ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 1 — Математика
Том 82, 1988

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 1 — Mathématiques
Tome 82, 1988

THE CREATION OF WEAK SINGULARITIES IN REFLECTING SEMILINEAR WAVES *

MARK WILLIAMS **

Марк Уилямс. ЗАРОЖДЕНИЕ СЛАБЫХ ОСОБЕННОСТЕЙ ДЛЯ ОТРАЖЕННЫХ ПОЛУЛИНЕЙНИХ ВОЛН. Рассматривается смешанная задача для полулинейного волнового уравнения. Дается описание процесса зарождения новых особенностей. Ставятся новые проблемы.

Mark Williams. THE CREATION OF WEAK SINGULARITIES IN REFLECTING SEMILINEAR WAVES. The process of creation of new singularities in mixed problems for semilinear wave equation is described. Some open problems are stated.

INTRODUCTION

For solutions to nonlinear hyperbolic equations, it is well-known that interactions between singularity-bearing rays can lead to the appearance of "anomalous" singularities, that is, singularities not present in the solutions to corresponding linear problems. The mechanisms by which this happens, crossing and self-spreading, have been understood for some time in the case when interactions occur in free space (e.g., [1], [2], [7]). Here we shall describe how anomalous singularities are produced in mixed problems for semilinear wave equations $\Box u = f(u)$ on the half-space $\mathbf{R}^{n+1}_+$, due to crossing and self-spreading at boundary points. Several new phenomena connected to the boundary appear. For example we will show that the analogue of Beals' "3s-theorem" fails for reflection in second-order mixed problems. Complete proofs of the results presented here may be found in [4], [8], [9].

*Invited lecture given on Sep. 25th 1987 at Varna conference on nonlinear PDE.
**Partially supported by NSF Grant DMS-8701654.

For $n \geq 2$ let $\mathbf{R}_+^n = \{(x,y) : x > 0\}$, $\Omega = (-\infty,+\infty) \times \overline{\mathbf{R}}_+^n$, $\Omega_T = (-T,T) \times \overline{\mathbf{R}}_+^n$, and $\mathrm{b}\Omega_T = \Omega_T \cap \{x = 0\}$. We state the following problems for a function $u(t,x,y) \in H^s_{\mathrm{loc}}(\Omega_T)$, $s > \frac{n+2}{2}$, which satisfies

$$\Box u = (D_t^2 - \Delta)u = \beta(t)f(u), \ u|_{\mathrm{b}\Omega_T} \in C^\infty, \tag{1}$$

where $\beta(t) \in C_0^\infty$ and $\mathrm{supp}\beta \subset \{|t| < \delta\}$ for some $\delta < T$. Here $z = (t,x,y)$ and $\zeta = (\tau,\xi,\eta)$ denotes the dual variables.

P r o b l e m I (*Crossing*). Let $\Gamma_1 = \{(z_1(t),\zeta_1)\}$, $\Gamma_2 = \{(z_2(t),\zeta_2)\}$ be incoming null bicharacteristics such that $\{z_1(t)\}$, $\{z_2(t)\}$ strike $\mathrm{b}\Omega_T$ transversally when $t = 0$ and cross there (i.e., $z_1(0) = z_2(0) \in \mathrm{b}\Omega_T$, $\zeta_1 \neq \pm\zeta_2$). Suppose that $\mathbf{WF}\, u|_{t<-\delta} = \Gamma_1|_{t<-\delta} \cup \Gamma_2|_{t<-\delta}$ (where $\Gamma_1|_{t<-\delta}$ means $\{(z_i(t), r\zeta_i) : r > 0,\ t < -\delta)\}$. Determine the location and strength of the anomalous singularities that may arise.

P r o b l e m II (*Self-spreading*). Let $\Gamma_\pm = \{(z(t),\pm\zeta)\}$ be incoming null bicharacteristics such that $\{z(t)\}$ strikes $\mathrm{b}\Omega_T$ transversally when $t = 0$. Suppose that $\mathbf{WF}\, u|_{t<-\delta} = \Gamma_+|_{t<-\delta} \cup \Gamma_-|_{t<-\delta}$. Determine the location and strength of the anomalous singularities that may arise.

