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NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS V. PREHISTORY OF MECHANICAL CONSTRAINTS

GEORGI CHOBANOV, IVAN CHOBANOV

> Not only the public press but also the precis of the paedagogues has taught us to gulp superlatives as daily narcotics. The professor who forgets to class, rate, and rank his subject as the first and finest something will fail to find it mentioned in the students' terminal examinations. The busy modern calls for culture in predigested quintessence pills, packaged in abridged paperbacks, explained by folksy prefaces in pellet-paragraphs of sugared baby-talk, lullabies to smugness. Suggestion to a general audience that historical facts must precede if not replace historical enthusiasm may expect only oblivion, the palm of dullness.
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> C. Truesdell: *The Mechanics of Leonardo da Vinci*

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. V. ПРЕДИСТОРИЯ МЕХАНИЧЕСКИХ СВЯЗЕЙ

Это есть пятая часть серии статьей, посвещенные динамических аксиом Ньютона и Эйлера; она естественное продолжение и развитие последней из них [17], в которой подробно дискутированы физические мотивировки понятия о механических связах, наложенных систем масовых точек и твердых тел. В работе приведены многочисленные автентические данные в связи с зачатием и преждевременном рождении понятия связи в ранной истории рациональной механики, причем особое внимание уделено сочинений *Discorsi e Dimostrazioni Matematiche Intorno à Due Nuove Scienze* Галилея, *Philosophiae Naturalis Principia Mathematica* Ньютона и *Traité de Dynamique* Даламбера.

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. V. PREHISTORY OF MECHANICAL CONSTRAINTS

This is the fifth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms; it is the natural continuation and development of the last of them [17], where the physical motivations of the notion of mechanical constraints imposed on mass-point and rigid body systems are discussed at length. Numerous authentical data are adduced in connection with the conception and premature birth of the constraint concept in the early history of rational mechanics, special stress being laid on Galileo's *Discorsi e Dimostrazioni Matematiche Intorno à Due Nuove Scienze*, Newton's *Philosophiae Naturalis Principia Mathematica*, and D'Alembert's *Traité de Dynamique*, where the germs of the constraint notion may be traced.

This fifth part of a series of articles dedicated to the Newtonian and Eulerian dynamical axioms is a natural continuation and development of the last of them [17], where the physical motivations of the notion mechanical constraints imposed on mass-points and rigid bodies are discussed at length and which is published in this volume of the *Annual*; that is why, in order to avoid reiterations, the literature quoted in this paper and in [17] has a unified numeration.

The motivations of the engineering praxis are as old as human race too. The *vectis*, *axis in peritrochio*, *trochlea seu polispastus*, *cochlea* and *cuneus* (that is to say the lever, axis and wheel, pulley, screw, and wedge, respectively) have been utilized already by the ancient to the fullest extent. In the *Auctoris praefacio ad lectorem* of his *Principia* [7] Newton states:

"Pars haec mechanicae a veteribus in potentiis quinque ad artes manuales spectantibus exculta fuit, qui gravitatem (cum potentia manualis non sit) vix aliter quam in ponderibus per potentis illas movendis considerarunt."

As Krilov observes in his Russian version [18] of *Principia*, the term *potentia* is used here in two ways: the first time as a synonym of *machina*, and the second time as a synonym of *power*. As an illustration he adduces the following excerpt from Maclaurin's book [19]:

"It is distinguished by Sir I. Newton into *practical* and *rational* mechanics; the former treats of the *mechanical powers* viz. the *lever*, the *axis* and *wheel*, the *pulley*, the *wedge* and the *screw* to which the *inclined* plane is to be added and of the various combinations together. Rational mechanics comprehends the whole theory of motion and shews when the *powers* of *forces* are given how to determine the motions that are produced by them ... in tracing the powers that operate in nature from the phenomena we proceed by analysis and deducing the phenomena from the powers or causes that produce them we proceed by synthesis."

The close relations of early mechanicians with engineering experience is reflected in an excellent manner in Galileo's *Dialoghi delle nuove scienze* [20], *Giornata prima* of which begins with the following inferences of Salviati and Sagredo:

SAL. Largo campo di filosofare a gl'intelletti specolativi parmi che porga la frequente pratica del famoso arsenale di voi, Signori Veneziani, ed in particolare in quella parte che mecanica si dominanda; atteso che quivi ogni sorte di strumento e di machina vien continuamente posta in opera da numero grande d'artefici, tra i quali, e per l'osservazioni fatte dai loro antecessori, e per quelle che di propria avvertenza vanno continuamente per se stessi facendo, è forza che ve ne siano de i pertissimi e di finissimo discorso.

SAGR. V. S. non s'inganna punto: ed io, come per natura curioso, frequento per mio diporte la visita di questo luogo e la pratica di questi che noi, per certa preminenza che tengono sopra 'l resto della maestranza, domandiamo proti; la conferenza de i quali mi ha piu volte aiutato nell'investigazione della ragione di effetti non solo maravigliosi, ma reconditi ancora e quasi inopinabili. E vero che tal volta anco mi ha messo in confusione ed in disperazione di poter penetrare comme possa seguire quello che, lontano da ogni mio concetto, mi dimostra il senso esser vero ... [21, II, p. 81].

The question now quite reasonably arises: are there in Galileo's mechanical writings solutions of dynamical problems concerning motions of constrained mass-points or rigid bodies? Before answering this question we may point out that it is by no means groundless, since motions of mass-points along inclined (to the vertical) lines or circumferences are *par excellence* constrained motions, and such problems are abundant in *Giornata terza* of *Discorsi.* As a matter of fact, all theorems, propositions, corollaries, problems, and scholiums of section *De motu naturaliter accelerato* of *Giornata terza*, as well as all concomitant commentaries of Salviati, Sagredo, and Simplicio, beginning with *Theorema III*, *Propositio III*, are concerned with *motus naturalis* along inclined lines. On that ground, formally at least, one may expect that germs leastwise of constraint dynamics may be found in Galileo's works.

Alas, those are blighted hopes, and the reason is a quite simple one. In spite of all traditional physical folklore there is no dynamics at all in anything Galileo has written on mechanics. In vain will remain all Lagrange's efforts to render *quae sunt Caesaris Deo et quae sunt Dei Caesari*:

"La Dynamique est la science des forces accélératrices ou retardatrices et des mouvements variés qu'elles doivent produire. Cette science est due entièrement aux modernes, et Galilée est celui qui en a jeté les premiers fondements. Avant lui on n'avait considéré les forces qui agissent sur les corps que dans l'état d'équilibre; et quoiqu'on ne pût attribuer l'accélération des corps pesants et le mouvement curviligne des projectiles qu'à l'action constante de la gravité, personne n'avait encore réussi à déterminer les lois de ces phénomènes journaliers, d'après une cause si simple. Galilée a fait le premier ce pas important et a ouvert par là une carrière nouvelle et immense à l'avancement de la Méchanique. Cette découverte est exposée et developpée dans l'Ouvrage intitulé: *Discorsi e dimostrazioni matematiche intorno a due nuove scienze*, lequel parut, pour la première fois, à Leyde, en 1638. Elle ne procura pas à Galilée, de son vivant, autant de célébrité que celles qu'il avait faites dans le ciel; mais elle fait aujourd'hui la partie la plus solide et la plus reelle de la gloire de ce grand homme" [11, p. 237].

This brilliant appraisal of Galileo's mechanical performances has been multiplied in the course of two clear centuries in a myriad physical and mechanical text-books, treatises, monographs, articles, journals, newspapers, etceteras in precise conformity with the renowned verse of Vergilius *Fama mobilitate viget viresque adquirit eundo.* As a result of the loathsome aptitude of human mind to idolatry the image of Galileo as the founder of non-peripatetic dynamics is rooted in public spiritedness as tight as the image of Marx as the founder of non-capitalistic economics. Fortunately, the first case is not this far sinister.

The quoted excerpt from [11] may be written only by someone who has not read *Discorsi*; or by someone who has read it carelessly; or by someone who has read it attentively and has understood nothing. Otherwise one cannot explain why the adjective *dynamical* is ascribed to purely *kinematical* investigations.

For the whole of the content of *Giornata terza* & *quarta* is kinematics, only kinematics, and nothing save kinematics.

This is a point that needs a close attention. In order to make things transparent and to leave no room for gratuitous and subjective interpretations, let us drink one sip or two out of the very spring:

"Quae in motu aequabili contingunt accidentia, in praecedenti libro considerata sunt: modo de motu accelerato pertractandum.

Et primo, definitionem ei, quo utitur natura, apprime congruentem investigare atque explicare convenit. Quamvis enim aliquam lationis speciem ex arbitrio confingere, et consequentes eius passiones contemplari, non sit inconveniens ..., tamen, quandoquidem quadam accelerationis specie gravium descendentium utitur natura, eorundem speculari passiones decrevimus, si eam, quam allaturi sumus de nostro motu accelerato definitionem, cum essentia motus naturaliter accelerati congruere contigerit" [21, II, p. 254].

