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# INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, II

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*Das Wesen der Mathematik liegt eben in ihrer Freiheit.*

*Georg Cantor*

*Иван Чобанов.* ВВЕДЕНИЕ В АЛГЕБРАИЧЕСКУЮ ТЕОРИЮ СКОЛЬЗЯЩИХ ВЕКТОРОВ, II. Эта работа является второй частью статьи [1] под тем же наименованием, опубликованной несколько лет тому назад в том же *Ежегоднике*, в которой предложена алгебраическая теория реальных скользящих векторов на основе аксиоматически введенных реальных стандартных векторов. Между тем автором опубликована комплексная версия [2] реалных стандартных векторных пространств и развита соответствующая комплексная версия [3] реальной трехмерной линейной аналитической геометрии [4]. Настоящая работа является комплексной версией конструкций, изложенных в работе [1]. Как известно, традиционной механической интерпретацией реальных скользящих векторов являются концентрированные силы аналитической статики и аналитической динамики. Возможность построения комплексных скользящих векторов имеет глубокие последствия. Ее главным результатом является возможность построения комплексной аналитической механики со всеми проистекающими от этого консеквенциями для логического фундамента этой науки и для решения шестой проблемы Гильберта об ее аксиоматической консолидации.

*Ivan Chobanov.* INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, II. This paper represents the second part of the article [1] under the same title published some years ago in this *Annual*, in which an algebraic theory of arrows or sliding vectors has been proposed, based on the axiomatically defined real standard vectors. Meanwhile the author has proposed a complex version [2] of the real standard vector spaces and has developed the corresponding complex version [3] of the real 3-dimensional linear analytic geometry [4]. This paper represents a complex version of the constructions exposed in [1]. As it is wellknown, the traditional mechanical interpretation of the real sliding vectors are the concentrated forces in analytical statics and analytical dynamics. The possibility of defining complex arrows has far reaching consequences. Its main result consists

**in the potenciality to develop a complex analytical mechanics with all the after-effects this fact implies for the logical foundations of this science and for the solution of Hilbert's sixth problem concerning its axiomatical consolidation.**

The present paper represents the second part of the article [1] under the same title published in this *Annual* about ten years ago. In the latter an algebraic theory of the real arrows (or sliding vectors, *vecteurs glissants, gleitende Vectoren, скользящие векторы*) has been proposed, based on the axiomatically defined real standard vectors.

Meanwhile some important development has taken place. It has been discovered [2] that, by the aid of *mot-á-mot* the same system of 15 axioms, by means of which the real standard vector space may be described axiomatically, it is possible to define a complex standard vector space (infinitely many such spaces, as a matter of fact), by introducing a fourth operation vector multiplication in an Hermitean space (which turns out to be probably 3-dimensional), this operation being characterized by two only specific axioms. At that, as it turned out to be, these complex standard vector spaces possess verbatim the same algebraic properties as the real one, *mutatis mutandis*, as it is clear by itself.

This mathematical phenomenon has far-reaching consequences.

First of all, analytic geometries in complex standard vector spaces may be developed, as it has been manifested in the article [3]. This mathematical process provides the geometry, necessary as well as sufficient, for all the following constructions.

Second, an algebraic theory of arrows in complex standard vector spaces may be developed, as this paper and its continuations display. This fact is important in the following two respects.

On the one hand, the real arrows interpret mathematically the (sometimes) so-called *concentrated forces*, i.e. those active and passive forces that are specific for analytical statics and analytical dynamics. Now the possibility to define complex forces is a *conditio sine qua non* for the potentiality to develop a *complex analytical mechanics* (i.e. an analytical mechanics in complex standard vector spaces), and this condition is satisfied by the complex arrows proposed in this paper.

On the other hand, by means of the so-called *statical-kinematical analogy*, the arrows have a direct relationship with the rigid body kinematics. Strictly speaking, a dictionary may be composed (that may be called the *statical-kinematical dictionary*), by the aid of which a bijection may be established between the mathematical facts in the algebra of arrows, on the one hand, and of the kinematics of rigid bodies, on the other hand. In the presence of this dictionary it is out and out superfluous to seek and prove theorems of rigid body kinematics which concern the velocity distribution in a moving rigid body: it is perfectly sufficient to point out the terms of the arrow-algebra that correspond to the respective kinematical terms involved in the kinematical theorems in question, and to prove corresponding theorems for these arrow-algebraic terms. Afterwards, the conclusions of these theorems have to be translated into the kinematical language by means of the statical-kinematical dictionary. In such a way, to any proposition of the algebra of arrows there corresponds automatically a true proposition of the analysis of rigid body kinematics. At that, theorems are discovered and proved out and out easier in the former than in the latter. In such a manner, the possibility to define complex arrows has a direct bearing to the potentiality to develop a rigid body kinematics in complex standard vector spaces (established, by the way, earlier, in the more general mathematical situation proposed by Hermitean spaces, at that, see [5]), and a complex analytical dynamics in the long run, as it will be immediately seen.

Namely, complex forces and rigid body kinematics in complex standard vector spaces being once developed, all that remains to be done for the building-up of a complex rigid body dynamics is to form those fundamental for this science quantities, the *momentum* and the *moment of momentum* (alias *kinetical moment*) of a rigid body in complex standard vector spaces, and to formulate the corresponding *Eulerian dynamical axioms* (or *laws*, or *principles*), viz. those *of momentum* and *of moment of momentum* of rigid bodies, without which in analytical dynamics *terra, aqua, aere et igni interdicti sumus.*

The importance of all these constructions is predominantly an ideological one, since all they result in a mathematical *Weltanschauung* which affects profoundly the logical foundations of the great science of analytical mechanics.

A wide-spread prejudice even today and even among professional mathematicians is that rational mechanics, in general, and analytical mechanics, in particular, are not mathematics — al least not in the sense this term is accepted nowadays. This bias is supported bilaterally.

On the one hand, there is the multitudinous army of mechanicians and of the ingratiated themselves to mechanics physicists and engineers with such a level of mathematical schooling that, at the best, cannot but call forth condescending smiles on the part of the modern professional mathematicians. For these mechanicians the standpoint that rational mechanics is mathematics, and not applied mathematics at that, is a rather disadvantageous — we should say, unprofitable, unproductive, even contraproductive — one. This attitude once adopted, all mechanical writings should unconditionally satisfy the severe modern mathematical criteria of logical rigour — a demand that goes many times beyond the possibilities of their authors. Herefrom the myth of those would-be specific peculiarities of rational mechanics that presumably does not permit its insertion in the confined frames of pure mathematics, in the downright sense of the word. If we try to persuade these people in the contrary, then we purely and simply *canimus surdis*, putting it mildly.

On the other hand, there is the not lesser army of those mathematicians who have wound up with rational mechanics on the very student's desks and, in their *horror vacui*, regard it as a little short of a *monstrum horrendum, informe, ingens.* The fear of the unknown is instinctive; it is proverbial, too: *ignoti nulla supido ... damnant quod non intellegunt.* As to mechanical ignorance of some pure mathematicians, it is comparable only with the mathematical ignorance of some applied mechanicians: Banach, for instance, went as far as to write in his *Mechanics* [6] neither more nor less than "if a rigid body is at rest, we shall say that it is in equilibrium" (p. 234) ?!?

Things being as they are, is it strange that rational mechanics is nowadays a *persona non grata* in the United Kingdom of Mathematical Sciences?

There are lucky exceptions, though. One of them was Hilbert. Another one is Truesdell.

As a pure mathematician, Hilbert needs no recommendations. *Corvo quoque rarior albo*, however, he was one of those few pure mathematicians who are complete strangers to the very idea of mathematical chauvinism. Hilbert was a mathematical cosmopolitan. The final chord of his famous *Mathematische Probleme* [7] is a vivid incarnation of his mathematical credo:

"... und es drängt sich uns die Frage an, ab der Mathematik einst bevorsteht, was anderen Wissenschaften längst widerfahren ist, nämlich daß sie in einzelne Teilwissenschaften zerfällt, deren Vertreter sich kaum noch einmal verstehen und deren Zusammenhang daher immer loser wird. Ich glaube und wünsche dies nicht. Die mathematische Wissenschaft ist meiner Ansicht nach ein unteilbares Ganzes, ein Organismus, dessen Lebensfähigkeit durch den Zusammenhang seiner Teile bedingt wird. Denn bei aller Verschiedenheiten des mathematischen Wissenstoffes im einzelnen, gewahren wir doch sehr deutlich die Gleichheit der logischen Hilfsmitteln, die Verwandschaft der Ideenbildungen in der ganzen Mathematik, und die zahlreichen Analogien in ihren verschiedenen Wissensgebieten. Auch bemerken wir: je weiter eine mathematische Theorie ausgebildet wird, desto harmonischer und einheitlicher gestaltet sich ihr Aufbau, und ungeahnte Beziehungen zwischen bisher getrennten Wissenszweigen werden entdeckt. So kommt es, daß mit der Ausdehnung der Mathematik ihr einheitlicher Charakter nicht verlorengeht, sondern desto deutlicher offenbar wird.

Aber — so fragen wir — wird es bei der Ausdehnung des mathematischen Wissens für den einzelnen Forscher nicht schließlich unmöglich, alle Teile dieses Wissens zu umfassen? Ich möchte als Antwort darauf hinweisen, wie sehr es im Wesen der mathematischen Wissenschaft liegt, daß jeder wirkliche Fortschritt stets Hand in Hand geht mit der Auffindung schärferen Hilfsmittel und einfacheren Methoden, die zugleich das Verständnis früheren Theorien erleichtern und umständliche ältere Entwicklungen beseitigen, and daß es daher dem einzelnen Forscher, indem er sich diese schärferen Hilfsmitter und einfacheren Methoden zu eigen macht, leichter gelingt, sich in den verschiedenen Wissenszweigen der Mathematik zu orientieren, als dies für irgend eine andere Wissenschaft der Fall ist.

Der einheitliche Charakter der Mathematik liegt im inneren Wesen dieser Wissenschaft begründet ..."

Like Archimedes, Hilbert stood firm on his physical legs. Although he has never taught mechanics and has written not a single specific line on rational mechanics, he nevertheless did not deny its purely mathematical core and believed steadily in its potentialities of being developable as an axiomatically deducible structure. In point of fact, Hilbert included in his list of 23 mathematical problems [7] the nineteenth century bequeaths to the twentieth to solve, as problem number six, that of the axiomatical foundation of rational mechanics.

As a pure mechanician Truesdell needs no recommendations either. Thirty years ago he performed such a bright mathematical apology of rational mechanics that I shall never get tired of quoting it over and over again:

"... *rational mechanics is a part of mathematics.* It is a mathematical science, and in its relations to experience, intuition, abstraction, and everyday life it does not differ in essence from other branches of mathematics ...

Is rational mechanics a part of pure mathematics? To most mathematicians today pure mathematics means topology, abstract algebra, or analysis in abstract spaces. These, most certainly, rational mechanics makes no attempt to imitate. While in spirit it is nearest to geometry, its problems, its aims, and its methods bear little evident similarity to those of other parts of mathematics. A theorem in topology is not evaluated in terms of its bearing on the theory of numbers. It is equally ridiculous, though unfortunately not infrequent, to deprecate theorems of rational mechanics when they do not also contribute to the more popular branches of pure mathematics.

Is rational mechanics a part of applied mathematics? Most certainly not" [8, p. 335, 337].

Tormented words, it is true. *Nuda veritas,* though. Alas, one swallow does not make a summer. *Sunt verba et voces, praetereque nihil*: the physical, the engineering, the antimathematical mental constitution — mathematics in no wise means formulas only — of the prevalent majority of contemporary mechanicians that has driven *Ilias malorum* in rational mechanics, persevers in being the predominating ideology, the retrograde philosophy in this great science, in which the *certamen pro aris et focis* has not yet begun.

However modest, the present paper contributes my mite in the noble struggle against present-day obscurantism in mechanics conceived by Lagrange's *ipsissima verba* "la maniere dont j'ai tâché de remplir cet objet ne laissera rien à desirer". At the same time, it incarnates Hilbert's dictim in [7], *nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien berücksichtigen zu haben.*

## § 1. PRAELIMINARIA

The complex standard vector space being *principium ab Jove* for the whole following exposition, the most important moments of their introduction will be now reminded.

The following notations are permanently used throughout this paper.

The symbols Ax, Df, Pr, Dm, Sch, Sgn, and sgn: replace the words *axiom, definition, proposition, proof, scholium, notation* and *denote* respectively.

The letters $R$ and $C$ are reserved for the *fields of all real* and *all complex numbers* respectively.

The letters $F$ amd $P$ are reserved for any *ordered field* and for any *Pythagorean field* respectively. An ordered field $P$ is called *Pythagorean* iff $0 \leqq \alpha \in P$ implies the existance of a $\beta \in P$ with $0 \leqq \beta$ and $\beta^2 = \alpha$. Then $\beta$ is called the *square root* of $\alpha$ and is denoted by $\sqrt{\alpha}$.

The symbol $C(F)$ is reserved for the *complex extension* of $F$. In other words, $C(F) = F^2$ supplied with the two operations

$$(1) \qquad (x_1, x_2) + (y_1, y_2) \quad \text{sgn}: \quad (x_1 + y_1, x_2 + y_2)$$

(*addition* in $C(F)$) and

$$(2) \qquad (x_1, x_2)(y_1, y_2) \quad \text{sgn}: \quad (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$$

(*multiplication* in $C(F)$). Besides, by definition

$$(3) \qquad \overline{(x_1, x_2)} \quad \text{sgn}: \quad (x_1 - x_2) \qquad ((x_1, x_2) \in F^2),$$

$$(4) \qquad |(x_1, x_2)| \quad \text{sgn}: \quad \sqrt{x_1^2 + x_2^2} \qquad ((x_1, x_2) \in P^2).$$

The symbols in the left-hand sides of (3) and (4) are called the *conjugate number* of $(x_1, x_2)$ and the *module* of $(x_1, x_2)$ respectively. Obviously, (2)–(4) imply

$$(5) \qquad |(x_1, x_2)|^2 = (x_1, x_2)\overline{(x_1, x_2)} \qquad ((x_1, x_2) \in P^2),$$

the "real" element $(x, 0)$ of $C(F)$ being identified with the element $x$ of $F$ by means of the traditional convention

(6) $\qquad x \quad \text{sgn}: \quad (x, 0).$

Quotations are made in the following manner (the example is a fictitious one): Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and relation (6) of §7, for instance, are cited by Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and (6) respectively in §7 itself, but by 7Sgn 1, 7Ax 2, 7Df 3, 7Pr 4, 7Sch 5, and 7(6) respectively anywhere else.

