But, by (P2), $(L)_{z_j} \subseteq (L)_y$.

Let $\mathcal{T}(L)$ consists of all natural numbers y satisfying the condition: if $z \leq_L y$ and $|z|_L = 0$, then $z \in \{w_1, \ldots, w_r, x_1, \ldots, x_a\}$.

($\mathcal{T}$1) If $|y|_L = 0$, then $y \in \mathcal{T}(L) \iff y \in \{w_1, \ldots, w_r, x_1, \ldots, x_a\}$.

($\mathcal{T}$2) If $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$, then $y \in \mathcal{T}(L)$ iff $\forall j (1 \leqq j \leqq a_i \Rightarrow z_j \in \mathcal{T}(L))$.

Let $\mathrm{val}(w_i) = W_i$, $i = 1, \ldots, r$, and $\mathrm{val}(x_i) = X_i$, $i = 1, \ldots, a$.

For each $y \in \mathcal{T}(L)$, the term $\tau(y, L)$ of the language $\mathcal{L}$ is defined by means of the following inductive clauses:

If $|y|_L = 0$, then $\tau(y, L) = \mathrm{val}(y)$.

If $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$, then $\tau(y, L) = f_i(\tau(z_1, L), \ldots, \tau(z_{a_i}, L))$.

Notice that all variables which occur in $\tau(y, L)$ are among $W_1, \ldots, W_r$, $X_1, \ldots, X_a$.

Proposition 1. *Let F be a termal tree and $L \subseteq F$. Then the following is true for all natural numbers z and y:*

(a) *L is a termal tree.*

(b) $z \leq_L y \Rightarrow z \leq_F y$.

(c) $(L)_y \subseteq (F)_y$.

(d) $\mathcal{T}(L) \subseteq \mathcal{T}(F)$.

P r o o f. (a) is obvious. The proof of (b) is by induction on $|y|_L$. If $|y|_L = 0$ and $z \leq_L y$, then $z = y$. Hence $z \leq_F y$. Suppose that $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$. Then $z = y$ or for some j, $z \leq_L z_j$. Clearly $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in F$. From here using the induction hypothesis we obtain that $z \leq_F y$.

Now (c) follows easily from (b).

Let $y \in \mathcal{T}(L)$. We shall prove that $y \in \mathcal{T}(F)$ by induction on $|y|_L$. If $|y|_L = 0$ this follows from ($\mathcal{T}$1). Let $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$. Then, by ($\mathcal{T}$ 2), $z_j \in \mathcal{T}(L)$, $j = 1, \ldots, a_i$. By the induction hypothesis, $z_j \in \mathcal{T}(F)$, $j = 1, \ldots, a_i$. Clearly $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in F$. Then using once more ($\mathcal{T}$2) we obtain that $y \in \mathcal{T}(F)$.

Proposition 2. *Let L and F be termal trees and $(L)_y = (F)_y$. Then the following assertions are true for all natural z:*

(a) $z \leq_L y \iff z \leq_F y$.

(b) $z \leq_L y \Rightarrow (L)_z = (F)_z$.

(c) $y \in \mathcal{T}(L) \iff y \in \mathcal{T}(F)$.

(d) *If $y \in \mathcal{T}(L)$, then $\tau(y, L) = \tau(y, F)$.*

P r o o f. (a) follows from (P3).

Let $z \leq_L y$. We shall prove $(L)_z = (F)_z$ by induction on $|z|_L$. Let $|z|_L = 0$. Then $(L)_z = \emptyset$. Suppose that $(F)_z \neq \emptyset$. Then z is not F-prime and, hence, for some i and u, $(i,u,z) \in F$. By (a), $(i,u,z) \in (F)_y$ and, hence, $(i,u,z) \in L$. The last contradicts the fact that $|z|_L = 0$. Thus $(F)_z = \emptyset$. Suppose now that $(i,\langle z_1,\ldots,z_{a_i}\rangle,z) \in L$. By the induction hypothesis, $(L)_{z_j} = (F)_{z_j}$, $j = 1,\ldots,a_i$. Clearly, $(i,\langle z_1,\ldots,z_{a_i}\rangle,z) \in (L)_y$ and, hence, $(i,\langle z_1,\ldots,z_{a_i}\rangle,z) \in F$. Then $(L)_z = (F)_z$ follows from (P2).

The proof of (c) is by induction on $|y|_L$. Let $|y|_L = 0$. Then since $\emptyset = (L)_y = (F)_y$, $|y|_F = 0$. We have $y \in \mathcal{T}(L)$ iff $y \in \{w_1,\ldots,w_r,x_1,\ldots,x_a\}$ iff $y \in \mathcal{T}(F)$. Let $(i,\langle z_1,\ldots,z_{a_i}\rangle,y) \in L$. By (b), $(L)_{z_j} = (F)_{z_j}$, $j = 1,\ldots,a_i$. Clearly $(i,\langle z_1,\ldots,z_{a_i}\rangle,y) \in F$. Then, by the induction hypothesis and by ($\mathcal{T}$ 2), $y \in \mathcal{T}(L) \iff \forall j(1 \leqq j \leqq a_i \Rightarrow z_j \in \mathcal{T}(L)) \iff \forall j(1 \leqq j \leqq a_i \Rightarrow z_j \in \mathcal{T}(F)) \iff y \in \mathcal{T}(F)$.

The proof of (d) proceeds in a similar way.

Proposition 3. *Let F be a termal tree, $L \subseteq F$ and $S = L \cup (F)_y$. Then* $(S)_y = (F)_y$.