In such a manner, we come to know that Galileo:

1. Proceeds to study motions with impermanent velocity.

2. Realizes the possibility of an infinite variety of such motions.

3. Is interested in that special kind of accelerated motions Nature makes use of.

4. Does not know the definition of the *motus naturaliter acceleratus*.

How does Galileo solve the last problem? *Ipse dixit*:

"Postremo, ad investigationem motus naturaliter accelerati nos quasi manu duxit animadversio consuetudinis atque instituti ipsiusmet naturae in ceteris suis operibus omnibus, in quibus exercendis uti consuevit mediis primis, simplicissimis, facillimis. Neminem enim esse arbitror qui credat, natatum aut polatum simpliciori aut faciliori modo exerceri posse, quam eo ipso, quo piscem et aves instinctu naturali utuntur" [*ibid.*].

Ergo, Galileo:

5. Intends to discover the definition of naturally accelerated motion by observing (and possibly measuring) natural motions.

6. Proclaims a philosophical principle Nature obeys unquestingly: Simplicity.

What does, however, *simplicity* mean in the special case of natural motions? *Verba magistri*:

"Dum igitur lapidem, ex sublimi a quiete descendentem, nova deinceps velocitatis acquirere incrementa animadverto, cur talia additamenta, simplicissima atque omnibus magis obvia ratione, fieri non credam? Quod si attente inspiciamus, nullum additamentum, nullum incrementum, magis simplex inveniemus, quam illud, quod semper eodem modo superaddit ... Et sic a recta ratione absonum nequaquam esse videtur, si accipiamus, intentionem velocitatis fieri iuxta temporis extensionem; ex quo definitio motus, de quo acturi sumus, talis accipi potest: Motum aequabiliter, seu uniformiter, acceleratum dico illum, qui, a quiete recedens, temporibus aequalibus aequalia celeritatis momenta sibi superaddit" [*ibid.*, p. 254–255].

In such a manner, following a most natural, logical, and methodical course of thought, Galileo:

7. Arrives at the modern definition of uniformly accelerated motion.
8. Declares the latter the law of free fall.

Moreover, the author puts the last two items to the test of experimental control:

"In un regolo, o vogliàn dir corrente, di legno, lungo circa 12 braccia, e largo per un verso mezo braccio e per l'altro 3 dita, si era in questa minor larghezza incavato un canaletto, poco piu largo d'un dito; tiratolo drittissimo, e, per averlo ben pulito e liscio, incollativi dentro una carta pecora zannata e lustrata al possibile, si faceva in esso scendere una palla di bronzo durissimo, ben rotondata e pulita; constituito che si era il detto regolo pendente, elevando sopra il piano orizontale una delle sue estremitá un braccio o due ad arbitrio, si lasciava (come dico) scendere per il detto canale la palla, notando, nel modo che appresso dirò, il tempo che consumava nello scorrerlo tutto, replicando il medesimo atto molte volte per assicurarsi bene della quantitá del tempo, nel quale non si trovava mai differenza né anco della decima parte d'una battila di polso. Fatta e stabilita precisamente tale operazione, facemmo scender la medesima palla solamente per la quarta parte della lunghezza di esso canale; e misurato il tempo della sua scesa, si trovava sempre puntualissimamente esser la metá dell'altro: e facendo poi l'esperienze di altre parti, esaminando ora il tempo di tutta la lunghezza col tempo della metá, o con quello delli duo terzi o de i 3/4, o in conclusione con qualunque altra divisione, per esperienze ben cento volte replicate sempre s'incontrava, gli spazii passati esser tra di loro come i quadrati de i tempi, e questo in tutte le inclinazioni del piano, cioè del canale nel quale si faceva scender la palla; dove osservamo ancora, i tempi delle scese per diverse inclinazioni mantener esquisitamenre tra di loro quella proporzione che piú a basso troveremo essergli assegnata e dimostrata dall'Autore. Quanto poi alla misura del tempo, si teneva una gran secchia piena d'acqua, attaccata in alto, la quale per un sottil cannellino, sal da togli nel fondo, versava un sottil filo d'acqua, che s'andava ricevendo con un piccol bicchiero per tutto 'l tempo che la palla scendeva nel canale e nelle sue parti: le particelle poi dell'acqua, in tal guisa raccolte, s'andavano di volta in volta con esattissima bilancia pesando, dandoci le differenze e proporzioni de i pesi loro le differenze e proporzioni de i tempi; e questo con tal giustezza, che, come ho detto, tali operazioni, molte e molte volte replicata, giá mai non differivano d'un notabil momento" [*ibid.*, p. 274–275].

Once the definition of *motus naturaliter acceleratus* established, all that follows in *Giornata terza & quarta* of *Movimenti Locali* are kinematical exercises of free fall, sliding along a line inclined towards the horizon, and flying of projectiles. At that, the moving objects are points rather than bodies; moreover, those are geometrical — by no means mechanical — points that are moving on the pages of Galileo's *Discorsi*. The meaning of this statement is that the mass of the moving point is completely irrelevant to Galileo's meditations and calculations — there is no relationship between moving object and moving cause. As a mere child could say, the mass concept is void of sense if estranged from forces, and there are no forces at all in Galileo's mechanical studies — at least no such ones that would be found congenial today.

And yet, even only kinematically, no one can justly deny that there are constrained motions in Galileo's *Discorsi*. There may only be divergences of views on the degree of originality of the contributions of this notionalist to the field of rational mechanics. To Galileo's worshippers Truesdell's standpoint may seem blasphemy:

"Historians of letters had meanwhile created the myth of the 'Renaissance'. According to this myth, in the Middle Ages man hibernated beneath a pall of scholastic repetitions, borrowed from Aristotle and enforced by the Church; the Renaissance, casting all this aside, opened its eyes and discovered man and the world by personal sensation ...

... such historians of science ... found several of Galileo's ideas, more or less, in Leonardo's notes ... published in facsimile in the years 1881 to 1936 ... Previously historians had believed that Galileo thought these things out of 'genius' applied to thin air ... 'Discovery' of Leonardo transferred the point of application of this same theory a century backwards. He, too, had the same material to work with: 'genius' and thin air, and the remaining problem for this group of historians was only to see how Leonardo's ideas got to Galileo, thus making the latter a true grandson, if not son, of the Renaissance" [8, p. 25, 27].

To make matters worse, strange characters emerge from days long gone by:

"...the main kinematical properties of uniformly accelerated motions, still attributed to Galileo by the physics texts, were discovered and proved by scholars of Merton College — William Heytesbury, Richard Swineshead, and John of Dumbleton — between 1328 and 1350. Their work distinguished *kinematics*, the geometry of motion, from *dynamics*, the theory of the causes of motion. Their approach was mathematical. They succeeded in formulating a fairly clear concept of instantaneous speed, which means that they foreshadowed the concepts of function and derivative, and they proved that the space traversed by a uniformly accelerated motion in a given time is the same as that traversed by a uniform motion whose speed is the mean of the greatest and the least speeds in the accelerated motion. In principle, the qualities of Greek physics were replaced, at least for motions, by the numerical quantities that have ruled Western science ever since. This work was quickly diffused into France, Italy, and other parts of Europe. Almost immediately, Giovanni da Casale and Nicole Oresme found how to represent the results by geometrical graphs, introducing the connection between geometry and the physical world that became a second characteristic habit of Western thought — a habit so deep-seated that it is known to every carpenter and passes unremarked only in certain highly specialized professions ...

Clagett [22] has cited much evidence to show that these ideas, which originated in England and France in the early fourteenth century, were discussed back and forth in periods of varying activity and inactivity in France, the Empire, and Italy in the latter half of the same century and were taught in Italian universities in the next one, at the end of which a flood of printed books opened the subject to everyone — everyone who could understand Latin and mathematics" [*ibid.*, p. 30–31].

In the light of this information Galileo's pretensions in the introductory words of *De motu locali* seem a bit overdone:

"De subiecto vetustissimo novissimam promovemus scientiam. Motu nil forte antiquis in natura, et circa eum volumina nec pauca nec parva a philosophis conscripta reperiuntur; symptomatum tamen, quae complura et scitu digna insunt in eo, adhuc inobservata, necdum indemonstrata, comperio. Leviora quaedam adnotantur, ut, gratia exempli, naturalem motum gravium descendentium continue accelerari; verum, iuxta quam proportionem eius fiat acceleratio, proditum hucusque non est: nullus enim, quod sciem, demonstravit, spacia a mobili descendente ex quiete peracta in temporibus aequalibus, eam inter se retinere rationem, quam habent numeri impares ab unitate consequentes" [21, II, p. 247].