The whole of the following exposition is based on the following definition.

**Df 1**. $S$ denoting $F$ or $C(F)$, a *standard vector space over* $S$ (an $S$-*standard vector space*) is called any set $V_S$ for which mappings

(7) $\qquad m_1 : \quad V_S^2 \longrightarrow V_S$

(*addition* in $V_S$),

(8) $\qquad m_2 : \quad S \times V_S \longrightarrow V_S$

(*multiplication* of the elements of $S$ and $V_S$),

(9) $\qquad m_3 : \quad V_S^2 \longrightarrow S$

(*scalar multiplication* of the elements of $V_S$), and

(10) $\qquad m_4 : \quad V_S^2 \longrightarrow V_S$

(*vector multiplication* in $V_S$) are defined, such that, if

(11) $\qquad \boldsymbol{a} + \boldsymbol{b} \quad \text{sgn}: \quad m_1((\boldsymbol{a}, \boldsymbol{b}))$

(*sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$),

(12) $\qquad \alpha\boldsymbol{a} \quad \text{sgn}: \quad m_2((\alpha, \boldsymbol{a}))$

(*product* of $\alpha$ and $\boldsymbol{a}$),

(13) $\qquad \boldsymbol{ab} \quad \text{sgn}: \quad m_3((\boldsymbol{a}, \boldsymbol{b}))$

(*scalar product* of $\boldsymbol{a}$ and $\boldsymbol{b}$),

(14) $\qquad \boldsymbol{a} \times \boldsymbol{b} \quad \text{sgn}: \quad m_4((\boldsymbol{a}, \boldsymbol{b}))$

(*vector product* of $\boldsymbol{a}$ and $\boldsymbol{b}$), and

(15) $\qquad \boldsymbol{a} - \boldsymbol{b} \quad \text{sgn}: \quad \boldsymbol{a} + (-\boldsymbol{b})$

(*difference* of $\boldsymbol{a}$ and $\boldsymbol{b}$), then the following conditions are satisfied:

**Ax 1S**. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$.

**Ax 2S**. There exists $\boldsymbol{o} \in V_S$ with $\boldsymbol{a} \in V_S$ implies $\boldsymbol{a} + \boldsymbol{o} = \boldsymbol{a}$.

**Ax 3S**. $\boldsymbol{a} \in V_S$ implies: there exists $-\boldsymbol{a} \in V_S$ with $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{o}$.

**Ax 4S**. $\boldsymbol{a} \in V_S$ implies $1\boldsymbol{a} = \boldsymbol{a}$.

**Ax 5S**. $\lambda, \mu \in S$, $\boldsymbol{a} \in V_S$ imply $(\lambda\mu)\boldsymbol{a} = \lambda(\mu\boldsymbol{a})$.

**Ax 6S**. $\lambda, \mu \in S$, $\boldsymbol{a} \in V_S$ imply $(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$.

**Ax 7S**. $\lambda \in S$, $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$.

**Ax 8S.** $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{ab} = \overline{\boldsymbol{ba}}$.
**Ax 9S.** $\lambda \in S$, $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $(\lambda \boldsymbol{a})\boldsymbol{b} = \lambda(\boldsymbol{ab})$.
**Ax 10S.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{c} = \boldsymbol{ac} + \boldsymbol{bc}$.
**Ax 11S.** $\boldsymbol{a} \in V_S$ implies $\boldsymbol{aa} \succeq 0$.
**Ax 12S.** $\boldsymbol{a} \in V_S$, $\boldsymbol{aa} = 0$ imply $\boldsymbol{a} = \boldsymbol{o}$.
**Ax 13S.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{b} \times \boldsymbol{c} \cdot \boldsymbol{a}$.
**Ax 14S.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{ac})\boldsymbol{b} - (\boldsymbol{bc})\boldsymbol{a}$.
**Ax 15S.** There exist $\boldsymbol{a}, \boldsymbol{b} \in V_S$ with $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{o}$.

**Df 2.** The elements of $V_S$ are called *standard vectors over $S$ ($S$-standard vectors)*.

**Sch 1.** The conditions Ax 1S–15S are called *axioms of a standard vector space over $S$ (of a $S$-standard vector space)*.

**Sch 2.** The symbols 0 in Ax 11S, Ax 12S and 1 in Ax 4S denote the *zero-element* and the *unit-element* of $S$ respectively.

**Df 3.** $\boldsymbol{o}$ is called the *zero-vector.*
**Df 4.** $-\boldsymbol{a}$ is called the *opposite vector* of $\boldsymbol{a}$.
**Df 5.** $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$ is called the *right-hand compound product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
**Df 6.** $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$ is called the *left-hand compound product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
**Df 7.** $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is called the *right-hand double vector product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
**Df 8.** $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is called the *left-hand double vector product* of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
**Sgn 1.** $\boldsymbol{a}^2$ sgn: $\boldsymbol{aa}$ if $\boldsymbol{a} \in V_S$.
**Df 9.** $\boldsymbol{a}^2$ is called the *scalar square* of $\boldsymbol{a}$.
**Sgn 2.** $a$, $|\boldsymbol{a}|$, mod$\boldsymbol{a}$ sgn: $\sqrt{\boldsymbol{a}^2}$ if $\boldsymbol{a} \in V_P$ or $\boldsymbol{a} \in V_{C(P)}$.
**Df 10.** $a$, $|\boldsymbol{a}|$, mod$\boldsymbol{a}$ is called the *module* of $\boldsymbol{a}$.
**Sgn 3.** $V$ sgn: $V_R$.
**Df 11.** $V$ is called the *real standard vector space.*
**Df 12.** The elements of $V$ are called *real standard vectors.*
**Df 13.** $V_C$ is called the *complex standard vector space.*
**Df 14.** The elements of $V_C$ are called *complex standard vectors.*
**Sgn 4.** $\boldsymbol{a}^0$ sgn: $\frac{1}{a}\boldsymbol{a}$ if $\boldsymbol{a} \in V_P$ or $\boldsymbol{a} \in V_{C(P)}$ and $\boldsymbol{a} \neq \boldsymbol{o}$.
**Df 15.** $\boldsymbol{a}^0$ is called the *unit-vector* (the *ort*) of $\boldsymbol{a}$.

**Sch 2.** The basic algebraic properties of $V_S$ and especially of $V_{C(F)}$ and $V_{C(P)}$ are discussed at length in the article [**2**], see also [**9, 10**]. Therefrom we shall not dwell on this question in details here and, if necessary, we shall refer the reader to these sources. Yet, a compendium of the basic situations of $V_S$-algebra will be found to be useful. Therefore, such one is exposed immediately below.

**Pr 1.** The system of axioms Ax 1S–15S is consistent.

**Sch 3.** Pr 1 is proved by constructing a model of $V_S$. It is proposed by $S^3$, supplied with the following operations corresponding to the mappings (7)–(10) respectively:

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) \quad \text{sgn}: \quad (x_1 + y_1, x_2 + y_2, x_3 + y_3), \tag{16}$$

$$\lambda(x_1, x_2, x_3) \quad \text{sgn}: \quad (\lambda x_1, \lambda x_2, \lambda x_3), \tag{17}$$

$$(x_1, x_2, x_3)(y_1, y_2, y_3) \quad \text{sgn}: \quad \sum_{\nu=1}^{3} x_\nu \overline{y}_\nu, \tag{18}$$

(19) $(x_1, x_2, x_3) \times (y_1, y_2, y_3)$ sgn :

$$(\overline{x}_2\overline{y}_3 - \overline{x}_3\overline{y}_2, \overline{x}_3\overline{y}_1 - \overline{x}_1\overline{y}_3, \overline{x}_1\overline{y}_2 - \overline{x}_2\overline{y}_1)$$

Now it is verified that (16)–(19) satisfy Ax 1S–15S.

**Pr 2.** The system of axioms Ax 1S–15S is categorical.

**Sch 4.** Pr 2 is an immediate corollary from Pr 5 below and from the fact, well-known from the algebra of Hermitean spaces, that the theory of any finitedimensional Hermitean space is *categorical*, i.e. any two of its models are *isomorphic*

**Pr 3.** $V_S$ is a *group* with respect to the operation (7).

**Pr 4.** $V_S$ is a *linear space* over $S$ with respect to the mappings (7), (8).

**Pr 5.** $V_S$ is a *3-dimensional Hermitean space* over $S$ with respect to the mappings (7)–(9).

Pr 4 implies

**Pr 6.** $V_S$ is a *commutative group*.

**Sch 5.** Pr 3–6 represent, as the saying is, a global characteristic of the $S$-standard vector spaces. A local picture is proposed by the following propositions.

**Pr 7.** $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a}(\lambda \boldsymbol{b}) = \overline{\lambda(\boldsymbol{ab})}$.

**Pr 8.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \overline{\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}}$.

**Pr 9.** $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $(\boldsymbol{a} \times \boldsymbol{b})^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{ab})$.

**Pr 10.** $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply: $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent iff $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{o}$.

**Pr 11.** $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply

$$(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})\overline{(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})} = \begin{vmatrix} \boldsymbol{a}^2 & \boldsymbol{ab} & \boldsymbol{ac} \\ \boldsymbol{ba} & \boldsymbol{b}^2 & \boldsymbol{bc} \\ \boldsymbol{ca} & \boldsymbol{cb} & \boldsymbol{c}^2 \end{vmatrix} \quad (20)$$

**Pr 12.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply: $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent iff $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} \neq 0$.

**Pr 13.** $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$.

**Pr 14.** $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $(\lambda \boldsymbol{a}) \times \boldsymbol{b} = \overline{\lambda}(\boldsymbol{a} \times \boldsymbol{b})$.

**Pr 15.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$.

**Pr 16.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{ca})\boldsymbol{b} - (\boldsymbol{ba})\boldsymbol{c}$.

**Pr 17.** $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a} \times (\lambda \boldsymbol{b}) = \overline{\lambda}(\boldsymbol{a} \times \boldsymbol{b})$.

**Pr 18.** $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

**Sch 6.** A most important role in $V_S$-algebra play the so-called *Gibbs' vectors*. They are defined in the following manner.

Let

(21) $\boldsymbol{a}_\nu \in V_S \quad (\nu = 1, 2, 3),$

(22) $\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \neq 0.$

Then

(23) $\boldsymbol{a}_\nu^{-1}$ sgn : $\dfrac{\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}}{\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3} \quad (\nu = 1, 2, 3)$

provided

(24) $\boldsymbol{a}_{\nu+3}$ sgn : $\boldsymbol{a}_\nu \quad (\nu = 1, 2)$

are called *Gibbs'* or *reciprocal vectors* of the vectors (21).

The basic properties of (23) are described by the following propositions.

**Pr 19.** (21), (22) imply

(25) $$\boldsymbol{a}_\mu^{-1}\boldsymbol{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu = 1,\ 2,\ 3).$$

**Pr 20.** (21), (22) imply

(26) $$\boldsymbol{a}_1^{-1} \times \boldsymbol{a}_2^{-1} \cdot \boldsymbol{a}_3^{-1} \neq 0.$$

**Pr 21.** (21), (22) imply

(27) $$(\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a})(\boldsymbol{a}_1^{-1} \times \boldsymbol{a}_2^{-1} \cdot \boldsymbol{a}_3^{-1}) = 1.$$

**Pr 22.** (21), (22) imply

(28) $$(\boldsymbol{a}_\nu^{-1})^{-1} = \boldsymbol{a}_\nu \qquad (\nu = 1,\ 2,\ 3).$$

**Pr 23.** (21), (22) imply

(29) $$\boldsymbol{a}_\nu^{-1} = \boldsymbol{a}_\nu \qquad (\nu = 1,\ 2,\ 3).$$

iff

(30) $$\boldsymbol{a}_\mu\boldsymbol{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu,\ \nu = 1,\ 2,\ 3).$$

**Pr 24.** (21), (22),

(31) $$\boldsymbol{r} \in V_S$$

imply

(32) $$\boldsymbol{r} = \sum_{\nu=1}^{3}(\boldsymbol{r}\boldsymbol{a}_\nu^{-1})\boldsymbol{a}_\nu.$$

**Pr 25.** (21), (22), (31) imply

(33) $$\boldsymbol{r} = \sum_{\nu=1}^{3}(\boldsymbol{r}\boldsymbol{a}_\nu)\boldsymbol{a}_\nu^{-1}.$$

**Pr 26.** (21), (22),

(34) $$\alpha_\nu \in S \qquad (\nu = 1,\ 2,\ 3)$$

imply: there exists exactly one (31) with

(35) $$\boldsymbol{r}_\nu\boldsymbol{a}_\nu = \alpha_\nu \qquad (\nu = 1,\ 2,\ 3),$$

namely

$$r = \sum_{\nu=1}^{3} \alpha_\nu a_\nu^{-1}. \tag{36}$$

**Pr 27.** (21), (22) imply: there exists exactly one (31) with

$$r a_\nu = 0 \qquad (\nu = 1, 2, 3), \tag{37}$$

namely

$$r = o. \tag{38}$$

**Pr 28.** (31),

$$a_\nu \in V_S \qquad (\nu = 1, 2), \tag{39}$$

$$b_\nu \in V_S \qquad (\nu = 1, 2), \tag{40}$$

$$r \times a_\nu = b_\nu \qquad (\nu = 1, 2) \tag{41}$$

imply

$$a_\mu b_\nu + a_\nu b_\mu = 0 \qquad (\mu, \nu = 1, 2). \tag{42}$$

**Pr 29.** (31), (39),

$$a_1 \times a_2 \neq o, \tag{43}$$

$$r \times a_\nu = o \qquad (\nu = 1, 2) \tag{44}$$

imply (38).