From [4; Theorem 1.3] it is known that for u as in (1), microlocal H^r regularity for $r <\sim 2s - \frac{n}{2}$ propagates along generalized bicharacteristics. Thus, anomalous singularities in Problems I and II can have strength at most $\sim 2s - \frac{n}{2}$. But can singularities of this strength actually appear? When there is no boundary, Beals' 3s-theorem ([2], [3]) implies that a solution $u \in H^s_{\mathrm{loc}}$ to $\Box u = \beta f(u)$ can have anomalous singularities of strength at most $\sim 3s - n$. When the domain is a half-space, Theorem 1 below shows that for certain choices of $\beta(t)$, $f(u)$, and incoming singularities, new singularities of strength $\sim 2s - \frac{n}{2}$ do actually appear at the moment of reflection. So for u as in (1), although microlocal H^r-regularity for $r <\sim 3s - n$ propagates along null bicharacteristics in free space, for $r >\sim 2s - \frac{n}{2}$ it does not in general reflect (see Fig. 2).

MAIN RESULTS

Choose $\beta(t)$ as above, but now also such that $\widehat{\beta} \geq 0$ and $\widehat{\beta}(0) > 0$, and consider the mixed problem on Ω_T:

$$\Box u = \beta u^2, \ u|_{\mathrm{b}\Omega_T} \in C^\infty, \ u|_{t=-T} = w_0, \ u_t|_{t=-T} = w_1, \tag{2}$$

where w_0, w_1 are the Cauchy data of a function $w(t,x,y) \in H^s_{\mathrm{loc}}(\mathbf{R}^{n+1})$ defined as follows.

Fixing $\rho > 1$ and setting $\langle\xi,\eta\rangle = (1 + |\xi,\eta|^2)^{\frac{1}{2}}$, for any $\omega \in S^{n-1}$ we denote by $f_\omega(x,y)$ a function in $H^s(\mathbf{R}^n)$ such that

(3)
a) $\mathbf{WF}\, f_\omega = \{(0, r_\omega) : r > 0\}$;
b) $0 \leq \widehat{f}_\omega(\xi,\eta) \leq C\langle\xi,\eta\rangle^{-(s+\frac{n}{2}+a(\rho))}$, where $a(\rho) > 0$ and $a(\rho) = O(\rho - 1)$;
c) For (ξ,η) such that $|(\xi,\eta) - |\xi,\eta|\omega| \leq |\xi,\eta|^{\frac{1}{\rho}}$, $\widehat{f}_\omega(\xi,\eta) \geq C'\langle\xi,\eta\rangle^{-(s+\frac{n}{2}+a(\rho))}$.

(Such functions are constructed in [1].) Now define $f_1, f_2, g \in H^s(\mathbf{R}^n)$ by taking ω in (3) equal to $\omega_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\omega_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$, and $\tilde{\omega} = (1, 0)$ respectively. We shall consider three possibilities for w:

$$(4) \qquad \begin{aligned} &\text{a)} \quad w^\wedge(t,\xi,\eta) &=& \; e^{it|\xi,\eta|}[\widehat{f}_1(\xi,\eta) + \widehat{f}_2(\xi,\eta)];\\ &\text{b)} \quad w^\wedge(t,\xi,\eta) &=& \; e^{it|\xi,\eta|}\widehat{f}_1(\xi,\eta) + e^{-it|\xi,\eta|}\widehat{f}_2(-\xi,-\eta);\\ &\text{c)} \quad w^\wedge(t,\xi,\eta) &=& \; e^{it|\xi,\eta|}\widehat{g}(\xi,\eta) + e^{-it|\xi,\eta|}\widehat{g}(-\xi,-\eta). \end{aligned}$$