The first place in *Discorsi*, where inclined plane comes into view, is *Theorema III*, *Propositio III* of *De motu naturaliter accelerato* in *Giornata terza*, namely:

"Si super plano inclinato atque in perpendiculo, quorum eadem sit altitudo, feratur ex quiete idem mobile, tempora lationum erunt inter se ut plani ipsius et perpendiculi longitudines" [*ibid.*, p. 282].

Inclined planes are repeatedly used in well-nigh all the following theorems, propositions, corollaries, problems, scholia, and commentaries of Salviati, Sagredo, and Simplicio of *Giornata terza* of *Discorsi*. As a matter of fact, the whole content of this part of *Due scienze* consists of exercises on theme and variations point kinematics of uniformy accelerated motions.

It is quite immaterial to us whether Galileo's statements in the said propositions are true or false: the cold fact is, there is point kinematics of constrained motions in his book. This applies especially to an important problem — that of *lationem omnium velocissimam*, which later became the starting point of Johann Bernoulli's *Problemata novum, ad cuius solutionem geometrici invitantur*, as well as a stimulus for variational calculus. Formulated by Galileo in the form of a *Scholium*, it reads:

"Ex his quae demonstrata sunt, colligi posse videtur, lationem omnium velocissimam ex termino ad terminum non per brevissimam lineam, nempe per rectam, sed per circuli portionem, fieri" [*ibid.*, p. 333].

In such a way, Galileo:

9. Includes the circumference in the family of geometrical constraints.

10. Formulates a minimalization problem concerning constrained motions (for the first time in the history of mechanics, as far as our knowledge goes).

The importance of the last event is not diminished by the fact that Galileo's solution was wrong: another contingency was purely and simply impossible in his days. As Truesdell says:

" Now a mathematician has a matchless advantage over general scientists, historians, and exponents of other professions: He can be wrong. *A fortiori*, he can also be right. There are errors in Euclid, and, to within a set certainly of measure zero on the ordinary human scale, what Euclid proved to be true in ancient Greece is true even in the colossal, unprecedented, nucleospacial, totally welfared today. In the advance through the physical, social, historical, and other sciences, the demarcation between truth and falsehood grows vaguer, until in some areas truth can be rezoned as falsehood and falsehood enshrined into truth by consensus of "acknowledged experts and autorities" or even popular vote. One professor discussing the doctrines of Karl Marx may label them as grave errors; a second, equally qualified, may present them as problematic, partly true and partly not so; while a third, living in a different part of the world, may proclaim them as the quintessence of human knowledge. In the mathematical science as taught by the colleagues of these same three social scientists, there is no disagreement as to what is true and what is not. A mistake made by a mathematician, even a great one, is not a "difference of point of view" or "another interpretation of the data" or a "dictate of a conflicting ideology", it is a mistake. The greatest of all mathematicians, those who have discovered the greatest quantities of mathematical truths, are also those who have published the greatest number of lacunary proofs, insufficiently qualified assertions, and flat mistakes.

The mistakes made by a great mathematician are of two kinds: first, trivial slips that anyone can correct, and second, titanic failures reflecting the scale of the struggle which the great mathematician waged. Failures of this latter kind are often as important as successes, for they give rise to major discoveries by other mathematicians. One error of a great mathematician has often done more for science than a hundred little theorems proved by lesser men" [8, p. 140].

While Galileo did not have at his disposal the tool for solving dynamical problems involving constrained mass-points, Newton did. That is why it is interesting to the utmost degree (at least as far as our topic is concerned) to see what did he actually accomplish by its aid.

To this end it is sufficient to take a look at Newton's mechanical archives, his *Principia* [7]. The realization of the fact that this book is a treatise on point dynamics, not in the least on rigid dynamics, is as old as Euler:

"... while Newton had used the word 'body' vaguely and in at least three different meanings, Euler realized that the statements of Newton are generally correct only when applied to masses concentrated at isolated points ..." [8, p. 107].

Therefore, we must discover how far Newton has penetrated into the field of constrained mass-point dynamics. Already a mere glance at the contents of *Principia* at once displays that the only place of the work, where constrained motions may be treated, is *Sect. X*: *De Motu Corporum in Superficiebus datis, deq; Funipendulorum Motu reciproco* of *Liber Primus, De Motu Corporum*. It begins with *Prop. XLVI. Prob. XXXII*, namely:

"Posita cujuscunq; generis vi centripeta, datoq; tum virium centro tum plano quocunq; in quo corpus revolvitur, et concessis Figurarum curvilinearum quadraturis: requiritur motus corporis de loco dato data cum velocitate secundum Rectam in Plano illo datum egressi" [7, p. 145].

This is a constrained mass-point dynamical problem *par excellence*: it proposes to find the motion of a mass-point constrained to remain on a given plane and subjected to the action of an arbitrary central force, the pole of which is lying outside the plane.

Newton's Problem XXXII is extraordinary interesting with a view to our topic, namely the nascency of the idea of a mechanical constraint imposed on a mass-point or a rigid body. We shall therefore follow the train of thoughts of *Principia*'s author exposed in his solution of this problem. At that, with an eye to a greater clearness, we shall quote the English version of the work in Motte's translation rather than the original Latin text:

"Let S be the centre of force, SC the least distance of that centre from the given plane, P a body issuing from the place P in the direction of the right line PZ, Q the same body revolving in its curve, and PQR the curve itself which is required to be found, described in that given plane. Join CQ, QS, and if in QS we take SV proportional to the centripetal force with which the body is attracted towards the centre S, and draw VT parallel to CQ, and meeting SC in T; then will the force SV be resolved into two (by Cor. II of the Laws of Motion), the force ST, and the force TV; of which ST attracting the body in the direction of a line perpendicular to the plane, does not at all change its motion in that plane. But the action of the other force TV, coinciding with the position of the plane itself, attracts the body directly towards the given point C in that plane; and therefore causes the body to move in the plane in the same manner as if the force ST were taken away, and the body were to revolve in free space about the centre C by means of the force TV alone. But there being given the centripetal force TV with which the body Q revolves in free space about the given centre C, there is given (by Prop. XLII) the curve PQR which the body describes; the place Q, in which the body will be found at any given time; and, lastly, the velocity of the body in that place Q. And conversely, Q. E. I." [23, vol. T, p. 148–149].

In this solution two places of the book are quoted in the capacity of arguments: Corollary II of the introductory *Axiomata sive Leges Motus* and Proposition XLII. Problem XXIX. The corresponding texts of [23] read as follows:

"And hence is explained the composition of any one direct force AD, out of any two oblique forces AC and CD; and, on the contrary, the resolution of any one direct force AD into two oblique forces AC and CD: which composition and resolution are abundantly confirmed from mechanics" (p. 15).

"The law of centripetal force being given, it is required to find the motion of a body setting out from a given place, with a given velocity, in the direction of a given right line" (p. 133).

As it is immediately clear, Newton reduces his Problem XXXII to Problem XXIX. The fact itself is irrelevant to our concern, since we are interested in Newton's idea of a constraint imposed on a mass-point rather than in particular dynamical problems whichever concerning such constraints. That is why we shall present Newton's arguments in a modern form that will help us to expose the roots of the matter.

Using Newton's notations, let by definition $\boldsymbol{r} = \boldsymbol{SQ}$, $\boldsymbol{n} = \boldsymbol{SC}$, where it is supposed $\boldsymbol{n} \neq \boldsymbol{o}$, so that the unit vector $\boldsymbol{n}^0 = \frac{1}{n}\boldsymbol{n}$ exists. Under these notations, the equation of the plane π (that is to say CPQ is)

$$\boldsymbol{rn} = \nu \tag{1}$$

with an appropriate ν. On the other hand, the motion of the mass-point Q is governed by the equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{R}, \tag{2}$$

dots denoting, as traditionally in analytical dynamics, derivatives with respect to the time t, m — the mass of Q, $\boldsymbol{F} = \boldsymbol{VS}$ — the "centripetal force", acting on Q, and $\boldsymbol{R}$ — the reaction of the plane π on Q. Besides, let by definition $\bar{\rho} = CQ$, whence $\boldsymbol{r} = \boldsymbol{n} + \bar{\rho}$, and, since π is constant (ergo $\dot{\boldsymbol{n}} = \boldsymbol{o}$),

(3) $$\ddot{\boldsymbol{r}} = \ddot{\bar{\rho}}.$$

Now (2), (3) imply

(4) $$m\ddot{\bar{\rho}} = \boldsymbol{F} + \boldsymbol{R}.$$

Moreover, since obviously

(5) $$\boldsymbol{n}^2 = \nu$$

(the point C lying in the plane π), the relations (1), (5) imply

(6) $$\bar{\rho}\boldsymbol{n} = 0.$$

In such a manner, the problem of the *constrained* motion of the mass-point Q is reduced to that of the motion of the *free* mass-point Q under the action of the forces $\boldsymbol{F} + \boldsymbol{R}$.