**Pr 30.** (21), (22),

$$b_\nu \in V_S \qquad (\nu = 1, 2, 3), \tag{45}$$

$$a_\mu b_\nu + a_\nu b_\mu = 0 \qquad (\mu, \nu = 1, 2). \tag{46}$$

imply: there exists exactly one (31) with

$$r \times a_\nu = b_\nu \qquad (\nu = 1, 2, 3), \tag{47}$$

namely

$$r = \frac{1}{2} \sum_{\nu=1}^{3} a_\nu^{-1} \times b_\nu. \tag{48}$$

**Pr 31.** (39), (40), (42), (43) imply: there exists exactly one (31) with (41), namely (48) provided

$$a_3 \quad \mathrm{sgn}: \quad a_1 \times a_2, \tag{49}$$

$$b_3 \quad \mathrm{sgn}: \quad (b_1 \cdot a_2 \times a_1) a_1^{-1} + (b_2 \cdot a_2 \times a_1) a_2^{-1}. \tag{50}$$

**Sch 7.** There are four basic systems of vector-algebraic equations which are routinely applied to $V_S$-algebra and in its applications to various problems, mainly in geometry and mechanics. Three of them are the systems (35), (41), and (47). With regard to the fourth one, it is regarded in the following propositions Pr 33–Pr 35.

**Pr 32,** (31),

(51) $\boldsymbol{a}, \boldsymbol{b} \in V_S,$

(52) $\boldsymbol{a} \neq \boldsymbol{o},$

(53) $\boldsymbol{ab} = 0$

imply:

(54) $\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}$

iff there exists

(55) $\alpha \in S$

with

(56) $$\boldsymbol{r} = \alpha \boldsymbol{a} + \frac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a}^2}.$$

**Pr 33.** (51), (53), (55),

(57) $\boldsymbol{c} \in V_S,$

(58) $\boldsymbol{ac} \neq 0$

imply: there exists exactly one (31) with (54) and

(59) $\boldsymbol{rc} = \alpha,$

namely

(60) $$\boldsymbol{r} = \frac{\alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}}{\boldsymbol{ac}}.$$

**Pr 34.** (51), (55), (57),

(61) $\boldsymbol{ac} = 0,$

(62) $\alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} \neq \boldsymbol{o}$

imply: there exists no (31) with (54), (59).

**35.** (51)–(53), (55), (57), (61),

(63) $\boldsymbol{c} \neq \boldsymbol{o},$

(64) $\alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} = \boldsymbol{o}$

imply: any (31) satisfying (54) is satisfying (59) too, but there exists one at least (31) satisfying (59) which does not satisfy (54).

## § 2. BASIC DEFINITIONS

This paragraph contains the basic definitions relating mainly to a single arrow and its fundamental attributes.

**Sgn 1.** $W_S$ sgn: $\{(\boldsymbol{s}, \boldsymbol{m}) \in V_S^2 : \boldsymbol{s} \neq \mathbf{o}, \boldsymbol{sm} = 0 \vee \boldsymbol{s} = \boldsymbol{m} = \mathbf{o}\}$.

**Df 1.** The elements of $W_S$ are called *arrows in* $V_S$ or *S-arrows.*

**Df 2.** $\boldsymbol{s}$ is called the *basis* of $\vec{s}$ if

$$\vec{s} \in W_S, \tag{1}$$

$$\vec{s} = (\boldsymbol{s}, \boldsymbol{m}). \tag{2}$$

**Df 3.** $\boldsymbol{m}$ is called the *moment* of $\vec{s}$ if (1), (2).

**Sgn 2.** $\vec{o}$ sgn: $(\mathbf{o}, \mathbf{o})$.

**Df 4.** $\vec{o}$ is called the *zero-arrow.*

**Df 5.** $\vec{s}$ is called a *non-zero arrow* if (1),

$$\vec{s} \neq \vec{o}. \tag{3}$$

**Pr 1.** (2) implies (3) iff

$$\boldsymbol{s} \neq \mathbf{o}, \qquad \boldsymbol{sm} = 0. \tag{4}$$

**Dm.** Sgn 1, Sgn 2.

**Sgn 3.** $\Lambda_S$ sgn: $\{(\boldsymbol{s}, \boldsymbol{m}) \in V_S^2 : \boldsymbol{s} \neq \mathbf{o}, \boldsymbol{sm} = 0\}$.

**Pr 2.** (1) implies (3) iff $\vec{s} \in \Lambda_S$.

**Dm.** Sgn 1, Pr 1, Sgn 3.

**Pr 3.** If

$$\vec{o} \neq \vec{s} \in W_S, \tag{5}$$

then there exists exactly one $l \in L_S$ with

$$\vec{s} \mathbin{\&} l. \tag{6}$$

**Dm.** Pr 2, Sgn 3, [3] 1, Pr 19.

**Sgn 4.** dir $\vec{s}$ sgn: $l \in L_S$ with (6) if (5).

**Df 6.** dir $\vec{s}$ is called the *directrix* of $\vec{s}$.

**Pr 4.** If

$$\vec{s} \in \Lambda_S, \tag{7}$$

then

$$\vec{s} \mathbin{\&} \operatorname{dir} \vec{s}. \tag{8}$$

**Dm.** Sgn 3, Sgn 1, Pr 2, Pr 3, Sgn 4.

**Sgn 5.** $\operatorname{mom}_{\boldsymbol{r}} \vec{s}$ sgn: $\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}$ if (1), (2),

$$\boldsymbol{r} \in V_S. \tag{9}$$

**Df 7.** $\text{mom}_{\mathbf{r}}\overrightarrow{s}$ is called the ***r-moment*** of $\overrightarrow{s}$.
**Df 8.** $\boldsymbol{r}$ is called the *pole* of $\text{mom}_{\mathbf{r}}\overrightarrow{s}$.
**Pr 5.** (1), (2), (9) imply $\boldsymbol{s}\cdot\text{mom}_{\mathbf{r}}\overrightarrow{s} = 0$.
**Dm.** Sgn 5, Sgn 1.
**Pr 6.** (1), (2) imply $\boldsymbol{m} = \text{mom}_{\mathbf{o}}\overrightarrow{s}$.
**Dm.** Sgn 5.
**Pr 7.** (9) implies $\text{mom}_{\mathbf{r}}\overrightarrow{o} = \boldsymbol{o}$.
**Dm.** Sgn 5, Sgn 2.
**Pr 8.** (5), (9) imply

(10) $$\text{mom}_{\mathbf{r}}\overrightarrow{s} = \mathbf{o}$$

iff

(11) $$\boldsymbol{r}\ \mathrm{Z}\ \text{dir}\ \overrightarrow{s}.$$

**Dm.** Sgn 5 implies (10) iff

(12) $$\boldsymbol{r}\times\boldsymbol{s} = \boldsymbol{m}.$$

Pr 3, Sgn 4 imply: dir $\overrightarrow{s}$ exists. Now Sgn 4, **[3]** 4 Sgn 1 imply (11) iff (12).
**Pr 9.** (5), (9),

(13) $$\overline{\rho}\in V_S,$$

(14) $$\overline{\rho}\ \mathrm{Z}\ \text{dir}\ \overrightarrow{s}$$

imply

(15) $$\text{mom}_{\mathbf{r}}\overrightarrow{s} = (\overline{\rho}-\boldsymbol{r})\times\boldsymbol{s}.$$

**Dm.** (13), (14), Sgn 4, **[3]** 4Sgn 1 imply

(16) $$\overline{\rho}\times\boldsymbol{s} = \boldsymbol{m}.$$

Now (16), Sgn 5 imply (15).

**Sch 1.** Traditionally text-books on analytical mechanics define $\text{mom}_{\mathbf{r}}\overrightarrow{s}$ (in the case of $V$-arrows, naturally) by (15) rather than by

(17) $$\text{mom}_{\mathbf{r}}\overrightarrow{s} = \boldsymbol{m}+\boldsymbol{s}\times\boldsymbol{r}.$$

The definition (17) obviously surpasses the definition (15) in being more economical.
**Pr 10.** (1), (2),

(18) $$\boldsymbol{r}_\nu\in V_S \qquad (\nu = 1, 2)$$

imply

(19) $$\text{mom}_{\mathbf{r}_1}\overrightarrow{s} - \text{mom}_{\mathbf{r}_2}\overrightarrow{s} = \boldsymbol{s}\times(\boldsymbol{r}_1-\boldsymbol{r}_2).$$

**Dm.** Sgn 5.

**Sch 2.** The relation (19) is usually called the ***connection between the moments of an arrow with respect to two poles.***

**Sch 3.** (19) implies

$$(20) \qquad \boldsymbol{s} \cdot \mathrm{mom}_{\mathbf{r}_1} \overrightarrow{s} = \boldsymbol{s} \cdot \mathrm{mom}_{\mathbf{r}_2} \overrightarrow{s}.$$

The inference (20) from (19) is, however, a trivial one, in the light of Pr 5.

**Pr 11.** (1), (18) imply

$$(21) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_1} \overrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_2} \overrightarrow{s}.$$

**Dm.** Pr 10.

**Pr 12.** (1), (18),

$$(22) \qquad \boldsymbol{r}_1 \neq \boldsymbol{r}_2,$$

$$(23) \qquad S = P \quad \text{or} \quad S = C(P)$$

imply

$$(24) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \cdot \mathrm{mom}_{\mathbf{r}_1} \overrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \cdot \mathrm{mom}_{\mathbf{r}_2} \overrightarrow{s}$$

**Dm.** Pr 11, 1 Sgn 4.

**Sch 4.** The relation (24) gives an utterance of the fact that if (23) holds and if (18) are different poles, then the projections of $\mathrm{mom}_{\mathbf{r}_\nu} \overrightarrow{s}$ ($\nu = 1, 2$) on the line $l$ connecting them, i.e. defined by

$$(25) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{r}_2 \times \boldsymbol{r}_1) \,\&\, l,$$

are equal.

**Pr 13.** (1),

$$(26) \qquad \boldsymbol{r}_\nu \in V_S \qquad (\nu = 1, 2, 3),$$

$$(27) \qquad \boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1 \neq \mathbf{o},$$

$$(28) \qquad \mathrm{mom}_{\mathbf{r}_\nu} \overrightarrow{s} = \mathbf{o} \qquad (\nu = 1, 2, 3)$$

imply

$$(29) \qquad \overrightarrow{s} = \overrightarrow{o}.$$

**Dm.** (2), (28), Sgn 5 imply

$$(30) \qquad \boldsymbol{r}_\nu \times \boldsymbol{s} = \boldsymbol{m} \qquad (\nu = 1, 2, 3),$$

whence

$$(31) \qquad (\boldsymbol{r}_\nu - \boldsymbol{r}_3) \times \boldsymbol{s} = \mathbf{o} \qquad (\nu = 1, 2, 3).$$

On the other hand, (27) is equivalent to

(32) $\qquad (\boldsymbol{r}_1 - \boldsymbol{r}_3) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) \neq \mathbf{o}$

and (31), (32), 1 Pr 29 imply

(33) $\qquad \boldsymbol{s} = \mathbf{o}.$

Now (33), (30) imply

(34) $\qquad \boldsymbol{m} = \mathbf{o}$

and (2), (33), (34), Sgn 2 imply (29).

**Sch 5.** The condition (27) implies that (26) are not colinear, i.e. that there exists no line $l \in L_S$ with

(35) $\qquad \boldsymbol{r}_\nu \,\&\, l \qquad (\nu = 1, 2, 3).$

In such a manner, $\overrightarrow{s}$ is cetainly the zero-arrow if there exist three non-colinear poles (26) with (28). The inverse statement (i.e: that if (29), then there exist (26) and (27), (28)) is trivial in the light of Pr 7.

**Sch 6.** Pr 13 admits the following inversion.

**Pr 14.** (1), (26), (27),

(36) $\qquad \boldsymbol{n} \in V_S,$

(37) $\qquad \mathrm{mom}_{\mathbf{r}_\nu} \overrightarrow{s} = \boldsymbol{n} \qquad (\nu = 1, 2, 3)$

imply

(38) $\qquad \boldsymbol{n} = \mathbf{o}$

and (29).

**Dm.** (2), (37), Sgn 5 imply

(39) $\qquad \boldsymbol{r}_\nu \times \boldsymbol{s} = \boldsymbol{m} - \boldsymbol{n} \qquad (\nu = 1, 2, 3),$

whence (31). Since (27) is equivalent to (32), the relations (31), (32), 1 Pr 29 imply again (33). Now (33), (39) imply

(40) $\qquad \boldsymbol{m} = \boldsymbol{n}.$

On the orher hand, (2), (33), Sgn 1 imply (34), and (34), (40) imply (38).

**Sch 7.** Pr 13 and Pr 14 imply that the zero-arrow is the only arrow, the moments of which with respect to three non-colinear poles are invariable with respect to the latter.

The following two propositions give an idea of the distribution of the moments of an arrow in space.

**Pr 15.** (1),(2),(18),

(41) $\qquad \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) = \mathbf{o}$

imply

(42) $\qquad \mathrm{mom}_{\mathbf{r}_1} \overrightarrow{s} = \mathrm{mom}_{\mathbf{r}_2} \overrightarrow{s}.$

**Dm.** (41) implies

$$m + s \times r_1 = m + s \times r_2, \tag{43}$$

whence (42) (Sgn 5).

**Pr 16.** (1), (2), (18),

$$s \times (r_1 - r_2) \neq o \tag{44}$$

imply

$$\mathrm{mom}_{r_1} \overrightarrow{s} \neq \mathrm{mom}_{r_2} \overrightarrow{s}. \tag{45}$$

**Dm.** (44) implies

$$m + s \times r_1 \neq m + s \times r_2, \tag{46}$$

whence (45) (Sgn 6).