P r o o f. Clearly $S \subseteq F$ and, hence, $(S)_y \subseteq (F)_y$. By means of induction on $|z|_F$ we shall show that if $z \leq_F y$, then $(F)_z \subseteq (S)_z$. Let $|z|_F = 0$. Then $(F)_z = \emptyset$. Let $(i,\langle z_1,\ldots,z_{a_i}\rangle,z) \in F$ and $z \leq_F y$. Then $(i,\langle z_1,\ldots,z_{a_i}\rangle,z) \in (F)_y$ and, hence, $(i,\langle z_1,\ldots,z_{a_i}\rangle,z)$ belongs to S. From here $(F)_z \subseteq (S)_z$ follows from the induction hypothesis and (P2).

Let F be a termal tree. Then L is called a *subtree* of F, in symbols $L \prec F$, iff $L \subseteq F$ and

(ST) If $y \in \mathcal{T}(F)$, $(L)_y \neq (F)_y$ and $(i,z,y) \in F$, then $(i,z,y) \notin L$.

Proposition 4. *Let $L \prec F$ and $y \in \mathcal{T}(L)$. Then $(L)_y = (F)_y$.*

P r o o f. If y is F-prime then $(L)_y = (F)_y = \emptyset$. Suppose that $(i,z,y) \in F$. Assume that $(L)_y \neq (F)_y$. Clearly, $y \in \mathcal{T}(F)$. Then, by (ST), $(i,z,y) \notin L$. On the other hand, $y \notin \{w_1,\ldots,w_r,x_1,\ldots,x_a\}$ and $y \in \mathcal{T}(L)$. Then for some i_1 and z_1, $(i_1,z_1,y) \in L$. Since $L \subseteq F$, $i_1 = i$ and $z_1 = z$. Hence $(i,z,y) \in L$. A contradiction.

Proposition 5. *Let $L \prec F$ and $S = L \cup (F)_y$. Then $S \prec F$.*

P r o o f. Let $z \in \mathcal{T}(F)$, $(F)_z \neq (S)_z$ and $(i,u,z) \in F$. It should be clear that $(i,u,z) \notin (F)_y$. Since $(L)_z \subseteq (S)_z$, $(L)_z \neq (F)_z$. Then $(i,u,z) \notin L$. Thus $(i,u,z) \notin S$.

Let $R^1,\ldots,R^{n+k}$ be new predicate symbols. We shall suppose that each R^i is c_i+1-ary, where if $1 \leqq i \leqq n$, then $c_i = a_i$ and if $n+1 \leqq i \leqq n+k$, then $c_i = b_{i-n}$. Let E be a n.t. predicate of type $(a_1,\ldots,a_n,\ b_1,\ldots,b_k)$. Then E can be written in the form $P^1 \,\&\ldots\&\, P^m$, where $m \geqq 0$ and each P^l is *atomic* i.e. P^l is in the form $R^i(z_1,\ldots,z_{c_i},y)$ for some i, $1 \leqq i \leqq n+k$, and some $z_1,\ldots,z_{c_i}$, y of $\mathbb{N}$.

In what follows we shall consider only n.t. predicates and n.t. expressions of type $(a_1,\ldots,a_n,\ b_1,\ldots,b_k)$ without stating this explicit each time. We shall use $\langle \alpha,\ \mathfrak{B}\rangle$ to denote special enumerations of $\mathfrak{A}$. Given an enumeration $\langle \alpha,\ \mathfrak{B}\rangle$, we shall suppose that the structure $\mathfrak{B}$ is in the form $(\mathbb{N};\ \varphi_1,\ldots,\varphi_n,\ \sigma_1,\ldots,\sigma_k)$. By P^l, $l = 1,2,\ldots$ we shall denote atomic n.t. predicates.

Let $E = P^1 \,\&\ldots\&\, P^m$ be a n.t. predicate. Then E is said to be *correct* iff the following conditions are true for $l = 1,\ldots,m$:

1. If $P^l = R^i(z_1,\ldots,z_{c_i},y)$, $1 \leqq i \leqq n$, then

1.1. $y = \langle i,j,\langle z_1,\ldots,z_{c_i}\rangle\rangle$, for some $j \in \mathbb{N}$;

1.2. If $\varphi_i^1(z_1,\ldots,z_{c_i})$ is defined, then $\varphi_i^1(z_1,\ldots,z_{c_i}) \cong y$;

1.3. If $y \in \mathrm{dom}(\alpha_1)$, then $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$.

2. If $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, $n+1 \leqq i \leqq n+k$ then

2.1. $y \in \{0, 1\}$;

2.2. If $\sigma_{i-n}^1(z_1, \ldots, z_{c_i})$ is defined, then $\sigma_{i-n}^1(z_1, \ldots, z_{c_i}) \cong y$;

3. If $1 \leqq l_1 < l_2 \leqq m$, $P^{l_1} = R^i(z_1, \ldots, z_{c_i}, y_1)$ and $P^{l_2} = R^i(z_1, \ldots, z_{c_i}, y_2)$, then $y_1 = y_2$.

Clearly there exists an effective way of recognizing for each n.t. predicate E whether E is correct or not.

Proposition 6. *Let E be a n.t. predicate. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ and $E_{\mathfrak{B}} \cong 0$. Then E is correct.*

Proof. Let $E = P^1 \& \ldots \& P^m$. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ and $E_{\mathfrak{B}} \cong 0$. Let $1 \leqq l \leqq m$ and $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, where $1 \leqq i \leqq n$. Suppose that $y \in \mathrm{dom}(\alpha_1)$. Since $E_{\mathfrak{B}} \cong 0$, $P^l_{\mathfrak{B}} \cong 0$ and, hence, $\varphi_i(z_1, \ldots, z_{c_i}) \cong y$. Then by the definition of the relation "$\subseteq$" between finite parts and enumerations, $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$. So we obtain that the condition 1.3. is true. The other conditions are obvious.