As regards $\boldsymbol{F}$, Newton's decomposition $\boldsymbol{F} = \boldsymbol{VT} + \boldsymbol{TS}$ implies

(7) $$\boldsymbol{F} = F\bar{\rho}^0 + N\boldsymbol{n}^0,$$

$\bar{\rho}^0$ denoting the unit vector of $\bar{\rho}$, and F and N — the projections of $\boldsymbol{F}$ on $\bar{\rho}^0$ and $\boldsymbol{n}^0$, respectively. In such a manner, the relations (4), (7) imply

(8) $$m\ddot{\bar{\rho}} = F\bar{\rho}^0 + N\boldsymbol{n}^0 + \boldsymbol{R}.$$

Now, reducing Problem XXXII to Problem XXIX, Newton presupposes

(9) $$N\boldsymbol{n}^0 + \boldsymbol{R} = 0.$$

Why?

As regards the reaction $\boldsymbol{R}$ we know nothing save that it is acting on the mass-point Q, and this condition is satisfied by writing equation (4). Now Newton assumes on the sly that the plane π is *smooth*, in other words, that

(10) $$\boldsymbol{R} = R\boldsymbol{n}^0$$

with an appropriate R. Then (9), (10) imply

(11) $$N + R = 0.$$

As regards the equation of motion of Q as a free mass-point under the action of the central force $F\bar{\rho}^0$, namely

(12) $$m\ddot{\bar{\rho}} = F\bar{\rho}^0,$$

which is a corollary from (8) and (9), we shall not discuss the problem to what extend Newton could attack it by the aid of the mathematical artillery he had at his disposal in those times. (In the history of mechanics the solution of (12) is connected with the name of Binet, 1786–1856). As Truesdell says, "it is not the function of the historian to guess what Newton might have done of could have done" [8, p. 92]. The cold fact is that under the hypothesis (10) for a smooth plane π the condition (11) is necessary and sufficient for the plane motion of the mass-point Q.

We shall systematize our observations in connection with Newton's Problem XXXII in the form of several scholia.

Scholium 1. Newton's treatment does not make use of any system of reference.

Scholium 2. The reasons of the antecedent inference are rooted already in the formulation of *Lex II*, namely:

"Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur" [7, p. 12], where not a word is said about a system of reference with respect to which *mutationem motus* is calculated.

Scholium 3. This is a most regrettable circumstance since the validity of *Lex II* is not unconditional: *Lex II* holds only for the so-called inertial systems of reference.

Scholium 4. Newton makes no mention of any force acting on Q save $\boldsymbol{VS}$; in particular he does not even allude to the reaction $\boldsymbol{R}$.

Scholium 5. Therefore he does not require explicitly (10) expressing the smoothness of π.

Scholium 6. In the case of a non-smooth constraint π the whole of Newton's construction collapses.

Scholium 7. Newton's supposition "of which $\boldsymbol{ST}$ attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane" may be physically well-founded, but mathematically it is entirely groundless: a mathematical conclusion about forces and motions is legitimate if, and only if, it is derived from the equations of motion.

Scholium 8. Newton does not at all submit for discussion the question for the possibility of the geometrical constraint imposed on the mass-point Q: he considers this question apriorily settled.

Scholium 9. The foregoing conclusion stands in a causal connection with the *existence problem* in rational mechanics.

Scholium 10. All preceding ascertainments are by no means reprimands: *Impossibilium nulla est obligatio.* Juxtaposed with his epoch, Newton's performances seem superhuman. History in general, however, history of science, in particular, accepts no condolences. Our aim is to ascertain how the notion of mechanical constraint is conceived, born, and bred; and this goal cannot be achieved without the works of classics of mechanics, with all their merits and demerits. *Quod erat explicandum.*

It is pointless to expose the remaining propositions of section X of *Principia*. All of them concern particular motions of mass-points along given curve lines or surfaces; the treatment of any of them is imbued with the spirit of the age. It is true that the ratio of the mechanical content of *Principia* to its mythical fame is negligibly small. As Truesdell emphasizes, "except for certain simple if important special problems, Newton *gives no evidence of being able to set up differential equations of motion for mechanical systems* ... in Newton's *Principia* occur no equations of motion for systems of more than two free mass-points or more than one constrained mass-point" [8, p. 92–93]. At the same time, *quod sciamus*, this is the first mechanical work where constrained motions are considered in an almost modern way — in any case, by the aid of the momentum axiom.

Our goal has by no means been to propose a systematic historical investigation on the mechanical constraint notion. It is a hard nut to crack for a historian of rational mechanics who has made it his set purpose to track out the shady affair of development of dynamics of constrained systems. The hardships are due not only to the fact, properly explained in [17], that almost all motions Leonardo, Galileo, Tartaglia, and their successors had the chance to observe and, although extremely rarely, to measure, have been "impure" — that is to say, attended with the aftereffects of reactions now here of constraints and now there of resisting media; moreover, embarrassments come into being owing to the propensity of authors of mechanical writings to explain in fluent phrases, readily, ardently, and rather life-like to their readers somethings obscure to the authors in question themselves. Aggraviating the situation, the atmosphere becomes electrified by the fact that the mechanical constraint concept is, mathematically speaking, as yet in its historical phase; one cannot — as one can in other fields of mathematics — put one's finger on a certain line of a certain page in a certain book and pronounce the sacramental abracadabra: *this* is a mechanical constraint imposed on a rigid body.

Our occupations with related literary sources have confronted us with a non-incurious phenomenon. If one is apt to have faith in Truesdell's assessments of complicated mechanical situations — as we readily aknowledge we are — then one may find congenial the following opinion of this eminent author apropos of the early story of constrained mechanical systems:

"D'Alembert was the first to give a general rule for obtaining equations of motion of constrained systems. After decomposing the motion into two parts, one being 'natural' and the other due to the presence of the constraints, he asserts that the forces corresponding to the accelerations due to the constraints form a system in static equilibrium. Thus his principle is a development of one of the ideas of James Bernoulli's great paper of 1703; it is still closer to the principle stated even more obscurely by Daniel Bernoulli in his treatment of the hanging cord in 1732–1733 (published 1740). Like the older assertions of Descartes and Leibniz, it is a statement about the system as a whole, not about its parts, and it is insufficient to solve the general problems of dynamics; D'Alembert tacitly invoked other principles as well, but he got results; moreover, he was the first to derive a partial differential equation as the statement of a law of motion, the particular case being that of a heavy hanging cord" [8, p. 113].

Malum nullum est sine aliquo bono. When we for the first time were faced with these acknowledgements of D'Alembert's mechanical performances, we flat and plain could not co-ordinate them with the scientific image his mechanical writings have shaped in our consciousness. Using *vera rerum vocabula*, one cannot set at naught the fact that — more than half a century after *Principia* — D'Alembert declares in everyone's hearing in his *Traité* [9]:

"... j'ai, pour ainsi dire, détourné la vûe de dessus les *causes motrices*, pour n'envisager uniquement que le Mouvement qu'elles produisent; que j'aie entiérement proscrit les forces inhérentes au Corps en Mouvement, êtres obscurs & Métaphysiques, qui ne sont capables que de répandre les ténèbres sur une Science claire par elle-même" (p. XVI).

Moreover, as Truesdell *ibidem* makes out, "while Euler was soon to become the champion of Newton's approach to mechanics, D'Alembert started a new and opposed way of thinking. If the motion is known, he observed, then what we call forces are merely manifestations which may be calculated from it" (p. 113):

"Pourquoi donc aurions-nous recours à ce Principe dont tout le monde fait usage aujourd'hui, que la force accélératrice ou retardatrice est proportionelle à l'Elément de la vitesse; principe appuyé sur cet unique axiôme vague & obscur, que l'effet est proportionnel à sa cause. Nous n'examinerons point si ce Principe est de verité nécessaire; nous avouerons seulement que les prouves qu'on en a données jusqu'ici, ne nous paroissent pas fort convaincantes: nous ne l'adopterons pas non plus, avec quelques Geométres, comme de verité purement contingente, ce qui ruineroit la certitude de la Méchanique, & la réduiroit à n'être plus qu'une Science expérimentale: nous nous contenterons d'observer, que vrai ou doutreux, claire ou obscure, il est inutile à la Méchanique, & que per conséquent il doit en être banni [9, p. XI–XII].

Amicus Socrates, amicus Plato, sed magis amica veritas. In spite of our peerless veneration to Truesdell's erudition, independence of thought, and uprightness of judgements, let us penetrate the roots of matter of the problem of constrained systems of mass-points, in order to acquire an uninfluenced opinion on D'Alembert's dynamical performances. To this end, let us first see how the land lies as regards some indispensable definitions.