Hence $\Box w = 0$ and $\mathbf{WF}\, w$ in the three cases is, respectively,

$$(5) \qquad \begin{aligned} &\text{a)} \quad \bigcup_{i=1,2} \{(t, -t\omega_i, r, r\omega_i) : \ t \in \mathbf{R}, \ r > 0\};\\ &\text{b)} \quad \{(t, -t\omega_1, r, r\omega_1) : \ t \in \mathbf{R}, \ r > 0\}\\ &\qquad \bigcup \{(t, -t\omega_2, -r, -r\omega_2) : \ t \in \mathbf{R}, \ r > 0\};\\ &\text{c)} \quad \{(t, -t\tilde{\omega}, r, r\tilde{\omega}) : \ t \in \mathbf{R}, \ r \in \mathbf{R}\backslash 0\}. \end{aligned}$$

We proceed to define the sets that will carry anomalous singularities. For $w \in S^{n-1}$ let $K^\omega_\pm$ be the rays through $\pm(1, \omega)$ in $\mathbf{R}^{n+1}\backslash 0((\tau,\xi,\eta)$-space) and set

$$(6) \qquad \begin{aligned} &\text{a)} \quad B^{\omega_1,\omega_2} &=& \; \overline{K^{\omega_1}_+ + K^{\omega_2}_+} \text{ (closure in } \mathbf{R}^{n+1}\backslash 0);\\ &\text{b)} \quad B^{\omega_1,-\omega_2} &=& \; \overline{K^{\omega_1}_+ + K^{\omega_2}_-};\\ &\text{c)} \quad B^{\tilde{\omega}} &=& \; \{ \text{ tangent plane to } \tau^2 = |\xi,\eta|^2 \text{ at } \pm(1,\tilde{\omega})\}. \end{aligned}$$

With $\pi(\tau,\xi,\eta) = (\tau,\eta)$ set

$$(7) \qquad \begin{aligned} &\text{a)} \quad C_1 &=& \; \pi B^{\omega_1,\omega_2} \bigcap \{\tau^2 \geq |\eta|^2\} = \pi B^{\omega_1,\omega_2};\\ &\text{b)} \quad C_2 &=& \; \pi B^{\omega_1,-\omega_2} \bigcap \{\tau^2 \geq |\eta|^2\};\\ &\text{c)} \quad C_3 &=& \; \pi B^{\tilde{\omega}} \bigcap \{\tau^2 \geq |\eta|^2\} = \{\tau^2 \geq |\eta|^2\}. \end{aligned}$$

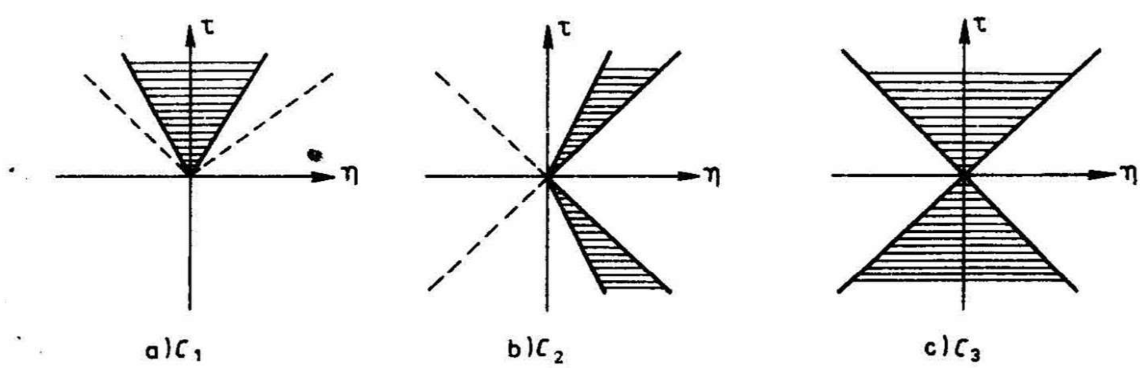

Fig. 1

Next, for $i = 1, 2, 3$ we define

$$(8) \qquad \Lambda_i = \{\lambda \doteq (t, x, y, \tau, \xi, \eta) : t > 0, \ (t, x, y) \in \mathring{\Omega}_T,$$

and λ lies on the outgoing $\left(\frac{dx}{dt} > 0\right)$ null bicharacteristic that passes over $(0, (\tau, \eta))$, for some $(\tau, \eta) \in \overset{\circ}{C}_i\}$.