A n.t. predicate $E = P^1 \& \ldots \& P^m$ is called *simple* iff whenever $1 \leqq l \leqq m$ and $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, then $n+1 \leqq i \leqq n+k$.

For each n.t. predicate E, $|E|$ is defined by the inductive clauses:

If E is simple, then $|E| = 0$;

If $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where $1 \leqq i \leqq n$, then $|E| = |E^1| + |E^2| + 1$.

Clearly if $|E| > 0$, then there exist unique n.t. predicates E^1, $R^i(z_1, \ldots, z_{c_i}, y)$ and E^2 such that E^2 is simple, $1 \leqq i \leqq n$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$.

Let us fix a correct n.t. predicate $E = P^1 \& \ldots \& P^m$. Define the finite subset L_E of $\mathbb{N}^3$ by the equivalence $(i, z, y) \in L_E \iff \exists i \exists z_1 \ldots \exists z_{c_i} \exists l (1 \leqq i \leqq n \,\&\, 1 \leqq l \leqq m \,\&\, z = \langle z_1, \ldots, z_{c_i} \rangle \,\&\, P^l = R^i(z_1, \ldots, z_{c_i}, y) \,\&\, y \notin \mathrm{dom}(\alpha_1))$.

It follows easily from the correctness of E that L_E is a termal tree.

A termal tree L is called *E-consistent* iff the following two conditions are true:

a) $L \cup L_E$ is a termal tree;

b) L is a subtree of $L \cup L_E$.

Notice that if $L = \emptyset$, then L is E-consistent. Notice also that if E is simple, then each termal tree is E-consistent.

Suppose now that $|E| > 0$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where $1 \leqq i \leqq n$ and E^2 is simple. Let L be an E-consistent termal tree. Denote $L \cup L_E$ by F and let $G = L \cup L_{E^1}$ and $S = L \cup (F)_y$.

Notice that $L \prec F$ and $S \prec F$. Notice also that $G \cup S = F$.

Proposition 7. *If $u \leq_F y$, and $u \neq y$, then $(G)_u = (F)_u$.*

Proof. Clearly $(G)_u \subseteq (F)_u$ for all u. Let $u \leq_F y$ and $u \neq y$. We shall show $(F)_u \subseteq (G)_u$ by induction on $|u|_F$. This is obvious if $|u|_F = 0$. Suppose that $(k, \langle u_1, \ldots, u_{c_k} \rangle, u) \in F$. Since $u \neq y$, $(k, \langle u_1, \ldots, u_{c_k} \rangle u) \in G$. Clearly $u_j \leq_F y$, $j = 1, \ldots, k$. Assume that for some j, $u_j = y$. Then $y \leq_F u$ and, hence, $y = u$. A contradiction. From here the inclusion $(F)_u \subseteq (G)_u$ follows from the induction hypothesis and (P2).

Proposition 8. *L is E^1-consistent.*

Proof. It is sufficient to show that $L \prec G$. Let $u \in T(G)$, $(L)_u \neq (G)_u$ and $(k, x, u) \in G$. Clearly $(L)_u \subset (G)_u \subseteq (F)_u$. Therefore $(L)_u \neq (F)_u$. Then, since $L \prec F$ and $(k, x, u) \in F$, $(k, x, u) \notin L$.

Proposition 9. *S is $E^1 \& E^2$-consistent.*

Proof. $S \cup L_{E^1 \& E^2} = F$.

Proposition 10. *If $u \in \mathcal{T}(S)$ and $u \neq y$, then $(S)_u = (G)_u$.*

Proof. Let $u \in \mathcal{T}(S)$. Since $S \prec F$, $(S)_u = (F)_u$. We shall show that if $u \neq y$, then $(G)_u = (F)_u$. If u is F-prime, then obviously $(G)_u = (F)_u = \emptyset$. Let $(k, x, u) \in F$. Clearly $(k, x, u) \in (F)_u$. Since, $(S)_u = (F)_u$, $(k, x, u) \in S$.

Let $u \neq y$. If $u \leq_F y$, then $(F)_u = (G)_u$ by Proposition 7. Otherwise, $(k, x, u) \notin (F)_y$ and, hence, $(k, x, u) \in L$. We have that $u \in \mathcal{T}(S)$ and, hence, $u \in \mathcal{T}(F)$. Since $L \prec F$, $(L)_u = (F)_u$. But $(L)_u \subseteq (G)_u \subseteq (F)_u$. Thus $(G)_u = (F)_u$.

Corollary 10.1. *If $u \in \mathcal{T}(S)$ and $u \neq y$, then $u \in \mathcal{T}(G)$ and $\tau(u, S) = \tau(u, G)$.*

Corollary 10.2. *If $y \in \mathcal{T}(S)$ and $y \notin \mathrm{dom}(\alpha_1)$, then $(S)_{z_j} = (G)_{z_j}$, $j = 1, \ldots, c_i$, and $\tau(y, S) = f_i(\tau(z_1, G), \ldots, \tau(z_{c_i}, G))$.*

Now we shall describe an algorithm $\mathcal{R}$ which transforms each correct n.t. predicate E and each E-consistent termal tree L into a termal predicate $\mathcal{R}(E, L)$.