From here further let $Oxyz$ denote an inertial orthonormal right-hand orientated Cartesian system of reference with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ of the axes Ox, Oy, Oz, respectively, and let all derivatives of vector functions be taken with respect to $Oxyz$.

A mass-point P is said to be *free* if it may, according to the conditions of the particular dynamical problem under consideration, take any position in space and move with any velocity. If P is free, $\boldsymbol{r} = \boldsymbol{OP}$, and

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{13}$$

then $\boldsymbol{r}$ and

$$\boldsymbol{v} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \tag{14}$$

may accept any conceivable values.

A mass-point P is said to be *non-free*, if it is not free. Instead of "non-free" the adjective *constrained* is often used. According to both definitions of *free* and *non-free* mass-points, P is non-free if some restrictions on the admissible values of $\boldsymbol{r}$ or $\boldsymbol{v}$ are imposed by the conditions of the particular dynamical problem under consideration.

There are two, and two only, *modi operandi*, sanctioned by the age-old mechanical tradition, to make a mass-point P constrained, and both are described immediately below.

The first one consists in the hypothesis that P is compelled, by the very conditions of the particular dynamical problem under consideration, to remain on a given surface

$$f(x, y, z, t) = 0, \tag{15}$$

"given" meaning "completely determined" by the said conditions of the problem. At that, it is supposed that the relation

(16) $$\operatorname{grad} f \neq \boldsymbol{o}$$

holds provided by definition

(17) $$\operatorname{grad} f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.$$

The second one consists in the hypothesis that P is compelled, by the very conditions of the particular dynamical problem under consideration, to remain on a given curve line

(18) $$f_\nu(x, y, z, t) = 0 \qquad (\nu = 1,\ 2),$$

"given" meaning "completely determined" by the said conditions of the problem. At that, it is supposed that the relation

(19) $$\operatorname{grad} f_1 \times \operatorname{grad} f_2 \neq \boldsymbol{o}$$

holds provided by definition

(20) $$\operatorname{grad} f_\nu = \frac{\partial f_\nu}{\partial x}\boldsymbol{i} + \frac{\partial f_\nu}{\partial y}\boldsymbol{j} + \frac{\partial f_\nu}{\partial z}\boldsymbol{k} \qquad (\nu = 1,\ 2).$$

Both the surface (15) and the line (18) are called *geometrical constraints* imposed on the mass-point. If a geometrical constraint is independent of the time t, it is called *scleronomic*; otherwise it is called *rheonomic*.

According to a dynamical axiom, any geometrical constraint, imposed on a mass-point P, generates a force $\boldsymbol{R}$ acting on P. It is called the *reaction* of the geometrical constraint and, along with other forces acting on P, it predestinates the mechanical behaviour of P.

The meaning of the last statement is as follows. Let $\boldsymbol{F}$ be the resultant of all active forces acting on P. This means that $\boldsymbol{F}$ is the sum of all forces acting on P in accordance with the conditions of the particular dynamical problem under consideration. In other words, $\boldsymbol{F}$ is a vector quantity, wholly determined by the said conditions for any position $\boldsymbol{r}$ of P, for any velocity $\boldsymbol{v}$ of P, and for any moment t. This implies that $\boldsymbol{F}$ belongs to the *data* of the dynamical problem concerned, being a completely determined function

(21) $$\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)$$

of $\boldsymbol{r}$, $\boldsymbol{v}$ and t. In such a manner, the term *active forces* is a synonym of the terms *given*, or *known*, or *determined* by the conditions of the dynamical problem. On the contrary, nothing is known about the reactions $\boldsymbol{R}$ of the geometrical constraints imposed on the mass-point P save that they are *acting* on P, the meaning of the last term being specified immediately. Therefore, in contrast to the term *active forces*, the reactions of the constraints are called also *passive forces*.

Acting means that the motion of P is governed by the equation

(22) $$\frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F} + \boldsymbol{R},$$

m denoting the mass of P and $m\boldsymbol{v}$ being by definition the *momentum* of P with respect to $Oxyz$. In such a manner, (22) is a mathematically formalized expression of Newton's *Lex II* already quoted above. Now there is much to be said about the quasi-differential equation (22).

Let us turn back to our constraints (15) and (18). Under certain hypotheses concerning the analytic nature of the left-hand sides of (15) and (18), let us suppose that:

1. If (15), then there exist certain functions

$$x = x(q_1, q_2, t), \quad y = y(q_1, q_2, t), \quad z = z(q_1, q_2, t) \tag{23}$$

of certain arguments q_1, q_2, satisfying (15) identically, i.e.

$$f(x(q_1, q_2, t),\ y(q_1, q_2, t),\ z(q_1, q_2, t),\ t) = 0 \tag{24}$$

for any values of q_1 and q_2 in their definitional domain. Therefore, no restrictions are imposed on the "velocities" of q_1 and q_2, i.e. on their derivatives $\dot{q}_1$ and $\dot{q}_2$ with respect to the time t.

2. If (18), then there exist certain functions

$$x = x(q, t), \quad y = y(q, t), \quad z = z(q, t) \tag{25}$$

of a certain argument q, satisfying (18) identically, i.e.

$$f_\nu(x(q, t),\ y(q, t),\ z(q, t),\ t) = 0 \qquad (\nu = 1,\ 2) \tag{26}$$

for any value of q in its definitional domain. Therefore, no restrictions are imposed on the "velocity" of q, i.e. on its derivative $\dot{q}$ with respect to the time t.

In both cases (15) and (18) there exists a number l $(1 \leqq l \leqq 2)$ and l quantities

$$q_\lambda \qquad (\lambda = 1, \ldots, l), \tag{27}$$

mutually independent, together with their velocities

$$\dot{q}_\lambda \qquad (\lambda = 1, \ldots, l), \tag{28}$$

such that any position of the mass-point P consistent with the geometrical constraints imposed on P is uniquely determined by (27). Under these notations the number l is called the *amount of the degrees of freedom* (or simply *degrees of freedom*) of P, and (27) are called the *independent parameters* (or simply *parameters*) of P; sometimes (27) are called the *generalized co-ordinates*, and (28) — the *generalized velocities* of P.

Introducing the acceleration $\boldsymbol{w} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}$ and supposing the mass m of P invariable in the course of the time t, one may write down (22) in the form

$$m\boldsymbol{w} = \boldsymbol{F} + \boldsymbol{R}. \tag{29}$$

The definition of $\boldsymbol{w}$ and (13), (14) imply

$$\boldsymbol{w} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}. \tag{30}$$

Let by definition

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}, \tag{31}$$

$$\boldsymbol{R} = R_x\boldsymbol{i} + R_y\boldsymbol{j} + R_z\boldsymbol{k}. \tag{32}$$

Now (30)–(32) imply that the equation (29) is equivalent with the system of equations

(33) $$m\ddot{x} = F_x + R_x, \quad m\ddot{y} = F_y + R_y, \quad m\ddot{z} = F_z + R_z.$$

With a view to generality, the efficiency of which will become clear later, let us work with l instead of 2 in the case (23) and of 1 in the case (25). In other words, let us compute the left-hand sides of (33) at an arbitrary l. Then we obviously obtain

(34) $$\dot{x} = \sum_{\lambda=1}^{l} \frac{\partial x}{\partial q_\lambda} \dot{q}_\lambda + \frac{\partial x}{\partial t},$$

(35) $$\ddot{x} = \sum_{\lambda=1}^{l} \frac{\partial x}{\partial q_\lambda} \ddot{q}_\lambda + \sum_{\lambda=1}^{l}\sum_{\mu=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu + 2\sum_{\lambda=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial t} \dot{q}_\lambda + \frac{\partial^2 x}{\partial t^2},$$

provided

(36) $$\frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} = \frac{\partial^2 x}{\partial q_\mu \partial q_\lambda}, \quad \frac{\partial^2 x}{\partial q_\lambda \partial t} = \frac{\partial^2 x}{\partial t \partial q_\lambda}$$

(λ, $\mu = 1, \ldots, l$), and two similar expressions for $\ddot{y}$ and $\ddot{z}$. Let us lay a special emphasis upon the fact that all coefficients of the quantities $\ddot{q}_\lambda$, $\dot{q}_\lambda \dot{q}_\mu$, $\dot{q}_\lambda$ (λ, $\mu = 1, \ldots, l$), as well as the free member $\frac{\partial^2 x}{\partial t^2}$ in (35) are completely determined functions of the parameters (27) of the mass-point, since by hypothesis the functions (23), as well as (25) of (27) are wholly certain.