Assuming T is small enough so that a solution $u \in H^s(\Omega_T)$ of (2) exists, we can now state

Theorem 1. Fix $\varepsilon > 0$.

i) *(Crossing)* Define w_0, w_1 in (2) as the Cauchy data at $t = -T$ of w as in (4a) (resp. (4b)). Then if ρ in (3) is chosen close enough to 1, $u \notin H^{2s-\frac{n}{2}+2+\varepsilon}(\lambda)$ for all $\lambda \in \Lambda_1$ (resp. Λ_2) (Fig.2a and 2b);

ii) *(Self-spreading)* The statement is the same, eexept w as in 4c is used and Λ_3 is substituted for Λ_1, Λ_2 (Fig. 2c).

The fact that u is C^∞ in the regions indicated in Fig. 2a and 2b is a consequence of Theorem 1.7 of [8]. This theorem shows that for solutions $u \in H^s(\Omega_T)$ whose incoming singularities are confined to proper cones in $T\star\overset{\circ}{\Omega}_T \cup T\star \mathrm{b}\Omega_T$, it is possible to propagate microlocal $H^{s',t'}$ regularity, for $s' < 2s + \frac{1}{2}$ and arbitrarily large t', outside an appropriate outgoing family of proper cones. This enables one to identify regions into which singularities arising from interactions at the boundary cannot spread.

For semilinear wave equations in free space, anomalous singular support is never produced by the interaction of fewer than three bicharacteristics, unless self-spreading occurs [1]. Part i) of Theorem 1 shows that when a boundary is involved, two crossing rays suffice.

Fig. 2

In the shaded areas $u \notin H^{2s-n/2+2+\varepsilon}$.

SKETCH OF THE PROOF OF THEOREM 1.

The proof is based on an idea like that used by Beals [1, 2] in his studies of spreading in free space. We write $u = v + R\beta u^2$ where $v \in H^s_{\mathrm{loc}}(\Omega)$ satisfies

$$(9) \qquad \Box v = 0,\ v|_{\mathrm{b}\Omega} \in C^\infty,\ v|_{t=-T} = w_0,\ v_t|_{t=-T} = w_1,$$

and R is the forward solution operator such that for any U with support in $t > -\delta$, RU satisfies (in Ω)

$$\square RU = U,\ RU|_{b\Omega} = 0,\ RU = 0 \text{ in } t < -\delta. \tag{10}$$

Rewriting u as $u = v + R\beta v^2 + R\beta(u^2 - v^2)$, we see that it will suffice to find singularities of strength $2s - \frac{n}{2} + 2 + \varepsilon$ in $R\beta v^2$ on Λ_i, as long as any singularities of $R\beta(u^2 - v^2)$ on Λ_i can be shown to be strictly weaker. So we proceed to study $R\beta v^2$ in the case where w as in (4a) is used to define w_0, w_1.

We can write

$$R\beta v^2 = (E\beta v^2)|_{\overset{\circ}{\Omega}} - C((E\beta v^2)|_{b\Omega}), \tag{11}$$

where E is the solution operator in $\mathbf{R}^{n+1}$ such that for any U with support in $t > -\delta$

$$\square EU = U \text{ and } EU = 0 \text{ in } t < -\delta, \tag{12}$$

and C is such that, for any V on $b\Omega$ with support in $t > -\delta$, CV satisfies (in Ω)

$$\square CV = 0,\ (CV)|_{b\Omega} = V,\ CV = 0 \text{ in } t < -\delta. \tag{13}$$

The term $(E\beta v^2)|_{\overset{\circ}{\Omega}}$ contributes nothing anomalous to $R\beta v^2$. To see this note first that v is simply the restriction to $x > 0$ of the function in $H^s_{\text{loc}}(\mathbf{R}^{n+1})$, which we will also call v, given by

$$v = v_a - v_b, \text{ where } v_a^\wedge(t,\xi,\eta) = w^\wedge(t,\xi,\eta) \text{ and } v_b^\wedge(t,\xi,\eta) = w^\wedge(t,-\xi,\eta). \tag{14}$$