Let $E = P^1 \& \ldots \& P^m$ be a correct n.t. predicate and let L be an E-consistent tree. We shall consider the following cases:

1. Let E be simple. Let $l_1 < l_2 < \ldots < l_h$ be all elements of $\{1, \ldots, m\}$ such that if $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then all $z_1, \ldots, z_{c_i}$ are elements of $\mathcal{T}(L)$.

Let $1 \leqq j \leqq h$ and let $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$. Clearly $y \in \{0, 1\}$ and $n + 1 \leqq i \leqq n + k$. Let $\Pi^{l_j} = T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 0$ and $\Pi^{l_j} = \neg T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 1$.

Let $\mathcal{R}(E, L) = \Pi^{l_1} \& \ldots \& \Pi^{l_h}$, if $h \geqq 1$ and let $\mathcal{R}(E, L) = T_0(X_1)$; otherwise.

2. Let $|E| > 0$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where E^2 is simple and $1 \leqq i \leqq n$. Denote by S the termal tree $L \cup (L \cup L_E)_y$. Clearly L is E^1-consistent and S is $E^1 \& E^2$-consistent.

Let $\mathcal{R}(E, L) = \mathcal{R}(E^1, L) \& \mathcal{R}(E^1 \& E^2, S)$, if $y \notin \mathcal{T}(S)$ and let $\mathcal{R}(E, L) = \mathcal{R}(E^1, L) \& T_0(\tau(y, S)) \& \mathcal{R}(E^1 \& E^2, S)$, if $y \in \mathcal{T}(S)$.

Here T_0 is the unary predicate symbol intended to represent the total unary predicate $\Sigma_0 = \lambda s.0$.

A simple induction on $|E|$ shows that whenever E is a correct n.t. predicate and L is an E-consistent termal tree, then $\mathcal{R}(E, L)$ is defined and $\mathcal{R}(E, L)$ is a termal predicate with variables among $W_1, \ldots, W_r$, $X_1, \ldots, X_a$.

If $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and L is a termal tree, then $\langle \alpha, \mathfrak{B} \rangle$ satisfies L, in symbols $\langle \alpha, \mathfrak{B} \rangle \vdash L$, iff $x_1, \ldots, x_a$ are elements of $\mathrm{dom}(\alpha)$ and whenever $(i, (z_1, \ldots, z_{c_i}), y) \in L$, then $\varphi_i(z_1, \ldots, z_{c_i}) \cong y$.

Let us suppose that $\alpha_1(w_i) \cong t_i$, $i = 1, \ldots, r$.

Proposition 11. *Let L be a termal tree, $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ and $\langle \alpha, \mathfrak{B} \rangle \vdash L$. Then $\mathcal{T}(L) \subseteq \mathrm{dom}(\alpha)$ and if $u \in \mathcal{T}(L)$, then*

$$(11.1) \qquad \alpha(u) \cong \tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)).$$

Proof. By means of induction on $|u|_L$ one can show that if $u \in \mathcal{T}(L)$, then $u \in \mathrm{dom}(\alpha)$ and (11.1) is true.

Let us fix a correct n.t. predicate E, an E-consistent termal tree L and let K be a finite subset of $\mathbb{N}$ containing all natural numbers which occur in E, all elements of $\{w_1, \ldots, w_r, x_1, \ldots, x_a\}$ and such that if $(i, (z_1, \ldots, z_{c_i}), y) \in L$, then $z_1, \ldots, z_{c_i}, y \in K$.

Proposition 12. *Suppose that E is simple. Let $\Pi = \mathcal{R}(E, L)$. Then the following is true:*

a) *If $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, $\langle \alpha, \mathfrak{B} \rangle \vdash L$ and $E_{\mathfrak{B}} \cong 0$, then $\Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$;*

b) *Let* $H_1 \cap T(L) = \varnothing$. *Let* $s_1, \ldots, s_a \in A$ *and for each* $u \in T(L)$, $\tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ *be defined. Then there exists a finite part* $\Delta_2 = (H_2, \alpha_2, \varphi_1^2, \ldots \varphi_n^2, \sigma_1^2, \ldots, \sigma_k^2)$ *such that* $\Delta_2 \supseteq \Delta$ *and the following conditions hold*:

(i) $\mathrm{dom}(\alpha_2) = T(L)$ *and* $H_2 = H_1 \cup (K \setminus T(L))$;

(ii) *If* $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, *then* $\langle \alpha, \mathfrak{B} \rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, *and* $E_{\mathfrak{B}} \cong \Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$.

P r o o f. Let $E = P^1 \& \ldots \& P^m$ and let $l_1 < \cdots < l_h$ be the elements of $\{1, \ldots, m\}$ such that if $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then all $z_1, \ldots, z_{c_i}$ belong to $T(L)$. If $1 \leq j \leq h$, and $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then let $\Pi^{l_j} = T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 0$ and let $\Pi^{l_j} = \neg T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, otherwise. Clearly $\Pi = T_0(X_1)$, if $h = 0$ and $\Pi = \Pi^{l_1} \& \ldots \& \Pi^{l_h}$, if $h > 0$.

Now a) follows easily from the definition of Π and from Proposition 11.

Suppose that $H_1 \cap T(L) = \varnothing$, $s_1, \ldots, s_a$ are elements of A and for each $u \in T(L)$, $\tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined.

Let $H_2 = H_1 \cup (K \setminus T(L))$. Clearly $H_2 \supseteq H_1$ and $T(L) \cap H_2 = \varnothing$. Define the mapping α_2 of T (L) into A by the equalities

$$\alpha_2(u) \cong \tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a), \quad u \in T(L).$$

In particular $\alpha_2(x_i) \cong s_i$, $i = 1, \ldots, a$, and $\alpha_2(w_i) \cong t_i$, $i = 1, \ldots, r$. Hence, $\alpha_1 \subseteq \alpha_2$.