On the other hand, as already underlined, the active forces (21) are entirely determined functions of $\boldsymbol{r}$, $\boldsymbol{v}$, and t. Now, with a view to (13), (14), (23), (25), (34) (and similar for $\dot{y}$ and $\dot{z}$), one arrives at the conclusion that (21) may be written in the form

(37) $$\boldsymbol{F} = \boldsymbol{F}(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t),$$

where the right-hand side is a completely determined function of the parameters (27), of the velocities (28), and possibly of the time t. Considering the decomposition (31), one may now quite lawfully state that the same holds for the projections F_x, F_y, F_z of $\boldsymbol{F}$ on the axes Ox, Oy, Oz, respectively.

Summing up, we may now state that the mechanical behaviour of the non-free mass-point P subjected to the geometrical constraint (15) or (18) is governed by the following system of differential-algebraic relations, qualified above as "quasi-differential equations":

(38) $$\sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + R_x, \quad \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + R_y, \quad \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + R_z,$$

where by definition

(39) $$X_\lambda = \frac{\partial x}{\partial q_\lambda}, \quad Y_\lambda = \frac{\partial y}{\partial q_\lambda}, \quad Z_\lambda = \frac{\partial z}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l),$$

$$
(40) \quad \begin{cases}
X = F_x - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 x}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 x}{\partial t^2}, \\
Y = F_y - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 y}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 y}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 y}{\partial t^2}, \\
Z = F_z - \sum_{\lambda=1}^{l} \sum_{\mu=1}^{l} \frac{\partial^2 z}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda \dot{q}_\mu - 2 \sum_{\lambda=1}^{l} \frac{\partial^2 z}{\partial q_\lambda \partial t} \dot{q}_\lambda - \frac{\partial^2 z}{\partial t^2}.
\end{cases}
$$

As it is immediately seen from (39), (40) and (37), (31), the equations (38) of motion of P involve unknown quantities of two entirely different kinds:

1. *Infinitesimal* unknowns, that is to say the parameters (27) of P together with their first and second derivatives with respect to the time t (the latters being present linearly).

2. *Finitesimal* unknowns, that is to say the projections of the reaction $\boldsymbol{R}$ of the constraint acting on P (the latters also being present linearly).

This circumstance is the reason calling the equations (38) *differential-algebraic* and *quasi-differential.*

The equations (38) represent the most adequate formal-mathematical expression of the dynamical problem under consideration. Therefore they deserve a special attention.

As any mathematical problem, the system of equations (38) engenders two challenges:

1. Existence?

2. Uniqueness?

It stands to reason, it would be an extravagant luxury to answer the second question before the first one is answered in the affirmative: it could be compared to taking down finger-prints of a ghost. And yet, the course of the solutions of mathematical problems is traditionally topsyturvied. Habitually first and foremost, disregarding the existence-problem, a provisional solution is sought by the problem solver, and only afterwards it is proved, commonly by means of an immediate check-up, that this potential solution is an actual one. At that, as a rule, the existence-problem is mathematically incomparably harder to solve than the uniqueness-problem.

Horribile visu, horribile dictu, horribile auditi — horresco referens: in rational mechanics, in general, and in rigid dynamics, in particular, the existence problem does not exist at all. Or, more correctly, it exists like the ozone-hole: everybody knows and nobody cares. Evidence? — Any treatise on analytical mechanics you like: the choice is yours. To express this statement in concrete form by indicating one particular from among countless amount of dynamical textbooks, books of problems, treatises, monographs, and articles would mean to do unjustice to the author of the selected work, converting him into a scapegoat for a widespread sin. And yet, under this reservation, we shall quote a practical example — solely in order not to be upbraided with groundless idle talk. As regards the pitiable absence of mind of ancient and modern mechanicians in connection with the existence-problem, it is in a full agreement with Seneca's observation *Quae fuerant vitia, mores sunt.*

The scapegoat in question is the Treatise [16] on rigid dynamics, published comparatively recently. Turning over its pages at random, we arrive at the problems of motion of a rod in a rotating plane (p. 119), rolling penny (p. 120–122), sleeping top (p. 152–156), sphere on turntable (p. 207–209), sphere on a rotating inclined plane (p. 209–211), sphere rolling on a fixed surface (p. 211–213), and so on, and so forth, etcetera. (As regards examples from other literary sources, ***nomen illis legio.***) Are in the solutions of all those problems in [16] answers of the existence-question? Not one jot! Not a whit! By no means! There is even not the least hint for such a thing. Incredible? — Incredible. Fact? — Fact. If somebody dares contest this statement, then there is a sole possible answer: *Hic Rhodus, hic salta.* That is to say, hic Pars' Treatise, *hic* points a finger at an existence proof.

Saeculi vitia, non hominis. The cause for this state of affairs in analytical dynamics is rooted in its dual nationality: down to the present day it is simultaneously a citizen both of United Kingdom's Mathematics and of United States' Physics. At least such is the mental disposition of most who work in this domain, in spite of the danger to fall between two stools. Indeed, mechanics is occupied studing motions, and motion is something that exists — isn't it? Then why worrying about such a nonsense as existence-problem?

Maybe. Maybe not. Do you remember the nursery rhymes:

> "For the want of a nail the shoe was lost,
> For the want of a shoe the horse was lost,
> For the want of a horse the rider was lost,
> For the want of a rider the battle was lost,
> For the want of a battle the kingdom was lost —
> And all for the want of a horseshoe nail."

Let us now make an *en gros* assessment of the situation. Our dynamical problem of the mechanical behaviour of the mass-point P, submitted to the geometrical constraint (15) or (18) and to the action of active forces with resultant (21), consists, first, in determining (if such exist) the parameters (27) of P as functions

(41) $$q_\lambda = q_\lambda(t) \qquad (\lambda = 1, \ldots, l)$$

of the time t, P starting from a fixed though wholly arbitrary initial position

(42) $$q_{\lambda_0} = q_\lambda(0) \qquad (\lambda = 1, \ldots, l)$$

with a fixed though completely arbitrary initial velocity

(43) $$\dot{q}_{\lambda_0} = \dot{q}_\lambda(0) \qquad (\lambda = 1, \ldots, l);$$

and, second, in determining (if such exists) the reaction $\boldsymbol{R}$ of the corresponding geometrical constraint, that is to say, the projections

(44) $$P_x,\ P_y,\ P_z$$

of $\boldsymbol{R}$ on the axes Ox, Oy, Oz of $Oxyz$, respectively, according to (32). Consequently, the unknown quantities of the mathematical problem under consideration are 5 in number in the case of constraint (15) (since then $l = 2$) and 4 in number in the case of constraint (18) (since then $l = 1$). Since the number of the equations (38) with (39), (40) we have at our disposal for the determination of those $l+3 > 3$ unknown quantities (41), (42) is exactly 3, our dynamical problem is, in the general case at least, mathematically indeterminate.

This conclusion is as two-faced as Ianus. Its favourable face is that one may hope that the existence-problem might be answered in the affirmative; its unfavourable face is that the answer might be as arbitrary as to seem meaningless. Both expectations are vindicated by reality.

There is one, and one only, way to make a constrained mass-point dynamical problem completely determined mathematically, and it consists in the hypothesis that the corresponding geometrical constraint is *smooth*. Physically the concept of *smoothness* is reduced to the idea that the corresponding surface or the corresponding curve line is *polished* like a mirror. The same physical idea suggests that the constraint generates *no friction*. Mathematically *smoothness* means that the reaction of the constraint is *normal* to the latter, i.e.

$$\boldsymbol{R} \times \operatorname{grad} f = 0 \tag{45}$$

in the case (15) and

$$\boldsymbol{R} . \operatorname{grad} f_1 \times \operatorname{grad} f_2 = 0 \tag{46}$$

in the case (18).

Indeed, (45) and (16) imply that there exists a scalar μ with

$$\boldsymbol{R} = \mu \operatorname{grad} f. \tag{47}$$

Now (47), (17), and (32) imply that the equations (38) take the form

$$\begin{cases} \sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + \mu \dfrac{\partial f}{\partial x}, \\ \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + \mu \dfrac{\partial f}{\partial y}, \\ \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + \mu \dfrac{\partial f}{\partial z}. \end{cases} \tag{48}$$

Similarly, (46) and (19) imply that there exist scalars μ_1 and μ_2 with

$$\boldsymbol{R} = \mu_1 \operatorname{grad} f_1 + \mu_2 \operatorname{grad} f_2. \tag{49}$$

Now (49), (20), and (32) imply that the equations (38) take the form

$$\begin{cases} \sum_{\lambda=1}^{l} X_\lambda \ddot{q}_\lambda = X + \mu_1 \dfrac{\partial f_1}{\partial x} + \mu_2 \dfrac{\partial f_2}{\partial x}, \\ \sum_{\lambda=1}^{l} Y_\lambda \ddot{q}_\lambda = Y + \mu_1 \dfrac{\partial f_1}{\partial y} + \mu_2 \dfrac{\partial f_2}{\partial y}, \\ \sum_{\lambda=1}^{l} Z_\lambda \ddot{q}_\lambda = Z + \mu_1 \dfrac{\partial f_1}{\partial z} + \mu_2 \dfrac{\partial f_2}{\partial z}. \end{cases} \tag{50}$$

In both cases (48) and (50) the number of the unknown quantities equals the number of the equations available for their determination, namely 3: in the case (15) the unknowns are q_1, q_2 and μ, and in the case (18) they are q, μ_1 and μ_2. *Q. E. D.*

Naturally, in both cases a horseshoe is still wanting: the solution of the existence-problem.