Observe that the singularities of v_a and v_b lie on incoming $\left(\frac{dx}{dt} < 0\right)$ and outgoing $\left(\frac{dx}{dt} > 0\right)$ rays, respectively. Recalling (4a), we may write $v_a = v_{a1} + v_{a2}$, $v_b = v_{b1} + v_{b2}$, where

$$v_{ai}^\wedge = e^{it|\xi,\eta|}\widehat{f}_i(\xi,\eta) \text{ and } v_{bi}^\wedge = e^{it|\xi,\eta|}\widehat{f}_i(-\xi,\eta). \tag{14'}$$

The assertion about $E\beta v^2$ follows immediately from the fact that since (with $\pi_2(t,x,y,\tau,\xi,\eta) = (\tau,\xi,\eta)$)

$$\pi_2 \mathbf{WF} v^2 \cap \text{char}\square = \pi_2 \mathbf{WF} v \cap \text{char}\square, \tag{15}$$

v^2 provides nothing new for E to propagate. To verify (15) just note that $\pi_2\mathbf{WF}\, v_{a1}^2 = K_+^{\omega_1}$; $\pi_2\mathbf{WF}\, v_{a1}v_{a2} \subset B^{\omega_1,\omega_2}$ (which satisfies $B^{\omega_1,\omega_2} \cap \text{char}\square = K_+^{\omega_1} \cup K_+^{\omega_2}$), and similarly for the remaining terms constituting v^2. So we have reduced to considering $C((E\beta v^2)|_{b\Omega})$. It is worth noting that the above argument indicates the main reason why singularities of strength $\sim 2s - \frac{n}{2}$ never appear in solutions to $\square u = f(u)$ due to interactions in free space.

Writing $v^2 = (v_a^2 - v_a v_b) + (v_b^2 - v_a v_b)$, we examine first

$$C((E\beta v_a^2)|_{b\Omega}) - C((E\beta v_a v_b)|_{b\Omega}). \tag{16}$$

After expanding (16) in terms of the v_{ai}, v_{bi}, one must consider differences like

$$C((E\beta v_{a1}v_{a2})|_{b\Omega}) - C((E\beta v_{a1}v_{b2})|_{b\Omega}) \tag{17}$$

We claim that each term in (17) has singularities of strength $\sim 2s - \frac{n}{2}$ on Λ_1. This illustrates an interesting feature of spreading at the boundary. Anomalous singularities of strength $\sim 2s - \frac{n}{2}$ arise at the boundary from three sources: interactions of incoming rays with incoming rays, incoming rays with outgoing (or reflected) rays, and outgoing rays with outgoing rays. We will see that the singularities produced by the incoming-outgoing interactions (represented in (17) by the second term) are strictly weaker than the other two types, so although some cancellations occur, singularities of strength $\sim 2s - \frac{n}{2}$ remain.

Return now to the first term in (17). It is not hard to show that $\beta v_{a1} v_{a2}$ and therefore also $E\beta v_{a1} v_{a2}$ have singularities of strength $\sim 2s - \frac{n}{2}$ throughout $\overset{\circ}{B}{}^{\omega_1,\omega_2}$ over $(t,x,y) = 0$. Consequently, $(E\beta v_{a1} v_{a2})|_{b\Omega}$ has singularities of strength $\sim 2s - \frac{n}{2}$ on $\overset{\circ}{C}_1 = \pi \overset{\circ}{B}{}^{\omega_1,\omega_2}$ over $0 \in b\Omega$. Since $C_1 \subset \{\tau^2 > |\eta|^2\}$, all of these singularities are propagated by the operator C. Hence $C((E\beta v_{a1} v_{a2})|_{b\Omega})$ has singularities of strength $\sim 2s - \frac{n}{2}$ throughout Λ_1. Since $\pi_2 \mathbf{WF} \beta v_{a1} v_{b2} \doteq B^{\omega_1,\omega_2^*}$, where $\omega_2^* = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$, and $\pi B^{\omega_1,\omega_2^*}$ also equals C_1, the same argument shows that the second term in (17) also has singularities of strength $\sim 2s - \frac{n}{2}$ throughout Λ_1. We will now show that these latter singularities are weaker by a factor strictly less than one.