Let $1 \leq i \leq n$. Define the partial function φ_i^2 on $H_2 \cup \mathrm{dom}(\alpha_2)$ by the condition $\varphi_i^2(z_1, \ldots, z_{c_i}) \cong y$ iff $(i, (z_1, \ldots, z_{c_i}), y) \in L$ or $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$. Since L is a termal tree, the definition of φ_i^2 is correct. Clearly $\varphi_i^2 \supseteq \varphi_i^1$.

Let $1 \leq j \leq k$. The partial predicate σ_j^2 on $(H_2 \cup \mathrm{dom}(\alpha_2))^{b_j} \setminus (\mathrm{dom}(\alpha_2))^{b_j}$ is defined by the equivalence:

$$\begin{aligned} &\sigma_j^2(z_1, \ldots, z_{b_j}) \cong y \\ &\quad \iff (\exists l(1 \leq l \leq m \,\&\, P^l = R^{n+j}(z_1, \ldots, z_{b_j}, y)) \text{ or } \sigma_j^1(z_1, \ldots, z_{b_j}) \cong y) \\ &\qquad \& \, (z_1, \ldots, z_{b_j}) \in (H_2 \cup \mathrm{dom}(\alpha_2))^{b_j} \setminus (\mathrm{dom}(\alpha_2))^{b_j}. \end{aligned}$$

Since E is a correct n.t. predicate, the definition of σ_j^2 is also correct.

Let $\Delta_2 = (H_2, \alpha_2, \varphi_1^2, \ldots, \varphi_n^2, \sigma_1^2, \ldots, \sigma_k^2)$. It follows easily from the definition of Δ_2 that Δ_2 is a finite part and $\Delta_2 \supseteq \Delta$.

Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$. We have to show that $\langle \alpha, \mathfrak{B} \rangle \vdash L$ and

$$E_{\mathfrak{B}} \cong \Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a). \tag{12.1}$$

The fact $\langle \alpha, \mathfrak{B} \rangle \vdash L$ follows immediately from the definition of the partial functions φ_i^2, $i = 1, \ldots, n$, and from $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$.

Let us turn to the proof of (12.1). From the definition of the predicates $\sigma_1^2, \ldots, \sigma_k^2$ and from $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$ it follows that if l is an element of $\{1, \ldots, m\} \setminus \{l_1, \ldots, l_h\}$, then $P^l_{\mathfrak{B}} \cong 0$. So to prove (12.1) it is sufficient to show that if $1 \leq j \leq h$, then $P^{l_j}_{\mathfrak{B}} \cong \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$. Let $1 \leq j \leq h$ and $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$. Then $z_1, \ldots, z_{c_j}$ are elements of $T(L) = \mathrm{dom}(\alpha_2)$ and

$\alpha(z_p) \cong \tau(z_p, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$, $p = 1, \ldots, c_i$. We have the following equivalencies:

$$\begin{aligned} P^{l_j}_{\mathfrak{B}} \cong 0 &\iff \sigma_{i-n}(z_1, \ldots, z_{c_i}) \cong y \\ &\iff \Sigma_{i-n}(\alpha(z_1), \ldots, \alpha(z_{c_i})) \cong y \\ &\iff \Sigma_{i-n}(\tau(z_1, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a), \ldots, \\ &\qquad \tau(z_{c_i}, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)) \cong y \\ &\iff \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0. \end{aligned}$$

The equivalence

$$P^{l_j}_{\mathfrak{B}} \cong 1 \iff \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 1$$

can be proved in a similar way.

Proposition 13. *Let* $\Pi = \mathcal{R}(E, L)$, $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, $\langle \alpha, \mathfrak{B} \rangle \vdash L$ *and* $E_{\mathfrak{B}} \cong 0$. *Then*

(13.1) $$\Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0.$$

P r o o f. Induction on $|E|$. If $|E| = 0$, then (13.1) follows from the previous proposition. Suppose that $|E| > 0$ and $E = E^1 \,\&\, R^i(z_1, \ldots, z_{c_i}, y) \,\&\, E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$, $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \,\&\, E^2, S)$. Since $E_{\mathfrak{B}} \cong 0$, $\langle \alpha, \mathfrak{B} \rangle \vdash L \cup L_E$ and, hence, $\langle \alpha, \mathfrak{B} \rangle \vdash S$. Then, by the induction hypothesis, $\Pi^1_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$ and $\Pi^2_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$

If $y \notin \mathcal{T}(S)$, then $\Pi = \Pi^1 \,\&\, \Pi^2$. Therefore (13.1) is true. Suppose that $y \in \mathcal{T}(S)$. Then $\Pi = \Pi^1 \,\&\, T_0(\tau(y, S)) \,\&\, \Pi^2$. Now we have to show that $\tau(y, S)_{\mathfrak{A}}(W_1/, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a))$ is defined. This follows from $\langle \alpha, \mathfrak{B} \rangle \vdash S$ and from Proposition 11.

Proposition 14. *Let* $\Pi = \mathcal{R}(E, L)$ *and* $s_1, \ldots, s_a$ *be elements of* A. *Suppose that* $\Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$ *and for each* $u \in \mathcal{T}(L)$, $\tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ *is defined. Then*

(14.1) $$\forall u (u \in \mathcal{T}(L \cup L_E) \Rightarrow \tau(u, L \cup L_E)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \quad \text{is defined}).$$

P r o o f. Induction on $|E|$. If $|E| = 0$, then $L \cup L_E = L$. The proposition is trivial. Let $|E| > 0$ and $E = E^1 \,\&\, R^i(z_1, \ldots, z_{c_i}, y) \,\&\, E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$. Clearly $S \cup L_{E^1 \& E^2} = L \cup L_E$. Let $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \,\&\, E^2, S)$. We have that $\Pi^1_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$ and $\Pi^2_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$.