**Sch 8.** Let (25) hold good. Then (41) implies that $l$ is coherent to dir $\overrightarrow{s}$ (Sgn 4, [3] 1 Sgn 6), i.e. $l$ is parallel or coincides with dir $\overrightarrow{s}$, while (44) implies that $l$ is non-coherent to dir $\overrightarrow{s}$ (Sgn 4, [3] 1 Sgn 7). Now Pr 15 and Pr 16 display that a necessary and sufficient condition for the equality of the moments of a non-zero arrow with respect to two different poles is the coincidence or the parallelism of dir $\overrightarrow{s}$ with the line incident with these poles.

**Pr 17.** (1), (2) imply $(-s, -m) \in W_S$.

**Dm.** Sgn 1.

**Sgn 6.** $-\overrightarrow{s}$ sgn: $(-s, -m)$ if (1), (2).

**Df 9.** $-\overrightarrow{s}$ is called the *opposite arrow* of $\overrightarrow{s}$.

**Pr 18.** (1) implies $-(-\overrightarrow{s}) = \overrightarrow{s}$.

**Dm.** Sgn 6.

**Pr 19.** $-\overrightarrow{o} = \overrightarrow{o}$.

**Dm.** Sgn 6, Sgn 2.

**Pr 20.** (1) implies (3) iff $-\overrightarrow{s} \neq \overrightarrow{o}$.

**Dm.** Pr 18, Pr 19.

**Pr 21.** (1) implies (3) iff $-\overrightarrow{s} \in \Lambda_S$.

**Dm.** Pr 20, Pr 2.

**Pr 22.** (1) implies (29) iff

$$-\overrightarrow{s} = \overrightarrow{s}. \tag{47}$$

**Dm.** Pr 19 and: (2), Sgn 6 imply that (47) is equivalent to

$$-s = s, \qquad -m = m. \tag{48}$$

Now (48) imply (33), (34), whence (29) (Sgn 2).

**Pr 23.** (1), (2),

$$\lambda \in S \tag{49}$$

imply $(\lambda s, \bar{\lambda} m) \in W_S$.

**Dm.** Sgn 1.

**Sgn 7.** $\lambda \overrightarrow{s}$ sgn: $(\lambda s, \bar{\lambda} m)$ if (1), (2), (49).

**Df 10.** $\lambda \overrightarrow{s}$ is called the ***product*** of $\lambda$ and $\overrightarrow{s}$.

**Pr 24.** (1) implies $0 \overrightarrow{s} = \overrightarrow{o}$.

**Dm.** Sgn 7, Sgn 2.

**Pr 25.** (49) implies $\lambda \overrightarrow{o} = \overrightarrow{o}$.

**Dm.** Sgn 7, Sgn 2.

**Pr 26.** (1) implies $1 \overrightarrow{s} = \overrightarrow{s}$.

**Dm.** Sgn 7.

**Pr 27.** (1) implies $(-1) \overrightarrow{s} = -\overrightarrow{s}$.

**Dm.** Sgn 7, Sgn 6.

**Pr 28.** (1), (49) imply $(-\lambda) \overrightarrow{s} = -(\lambda \overrightarrow{s})$.

**Dm.** Sgn 7, Sgn 6.

**Pr 29.** (1), (49) imply $\lambda(-\overrightarrow{s}) = -(\lambda \overrightarrow{s})$.

**Dm.** Sgn 6, Sgn 7.

**Pr 30.** (1), (49) imply $(-\lambda) \overrightarrow{s} = \lambda(-\overrightarrow{s})$.

**Dm.** Pr 28, Pr 29.

**Pr 31.** (1), (49) imply $(-\lambda)(-\overrightarrow{s}) = \lambda \overrightarrow{s}$.

**Dm.** Sgn 6, Sgn 7.

**Pr 32.** (1), (49),

(50) $\qquad \mu \in S$

imply $(\lambda \mu) \overrightarrow{s} = \lambda(\mu \overrightarrow{s})$.

**Dm.** Sgn 7.

**Pr 33.** (1),

(51) $\qquad 0 \neq \lambda \in S,$

(52) $\qquad \lambda \overrightarrow{s} = \overrightarrow{o}$

imply (29).

**Dm.** Pr 32, Pr 26, Pr 25.

**Pr 34.** (5), (49), (52) imply $\lambda = 0$.

**Dm.** Pr 33.

**Pr 35.** (5), (51) imply $\lambda \overrightarrow{s} \neq \overrightarrow{o}$.

**Dm.** Pr 33, Pr 34.

**Pr 36.** (5), (51) imply $\lambda \overrightarrow{s} \in \Lambda_S$.

**Dm.** Pr 35, Pr 2.

**Pr 37.** (51),

(53) $\qquad \overrightarrow{s}_\nu \in W_S \qquad (\nu = 1, 2),$

(54) $\qquad \overrightarrow{s}_1 = \lambda \overrightarrow{s}_2$

imply $\vec{s}_2 = \frac{1}{\lambda}\vec{s}_1$.

**Dm.** Pr 32, Pr 26.

**Pr 38.** (5), (51) imply $\vec{s} \sim \lambda\vec{s}$.

**Dm.** Pr 2, Pr 36, Sgn 7, [3] 1 Sgn 2.

**Pr 39.** (5), (51) imply: dir $\lambda\vec{s}$ exists.

**Dm.** Pr 35, Sgn 4.

**Pr 40.** (5), (51) imply dir $\lambda\vec{s}$ = dir $\vec{s}$.

**Dm.** Sgn 4, Pr 39, Pr 38, [3] 1 Ax 3.

**Pr 41.** (5) implies dir $(-\vec{s})$ = dir $\vec{s}$.

**Dm.** Pr 27, Pr 40.

**Pr 42.** If

(55) $$\vec{0} \neq \vec{s}_\nu \in W_S \qquad (\nu = 1, 2),$$

(56) $$\text{dir } \vec{s}_1 = \text{dir } \vec{s}_2,$$

then there exists (51) with (54).

**Dm.** (56), Sgn 4, Pr 2, [3] 1 Ax 5 imply

(57) $$\vec{s}_1 \sim \vec{s}_2.$$

Now (57), [3] 1 Sgn 2 imply that there exists (51) with (54).

**Pr 43.** (55) imply (56) iff (54).

**Dm.** Pr 40, Pr 42.

**Pr 44.** (49), (53), (54), (9) imply

(58) $$\text{mom}_{\mathbf{r}} \vec{s}_1 = \bar{\lambda} \text{mom}_{\mathbf{r}} \vec{s}_2$$

**Dm.** If

(59) $$\vec{s}_\nu = (s_\nu, m_\nu) \qquad (\nu = 1, 2),$$

then

(60) $$s_1 = \lambda s_2, \qquad m_1 = \bar{\lambda} m_2$$

(Sgn 7). Now (60), Sgn 5, 1 Pr 14 imply

(61) $$\begin{aligned} \text{mom}_{\mathbf{r}} \vec{s}_1 &= m_1 + s_1 \times r = \bar{\lambda} m_2 + (\lambda s_2) \times r \\ &= \bar{\lambda} m_2 + \bar{\lambda}(s_2 \times r) = \bar{\lambda}\, \text{mom}_{\mathbf{r}} \vec{s}_2. \end{aligned}$$

**Sgn 8.** mom $(\vec{s}_1, \vec{s}_2)$ sgn: $s_1 m_2 + s_2 m_1$ if (53), (59).

**Df 11.** mom $(\vec{s}_1, \vec{s}_2)$ is called the *mutual moment* of $\vec{s}_\nu$ $(\nu = 1, 2)$.

**Pr 45.** (53) imply mom $(\vec{s}_1, \vec{s}_2)$ = mom $(\vec{s}_2, \vec{s}_1)$.

**Dm.** Sgn 8.

**Pr 46.** (1) implies mom $(\vec{s}, \vec{o}) = 0$.

**Dm.** Sgn 2, Sgn 8.

**Pr 47.** (1) implies mom $(\vec{s}, \vec{s}) = 0$.
**Dm.** Sgn 8, Sgn 1.
**Pr 48.** (1), (49) imply mom $(\vec{s}, \lambda\vec{s}) = 0$.
**Dm.** Sgn 8, Sgn 7, Ax 9S, 1 Pr 7, Sgn 1.
**Pr 49.** (1) implies mom $(\vec{s}, -\vec{s}) = 0$.
**Dm.** Pr 48, Pr 27.
**Pr 50.** (55), (59), (18),

$$\mathbf{r}_\nu \ \mathrm{Z} \ \mathrm{dir}\ \vec{s}_\nu \qquad (\nu = 1, 2) \tag{62}$$

imply

$$\mathbf{s}_1 \cdot \mathrm{mom}_{\mathbf{r}_1} \vec{s}_2 = \mathbf{s}_2 \cdot \mathrm{mom}_{\mathbf{r}_2} \vec{s}_1. \tag{63}$$

**Dm.** Pr 9, 1 Pr 9, 1 Ax 8S, 1 Pr 13 imply

$$\mathbf{s}_1 \cdot \mathrm{mom}_{\mathbf{r}_1} \vec{s}_2 = \mathbf{s}_1 \cdot (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{s}_2 = \overline{\mathbf{s}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{s}_2} \tag{64}$$

$$= \mathbf{s}_2 \cdot \mathbf{s}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) = \mathbf{s}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{s}_1 = \mathbf{s}_2 \cdot \mathrm{mom}_{\mathbf{r}_2} \vec{s}_1.$$

**Pr 51.** (53), (59), (9) imply

$$\mathrm{mom}(\vec{s}_1, \vec{s}_2) = \mathbf{s}_1 \cdot \mathrm{mom}_{\mathbf{r}} \vec{s}_2 + \mathbf{s}_2 \cdot \mathrm{mom}_{\mathbf{r}} \vec{s}_1. \tag{65}$$

**Dm.** Sgn 5 implies

$$\mathbf{s}_1 \cdot \mathrm{mom}_{\mathbf{r}} \vec{s}_2 + \mathbf{s}_2 \cdot \mathrm{mom}_{\mathbf{r}} \vec{s}_1 = \mathbf{s}_1(\mathbf{m}_2 + \mathbf{s}_2 \times \mathbf{r}) + \mathbf{s}_2(\mathbf{m}_1 + \mathbf{s}_1 \times \mathbf{r}). \tag{66}$$

On the other hand, 1Pr 8, 1 Ax 8S, 1 Pr 13 imply

$$\mathbf{s}_1 \cdot \mathbf{s}_2 \times \mathbf{r} + \mathbf{s}_2 \cdot \mathbf{s}_1 \times \mathbf{r} = \overline{\mathbf{s}_1 \times \mathbf{s}_2 \cdot \mathbf{r}} + \overline{\mathbf{s}_2 \times \mathbf{s}_1 \cdot \mathbf{r}} \tag{67}$$

$$= \mathbf{r} \cdot \mathbf{s}_1 \times \mathbf{s}_2 + \mathbf{r} \cdot \mathbf{s}_2 \times \mathbf{s}_1 = \mathbf{r}(\mathbf{s}_1 \times \mathbf{s}_2 + \mathbf{s}_2 \times \mathbf{s}_1) = \mathbf{r}\mathbf{o} = 0.$$

Now (66), (67), Sgn 8 imply (65).
**Pr 52.** (18), (53), (59), (62) imply

$$\mathrm{mom}(\vec{s}_1, \vec{s}_2) = (\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{s}_1 \times \mathbf{s}_2. \tag{68}$$

**Dm.** (62), Sgn 4, [3] 4 Sgn 1 imply

$$\mathbf{r}_\nu \times \mathbf{s}_\nu = \mathbf{m}_\nu \qquad (\nu = 1, 2). \tag{69}$$

Now Sgn 8, (69), 1 Ax 8S, 1Pr 8, 1 Pr 13 imply (68).

**Sgn 1.** $\vec{s}_1 \mid \vec{s}_2$ sgn: $\boldsymbol{s}_1 \times \boldsymbol{s}_2 = \mathbf{o}$ if 2(53), 2(59).
**Df 1.** $\vec{s}_1$ is called *adherent* to $\vec{s}_2$ if $\vec{s}_1 \mid \vec{s}_2$.
**Sgn 2.** $\vec{s}_1 \overline{\mid} \vec{s}_2$ sgn: $\boldsymbol{s}_1 \times \boldsymbol{s}_2 \neq \mathbf{o}$ if 2(53), 2(59).
**Df 2.** $\vec{s}_1$ is called *non-adherent* to $\vec{s}_2$ if $\vec{s}_1 \overline{\mid} \vec{s}_2$.
**Pr 1.** 2(53) imply: exactly one of the relations

$$\vec{s}_1 \mid \vec{s}_2 \tag{1}$$

or

$$\vec{s}_1 \overline{\mid} \vec{s}_2 \tag{2}$$

holds.
**Dm.** Sgn 1, Sgn 2.
**Pr 2.** 2(53), (1) imply $\vec{s}_2 \mid \vec{s}_1$.
**Dm.** Sgn 1, 1 Pr 13.
**Pr 3.** 2(53), (2) imply $\vec{s}_2 \overline{\mid} \vec{s}_1$.
**Dm.** Pr 1, Pr 2.
**Pr 4.** 2(1) implies $\vec{s} \mid \vec{s}$.
**Dm.** Sgn 1.
**Pr 5.** 2(1) implies $\vec{s} \mid \vec{o}$.
**Dm.** Sgn 1, 2 Sgn 2.
**Pr 6.** 2(1) implies $\vec{s} \mid -\vec{s}$.
**Dm.** Sgn 1, 2 Sgn 6.
**Pr 7.** 2(1), 2(49) imply $\vec{s} \mid \lambda\vec{s}$.
**Dm.** Sgn 1, 2 Sgn 7.
**Sgn 3.** $\vec{s_1} \mid\mid\mid \vec{s}_2$ sgn: $\vec{s_1} \mid \vec{s}_2$ iff 2(55).
**Df 3.** $\vec{s}_1$ is called *coherent* to $\vec{s}_2$ if $\vec{s_1} \mid\mid\mid \vec{s}_2$.
**Sgn 4.** $\vec{s_1} \overline{\mid\mid\mid} \vec{s}_2$ sgn: $\vec{s_1} \overline{\mid} \vec{s}_2$ or $\vec{s}_\nu = \vec{o}$ $(1 \leqq \nu \leqq 2)$ iff 2(53).
**Df 4.** $\vec{s}_1$ is called *non-coherent* to $\vec{s}_2$ if $\vec{s_1} \overline{\mid\mid\mid} \vec{s}_2$.
**Pr 8.** 2(53) imply: exactly one of the relations

$$\vec{s}_1 \mid\mid\mid \vec{s}_2 \tag{3}$$

or

$$\vec{s}_1 \overline{\mid\mid\mid} \vec{s}_2 \tag{4}$$

holds.
**Dm.** Sgn 3, Sgn 4.
**Pr 9.** 2(55), (3) imply $\vec{s}_2 \mid\mid\mid \vec{s}_1$.
**Dm.** Sgn 3, Pr 2.
**Pr 10.** 2(53), (4) imply $\vec{s}_2 \overline{\mid\mid\mid} \vec{s}_1$.
**Dm.** Pr 8, Pr 9.

