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ON VOLUME TYPE FUNCTIONALS IN EUCLIDEAN GEOMETRY

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The paper is concerned with some aspects of the theory of volumes in Euclidean space. In this context, it is shown that there exists a solution of Cauchy's functional equation, which is absolutely nonmeasurable with respect to the class of all translation invariant measures on the real line **R**, extending the Lebesgue measure on **R**.

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The concept of volume for sufficiently simple geometric figures is one of the most important in classical geometry of Euclidean spaces. Discussions of this concept occupy a substantial place in all standard university lecture courses in Euclidean geometry. There are many text-books, manuals and monographs devoted to the subject (see, for example, [2], [3], [5], [6], [7], [8], [10], [11], [12]).

The deep notion of volume type functionals is closely tied with several interesting and important geometric topics, such as equidecomposability theory (including well-known paradoxes about partitions of certain geometric bodies), dissections of figures into finitely many other figures of a prescribed type, the behavior of the volume function under Minkowskis sum of point sets, etc.

One of the principal problems which arises here is to extend the function of elementary volume for simple geometric figures to a volume defined for a maximally large class of figures. This problem is successfully solved within framework of modern theory of invariant measures and its solution heavily depends on purely algebraic properties of a basic group of transformations of the Euclidean space.

These aspects are partially touched upon in H. Hadwiger's widely known monograph [3] in which the role of nontrivial solutions of Cauchy's functional equation is shown and stressed. As was proved by Frechet, all of those solutions are nonmeasurable in the Lebesgue sense. However, a much deeper result about such solutions can be established (see Theorem 2 below).

The present paper is devoted to some aspects highlighting profound connections between elementary theory of volume with general methods of the theory of additive functions having bad descriptive properties.

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{Q}$ is the set of all rational numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^n$ is the n-dimensional Euclidean space ($n \geq 1$);

dom(μ) is the domain of a given measure μ on $\mathbf{R}^n$.;

ran(f) is the range of a given function f;

λ_n is the classical Lebesgue measure on $\mathbf{R}^n$.

Let D_n be the group of all isometric transformations of $\mathbf{R^n}$ and let S_n be the ring of sets generated by the collection of all coordinate parallelepipeds of $\mathbf{R^n}$.

Let G be a subgroup of D_n. A functional V_n is called an elementary G-volume on $\mathbf{R^n}$ if the following four conditions hold:

(1) V_n is non-negative:

$$(\forall X)(X \in S_n \Rightarrow V_n(X) \geq 0);$$

(2) V_n is additive:

$$(\forall X)(\forall Y)(X \in S_n \wedge Y \in S_n \wedge X \cap Y = \emptyset \Rightarrow V_n(X \cup Y) = V_n(X) + V_n(Y));$$

(3) V_n is G-invariant:

$$(\forall g)(\forall X)(g \in G \wedge X \in S_n \Rightarrow V_n(g(X)) = V_n(X));$$

(4) $V_n(\triangle_n) = 1$, where $\triangle_n = [0,1]^n$ denotes the unit coordinate cube in $\mathbf{R^n}$.

The above-mentioned conditions are usually treated as Axioms of Invariant Finitely Additive Measure (see, for instance, [3], [5]).

If condition (2) is replaced by the countable additivity condition, then we obtain the definition of a G-measure (cf. [8]).

It is well known that the classical Jordan measure on $\mathbf{R^n}$ is a natural example of G-volume in $\mathbf{R^n}$. Respectively, a certain extension of Jordan measure to a sufficiently large class of subsets of $\mathbf{R^n}$ is the standard Lebesgue measure (see [3], [4]). In some sense, the latter class of sets is maximal, because within the framework of constructive methods it is impossible to further enlarge this class. This result is due to R. Solovay who was able to construct a model of set theory with a restricted (countable) version of the Axiom of Choice, in which all subsets of the space $\mathbf{R^n}$ turn out to be measurable in the Lebesgue sense (see [13]).

Notice that, by using the Zorn lemma, any D_n-volume on $\mathbf{R^n}$ ($n \geq 3$) can be extended to a maximal (by the inclusion relation) D_n-volume on $\mathbf{R^n}$, but the geometrical structure of the domain of such a maximal D_n-volume is not known and this problem seems to be of some interest.