Using the induction hypothesis we obtain that if $u \in \mathcal{T}(L \cup L_{E^1})$, then $\tau(u, L \cup L_{E^1})_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is defined.

Suppose that $y \notin \mathcal{T}(S)$. Then by Proposition 10, for $u \in \mathcal{T}(S)$, $(L \cup L_{E^1})_u = (S)_u$. Hence, by Proposition 2, if $u \in \mathcal{T}(S)$, then $\tau(u, S) = \tau(u, L \cup L_{E^1})$. Therefore, if $u \in \mathcal{T}(S)$, then $\tau(u, S)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is defined. Now applying the induction hypothesis for $E^1 \,\&\, E^2$ and S, we obtain (14.1).

Let us suppose that $y \in \mathcal{T}(S)$. Then $\Pi = \Pi^1 \,\&\, T_0(\tau(y, S)) \,\&\, \Pi^2$ and hence $\tau(y, S)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is defined. Using the same arguments as in the previous case, we obtain that if $u \in \mathcal{T}(S)$ and $u \neq y$, then

$\tau(u,S)\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)$ is defined. From here, applying the induction hypothesis for E^1 & E^2 and S, we obtain (14.1).

Proposition 15. *Let $H_1 \cap \mathcal{T}(L\cup L_E)=\varnothing$. Let $s_1,\ldots,s_a \in A$ and for each $u \in \mathcal{T}(L)$, $\tau(u,L)\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)$ be defined. Let $\Pi = \mathcal{R}(E,L)$ and $\Pi\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a) \cong 0$. Then there exists a finite part $\Delta_2=(H_2,\ \alpha_2,\ \varphi_1^2,\ldots,\varphi_n^2,\ \sigma_1^2,\ldots,\sigma_k^2)$ such that $\Delta_2 \supseteq \Delta$ and the following conditions hold:*

(i) $\mathrm{dom}(\alpha_2)=\mathcal{T}(L\cup L_E)$ *and* $H_2=H_1\cup(K\setminus\mathcal{T}(L\cup L_E))$;

(ii) *If* $\langle\alpha,\ \mathfrak{B}\rangle \supseteq \Delta_2$, *then* $\langle\alpha,\ \mathfrak{B}\rangle \vdash L\cup L_E$, $\alpha(x_i)\cong s_i$, $i=1,\ldots,a$, *and* $E_{\mathfrak{B}}\cong 0$.

Proof. Induction on $|E|$. If $|E|=0$, then the proposition follows from Proposition 12.

Let $|E|>0$ and $E=E^1\ \&\ R^i(z_1,\ldots,z_{c_i},y)\ \&\ E^2$, where $1\le i\le n$ and E^2 is simple. Let $S=L\cup(L\cup L_E)_y$, $\Pi^1=\mathcal{R}(E^1,L)$ and $\Pi^2=\mathcal{R}(E^1\ \&\ E^2,S)$. Clearly, $\Pi^1_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a) \cong \Pi^2_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)\cong 0$. Using the previous proposition, we obtain that if $u\in\mathcal{T}(L\cup L_{E^1})$, then

$$\tau(u,L\cup L_{E^1})\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)$$

is defined. From here, as in the proof of the previous proposition, it follows that for $u\in\mathcal{T}(S)$, $\tau(u,S)\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)$ is defined. Now applying the induction hypothesis for E^1 & E^2 and S, we obtain that there exists a finite part $\Delta_2=(H_2,\ \alpha_2,\ \varphi_1^2,\ldots,\varphi_n^2,\ \sigma_1^2,\ldots,\sigma_k^2)$ such that $\Delta_2\supseteq\Delta$ and the following conditions hold:

a) $\mathrm{dom}(\alpha_2)=\mathcal{T}(L\cup L_E)$ and $H_2=H_1\cup(K\setminus\mathcal{T}(L\cup L_E))$;

b) If $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$, then $\langle\alpha,\ \mathfrak{B}\rangle\vdash L\cup L_E$, $\alpha(x_i)\cong s_i$, $i=1,\ldots,a$, and $(E^1\&E^2)_{\mathfrak{B}}\cong 0$.

It remains to show that if $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$, then $E_{\mathfrak{B}}\cong 0$. Let $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$. If $(i,\langle z_1,\ldots,z_{c_i}\rangle,y)\in L_E$, then $E_{\mathfrak{B}}\cong 0$ follows from $(E^1\ \&\ E^2)_{\mathfrak{B}}\cong 0$ and $\langle\alpha,\ \mathfrak{B}\rangle\vdash L\cup L_E$. Otherwise, $y\in\mathrm{dom}(\alpha_1)$. Then, by the correctness of E, $\varphi_i^1(z_1,\ldots,z_{c_i})\cong y$. Hence, $\varphi_i(z_1,\ldots,z_{c_i})\cong y$. From here and from $(E^1\&E^2)_{\mathfrak{B}}\cong 0$ it follows that $E_{\mathfrak{B}}\cong 0$.