**Pr 11.** 2(5) implies $\vec{s} \; ||| \; \vec{s}$.
**Dm.** Sgn 3, Pr 4.
**Pr 12.** 2(5) implies $\vec{s} \; ||| \; -\vec{s}$.
**Dm.** Sgn 3, 2 Pr 21, Pr 4.
**Pr 13.** 2(5), 2(51) imply $\vec{s} \; ||| \; \lambda\vec{s}$.
**Dm.** Sgn 3, 2 Pr 35, Pr 7.
**Pr 14.** 2(1) implies $\vec{s} \; \overline{|||} \; \vec{o}$.
**Dm.** Sgn 4.
**Sgn 5.** $\vec{s_1} \; || \; \vec{s}_2$ sgn: $\vec{s_1} \; ||| \; \vec{s}_2$, $\vec{s}_1 \neq \lambda\vec{s}_2$ iff 2(53), 2(51).
**Df 5.** $\vec{s}_1$ is called *parallel* to $\vec{s}_2$ if $\vec{s_1} \; || \; \vec{s}_2$.
**Sgn 6.** $\vec{s_1} \; \overline{||} \; \vec{s}_2$ sgn: $\vec{s_1} \; \overline{|||} \; \vec{s}_2$ or $\vec{s}_1 = \lambda\vec{s}_2$ iff 2(53), 2(51).
**Df 6.** $\vec{s}_1$ is called *non-parallel* to $\vec{s}_2$ if $\vec{s_1} \; \overline{||} \; \vec{s}_2$.
**Pr 15.** 2(53) imply: exactly one of the relations

(5) $$\vec{s}_1 \; || \; \vec{s}_2$$

or

(6) $$\vec{s}_1 \; \overline{||} \; \vec{s}_2$$

holds.
**Dm.** Sgn 5, Sgn 6.
**Pr 16.** 2(53), (5) imply $\vec{s}_2 \; || \; \vec{s}_1$.
**Dm.** Sgn 5, Pr 9, 2 Pr 37.
**Pr 17.** 2(53), (6) imply $\vec{s}_2 \; \overline{||} \; \vec{s}_1$.
**Dm.** Pr 15, Pr 16.
**Pr 18.** 2(1) implies $\vec{s} \; \overline{||} \; \vec{s}$.
**Dm.** Sgn 6, 2 Pr 26.
**Pr 19.** 2(1) implies $\vec{s} \; \overline{||} \; -\vec{s}$.
**Dm.** Sgn 6, 2 Pr 27.
**Pr 20.** 2(1) implies $\vec{s} \; \overline{||} \; \vec{o}$.
**Dm.** Sgn 6, Pr 14.
**Sgn 7.** $\vec{s_1} \; \uparrow\downarrow \; \vec{s}_2$ sgn: $s_1 + s_2 = o$, $m_1 + m_2 \neq o$ iff 2(53), 2(59).
**Df 7.** $\vec{s}_1$ is called *dipolar* to $\vec{s}_2$ if $\vec{s_1} \; \uparrow\downarrow \; \vec{s}_2$.
**Sgn 8.** $\vec{s_1} \; \overline{\uparrow\downarrow} \; \vec{s}_2$ sgn: $s_1 + s_2 \neq o$ or $m_1 + m_2 = o$ iff 2(53), 2(59).
**Df 8.** $\vec{s}_1$ is called *non-dipolar* to $\vec{s}_2$ if $\vec{s_1} \; \overline{\uparrow\downarrow} \; \vec{s}_2$.
**Pr 21.** 2(53) imply: exactly one of the relations

(7) $$\vec{s}_1 \; \uparrow\downarrow \; \vec{s}_2$$

or

(8) $$\vec{s}_1 \; \overline{\uparrow\downarrow} \; \vec{s}_2$$

holds.

**Dm.** Sgn 7, Sgn 8.

**Pr 22.** 2(53), (7) imply $\vec{s}_2 \uparrow\downarrow \vec{s}_1$.

**Dm.** Sgn 7.

**Pr 23.** 2(53), (8) imply $\vec{s}_2 \overline{\uparrow\downarrow} \vec{s}_1$.

**Dm.** Pr 21, Pr 22.

**Pr 24.** 2(53), (7) imply (1).

**Dm.** Sgn 7, Sgn 1.

**Pr 25.** 2(53), (7) imply 2(55).

**Dm.** Let 2(59) hold and let, for instance, $\vec{s}_1 = \vec{o}$. Then 2 Sgn 2 implies $\boldsymbol{s}_1 = \boldsymbol{m}_1 = \mathbf{o}$. Now $\boldsymbol{s}_1 = \mathbf{o}$ and Sgn 7 imply $\boldsymbol{s}_2 = \mathbf{o}$, whence $\boldsymbol{m}_2 = \mathbf{o}$ according to 2 Sgn 1. Then $\boldsymbol{m}_1 + \boldsymbol{m}_2 = \mathbf{o}$ contrary to Sgn 7.

**Pr 26.** 2(1) implies $\vec{s} \overline{\uparrow\downarrow} \vec{o}$.

**Dm.** Pr 25, Pr 21.

**Pr 27.** 2(1) implies $\vec{s} \overline{\uparrow\downarrow} \vec{s}$.

**Dm.** Pr 26, 2 Pr 1, Sgn 8.

**Pr 28.** 2(1) implies $\vec{s} \overline{\uparrow\downarrow} -\vec{s}$.

**Dm.** Sgn 6, Sgn 8.

**Pr 29.** 2(1), $-1 \neq \lambda \in S$ imply $\vec{s} \overline{\uparrow\downarrow} \lambda\vec{s}$.

**Dm.** If $\vec{s} = \vec{o}$, then 2 Pr 25, Pr 26. If $\vec{s} \neq \vec{o}$, then 2(2) implies $\boldsymbol{s} \neq \mathbf{o}$ (2 Pr 1) whence $\boldsymbol{s} + \lambda\boldsymbol{s} \neq \mathbf{o}$ by virtue of $\lambda \neq -1$. Now Sgn 8.

**Pr 30.** 2(53), (7) imply (3).

**Dm.** Sgn 3, Pr 24, Pr 25.

**Pr 31.** 2(53), (7) imply (5).

**Dm.** Sgn 5, Pr 28–Pr 30, 2 Pr 24.

**Pr 32.** 2(18), 2(53), (7), 2(62) imply

$$\text{(9)} \qquad \text{mom}_{\mathbf{r}_1} \vec{s}_2 = \text{mom}_{\mathbf{r}_2} \vec{s}_1.$$

**Dm.** 2(62), 2 Sgn 4, [3] 4 Sgn 1 imply 2(69). 2 Sgn 5 implies

$$\text{(10)} \qquad \text{mom}_{\mathbf{r}_1} \vec{s}_2 = \boldsymbol{m}_2 + \boldsymbol{s}_2 \times \boldsymbol{r}_1,$$

$$\text{(11)} \qquad \text{mom}_{\mathbf{r}_2} \vec{s}_1 = \boldsymbol{m}_1 + \boldsymbol{s}_1 \times \boldsymbol{r}_2.$$

Sgn 7, (7) imply

$$\text{(12)} \qquad \boldsymbol{s}_1 + \boldsymbol{s}_2 = \mathbf{o}.$$

Now 2(69) and (10)–(12) imply (9).

**Pr 33.** 2(53), (1) imply

$$\text{(13)} \qquad \text{mom}(\vec{s}_1, \vec{s}_2) = 0.$$

**Dm.** 2 Pr 46 or 2 Pr 52, Sgn 1.

**Pr 34.** 2(53), (3) imply (13).

**Dm.** Sgn 3, Pr 33.

**Pr 35.** 2(53), (5) imply (13).
**Dm.** Sgn 5, Pr 33.
**Pr 36.** 2(53), (7) imply (13).
**Dm.** Sgn 7, Pr 33.

## § 4. PERPENDICULARITY

**Sgn 1.** $\vec{s}_1 \top \vec{s}_2$ sgn: $s_1 s_2 = 0$ iff 2(53), 2(59).
**Df 1.** $\vec{s}_1$ is called *normal* to $\vec{s}_2$ if $\vec{s}_1 \top \vec{s}_2$.
**Sgn 2.** $\vec{s}_1 \overline{\top} \vec{s}_2$ sgn: $s_1 s_2 \neq 0$ iff 2(53), 2(59).
**Df 2.** $\vec{s}_1$ is called *non-normal* to $\vec{s}_2$ if $\vec{s}_1 \overline{\top} \vec{s}_2$.
**Pr 1.** 2(53) imply: exactly one of the relations

$$\vec{s}_1 \top \vec{s}_2 \tag{1}$$

or

$$\vec{s}_1 \overline{\top} \vec{s}_2 \tag{2}$$

holds.

**Dm.** Sgn 1, Sgn 2.
**Pr 2.** 2(53), (1) imply $\vec{s}_2 \top \vec{s}_1$.
**Dm.** Sgn 1.
**Pr 3.** 2(53), (2) imply $\vec{s}_2 \overline{\top} \vec{s}_1$.
**Dm.** Pr 1, Pr 2.
**Pr 4.** 2(53), (1) imply 3(4).
**Dm.** If $\vec{s}_\nu = \vec{o}$ $(1 \leqq \nu \leqq 2)$, then 3 Sgn 4. If $\vec{s}_\nu \neq \vec{o}$ $(\nu = 1, 2)$, then $s_\nu \neq \mathbf{o}$ $(\nu = 1, 2)$ (2 Pr 1). Now $s_1 s_2 = 0$ implies $(s_1 \times s_2)^2 = s_1^2 s_2^2$ (1 Pr 9) whence $s_1 \times s_2 \neq \mathbf{o}$ (1 Ax 12S). Then 3 Sgn 2, 3 Sgn 4.
**Pr 5.** 2(53), (1) imply 3(6).
**Dm.** Pr 4, Sgn 6.
**Pr 6.** 2(53), (1) imply 3(8).
**Dm.** 3 Pr 21, 3 Pr 31, Pr 5, 3 Pr 15.
**Pr 7.** 2(1) implies $\vec{s} \top \vec{o}$.
**Dm.** 2 Sgn 2, Sgn 1.
**Pr 8.** 2(5) implies $\vec{s} \overline{\top} \vec{s}$.
**Dm.** 2 Pr 1, 1 Ax 12S, Sgn 2.
**Pr 9.** 2(5) implies $\vec{s} \overline{\top} -\vec{s}$.
**Dm.** 2 Pr 1, 2 Sgn 6, 1 Ax 12S, Sgn 2.
**Pr 10.** 2(5), 2(51) imply $\vec{s} \overline{\top} \lambda\vec{s}$.
**Dm.** 2 Pr 1, 2 Sgn 7, 1 Ax 12S, Sgn 2.
**Pr 11.** 2(5), 2(53), $\vec{s} \mid \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).
**Dm.** 2(2), 2(59) imply $s \neq \mathbf{o}$ (2 Pr 1), $s \times s_1 = o$ (2 Sgn 1), $s s_2 = 0$ (Sgn 1). Then there exists 1(55) with $s_1 = \alpha s$ (1 Pr 13, 1 Pr 32), hence $s_1 s_2 = 0$. Now Sgn 1.
**Pr 12.** 2(1), 2(53), $\vec{s} \parallel\mid \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).

**Dm.** 2 Sgn 3, Pr 11.

**Pr 13.** 2(1), 2(53), $\vec{s} \parallel \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).

**Dm.** 2 Sgn 5, Pr 12.

**Pr 14.** 2(1), 2(53), $\vec{s} \uparrow\downarrow \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).

**Dm.** 2 Pr 31, Pr 13.

**Sgn 3.** $\vec{s_1} \perp \vec{s}_2$ sgn: $s_1 s_2 = 0$ iff 2(55), 2(59).

**Df 3.** $\vec{s}_1$ is called *perpendicular* to $\vec{s}_2$ if $\vec{s_1} \perp \vec{s}_2$.

**Sgn 4.** $\vec{s_1} \overline{\perp} \vec{s}_2$ sgn: $s_1 s_2 \neq 0$ or $s_\nu = \mathbf{o}$ $(1 \leqq \nu \leqq 2)$ iff 2(53), 2(59).

**Df 4.** $\vec{s}_1$ is called *non-perpendicular* to $\vec{s}_2$ if $\vec{s_1} \overline{\perp} \vec{s}_2$.

**Pr 15.** 2(53) imply: exactly one of the relations

$$\vec{s}_1 \perp \vec{s}_2 \tag{3}$$

or

$$\vec{s}_1 \overline{\perp} \vec{s}_2 \tag{4}$$

holds.

**Dm.** Sgn 3, Sgn 4.

**Pr 16.** 2(53), (3) imply $\vec{s}_2 \perp \vec{s}_1$.

**Dm.** Sgn 3.

**Pr 17.** 2(53), (4) imply $\vec{s}_2 \overline{\perp} \vec{s}_1$.

**Dm.** Pr 15, Pr 16.

**Pr 18.** 2(53), (3) imply (1).

**Dm.** Sgn 3, Sgn 1.

**Pr 19.** 2(53), (3) imply 3(2).

**Dm.** 2(59) imply $s_\nu \neq \mathbf{o}$ $(\nu = 1, 2)$ (Sgn 3, 2 Pr 1). Now $s_1 s_2 = 0$ implies $(s_1 \times s_2)^2 = s_1^2 s_2^2$ (1 Pr 9) whence $s_1 \times s_2 \neq \mathbf{o}$ (1 Ax 12S). Then 3 Sgn 2.