The situation around a single mass-point being, in such a manner, settled on principle, let us now turn back to D'Alembert's "Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une maniére quelconque". Let us carry ourselves mentally in the age when he wrote his *Traité*. Truesdell might be helpful again:

"... a large part of the literature of mechanics for sixty years following the *Principia* searches various principles with a view to finding the equations of motion for the systems Newton had studied and for other systems nowadays thought of as governed by the 'Newtonian' equations" [8, p. 92–93].

Now all mathematicians of many decades after *Principia* passionately strove for disclosing the mysteries mystifying the motions of the most enigmatic of all mechanical systems called *rigid bodies*. Most of them, D'Alembert in the first place, chose the most natural, most obvious, and most wrong way: the idea that a rigid body is an aggregate of mass-points, constrained in such a manner that their mutual distances remain invariable. The rise and fall of this idea is reflected in the *Traité de Dynamique* and *Méchanique Analitique*. But let us not go so far. Let us first formulate the basic notions of a system of constrained mass-points.

Let Σ be such a system, i.e. a set of n mass-points P_ν with masses m_ν and radius-vectors $\boldsymbol{r}_\nu = \boldsymbol{OP}_\nu$ ($\nu = 1, \ldots, n$). Some of the points of Σ may be free, some may be constrained to remain on certain surfaces, and some may be compelled to slide on certain curve lines. If one applies to any of these points the arguments used in the case of a single mass-point adduced above, one sees at once that for any of them there exists a number, at least 1 and at most 3, of mutually independent parameters determining its admissible by the corresponding geometrical constraints positions in space; let (27) be those parameters for all the points of Σ arranged in a definite order, say according to the increasing number ν of the point P_ν.

Besides, let $\boldsymbol{F}_\nu$ and $\boldsymbol{R}_\nu$ be the resultants respectively of the active forces and the reaction of the constraint imposed on the ν-th point of Σ, and let $\boldsymbol{w}_\nu = \ddot{\boldsymbol{r}}_\nu$ ($\nu = 1, \ldots, n$) be its acceleration with respect to $Oxyz$. Then, obviously, according to Newton's *Lex II*, the motion of P_ν will be governed by the equation

(51) $$m_\nu \boldsymbol{w}_\nu = \boldsymbol{F}_\nu + \boldsymbol{R}_\nu \qquad (\nu = 1, \ldots, n).$$

The dynamical problem we are faced with in such a manner, concerning the mechanical behaviour of Σ, consists in solving the system of equations (51) under entirely arbitrary, though fixed, initial conditions (42), (43), i.e. in discovering such functions (41) and such linear unknown quantities

(52) $$R_{\nu x},\ R_{\nu y},\ R_{\nu z} \qquad (\nu = 1, \ldots, n)$$

provided

(53) $$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n),$$

that satisfy (51) identically, taking (42), (43) into account.

Does D'Alembert solve this problem in [9]? Is he "the first to give a general rule for obtaining equations of motion of constrained systems" as Truesdell generously states? Do we find in [9] the system (51) or at least some semblance, some similarity, some likeness of it?

Certainly not. Nothing of the kind. Never a whit. *Traité de Dynamique* is as far from (51) as Stahl from Lavoisier.

Let us lay special emphasis on the fact of extraordinary importance on principle that to describe mathematically the mechanical behaviour of the system Σ means to determine the dynamical demeanour of any mass-point entering into the composition of Σ. This means to know the functions

$$(54) \qquad \boldsymbol{r}_\nu = \boldsymbol{r}_\nu(t) \qquad (\nu = 1, \ldots, n)$$

if the initial conditions

$$(55) \qquad \boldsymbol{r}_{\nu 0} = \boldsymbol{r}_\nu(0), \qquad \boldsymbol{v}_{\nu 0} = \boldsymbol{v}_\nu(0) \qquad (\nu = 1, \ldots, n)$$

are prescribed provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$, as well as $\boldsymbol{R}_\nu$ for any $\nu = 1, \ldots, n$. Now in D'Alembert's *Traité* there is not the slightest trace of a solution of this problem even in its most elementary case $n = 2$.

Extending our analysis in connection with the system (51), let us note that, in contrast to the case of a single mass-point P, when the active force $\boldsymbol{F}$ acting on P may, according to (21), depend only on the position $\boldsymbol{r}$ and the velocity $\boldsymbol{v}$ of P itself, in the case of a system Σ of mass-points the active force $\boldsymbol{F}_\nu$ acting on P_ν may depend on the positions $\boldsymbol{r}_\mu$ and the velocities $\boldsymbol{v}_\mu$ of all the points P_μ $(\mu = 1, \ldots, n)$ of Σ. In other words, in the general case it is supposed that the active forces $\boldsymbol{F}_\nu$ are completely determined functions

$$(56) \qquad \boldsymbol{F}_\nu = \boldsymbol{F}_\nu(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n; \boldsymbol{v}_1, \ldots \boldsymbol{v}_n; t)$$

of all $\boldsymbol{r}_\nu$, $\boldsymbol{v}_\nu$ $(\nu = 1, \ldots, n)$ and possibly of the time t. In such a manner, although the solution of the system (51) requires the determination of (54) provided (55), and of (52) provided (53) for any particular $\nu = 1, \ldots, n$, the integration of the system (51) of quasi-differential equations cannot be accomplished separately for any particular ν, since (51) represents a system of interdependent relations.

We proceed now to one of the greatest mistifications in all the history of rational mechanics. *Chapitre Premier. Exposition du Principe of Second Partie. Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres d'une maniére quelconque, avec plusieurs applications de ce Principe* of [9] begins with the following declaration:

"Les Corps n'agissent les uns sur les autres que de troi maniéres différentes qui nous soient connus: ou par impulsion immédiate, comme dans le choc ordinaire, ou par le moyen de quelque Corps interposé entr'eux, & auquel ils sont attachés, ou enfin, par une vertu d'attraction réciproque, comme sont dans le systême Newtonien le Soleil & les Planetes. Les effets de cette derniére espece d'action ayant été suffisamment examinés, je me bornerai à traiter ici du Mouvement des Corps qui se choquent d'une maniére quelconque, ou de ceux qui se tirent par des fils ou des verges inflexibles. Je m'arrêterai d'autant plus volontiers sur ce sujet, que les plus grands Géométres ne nous ont donné jusqu'à présent qu'un très petit nombre de Problêmes de ce genre, & que j'espere, par la Méthode générale qui je vais donner, mettre tous ceux qui sont au fait du calcul & des Principes de la Mécanique, en état de résoudre les plus difficiles Problêmes de cette espece" (p. 49–50).

A *Définition* follows:

"J'appelerai dans la suite *Mouvement* d'un Corps, la vitesse de ce même Corps considerée en ayant égard à sa direction, & par *quantité de Mouvement*, j'entendrai à l'ordinaire le produit de la masse par la vitesse" (p. 50).

The formulation of *Problême general* reads:

"*Soi donné un systême de Corps disposés les uns par rapport aux autres d'une maniére quelconque; et supposons qu'on imprime à chacun de ces Corps un Mouvement particulier, qu'il ne puisse suivre à cause de l'action des autres Corps, trouver le Mouvement que chaque Corps doit prendre*" (*ibid.*).