Let us focus attention on $\gamma_+(0,(\overline{\tau},\overline{\eta}))$, the outgoing null bicharacteristic passing over a fixed point $(0,(\overline{\tau},\overline{\eta})) \in T^\star b\Omega \backslash 0$ with $(\overline{\tau},\overline{\eta}) \in \overset{\circ}{C}_1$ (so $\gamma_+(0,(\overline{\tau},\overline{\eta})) \subset \Lambda_1$). Setting $\overline{\mu} = (\overline{\tau},\overline{\eta})$, without loss of generality we may take $\overline{\mu} = (1,0)$. Our starting point is the observation that for the purpose of studying the singularities on $\gamma_+(0,\overline{\mu})$ of either term in (17), the rather awkward operator E can be replaced by a simple multiplier.

L e m m a 1. Let $H(\tau,\xi,\eta)$ be the characteristic function of a small conic neighborhood of $A \equiv (1, \frac{1}{\sqrt{2}}, 0) = \frac{1}{2}(1,\omega_1) + \frac{1}{2}(1,\omega_2) \in B^{\omega_1,\omega_2}$, and set $\widehat{U}(\tau,\xi,\eta) = \frac{H(\tau,\xi,\eta)}{\tau^2 - |\xi,\eta|^2} \beta \widehat{v_{a1} v_{a2}}$. Then $\mathbf{WF}[C((E\beta v_{a1} v_{a2})|_{b\Omega}) - C(U|_{b\Omega})] \cap \gamma_+(0,\overline{\mu}) = \phi$.

Proof. Set $V = E\beta v_{a1} v_{a2} - C((E\beta v_{a1} v_{a2})|_{b\Omega})$ and $\tilde{V} = U - C(U|_{b\Omega})$. Thus $V - \tilde{V}$ satisfies (in Ω)

$$(18) \qquad \Box(V - \tilde{V}) = (1 - H(D))\beta v_{a1} v_{a2}, \ (V - \tilde{V})|_{b\Omega} = 0,$$

$$V - \tilde{V} \in C^\infty \text{ in } t < -\delta.$$

Now $\beta \widehat{v_{a1} v_{a2}}$ is rapidly decreasing outside B^{ω_1,ω_2}. Thus the fact that $H \equiv 1$ near A implies $\mathbf{WF}(1 - H(D))\beta v_{a1} v_{a2} \cap [\gamma_-(0,\overline{\mu}) \cup \gamma_+(0,\overline{\mu})] = \phi$ (where γ_- denotes the incoming bicharacteristic) and $(0,\overline{\mu}) \notin \mathbf{WF}_b(1 - H(D))\beta v_{a1} v_{a2}$. Classical results on reflection of $\mathbf{WF}_b$ (e.g. [5]) then imply $\mathbf{WF}(V - \tilde{V}) \cap \gamma_+(0,\overline{\mu}) = \phi$, since this clearly holds for γ_-. Because $E\beta v_{a1} v_{a2}$ and U have no singularities on γ_+, the lemma follows.

Of course an analogous result holds for $C((E\beta v_{a1} v_{a2})|_{b\Omega})$, where now one takes $\tilde{H}(\tau,\xi,\eta)$ to be the characteristic function of a small conic neighborhood of $B \equiv (1,0,0) \in B^{\omega_1,\omega_2^*}$ (recall, $\omega_2^* = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$) and sets $\widehat{\tilde{U}} = \frac{\tilde{H}(\tau,\xi,\eta)}{\tau^2 - |\xi,\eta|^2} \beta \widehat{v_{a1} v_{b2}}$. In Fig. 3 we have drawn the crosssection of $\tau^2 = |\xi,\eta|^2$ at $\tau = 1$, indicating A, B, B^{ω_1,ω_2}, B^{ω_1,ω_2^*} and the singular directions of the v_{ai}, v_{bi}.