Proposition 16. *Let $H_1\cap\mathcal{T}(L\cup L_E)=\varnothing$. Let $s_1,\ldots,s_a\in A$ and for each $u\in\mathcal{T}(L)$, $\tau(u,L)\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ be defined. Let $\Pi=\mathcal{R}(E,L)$ and $\Pi\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ be not defined. Then there exists a finite part $\Delta_2\supseteq\Delta$ such that if $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$, then $\langle\alpha,\ \mathfrak{B}\rangle\vdash L$, $\alpha(x_i)\cong s_i$, $i=1,\ldots,a$, and $E_{\mathfrak{B}}$ is not defined.*

Proof. Induction on $|E|$. If $|E|=0$, then the proposition follows from Proposition 12. Let $E=E^1\&R^i(z_1,\ldots,z_{c_i},y)\&E^2$, where $1\le i\le n$ and E^2 is simple. Let $S=L\cup(L\cup L_E)_y$ and $\Pi^1=\mathcal{R}(E^1,L)$ and $\Pi^2=\mathcal{R}(E^1\&E^2,S)$. We shall consider the following cases:

1. $\Pi^1_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is not defined. By the induction hypothesis, there exists a finite part $\Delta_2\supseteq\Delta$ such that if $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$, then $\langle\alpha,\ \mathfrak{B}\rangle\vdash L$, $\alpha(x_i)\cong s_i$, $i=1,\ldots,a$, and $E^1_{\mathfrak{B}}$ is not defined. Clearly if $\langle\alpha,\ \mathfrak{B}\rangle\supseteq\Delta_2$, then $E_{\mathfrak{B}}$ is also undefined.

2. $\Pi^1_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is defined. Hence, $\Pi^1_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)\cong 0$. From Proposition 14 and from Proposition 10 it follows that if $u\in\mathcal{T}(S)$ and $u\ne y$, then

(16.1) $\quad\tau(u,S)\mathfrak{A}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is defined.

2.1. Let $y \notin \mathcal{T}(S)$. Then $\Pi = \Pi^1 \& \Pi^2$ and , hence, $\Pi^2(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is not defined. By (16.1) and by the induction hypothesis applied for $E^1\&E^2$ and S, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle\alpha, \mathfrak{B}\rangle \vdash S$, and hence $\langle\alpha, \mathfrak{B}\rangle \vdash L$, $\alpha(x_i) = s_i$, $i = 1,\ldots,a$, and $(E^1\&E^2)_{\mathfrak{B}}$ is not defined. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$. Using the same arguments as in the proof of the previous proposition, we obtain that $\varphi_i(z_1,\ldots,z_{c_i}) \cong y$. Then, since $(E^1\&E^2)_{\mathfrak{B}}$ is not defined, $E_{\mathfrak{B}}$ is also not defined.

2.2. Let $y \in \mathcal{T}(S)$. Then $\Pi = \Pi^1 \& T_0(\tau(y,S)) \& \Pi^2$. Suppose that $\tau(y,S)_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is defined. Then using (16.1), we obtain that if $u \in \mathcal{T}(S)$, then $\tau(u,S)_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is defined. Now the proof proceeds as in the previous case.

Suppose now that $\tau(y,S)_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,X_1/s_1,\ldots,X_a/s_a)$ is not defined. Then $y \notin \mathrm{dom}(\alpha_1)$ and, hence, by Corollary 10.1 and Corollary 10.2, $z_1,\ldots,z_{c_i}$ are elements of $\mathcal{T}(L \cup L_{E^1})$ and $\tau(y,S) = f_i(\tau(z_1,L\cup L_{E^1}),\ldots,\tau(z_{c_i},L\cup L_{E^1}))$. By Proposition 15, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle\alpha, \mathfrak{B}\rangle \vdash L \cup L_{E^1}$, $E^1_{\mathfrak{B}} \cong 0$ and $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$. From here by Proposition 11, we obtain that $\mathcal{T}(L \cup L_{E^1}) \subseteq \mathrm{dom}(\alpha)$ and

(16.2) for $u \in \mathcal{T}(L\cup L_{E^1})$,
$$\alpha(u) \cong \tau(u,L\cup L_{E^1})_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a).$$

Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$. Clearly $\langle\alpha, \mathfrak{B}\rangle \vdash L$. From (16.2) it follows $z_1,\ldots, z_{c_i} \in \mathrm{dom}(\alpha)$ and for $j = 1,\ldots,c_i$, $\alpha(z_j) \cong \tau(z_j,L\cup L_{E^1})_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$. We shall prove that $\varphi_i(z_1,\ldots,z_{c_i})$ is not defined. Indeed,

$$\begin{aligned}
&\varphi_i(z_1,\ldots,z_{c_i}) \text{ is defined } \iff \theta_i(\alpha(z_1),\ldots,\alpha(z_{c_i})) \text{ is defined}\\
&\iff \theta_i(\tau(z_1,L\cup L_{E^1})_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a),\ldots,\\
&\qquad \tau(z_{c_i},L\cup L_{E^1})_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)) \text{ is defined}\\
&\iff \tau(y,S)_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a) \text{ is defined.}
\end{aligned}$$

From here, since $E^1_{\mathfrak{B}} \cong 0$, it follows that $E_{\mathfrak{B}}$ is not defined.