Before proceeding to D'Alembert's "Solution" let us fix our eyes on the formulation of *Problême general.* First of all, it sticks out a mile that D'Alembert's *Corps* are, as it is, purely and simply *mass-points* and *no bodies* at all: the formulation of *Problême general* attaches *Mouvement* to *chaque Corps*, that is to say *vitesse* according to D'Alembert's "Définition", and velocity is a mechanical entity that becomes wholly meaningless when attached to rigid bodies — it is meaningful only when localized to points. The second circumstance that cannot slip anybody's attention is that D'Alembert's *Problême general* is, when all is said and done, a purely kinematical proposition with not an atom of dynamics. Now one is at a loss how could D'Alembert, on the basis of a purely kinematical *Principe general*, redeem his promise made with such an aplomb in the title of the *Seconde Partie* of the work, namely to "trouver le Mouvemens de plusieurs Corps qui agissent les uns sur les autres d'une maniére quelconque"? Be that as it may, let us proceed, after these remarks, to D'Alembert's *Solution*:

"Soient *A*, *B*, *C*, & c. les Corps qui composent le systême, & supposons qu'on leur ait imprimé les Mouvemens *a*, *b*, *c*, & c. qu'ils soient forces, a cause de leurs action mutuelle, de changer dans les Mouvemens a, b, c, & c. Il est clair qu'un peut regarder le Mouvement *a* imprimé au Corps *A* comme composé du Mouvement a qu'il a pris, & d'un autre Mouvement α; qu'on peut de même regarder les Mouvemens *b*, *c*, & c. comme composés des Mouvemens b, β; c, $\varkappa$; & c. d'ou il s'ensuit que le Mouvement des Corps *A*, *B*, *C*, & c. entr'eux auroit été le même, si au lieu de leur donner les impulsions *a*, *b*, *c*, on leur eût donné à la fois les doubles impulsions a, α; b, β; c, $\varkappa$, etc. Or par la supposition, les Corps *A*, *B*, *C*, & c. ont pris d'eux-mêmes les Mouvemens a, b, c; etc. Donc les Mouvemens α, β, $\varkappa$ & c. doivent être tels qu'ils ne dérangent rien dans les Mouvemens a, b, c, etc. c'est-à-dire, que si les Corps n'avoient reçu que les Mouvemens α, β, $\varkappa$ & c. ces Mouvemens auroient dû se détruire mutuellement, & le systême demeurer en repos.

Delà résulte le Principe suivant, pour trouver le Mouvement de plusieurs Corps qui agissent les unes sur les autres. *Décomposés les Mouvemens a, b, c* & c. *imprimés a chaque Corps, chacun en deux autres* a, α; b, β; c, $\varkappa$; & c. *qui soient tels, que si l'on n'eût imprimé aux Corps que les Mouvemens* a, b, c, & c. *ils eussent pû conserver ces Mouvemens sans se nuire réciproquement*; *et que si on ne leur eût* *imprimé que les Mouvemens α, β, ϰ, & c. le système fut demeuré en repos*; il est clair que a, b, c seront les Mouvemens que ces Corps prendront en vertu de leur action. Ce *Q. F. Trouver*" (*ibid.*, p. 50–51).

This "Solution" of D'Alembert's provides the occasion for quite a lot of commentaries, all of them curious, instructive, and beneficial. We shall, however, spare them for the time being, postponing a detailed discussion of the preceding text for immediate future. For the time being we shall restrict our attention on the sequels this *Principe général* of D'Alembert has had in the subsequent development of rigid dynamics.

Disregarding the *Eigenwerte* D'Alembert himself placed on his principle in the *Préface* of [9] and in its application to various problems of mechanics in this very work, let us first note that some decades later the same principle has been rediscovered and brought back to life by Lagrange, D'Alembert's true spiritual son. Meanwhile, let us read the commentary of the Russian translator of [9], made immediately after the principle is announced in the book:

"В настоящем n° Даламбером формулируется то правило, которое ныне называется „принципом Даламбера". Как видно, этот „принцип" выглядит у его автора совсем не так, как он излагается ныне в учебниках. Форма, близкая к современной, придана была принципу Даламбера Лагранжем в его „Аналитической механике".

Даламбер дал изложение своего „принципа" и в „Энциклопедии", в статье „Dynamique" (Динамика). Приведем здесь это изложение буквально ..." [24, с. 333–334].

The author of this quite equitable finding takes into consideration several somethings, the first of which is the singing praise to the skies of D'Alembert's *Principe général* in *Section Première. Sur les différents principes de la dynamique* of *Second Partie, La Dynamique* of [10] by Lagrange, who has been 7 years old when D'Alembert published his *Principe* and, as regards the penetrating into the roots of matter, did not fledge much since.

For the time being at least we shall wind up our exposition by a mathematical *coup de grâce, in arenam cum aequalibus descendi.*

Let us rewrite (51) in the form

$$m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu = \boldsymbol{o} \qquad (\nu = 1, \ldots, n), \tag{57}$$

and let regard the formal expressions

$$A_\lambda = \sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu) \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l) \tag{58}$$

and

$$\delta A = \sum_{\lambda=1}^{l} A_\lambda \delta q_\lambda, \tag{59}$$

δq_λ denoting arbitrary infinitesimal variations of q_λ ($\lambda = 1, \ldots, l$), respectively, not necessarily co-ordinated with the dynamical equations (51), the $\boldsymbol{r}_\nu$ ($\nu = 1, \ldots, n$)

in (58) being subordinated to the geometrical constraints imposed on the system Σ of mass-points. The following computations are traditional. The identities

$$\boldsymbol{v}_\nu = \sum_{\lambda=1}^{l} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \dot{q}_\lambda + \frac{\partial \boldsymbol{r}_\nu}{\partial t} \qquad (\nu = 1, \ldots, n) \tag{60}$$

imply

$$\frac{\partial \boldsymbol{v}_\nu}{\partial \dot{q}_\lambda} = \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l;\ \nu = 1, \ldots, n) \tag{61}$$

and

$$\frac{\partial \boldsymbol{v}_\nu}{\partial q_\mu} = \sum_{\lambda=1}^{l} \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial q_\lambda} \dot{q}_\lambda + \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial t} \tag{62}$$

$(\mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$. Now (62) and

$$\frac{d}{dt} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\mu} = \sum_{\lambda=1}^{l} \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\lambda \partial q_\mu} \dot{q}_\lambda + \frac{\partial^2 \boldsymbol{r}_\nu}{\partial t \partial q_\mu} \tag{63}$$

$(\mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$ imply

$$\frac{d}{dt} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\mu} = \frac{\partial \boldsymbol{v}_\nu}{\partial q_\mu} \qquad (\mu = 1, \ldots, l;\ \nu = 1, \ldots, n) \tag{64}$$

provided

$$\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\lambda \partial q_\mu} = \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial q_\lambda}, \qquad \frac{\partial^2 \boldsymbol{r}_\nu}{\partial t \partial q_\mu} = \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q_\mu \partial t} \tag{65}$$

$(\lambda, \mu = 1, \ldots, l;\ \nu = 1, \ldots, n)$. If by definition

$$Q_\lambda^{(a)} = \sum_{\nu=1}^{n} \boldsymbol{F}_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda}, \qquad Q_\lambda^{(p)} = \sum_{\nu=1}^{n} \boldsymbol{R}_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \tag{66}$$

$(\lambda = 1, \ldots, l)$ and

$$T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu v_\nu^2, \tag{67}$$

then (58), (61), (64), (66), (67) imply

$$A_\lambda = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} - Q_\lambda^{(p)} \qquad (\lambda = 1, \ldots, l). \tag{68}$$

If the constraints imposed on the system Σ of mass-points are *smooth*, then the second definition (66) implies

$$Q_\lambda^{(p)} = 0 \qquad (\lambda = 1, \ldots, l) \tag{69}$$

and (68), (69), (58) imply

$$\sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu) \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} \tag{70}$$

($\lambda = 1, \ldots, l$), whence it is immediately seen that Newton's *Lex II* (57) applied on Σ automatically leads to Lagrange's dynamical equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda^{(a)} = 0 \qquad (\lambda = 1, \ldots, l). \tag{71}$$

As a matter of fact, the *fundamental identities of Lagrangean formalism* (70) at once display that the left-hand sides of Lagrange's dynamical equations (71) are, purely and simply, linear combinations of the projections of the left-hand sides of Newton's *Lex II* (57) on axes, defined by the directions

$$\frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l;\ \nu = 1, \ldots, n) \tag{72}$$

co-ordinated with the geometrical constraints imposed on Σ.

If by definition

$$\delta \boldsymbol{r}_\nu = \sum_{\lambda=1}^{l} \frac{\partial \boldsymbol{r}_\nu}{\partial q_\lambda} \delta q_\lambda \qquad (\nu = 1, \ldots, n), \tag{73}$$

then (58), (59) imply

$$\delta A = \sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu), \tag{74}$$

and the relation $\delta A = 0$, i.e.

$$\sum_{\nu=1}^{n} (m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu - \boldsymbol{R}_\nu)\delta \boldsymbol{r}_\nu = 0, \tag{75}$$

is usally accepted in the traditional dynamical literature as a modern mathematical expression of D'Alembert's original *Principe général*. For the time being at least we shall refrain from commentaries as to the degree of adequacy of such an interpretation, in accordance with *Davus sum, non Oedipus* of Terentius.

As far as our experience goes, the composing of the true history of the theory of mechanical constraints is as yet postponed *ad Calendas Graecas*. As already emphasized and as maybe it becomes transparent from our exposition, this is a back-breaking task. Neither shall we dare penetrate imprudently the vast white fields of this *terra incognita*. One thing is certain: before one sets one's foot in its Arcadia, one must cross the rocky mountains of Lagrangean formalism.

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