**Pr 20.** 2(53), (3) imply 3(4).

**Dm.** Pr 18, Pr 4.

**Pr 21.** 2(53), (3) imply 3(6).

**Dm.** Pr 18, Pr 5.

**Pr 22.** 2(53), (3) imply 3(8).

**Dm.** Pr 18, Pr 6.

**Pr 23.** 2(1) implies $\vec{s} \overline{\perp} \vec{o}$.

**Dm.** Sgn 4, 2 Sgn 2.

**Pr 24.** 2(1) implies $\vec{s} \overline{\perp} \vec{s}$.

**Dm.** Sgn 4, 2 Sgn 2, 2 Pr 1, 1 Ax 12S.

**Pr 25.** 2(1) implies $\vec{s} \overline{\perp} -\vec{s}$.

**Dm.** Sgn 4, 2 Sgn 6, 2 Sgn 2, 2 Pr 1, 1 Ax 12S.

**Pr 26.** 2(1), 2(49) imply $\vec{s} \overline{\perp} \lambda\vec{s}$.

**Dm.** Sgn 4, 2 Pr 24, 2 Pr 25, 2 Pr 1, 1 Ax 12S.

**Pr 27.** 2(1), 2(53), $\vec{s} \mid \vec{s}_1$, $\vec{s} \perp \vec{s}_2$ imply (1).

**Dm.** Pr 18, Pr 11.

**Pr 28.** 2(1), 2(53), $\vec{s} \mid\mid\mid \vec{s}_1$, $\vec{s} \perp \vec{s}_2$ imply (3).

**Dm.** 3 Sgn 3, Pr 12, Sgn 1, Sgn 3.

**Pr 29.** 2(1), 2(53), $\vec{s} \parallel \vec{s}_1$, $\vec{s} \perp \vec{s}_2$ imply (3).
**Dm.** 2 Sgn 5, Pr 28.
**Pr 30.** 2(1), 2(53), $\vec{s} \uparrow\downarrow \vec{s}_1$, $\vec{s} \perp \vec{s}_2$ imply (3).
**Dm.** 2 Pr 31, Pr 29.

## § 5. OTHER RELATIONS

**Sgn 1.** $\vec{s_1} \wedge \vec{s}_2$ sgn: $s_1 \times s_2 \neq o$, $s_1 m_2 + s_2 m_1 = 0$ iff 2(53), 2(59).
**Df 1.** $\vec{s}_1$ is called *intersecting* $\vec{s}_2$ if $\vec{s_1} \wedge \vec{s}_2$.
**Sgn 2.** $\vec{s_1} \barwedge \vec{s}_2$ sgn: $s_1 \times s_2 = 0$, $s_1 m_2 + s_2 m_1 \neq 0$ iff 2(53), 2(59).
**Df 2.** $\vec{s}_1$ is called *non-intersecting* $\vec{s}_2$ if $\vec{s_1} \barwedge \vec{s}_2$.
**Pr 1.** 2(53) imply: exactly one of the relations

(1) $$\vec{s}_1 \wedge \vec{s}_2$$

or

(2) $$\vec{s}_1 \barwedge \vec{s}_2$$

holds.
**Dm.** Sgn 1, Sgn 2.
**Pr 2.** 2(53), (1) imply $\vec{s}_2 \wedge \vec{s}_1$.
**Dm.** Sgn 1.
**Pr 3.** 2(53), (2) imply $\vec{s}_2 \barwedge \vec{s}_1$.
**Dm.** Pr 1, Pr 2.
**Pr 4.** 2(53), (1) imply

(3) $$\vec{s}_\nu \neq \vec{o} \qquad (\nu = 1, 2).$$

**Dm.** Sgn 1, 2 Pr 1.
**Pr 5.** 2(53), (1) imply 3(2).
**Dm.** Sgn 1, 3 Sgn 2.
**Pr 6.** 2(53), (1) imply 3(4).
**Dm.** 3 Sgn 4, Pr 5.
**Pr 7.** 2(53), (1) imply 3(6).
**Dm.** 3 Sgn 6, Pr 6.
**Pr 8.** 2(53), (1) imply 3(8).
**Dm.** Sgn 1, 3 Sgn 8.
**Pr 9.** 2(1) implies $\vec{s} \barwedge \vec{s}$.
**Dm.** Sgn 2.
**Pr 10.** 2(1) implies $\vec{s} \barwedge \vec{o}$.
**Dm.** Pr 4.
**Pr 11.** 2(1) implies $\vec{s} \barwedge -\vec{s}$.
**Dm.** 2 Sgn 6, Sgn 2.
**Pr 12.** 2(1), 2(49) imply $\vec{s} \barwedge \lambda\vec{s}$.
**Dm.** 2 Sgn 7, Sgn 2.
**Pr 13.** 2(53), (1) imply $\vec{s}_1 \wedge -\vec{s}_2$.

**Dm.** 2 Sgn 6, Sgn 1.
**Pr 14.** 2(53), (1), 2(51) imply $\vec{s}_1 \ \wedge \ \lambda\vec{s}_2$.
**Dm.** 2 Sgn 7, Sgn 1.
**Sgn 3.** $\vec{s}_1 \ \otimes \ \vec{s}_2$ sgn: $\mathbf{s}_1 \times \mathbf{s}_2 \neq \mathbf{o}, \mathbf{s}_1\mathbf{m}_2 + \mathbf{s}_2\mathbf{m}_1 \neq 0$ iff 2(53), 2(59).
**Df 3.** $\vec{s}_1$ is called *crossed* with $\vec{s}_2$ if $\vec{s}_1 \ \otimes \ \vec{s}_2$.
**Sgn 4.** $\vec{s}_1 \ \overline{\otimes} \ \vec{s}_2$ sgn: $\mathbf{s}_1 \times \mathbf{s}_2 = \mathbf{o}$ or $\mathbf{s}_1\mathbf{m}_2 + \mathbf{s}_2\mathbf{m}_1 = 0$ iff 2(53), 2(59).
**Df 4.** $\vec{s}_1$ is called *non-crossed* with $\vec{s}_2$ if $\vec{s}_1 \ \overline{\otimes} \ \vec{s}_2$.
**Pr 15.** 2(53) imply: exactly one of the relations

(4) $$\vec{s}_1 \otimes \vec{s}_2$$

or

(5) $$\vec{s}_1 \,\overline{\otimes}\, \vec{s}_2$$

holds.
**Dm.** Sgn 3, Sgn 4.
**Pr 16.** 2(53), (4) imply $\vec{s}_2 \ \otimes \ \vec{s}_1$.
**Dm.** Sgn 3.
**Pr 17.** 2(53), (5) imply $\vec{s}_2 \ \overline{\otimes} \ \vec{s}_1$.
**Dm.** Pr 15, Pr 16.
**Pr 18.** 2(53), (4) imply (3).
**Dm.** Sgn 3, 2 Pr 1.
**Pr 19.** 2(53), (4) imply 3(2).
**Dm.** Sgn 3, 3 Sgn 2.
**Pr 20.** 2(53), (4) imply 3(4).
**Dm.** 3 Sgn 4, Pr 19.
**Pr 21.** 2(53), (4) imply 3(6).
**Dm.** 3 Sgn 6, Pr 20.
**Pr 22.** 2(53), (4) imply 3(8).
**Dm.** 3 Pr 21, 3 Sgn 7, Sgn 3.
**Pr 23.** 2(53), (4) imply (2).
**Dm.** Sgn 1, Sgn 4.
**Pr 24.** 2(53), (1) imply (5).
**Dm.** Sgn 1, Sgn 4.
**Pr 25.** 2(1) implies $\vec{s} \ \overline{\otimes} \ \vec{s}$.
**Dm.** 2 Sgn 1, Sgn 4.
**Pr 26.** 2(1) implies $\vec{s} \ \overline{\otimes} \ \vec{o}$.
**Dm.** 2 Sgn 2, Sgn 4.
**Pr 27.** 2(1) implies $\vec{s} \ \overline{\otimes} \ -\vec{s}$.
**Dm.** 2 Sgn 6, Sgn 4.
**Pr 28.** 2(1), 2(49) imply $\vec{s} \ \overline{\otimes} \ \lambda\vec{s}$.
**Dm.** 2 Pr 24, 2 Pr 25, Pr 26, 2 Sgn 7, Sgn 4.
**Pr 29.** 2(53), (4) imply $\vec{s}_1 \ \otimes \ -\vec{s}_2$.
**Dm.** 2 Sgn 6, Sgn 3.
**Pr 30.** 2(53), (4), 2(51) imply $\vec{s}_1 \ \otimes \ \lambda\vec{s}_2$.
**Dm.** 2 Sgn 7, Sgn 3.
**Pr 31.** 2(53), (1) imply: there exists exactly one 2(9) with

(6) $\quad r \text{ Z dir } \overrightarrow{s}_\nu \quad (\nu = 1, 2),$

namely

(7) $$r = \frac{1}{2}\sum_{\nu=1}^{3} s_\nu^{-1} \times m_\nu$$

provided 2(59),

(8) $\quad s_3 \text{ sgn} : s_1 \times s_2,$

(9) $\quad m_3 \text{ sgn} : (m_1 \cdot s_2 \times s_1)s_1^{-1} + (m_1 \cdot s_2 \times s_1)s_2^{-1}.$

**Dm.** dir $\overrightarrow{s}_\nu$ $(\nu = 1, 2)$ exist by virtue of Pr 4, 2 Sgn 4. The relations (6) are equivalent to

(10) $\quad r \times s_\nu = m_\nu \quad (\nu = 1, 2)$

respectively, provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). In view of Sgn 1, 1 Pr 31 the system (10) has exactly one solution 2(9), namely (7) provided (8), (9).

**Pr 32.** 2(53), (4) imply: there exists no 2(9) with (6).

**Dm.** dir $\overrightarrow{s}_\nu$ $(\nu = 1, 2)$ exist by virtue of Pr 18, 2 Sgn 4. The relations (6) are equivalent to (10) respectively provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). In view of Sgn 3, 1 Pr 28, there exists no 2(9) with (10), since the necessary for the consistency of the system of vector equations (10) condition $s_1 m_2 + s_2 m_1 = 0$ is violated.

## § 6. ARMS & FEET

**Sch 1.** Let $r \in V_S$ and $\overrightarrow{s} = (s, m) \in W_S$ be given, $\overrightarrow{s} \neq \overrightarrow{o}$. Let $\overline{\rho} \in V_S$ be wanted satisfying

(1) $\quad \overline{\rho} \text{ Z dir } \overrightarrow{s},$

(2) $\quad (\overline{\rho} - r)s = 0.$

The condition ( 1) is equivalent to

(3) $\quad \overline{\rho} \times s = m$

(2 Sgn 4, [3] 4 Sgn 1). In other words, $\overline{\rho}$ is sought as a solution of the system of vector equations (3) and

(4) $\quad \overline{\rho} s = r s$

provided $s \neq o$, $sm = 0$. According to 1 Pr 33, this system has exactly one solution

(5) $$\overline{\rho} = \frac{(rs)s + s \times m}{s^2}.$$

It is called the *foot* or $r$ on $\overrightarrow{s}$ and is denoted by $\text{foot}_{\mathbf{r}}\overrightarrow{s}$. On the other hand, (5) implies

$$\bar{p} - r = \frac{s \times (m + s \times r)}{s^2} \tag{6}$$

The right-hand side of (6) is called the *arm* of $\overrightarrow{s}$ with respect to $r$ and is denoted by $\text{arm}_{\mathbf{r}}\overrightarrow{s}$. These and other circumstances are formalized in the present paragraph.

**Sgn 1.** $\text{arm}_{\mathbf{r}}\overrightarrow{s}$ sgn: $\dfrac{s \times (m + s \times r)}{s^2}$ iff 2(5), 2(2), 2(9).

**Df 1.** $\text{arm}_{\mathbf{r}}\overrightarrow{s}$ is called the *arm* of $\overrightarrow{s}$ with respect to $r$ (the *r-arm* of $\overrightarrow{s}$).

**Pr 1.** 2(5), 2(2), 2(9) imply

$$\text{arm}_{\mathbf{r}}\overrightarrow{s} = \frac{s \times \text{mom}_{\mathbf{r}}\overrightarrow{s}}{s^2}. \tag{7}$$

**Dm.** Sgn 1, 2 Sgn 5.

**Pr 2.** 2(5), 2(2), 2(9) imply $s \cdot \text{arm}_{\mathbf{r}}\overrightarrow{s} = 0$.

**Dm.** Sgn 1.

**Pr 3.** 2(5), 2(2), 2(9) imply $\text{mom}_{\mathbf{r}}\overrightarrow{s} \cdot \text{arm}_{\mathbf{r}}\overrightarrow{s} = 0$.

**Dm.** Pr 1.

**Pr 4.** 2(5), 2(2), 2(9) imply $s \times \text{mom}_{\mathbf{r}}\overrightarrow{s} = s^2 \text{arm}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** Pr 1.

**Pr 5.** 2(5), 2(2), 2(9) imply $\text{mom}_{\mathbf{r}}\overrightarrow{s} = \text{arm}_{\mathbf{r}}\overrightarrow{s} \times s$.

**Dm.** Pr 1, Pr 5.

**Pr 6.** 2(5), 2(2), 2(9) imply

$$\text{arm}_{\mathbf{r}}\overrightarrow{s} = \frac{s \times m + (rs)s}{s^2} - r. \tag{8}$$

**Dm.** Sgn 1, Pr 16.

**Pr 7.** 2(5), 2(9) imply $\text{arm}_{\mathbf{r}}(-\overrightarrow{s}) = \text{arm}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** 2 Sgn 6, Sgn 1.