It is well known that if an additive function

$$f : \mathbf{R} \to \mathbf{R}$$

satisfied one of the following conditions, then there exists a real constant k such that $f(x) = k \cdot x$ for all $x \in \mathbf{R}$:

(a) f is continuous at a point of $\mathbf{R}$;

(b) f is monotone on an interval of positive length;

(c) f is bounded from above (or below) on an interval of positive length;

(d) f is locally integrable in the Lebesgue sense;

(e) f is Lebesgue measurable;

(f) f is a Borel function;

(g) f has the Baire property.

It is clear that if a functional V_1 satisfied the condition (1) (that is, $V_n \geq 0$), then

$$V_1 = kx, \qquad (k \in \mathbf{R}).$$

From the measure-theoretical point of view, there are many interesting and important facts concerning G-volumes. The most famous among them is due to Banach.

Theorem 1 (Banach). *In the cases $n = 1$ and $n = 2$ there exists a non-negative additive functional defined on the family of all bounded subsets of the Euclidean space $\mathbf{R}^n$, invariant under the group of all isometries of $\mathbf{R}^n$ and extending the Lebesgue measure λ_n.*

The proof of Theorem 1 can be found e.g. in [1], [8].

It directly follows from this theorem that if X and Y are two Lebesgue measurable subsets of $\mathbf{R}^n$ (n = 1,2) such that $\lambda_n(X) \neq \lambda_n(Y)$, then X and Y are not finitely equidecomposable subsets of $\mathbf{R}^n$.

Example 1. In the case $n \geq 3$, we have no analogous result because of the famous Banach-Tarski paradox. As a remark, notice that if $n \geq 3$, then the group D_n possesses paradoxical properties which are implied by the fact that this group contains a free subgroup generated by two independent rotations. Actually, just from the latter circumstance follows the Banach-Tarski paradox stating that any two bounded subsets of the Euclidean space $\mathbf{R^n}$, $n \geq 3$ with nonempty interiors are equivalent by finite decompositions. Namely, if A and B are two bounded subsets of $\mathbf{R}^n$ $(n \geq 3)$, both of which have a nonempty interior, then there are partitions of A and B into a finite number of disjoint subsets

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k,$$

$$B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k,$$

such that for each $i \in [1, k]$ the sets A_i and B_i are D_n-congruent.

The proof of this paradox is essentially based on the Hausdorff theorem (see [1]) which states that if one removes a certain countable subset of the sphere S^2 in $\mathbf{R^3}$, then the remainder can be divided into three disjoint subsets A, B and C such that A, B, C and $B \cup C$ are mutually congruent under the group of all rotations of $\mathbf{R}^3$ about its origin. In particular, it follows from the above theorem that on S^2 there is no finitely additive non-negative normalized functional defined on all of its subsets such that the values of this functional on congruent sets are equal to each other.

From the Hausdorff theorem also follows that on the Euclidean space $\mathbf{R}^n$ $(n \geq 3)$ there exists no D_n-volume defined on the family of all subsets of $\mathbf{R}^n$.

Example 2. There is also a somewhat paradoxical result in the case of the plane $\mathbf{R}^2$. Namely, let G denote the group of all those affine transformations of $\mathbf{R}^2$ which preserve the area, i.e., g belongs to G if and only if $|det(g)| = 1$ (this group is much wider than D_2). According to the theorem of von Neumann, if A and B are two bounded subsets of the plane $\mathbf{R}^2$, both of which have nonempty interiors, then there are partitions of A and B into a finite number of disjoint subsets

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k,$$

$$B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k,$$

such that for each $i \in [1, k]$ the sets A_i and B_i are G-congruent.

A detailed discussion of Example 2 see in [3].

Let us restrict our further considerations to the ring of all polyhedrons in the space $\mathbf{R}^n$ (see, e. g., [3]).

Hilbert's third problem is formulated as follows:

Given any two polyhedrons of equal volume, is it always possible to cut the first into finitely many polyhedrons which can be reassembled to yield the second?