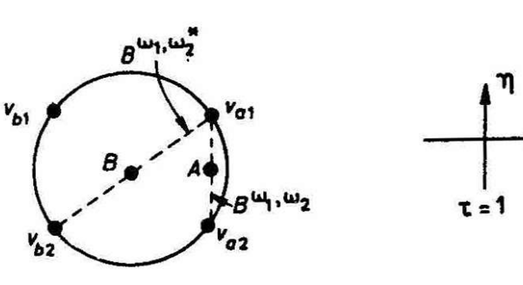

Fig. 3

In view of Lemma 1 it suffices to compare $C(U|_{b\Omega})$ with $C(\tilde{U}|_{b\Omega})$. Now $\widehat{U|_{b\Omega}} = \int \frac{H}{\tau^2-|\xi,\eta|^2}\widehat{\beta v_{a1} v_{a2}}\, d\xi$ while $\widehat{\tilde{U}|_{b\Omega}} = \int \frac{\tilde{H}}{\tau^2-|\xi,\eta|^2}\widehat{\beta v_{a1} v_{b2}}\, d\xi$. Since supp$H$ lies closer to $\tau^2 = |\xi,\eta|^2$ than supp$\tilde{H}$ does, it follows that for $(\tau,\xi,\eta) \in \text{supp}H (\tau,\xi',\eta) \in \text{supp}\tilde{H}$, and $\langle\tau,\eta\rangle$ sufficiently large, we have

$$(19) \qquad (\tau^2 - |\xi',\eta|^2)^{-1} \leq \varepsilon(\tau^2 - |\xi,\eta|^2)^{-1} \text{ for an } \varepsilon < 1.$$

A comparison of the above two integrals using (19) yields easily that singularities of $C(\tilde{U}|_{b\Omega})$ on $\gamma_+(0,\overline{\mu})$ are strictly weaker than those of $C(U|_{b\Omega})$ ([9], Lemma 3.18), so the difference still has singularities of strength $\sim 2s - \frac{n}{2}$ on γ_+. Since the same argument applies to the terms like $C((E\beta v_{b1}v_{b2})|_{b\Omega}) - C((E\beta v_{a1}v_{b2})|_{b\Omega})$ in $R\beta v^2$, we can summarize the above as follows:

(20) **The anomalous singularities on a fixed ray in Λ_1 produced by interactions of incoming rays with incoming rays (v_a, v_a), or by interactions of outgoing rays with outgoing rays (v_b, v_b), are strictly stronger than those produced by incoming-outgoing interactions (v_a, v_b). Moreover, as the incoming rays approach being gliding rays, the difference in strength decreases, and so cancellations become increasingly significant.**

Essentially the same arguments applied to the remaining terms constituting $R\beta v^2$ show that they contribute singularities of the same strength and sign as those of (17). So this ends our discussion of $R\beta v^2$ in the case when v is defined using w as in (4a). When w as in (4b) (resp.(4c)) is used, the analysis follows the same outline with the plane $B^{\omega_1,-\omega_2}$ (resp. $B^{\tilde{\omega}}$) in the role of B^{ω_1,ω_2}. Note that C_2 (resp. C_3) is properly contained in $\pi B^{\omega_1,-\omega_2}$ (resp. $\pi B^{\tilde{\omega}}$). We restrict to $\tau^2 \geq |\eta|^2$, of course, because the only singularities in $(E\beta v^2)|_{b\Omega}$ that C can propagate are those at points $(0,(\tau,\eta))$ with $\tau^2 \geq |\eta|^2$. In each of the three cases one obtains $R\beta v^2 \notin H^{2s-\frac{n}{2}+2+\epsilon}$ on the appropriate Λ_i.

THE REMAINDER $R\beta(u^2 - v^2)$.

Recalling that $u = v + R\beta u^2$, we have $R\beta(u^2 - v^2) = R\beta(2vR\beta u^2) + R\beta(R\beta u^2)^2$. Since each application of R smooths by one derivative, it is reasonable to expect that any singularities of $R\beta(u^2 - v^2)$ are strictly weaker than those of $R\beta v^2$ on Λ_i. The proof that this is so (see[9]) uses the following results:

(21) a) a microlocal H^s algebra lemma for interior points (namely, Rauch's Lemma [6]);

b) a microlocal $H^{s,s'}$ algebra lemma for boundary points analogous to Rauch's Lemma ([8, Lemma 1.10]);

c) the fact that for u as in (1), microlocal H^r-regularity for $r < 2s - \frac{n}{2} + \frac{1}{2}$ propagates along generalized bicharacteristics ([4, Theorem 1.3]);

d) a theorem describing propagation of microlocal $H^{s,s'}$ regularity along generalized bicharacteristics for second-order, linear, mixed problems with Dirichlet boundary conditions ([8, Theorem 1.3]).