**Pr 8.** 2(5), 2(9), 2(51) imply $\text{arm}_{\mathbf{r}}(\lambda\overrightarrow{s}) = \text{arm}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** 2 Sgn 7, Sgn 1.

**Pr 9.** 2(5), 2(2), 2(9) imply $\left(\text{mom}_{\mathbf{r}}\overrightarrow{s}\right)^2 = s^2\left(\text{arm}_{\mathbf{r}}\overrightarrow{s}\right)^2$.

**Dm.** Pr 5, Pr 2, 1 Pr 9.

**Pr 10.** 2(5), 2(9) imply 2(10) iff

$$\text{arm}_{\mathbf{r}}\overrightarrow{s} = \mathbf{o}. \tag{9}$$

**Dm.** Pr 9, 1 Ax 12S.

**Pr 11.** 2(5), 2(9) imply (1) iff (9).

**Dm.** Pr 10, 2 Pr 8.

**Pr 12.** 2(53), 3(5), 2(18), 2(62) imply

$$\text{arm}_{\mathbf{r}_2}\overrightarrow{s}_1 + \text{arm}_{\mathbf{r}_1}\overrightarrow{s}_2 = \mathbf{o}. \tag{10}$$

**Dm.** 3(5) imply 5(3) (3 Pr 20, 2 Pr 15), whence dir $\overrightarrow{s}_\nu$ and $\mathrm{arm}_{\mathbf{r}}\overrightarrow{s}_\nu$ ($\nu = 1, 2$) exist for any 2(9) (2 Sgn 4, Sgn 1). The relations 2(62) are equivalent with 2(69) provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). On the other hand, Sgn 1 implies

$$\text{(11)} \qquad \mathrm{arm}_{\mathbf{r}_2}\overrightarrow{s}_1 = \frac{s_1 \times (m_1 + s_1 \times r_2)}{s_1^2},$$

$$\text{(12)} \qquad \mathrm{arm}_{\mathbf{r}_1}\overrightarrow{s}_2 = \frac{s_2 \times (m_2 + s_2 \times r_1)}{s_2^2}$$

and (11), (12), 2(69) imply

$$\text{(13)} \qquad \mathrm{arm}_{\mathbf{r}_2}\overrightarrow{s}_1 = \frac{s_1 \times ((r_1 - r_2) \times s_1)}{s_1^2},$$

$$\text{(14)} \qquad \mathrm{arm}_{\mathbf{r}_1}\overrightarrow{s}_2 = \frac{s_2 \times ((r_2 - r_1) \times s_2)}{s_2^2}$$

respectively. At last, 3(5) and 2(59) imply

$$\text{(15)} \qquad s_1 \times s_2 = \mathbf{o}$$

(3 Sgn 5, 3 Sgn 3, 3 Sgn 1) and (15), 5(3), 2 Pr 1 imply that there exists (51) with

$$\text{(16)} \qquad s_2 = \lambda s_1.$$

Now (16), (14), 1Ax 8S, 1 Pr 7, 1 Sgn 1, 1 Pr 14, 1 Pr 17 imply

$$\text{(17)} \qquad \mathrm{arm}_{\mathbf{r}_1}\overrightarrow{s}_2 = \frac{s_1 \times ((r_2 - r_1) \times s_1)}{s_1^2}$$

and (13), (17) imply (10).

**Pr 13.** 2(53), 3(7), 2(18), 2(62) imply (10).
**Dm.** 3 Pr 31, Pr 12.

**Sgn 2.** $\mathrm{foot}_{\mathbf{r}}\overrightarrow{s}$ sgn: $\dfrac{s \times m + (rs)s}{s^2}$ iff 2(5), 2(2), 2(9).

**Df 2.** $\mathrm{foot}_{\mathbf{r}}\overrightarrow{s}$ is called the *foot* of $r$ on $\overrightarrow{s}$.

**Pr 14.** 2(5), 2(2), 2(9) imply $\mathrm{foot}_{\mathbf{r}}\overrightarrow{s} = r + \mathrm{arm}_{\mathbf{r}}\overrightarrow{s}$.
**Dm.** Sgn 2, Pr 6.

**Pr 15.** 2(5), 2(9) imply $\mathrm{foot}_{\mathbf{r}}(-\overrightarrow{s}) = \mathrm{foot}_{\mathbf{r}}\overrightarrow{s}$.
**Dm.** Pr 14, Pr 7.

**Pr 16.** 2(5), 2(9), 2(51) imply $\mathrm{foot}_{\mathbf{r}}(\lambda\overrightarrow{s}) = \mathrm{foot}_{\mathbf{r}}\overrightarrow{s}$.
**Dm.** Pr 14, Pr 8.

**Pr 17.** 2(5), 2(9),

$$\text{(18)} \qquad r \ \mathrm{Z}\ \mathrm{dir}\ \overrightarrow{s}$$

imply

$$(19)\qquad \left(\operatorname{arm}_{\mathbf{r}}\overrightarrow{s},\ \boldsymbol{r}\times\operatorname{arm}_{\mathbf{r}}\overrightarrow{s}\right)\in\Lambda_S.$$

**Dm.** 2(5) implies: dir $\overrightarrow{s}$ exists (2 Sgn 4). Now (18) is equivalent to

$$(20)\qquad \operatorname{arm}_{\mathbf{r}}\overrightarrow{s}\neq\mathbf{o}$$

(Pr 11) whence (19) (2 Sgn 3).

**Pr 18.** 2(5), 2(9), (18) imply: there exists exactly one $l\in L_S$ with

$$(21)\qquad \left(\operatorname{arm}_{\mathbf{r}}\overrightarrow{s},\ \boldsymbol{r}\times\operatorname{arm}_{\mathbf{r}}\overrightarrow{s}\right)\ \&\ l.$$

**Dm.** Pr 17, [3] 1 Pr 19.

**Sgn 3.** $\operatorname{axis}_{\mathbf{r}}\overrightarrow{s}$ sgn: $l\in L_S$ with (21) iff 2(5), 2(9), (18).

**Df 3.** $\operatorname{axis}_{\mathbf{r}}\overrightarrow{s}$ is called the *$\boldsymbol{r}$-axis* of $\overrightarrow{s}$.

**Pr 19.** 2(5), 2(9), (18) imply $\boldsymbol{r}\ \mathrm{Z}\ \operatorname{axis}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** [3] 4 Sgn 1.

**Pr 20.** 2(5), 2(9), (18) imply $\operatorname{axis}_{\mathbf{r}}(-\overrightarrow{s})=\operatorname{axis}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** Sgn 3, Pr 7.

**Pr 21.** 2(5), 2(9), (18), 2(51) imply $\operatorname{axis}_{\mathbf{r}}(\lambda\overrightarrow{s})=\operatorname{axis}_{\mathbf{r}}\overrightarrow{s}$.

**Dm.** Sgn 3, Pr 8.

**Sch 2.** Let

$$(22)\qquad (\boldsymbol{a}_\nu,\boldsymbol{b}_\nu)\in\Lambda_S,\qquad l_\nu\in L_S,\qquad (\boldsymbol{a}_\nu,\boldsymbol{b}_\nu)\ \&\ l_\nu,\qquad (\nu=1,2),$$

$$(23)\qquad \boldsymbol{a}_1\times\boldsymbol{a}_2\neq\mathbf{o}.$$

Then

$$(24)\qquad \left(\boldsymbol{a}_1\times\boldsymbol{a}_2,\ \frac{(\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{b}_2)\boldsymbol{a}_1+(\boldsymbol{a}_2\times\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{a}_2}{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2}\times(\boldsymbol{a}_1\times\boldsymbol{a}_2)\right)\in\Lambda_S.$$

Let $l\in L_S$ be defined by

$$(25)\qquad \left(\boldsymbol{a}_1\times\boldsymbol{a}_2,\ \frac{(\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{b}_2)\boldsymbol{a}_1+(\boldsymbol{a}_2\times\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{a}_2}{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2}\times(\boldsymbol{a}_1\times\boldsymbol{a}_2)\right)\ \&\ l.$$

Then it is proved [3, p. 122] that there exists exactly one couple $\boldsymbol{r}_\nu\in V_S$ $(\nu=1,2)$ with $\boldsymbol{r}_\nu\ \mathrm{Z}\ l$, $\boldsymbol{r}_\nu\ \mathrm{Z}\ l_\nu$ $(\nu=1,2)$ respectively, namely

$$(26)\qquad \boldsymbol{r}_1=\frac{(\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{b}_2)\boldsymbol{a}_1+(\boldsymbol{a}_2\times\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{a}_2+(\boldsymbol{a}_2\boldsymbol{b}_1)\boldsymbol{a}_1\times\boldsymbol{a}_2}{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2},$$

$$(27)\qquad \boldsymbol{r}_2=\frac{(\boldsymbol{a}_1\times\boldsymbol{a}_2\cdot\boldsymbol{b}_2)\boldsymbol{a}_1+(\boldsymbol{a}_2\times\boldsymbol{a}_1\cdot\boldsymbol{b}_1)\boldsymbol{a}_2-(\boldsymbol{a}_1\boldsymbol{b}_2)\boldsymbol{a}_1\times\boldsymbol{a}_2}{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2}.$$

Now (26), (27) imply

$$(28)\qquad \boldsymbol{r}_1-\boldsymbol{r}_2=\frac{\boldsymbol{a}_1\boldsymbol{b}_2+\boldsymbol{a}_2\boldsymbol{b}_1}{(\boldsymbol{a}_1\times\boldsymbol{a}_2)^2}\boldsymbol{a}_1\times\boldsymbol{a}_2.$$

These geometrical facts give rise to the following considerations concerning $V_S$-arrows.

**Pr 22.** $s_\nu, m_\nu \in V_S$ $(\nu = 1, 2)$,

(29) $$s_1 \times s_2 \neq o,$$

(30) $$s \text{ sgn} : \frac{s_1 m_2 + s_2 m_1}{(s_1 \times s_2)^2} s_1 \times s_2,$$

(31) $$m \text{ sgn} : \frac{s}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2)s_1 + (s_1 \times s_2 \cdot m_1)s_2),$$

2(2) imply 2(1).

**Dm.** 2 Sgn 1.

**Sgn 4.** $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2)$ sgn: $(s, m)$ iff 1(53), 2(59), (29) — (31).

**Df 4.** $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2)$ is called the *axis* of $\overrightarrow{s}_1, \overrightarrow{s}_2$.

**Pr 23.** 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \neq \overrightarrow{o}$ iff 5(4).

**Dm.** 2Pr 1, Sgn 4, 5 Sgn 3.

**Pr 24.** 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) = -\text{ax}(\overrightarrow{s}_2, \overrightarrow{s}_1)$.

**Dm.** Sgn 4, 2 Sgn 6.

**Pr 25.** 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \top \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

**Dm.** Sgn 4, 4 Sgn 1.

**Pr 26.** 2(53), 2(59), 5(4) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \perp \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

**Dm.** Sgn 4, 4 Sgn 3, Pr 23, 5 Pr 18.

**Pr 27.** 2(53), 2(59), 5(4) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \wedge \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

**Dm.** Sgn 4, 5 Sgn 1, [3] 1 Pr 106.

**Pr 28.** 2(53), 2(59), 5(4) imply: there exists exactly one couple $r_\nu \in V_S$ $(\nu = 1, 2)$ with 2(62) and

(32) $$r_\nu \text{ Z dir ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \qquad (\nu = 1, 2),$$

namely

(33) $$r_1 = \frac{1}{(s_1 \times s_2)^2} ((s_1 \times s_2 m_2)s_1 + (s_2 \times s_1 m_1)s_2 + (s_2 m_1)s_1 \times s_2),$$

(34) $$r_2 = \frac{1}{(s_1 \times s_2)^2} ((s_1 \times s_2 m_2)s_1 + (s_2 \times s_1 m_1)s_2 - (s_1 m_2)s_1 \times s_2).$$

**Dm.** [3] 4 Pr 47.

**Pr 29.** 2(53), 2(59), 5(4), (30), (38), (39) imply $s = r_1 - r_2$.

**Dm.** Clear.

## § 7. ADDITION

**Pr 1.** If

(1) $$(s_\nu, m_\mu) \in W_S \qquad (\nu = 1, 2),$$

then

(2) $$(s_1 + s_2, m_1 + m_2) \in W_S$$

iff

(3) $$s_1 + s_2 = o, \qquad m_1 + m_2 = o$$

or

(4) $$s_1 + s_2 \neq o, \qquad s_1 m_2 + s_2 m_1 = 0:$$

**Dm.** 2 Sgn 1.

**Pr 2.** (1) imply (2) iff one of the following conditions is satisfied:

(5) $$s_1 = -s_2, \qquad m_1 = -m_2$$

or

(6) $$s_1 \neq -s_2, \qquad s_1 \times s_2 = o$$

or

(7) $$s_1 \times s_2 \neq o, \qquad s_1 m_2 + s_2 m_1 = 0.$$

**Dm.** By virtue of Pr 1 it must be proved that the systems of conditions (3) and (4), on the one hand, and (5) — (7), on the other hand, are equivalent. Since (3) and (5) are equivalent, it remains to be proved that the conditions (4), on the one hand, and (6) and (7), on the other hand, are equivalent.

The first condition (4), i.e.

(8) $$s_1 + s_2 \neq o,$$

is consistent with the alternative

(9) $$s_1 \times s_2 = o$$

or

(10) $$s_1 \times s_2 \neq o.$$

The case (8), (10) with

(11) $$s_1 m_2 + s_2 m_1 = 0,$$

i.e. the case (7), is equivalent to (4).

As regards the case (8), (9), the following two subcases are possible: at least one of the vectors $s_1$ and $s_2$ is zero; or none is zero.

In the first subcase, let for instance

(12) $$s_1 = o.$$

Now (12), (1), 2 Sgn 1 imply $m_1 = o$, and (11) is satisfied trivially.