Proposition 17. *There exists an effective way to define for each n.t. expression $E \supset y$ of type $(a_1,\ldots,a_n,b_1,\ldots,b_k)$ a conditional term Q with variables among $W_1,\ldots,W_r,X_1,\ldots,X_a$ such that for all elements $s_1,\ldots,s_a$ and t of A, the following conditions are satisfied:*

(1) *If $Q_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a) \cong t$, then there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle\alpha, \mathfrak{B}\rangle$ is a special enumeration and $\Delta_2 \subseteq \langle\alpha, \mathfrak{B}\rangle$, then $\alpha(x_1) \cong s_1,\ldots,\alpha(x_a) \cong s_a$, $\alpha(y) \cong t$ and $E_{\mathfrak{B}} \cong 0$;*

(2) *If $Q_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a) \not\cong t$, then at least one of the following is true:*

(2.1) *There exists a finite part $\Delta_2 \supseteq \Delta$, such that if $\langle\alpha, \mathfrak{B}\rangle$ is a special enumeration and $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $x_1,\ldots,x_a$ belong to $\mathrm{dom}(\alpha)$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \mathrm{dom}(\alpha)$;*

(2.2) *For each special enumeration $\langle\alpha, \mathfrak{B}\rangle$, if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$, $\alpha(x_1) \cong s_1,\ldots, \alpha(x_a) \cong s_a$ and $E_{\mathfrak{B}} \cong 0$, then $\alpha(y) \not\cong t$;*

(3) *If $Q_{\mathfrak{A}}(W_1/t_1,\ldots,W_r/t_r,\ X_1/s_1,\ldots,X_a/s_a)$ is not definite, then there exists a finite part $\Delta_2 \supseteq \Delta$, such that if $\langle\alpha, \mathfrak{B}\rangle$ is a special enumeration and $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$, and $E_{\mathfrak{B}}$ is not defined.*

P r o o f. Let E be a n.t. predicate and $y \in \mathbb{N}$. Suppose that E is not correct. Let $Q = (\neg T_0(X_1) \supset X_1)$. Then (1) and (3) are obvious. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Then, by Proposition 6, $E_{\mathfrak{B}} \not\cong 0$. So we receive that (2.2) is true.

Suppose now that E is correct. Let L be the empty termal tree. Let K be the finite set of natural numbers consisting of the elements of $\{w_1, \ldots, w_r, x_1, \ldots, x_a, y\}$ and of those natural numbers which occur in E. Let $\Pi = \mathcal{R}(E, L)$. Let $Q = (\Pi \supset \tau(y, L_E))$, if $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$. Let $Q = (\neg T_0(X_1) \supset X_1)$, otherwise. We shall show that Q satisfies the conditions (1), (2) and (3).

Suppose that $s_1, \ldots, s_a$ and t are elements of A.

Let $Q\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong t$. Then Q is not in the form $(\neg T_0(X_1) \supset X_1)$ and, hence, $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$. Clearly $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$ and $\tau(y, L_E)\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,$ $X_1/s_1, \ldots, X_a/s_a) \cong t$.

Obviously $\mathcal{T}(L) = \{w_1, \ldots, w_r,\ x_1, \ldots, x_a\}$. Hence, for $u \in \mathcal{T}(L)$, $\tau(u, L)\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is defined.

By Proposition 15, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle\alpha, \mathfrak{B}\rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, and $E_{\mathfrak{B}} \cong 0$. Using Proposition 11, we obtain $\alpha(y) \cong t$. So we receive that (1) is true.

Let us suppose that $Q\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \not\cong t$.

We shall consider the following cases:

1. $H_1 \cap \mathcal{T}(L_E) \neq \varnothing$. We shall show the validity of (2.2). Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Assume that $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, and $E_{\mathfrak{B}} \cong 0$. Then $\langle\alpha, \mathfrak{B}\rangle \vdash L_E$. From here, by Proposition 11, it follows that $\mathcal{T}(L_E) \subseteq \mathrm{dom}(\alpha)$. Hence $H_1 \cap \mathrm{dom}(\alpha) \neq \varnothing$. The last contradicts $\Delta \subseteq \langle\alpha, \mathfrak{B}\rangle$. So we receive that (2.2) is true.

2. Let $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $y \notin \mathcal{T}(L_E)$. Then $y \in K \setminus \mathcal{T}(L_E)$. Suppose that $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \not\cong 0$. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$, $\alpha(x_i) \cong s_i$, $i = 1, \ldots, n$, and $E_{\mathfrak{B}} \cong 0$. Since $L = \varnothing$, $\langle\alpha, \mathfrak{B}\rangle \vdash L$. By Proposition 13, $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 0$. A contradiction. So we obtain again that (2.2) is true.

Suppose now that $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$. By Proposition 15, there exists a finite part $\Delta_2 = (H_2,\ \alpha_2,\ \varphi_1^2, \ldots, \varphi_n^2, \sigma_1^2, \ldots, \sigma_k^2)$ such that $\Delta_2 \supseteq \Delta$, $H_2 = H_1 \cup K \setminus \mathcal{T}(L_E)$ and if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $E_{\mathfrak{B}} \cong 0$ and $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$. So we have that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \mathrm{dom}(\alpha)$. So the validity of (2.1) is proved.

3. Let $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $y \in \mathcal{T}(L_E)$. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Assume that $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, $E_{\mathfrak{B}} \cong 0$ and $\alpha(y) \cong t$. By Proposition 13, $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,$ $X_1/s_1, \ldots, X_a/s_a) \cong 0$. Using Proposition 11 and $E_{\mathfrak{B}} \cong 0$, we obtain that $\alpha(y) \cong \tau(y, L_E)\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong t$. Therefore $Q\mathfrak{a}(W_1/t_1, \ldots,$ $W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong t$. A contradiction. So we have proved (2.2).

Suppose now that $Q\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is not definite. Then Q is not in the form $(\neg T_0(X_1) \supset X_1)$. Hence $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $Q = (\Pi \supset \tau(y, L_E))$. Assume that $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong 0$ and $\tau(y, L_E)\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is not defined. The last contradicts Proposition 14. Hence $\Pi\mathfrak{a}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is not defined. From here (3) follows from Proposition 16.

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