The first step is to obtain a careful estimate of the regularity of v^2 on Λ_i. With this (21 a-d) yield fairly easily that the term $R\beta(R\beta u^2)^2$ is strictly weaker than $R\beta v^2$ on Λ_i, but such an argument does not quite yield sufficient regularity of $R\beta(vR\beta u^2)$. (The difference arises because $R\beta u^2 \in H^{s+1}_{\mathrm{loc}}$, while $v \in H^s_{\mathrm{loc}}$.) To find the extra smoothness needed, we write $R\beta u^2 = R\beta A u^2 + R\beta(I - A)u^2$, where A is a tangential pseudo-differential operator equal to 1 and supported near the strong (s) singularities of u^2. Since the incoming singularity-bearing rays meet $\mathrm{b}\Omega$ transversally, A can be chosen with support in the hyperbolic region of $T \star \mathrm{b}\Omega$. This permits one to extend $R\beta A u^2$ (originally defined only in $x \geq 0$) across the boundary as the solution U of a wave equation in free space. Using this property of U, estimates like $\langle \tau, \xi, \eta \rangle^{s+1} \phi \widehat{R\beta u^2} \in L^2$ can be improved to estimates like $\langle \tau - |\xi, \eta| \rangle \langle \tau, \xi, \eta \rangle^{s+1} \hat{U} \in L^2$. These in turn are used to obtain a more refined estimate of $vR\beta u^2$ on Λ_i. A final application of (21d) then shows that $R\beta(vR\beta u^2)$ is strictly weaker than $R\beta v^2$ on Λ_i.

OPEN PROBLEMS

It is interesting to consider the analogues of Problems I and II when the incoming rays are gliding rays. The remark (20) suggests, if gliding rays are thought of as limits of transversal rays, that anomalous singularities arising from the crossing of two gliding rays may be significantly weaker than $\sim 2s - \frac{n}{2}$. In Problem II note that if $\Gamma_\pm = \{(z(t), \pm\zeta)\}$ are gliding rays, the tangent plane to char$\Box$ at $\pm\zeta$ projects under $(\tau, \xi, \eta) \to (\tau, \eta)$ to a set that misses the hyperbolic region. In view of the role played by $B^{\hat{\omega}}$ in the preceding analysis, this suggests that new singularities due to the self-spreading of a gliding ray may be much weaker than in the transversal case, perhaps only of strength $\sim 3s - n$.

REFERENCES

1. B e a l s, M. Spreading of singularities for a semilinear wave equation. — Duke Math. J., **49**, 1982, 275-286.
2. B e a l s, M. Self-spreading and strength of singularities for solutions to semilinear wave equation. — Annals of Math., **118**, 1983, 187-214.
3. B e a l s, M. Propagation of smoothness for nonlinear second-order strictly hyperbolic differential equations. — AMS Proceedings of Symposia in Pure Math., **43**, 1985, 21-44.
4. D a v i d, F., M. W i l l i a m s. Singularities of solutions to semilinear boundary value problems. — Amer. J. Math. (to appear).

5. N i r e n b e r g, L. Lectures on linear partial differential equations. — AMS Regional Conf. in Math., 17, 1972, 1-58.
6. R a u c h, J. Singularities of solutions to semilinear wave equations. — J. Math. Pures et Appl., 58, 1979, 299-308.
7. R a u c h, J., M. R e e d. Propagation of singularities for semilinear hyperbolic equations in one space variable. — Annals of Math., 111, 1980, 531-552.
8. W i l l i a m s., M. Spreading of singularities at the boundary in semilinear hyperbolic mixed problems I: microlocal $H^{s,s'}$ regularity. — Duke Math. J. (to appear).
9. W i l l i a m s, M. Spreading of singularities at the boundary in semilinear hyperbolic mixed problems II: crossing and self-spreading. — Submitted.

Received 16.II.1989