Two polyhedrons are equidecomposable if the first of them admits a cutting into finitely many polyhedrons which can be reassembled to yield the second one. Obviously, any two equidecomposable polyhedrons have the same volume. The converse assertion is not true. For example, the unit cube in $\mathbf{R}^3$ and a regular tetrahedron of volume 1 are not equidecomposable, which gives a negative solution of Hilbert's third problem.

In connection with this problem, for every polyhedron P, Dehn introduced some kinds of functionals, now widely known as the Dehn invariants, which are defined as follows:

$$D_f(P) = \sum l(e)f(\alpha), \qquad (*)$$

where $l(e)$ is the length of an edge e of P, α is the value of the dihedral angle of P between the two faces meeting at e, and f is an additive function such that $f(\pi) = 0$.

In other words, a function f is any solution of the Cauchy functional equation

$$f(x + y) = f(x) + f(y),$$

such that $f(\pi) = 0$.

Notice also that, the sum in $(*)$ is taken over all edges of the polyhedron P.

It is well known that any nonzero additive function f which participates in Dehn invariants is nonmeasurable in the Lebesgue sense (see [3], [9]).

Let M be the class of all those measures on $\mathbf{R}$ which are translation invariant and extend the Lebesgue measure λ_1.

We shall say that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to M if, for every measure $\mu \in M$, this f is not μ-measurable.

Theorem 2. *There exists an additive function $f : \mathbf{R} \to \mathbf{Q}$ which is absolutely nonmeasurable with respect to the class of all translation invariant measures on the real line $\mathbf{R}$, extending the Lebesgue measure λ_1.*

Proof. For establishing this fact, consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$. Take an arbitrary element $e \in \mathbf{Q}\setminus\{0\}$. It is well known that, one-element set $\{e\}$ can be extended to a basis of $\mathbf{R}$, that is there exists a Hamel basis $\{e_i : i \in I\}$ for $\mathbf{R}$, containing e. The latter means that $\{e_i : i \in I\}$ is a maximal (with respect to inclusion) linearly independent (over $\mathbf{Q}$) family of elements of $\mathbf{R}$ and $e \in \{e_i : i \in I\}$. Now, find the index $i_0 \in I$ for which $e_{i_0} = e$ and consider the vector subspace V of $\mathbf{R}$ generated by the family $\{e_i : i \in I\setminus\{i_0\}\}$. It is obvious that V turns out to be a vector space in $\mathbf{R}$, complementary to the vector subspace $\mathbf{Q}$. In other words, we have the representation

$$\mathbf{R} = V + \mathbf{Q}, \qquad (V \cap \mathbf{Q} = \{0\})$$

of the space $\mathbf{R}$ in the form of a direct sum of its two vector subspaces. In particular, for each $x \in \mathbf{R}$, the relation

$$\mathrm{card}(\mathrm{V} \cap (\mathrm{x} + \mathbf{Q})) = 1$$

is true, from which it follows that V is a certain Vitali subset of $\mathbf{R}$ (see, e. g., [1], [5]).

For any $x \in \mathbf{R}$, we have the unique representation

$$x = v + q \qquad (v \in V, q \in \mathbf{Q}).$$

Consider a function

$$f : \mathbf{R} \to \mathbf{Q}$$

defined by the formula:

$$f(x) = q \qquad (x \in \mathbf{R}).$$

Obviously,

$$f(x + y) = f(x) + f(y) \qquad (x \in \mathbf{R}, y \in \mathbf{R}).$$

We thus conclude that f turns out to be an additive functional on $\mathbf{R}$ and $ran(f) = \mathbf{Q}$. Also, a straightforward verification shows that f is not measurable with respect to every translation invariant measure on the real line $\mathbf{R}$, extending the Lebesgue measure. This follows from the fact, that

$$f^{-1}(0) = V$$

and V is nonmeasurable with respect to every translation invariant measure μ on the real line $\mathbf{R}$, extending the Lebesgue measure λ_1. In other words, we always have $f^{-1} \notin dom(\mu)$ (compare with Theorem 1).

This finishes the proof of the Theorem 2. □

Example 3. There exist many nontrivial solutions of Cauchy functional equation which are not absolutely nonmeasurable with respect to the class M. Moreover, most of solutions of Cauchy functional equation are not absolutely nonmeasurable with respect to M.

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