In the second subcase, (9) implies that there exist $\lambda_\nu \in S$ $(\nu = 1, 2)$ with $\lambda_\nu \neq 0$ $(\nu = 1, 2)$ and

$$(13) \qquad s_1 = \lambda_1 s_2, \qquad s_2 = \lambda_2 s_1.$$

On the other hand, (1) and 2 Sgn 1 imply

$$(14) \qquad s_\nu m_\nu = 0 \qquad (\nu = 1, 2).$$

Now (13), (14) imply (11) again.

In such a manner, it is proved that the conditions (4) are equivalent to the conditions (6) in the case (9), and with (7), in the case (10), *q.e.d.*

**Pr 3.** (1) imply

$$(15) \qquad (s_1 + s_2, m_1 + m_2) \notin W_S$$

iff

$$(16) \qquad s_1 = -s_2, \qquad m_1 \neq -m_2$$

or

$$(17) \qquad s_1 m_2 + s_2 m_1 \neq 0.$$

**Dm.** Pr 1, Pr 2.

**Pr 4.** (1) imply

$$(18) \qquad (s_1 + s_2, m_1 + m_2) \in \Lambda_S$$

iff (4).

**Dm.** 2Sgn 3, 2 Sgn 1.

**Pr 5.** (1) imply (18) iff (6) or (7).

**Dm.** Pr 2, 2 Sgn 3.

**Sgn 1.** $\vec{s}_1 + \vec{s}_2$ sgn: $\vec{s}$ with

$$(19) \qquad \vec{s} = (s_1 + s_2, m_1 + m_2)$$

provided

$$(20) \qquad \vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2)$$

iff (2).

**Df 1.** $\vec{s}_1 + \vec{s}_2$ is called the *sum* of $\vec{s}_1$ and $\vec{s}_2$.

**Df 2.** The operation in $W_S$ defined by means of Sgn 1, is called *addition* in $W_S$.

**Pr 6.** (20), (2) imply

$$(21) \qquad (s_1, m_1) + (s_2, m_2) = (s_1 + s_2, m_1 + m_2).$$

**Dm.** Sgn 1.

**Pr 7.** 2(53) imply:

$$(22) \qquad \vec{s}_1 + \vec{s}_2 \quad \text{exists}$$

iff

$$(23)\qquad \vec{s}_1 | \vec{s}_2,\ \vec{s}_1 \overline{\uparrow\downarrow} \vec{s}_2$$

or

$$(24)\qquad \vec{s}_1 \wedge \vec{s}_2.$$

**Dm.** Sgn 1, Pr 2, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1.

**Pr 8.** 2(53) imply:

$$(25)\qquad \vec{s}_1 + \vec{s}_2 \quad \text{does not exist}$$

iff

$$(26)\qquad \vec{s}_1 \uparrow\downarrow \vec{s}_2$$

or

$$(27)\qquad \vec{s}_1 \otimes \vec{s}_2.$$

**Dm.** Sgn 1, Pr 3, 3 Sgn 7, Pr 2, 5 Sgn 3.

**Pr 9.** 2(53), (22) imply

$$(28)\qquad \vec{s}_1 + \vec{s}_2 = \vec{s}_2 + \vec{s}_1.$$

**Dm.** The right-hand side of (28) exists (Pr 7, 3 Pr 2, 3 Pr 23, 5 Pr 16). Then Sgn 1.

**Pr 10.** 2(1) implies

$$(29)\qquad \vec{s} + \vec{o} = \vec{s}.$$

**Dm.** The left-hand side of (29) exists (2 Sgn 2, Sgn 1, 2 Sgn 1). Then Sgn 1.

**Pr 11.** 2(1) implies

$$(30)\qquad \vec{s} + (-\vec{s}) = \vec{o}.$$

**Dm.** The left-hand side of (30) exists (2 Sgn 6, Sgn 1, 2 Sgn 1). Then Sgn 1, 2 Sgn 2.

**Pr 12.** If

$$(31)\qquad \vec{s}_\nu \in W_S \qquad (\nu = 1, 2, 3),$$

then

$$(32)\qquad (\vec{s}_1 + \vec{s}_2) + \vec{s}_3 = \vec{s}_1 + (\vec{s}_2 + \vec{s}_3),$$

provided all sums exist.

**Dm.** Sgn 1, 1 Ax 1S.

**Sgn 2.** $L(\vec{s})$ sgn: $\{\lambda\vec{s} : \lambda \in S\}$ iff $\vec{o} \neq \vec{s} \in W_S$.

**Df 3.** $L(\vec{s})$ is called the *linear span* of $\vec{s}$.

**Pr 13.** 2(55), 2(49), $\vec{s}_1 = \lambda\vec{s}_2$ imply $L(\vec{s}_1) = L(\vec{s}_2)$.

**Dm.** Sgn 2, 2 Sgn 7.

**Pr 14.** If

$$\overrightarrow{s}_\nu \in L(\overrightarrow{s}) \qquad (\nu = 1, 2), \tag{33}$$

then (22).

**Dm.** (33), Sgn 2 imply: there exist

$$\lambda_\nu \in S \qquad (\nu = 1, 2) \tag{34}$$

with

$$\overrightarrow{s}_\nu = \lambda_\nu \overrightarrow{s} \qquad (\nu = 1, 2). \tag{35}$$

Then (35) and 2(2) imply

$$\overrightarrow{s}_\nu = (\lambda_\nu s, \overline{\lambda}_\nu m) \qquad (\nu = 1, 2), \tag{36}$$

in view of 2 Sgn 7, and (36), 2(59) imply

$$s_1 + s_2 = (\lambda_1 + \lambda_2)s, \qquad m_1 + m_2 = (\overline{\lambda}_1 + \overline{\lambda}_2)m. \tag{37}$$

If $\lambda_1 + \lambda_2 = 0$, then (37) imply (3) whence (22) (Sgn 1, Pr 1). If $\lambda_1 + \lambda_2 \neq 0$, then (8) (Sgn 2). On the other hand, (36), 2(59) imply

$$s_1 m_2 + s_2 m_1 = 2\lambda_1 \overline{\lambda}_2 (sm) = 0. \tag{38}$$

(1 Ax 9S, 1 Pr 7, 2 Sgn 1), i.e. (11), and (8), (11) imply (22) (Pr 1, Sgn 1).

**Pr 15.** $L(\overrightarrow{s})$ is a group with respect to the addition in $W_S$.

**Dm.** Pr 14, Pr 12, Sgn 2 with $\lambda = 0$, 2 Pr 24, Pr 10, Sgn 2 with $\lambda = -1$, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\overrightarrow{s})$ instead of $V_S$).

**Pr 16.** 2(1),

$$\lambda, \mu \in S \tag{39}$$

imply

$$(\lambda + \mu)\overrightarrow{s} = \lambda \overrightarrow{s} + \mu \overrightarrow{s}. \tag{40}$$

**Dm.** Both sides of (40) exist: in case of $\overrightarrow{s} = \overrightarrow{o}$ all members in (40) are the zero-arrows (2 Pr 25); in case of $\overrightarrow{s} \neq \overrightarrow{o}$ see Pr 14. Now (40) follows from 2 Sgn 7 and Sgn 1.

**Pr 17.** $\lambda \in S$, 2(53), (22) imply

$$\lambda(\overrightarrow{s}_1 + \overrightarrow{s}_2) = \lambda \overrightarrow{s}_1 + \lambda \overrightarrow{s}_2. \tag{41}$$

**Dm.** Both sides of (41) exist: in case of $\lambda = 0$, all members in (41) are the zero-arrows (2 Pr 24); in case of $\lambda \neq 0$ see Pr 1 and 2 Sgn 7. Now (41) follows from 2 Sgn 7 and Sgn 1.

**Pr 18.** $L(\overrightarrow{s})$ is a 1-dimensional linear space over $S$ with respect to the addition in $W_S$ and to the multiplication 2 Sgn 7 of the elements of $S$ and $W_S$.

**Dm.** As regards the addition see Pr 15. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied (with $L(\vec{s})$ instead of $V_S$).

As regards the dimension of the linear space $L(\vec{s})$ over $S$, let uz note, first, that there exists a linearly independent element of $L(\vec{s})$, namely $\vec{s}$; and, second, that any two elements of $L(\vec{s})$, are linearly dependent. Indeed, let (33) hold. Then Sgn 2 implies that there exist (34) with (35). If both of (34) are zeroes, then both of (33) are zeroes too (2 Pr 24), and they are, therefrom, trivially linearly dependent; if at least one of (34) iz non-zero, then the linear combination

$$\lambda_2 \vec{s}_1 + (-\lambda_1)\vec{s}_2 = \lambda_1\lambda_2(\vec{s} - \vec{s}) = \lambda_1\lambda_2 \vec{o} \tag{42}$$

vanishes with non-zero coefficients (34): consequently, (33) are linearly dependent.

**Sch 1.** The record (42) is not a quite orthodoxal one: it exploits the undefined still notion of a *difference* of two arrows (the difference $\vec{s} - \vec{s}$, as a matter of fact). In actually, (42) may be rewritten in the almost equivalent form

$$\lambda_3 \vec{s}_1 + (-\lambda_1)\vec{s}_2 = \lambda_2(\lambda_1 \vec{s}) + (-\lambda_1)(\lambda_2 \vec{s}) \tag{43}$$

$$= (\lambda_2\lambda_1)\vec{s} + (-\lambda_1\lambda_2)\vec{s}_2 = (\lambda_2\lambda_1 + (-\lambda_1\lambda_2))\vec{s} = 0\vec{s} = \vec{o}$$

(2 Pr 32, Pr 16, 2 Pr 24). As it is, the *difference* of two arrows (if it exists) is defined immediately below.

**Sgn 3.** $\vec{s}_1 - \vec{s}_2$ sgn: $\vec{s}_1 + (-\vec{s}_2)$ iff 2(53) and $\vec{s}_1 + (-\vec{s}_2)$ exist.

**Df 4.** $\vec{s}_1 - \vec{s}_2$ is called the *difference* of $\vec{s}_1$ and $\vec{s}_2$.

**Df 5.** The operation in $W_S$ defined by means of Sgn 3 is called *subtraction* in $W_S$.

**Pr 19.** 2(53) imply

$$\vec{s}_1 - \vec{s}_2 \quad \text{exists} \tag{44}$$

iff

$$\vec{s}_1 | \vec{s}_2, \qquad \vec{s}_1 \uparrow\downarrow - \vec{s}_2 \tag{45}$$

or

$$\vec{s}_1 \wedge \vec{s}_2. \tag{46}$$

**Dm.** Sgn 3, Pr 7, 2 Sgn 6, 3 Sgn 1, 5 Sgn 1.

**Pr 20.** 2(53) imply

$$\vec{s}_1 - \vec{s}_2 \quad \text{does not exist} \tag{47}$$

iff

$$\vec{s}_1 | \vec{s}_2, \qquad \vec{s}_1 \uparrow\downarrow - \vec{s}_2 \tag{48}$$

or

$$\vec{s}_1 \otimes \vec{s}_2. \tag{49}$$

**Dm.** Sgn 3, Pr 8, 2 Sgn 6, 5 Sgn 3.
**Pr 21.** (20)

$$(s_1 - s_2, m_1 - m_2) \in W_S \tag{50}$$

imply

$$(s_1, m_1) - (s_2, m_2) = (s_1 - s_2, m_1 - m_2). \tag{51}$$

**Dm.** Sgn 3, 2 Sgn 6, Pr 6.
**Pr 22.** 2(53), (44) imply

$$\vec{s}_1 - \vec{s}_2 = -\vec{s}_2 + \vec{s}_1. \tag{52}$$

**Dm.** Sgn 3, Pr 9.
**Pr 23.** 2(1)) implies

$$\vec{s} - \vec{o} = \vec{s}. \tag{53}$$

**Dm.** Sgn 3, 2 Pr 19, Pr 10
**Pr 24.** 2(1) implies

$$\vec{s} - \vec{s} = \vec{o}. \tag{54}$$

**Dm.** Sgn 3, Pr 11.
**Sch 2.** Pr 24 has been used on the sly in (42).
**Pr 25.** 2(53), (22) imply

$$-(\vec{s}_1 + \vec{s}_2) = -\vec{s}_1 - \vec{s}_2. \tag{55}$$

**Dm.** The right-hand side of (55) exists (Pr 7, Sgn 3, 2 Sgn 6, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1). Then Pr 6, 2 Sgn 6, Pr 21.
**Pr 26.** 2(53), (22) imply

$$-(\vec{s}_1 - \vec{s}_2) = -\vec{s}_1 + \vec{s}_2. \tag{56}$$

**Dm.** Pr 25, Sgn 3, 2 Sgn 6.
**Pr 27.** (31) imply

$$(\vec{s}_1 - \vec{s}_2) + \vec{s}_3 = \vec{s}_1 + (-\vec{s}_2 + \vec{s}_3) \tag{57}$$

provided all sums and differences exist.
**Dm.** Sgn 3, Pr 12.
**Pr 28.** (31) imply

$$(\vec{s}_1 + \vec{s}_2) - \vec{s}_3 = \vec{s}_1 + (\vec{s}_2 - \vec{s}_3) \tag{58}$$

provided all sums and differences exist.
**Dm.** Sgn 3, Pr 12.
**Pr 29.** (31) imply

$$(59)\qquad (\vec{s}_1-\vec{s}_2)-\vec{s}_3=\vec{s}_1-(\vec{s}_2+\vec{s}_3)$$

provided all sums and differences exist.

**Dm.** Sgn 3, Pr 12.

**Pr 30.** (39), 2(1) imply

$$(60)\qquad (\lambda-\mu)\vec{s}=\lambda\vec{s}-\mu\vec{s}.$$

**Dm.** If $\vec{s}=\vec{o}$, then all members in (60) are zeroes (2 Pr 25). If $\vec{s}\neq\vec{o}$, then (60) is simplified by Pr 18.

**Pr 31.** 2(49), 2(53), (44) imply

$$(61)\qquad \lambda(\vec{s}_1-\vec{s}_2)=\lambda\vec{s}_1-\lambda\vec{s}_2.$$

**Dm.** The right-hand side of (61) exists: if $\lambda=0$, then all members in (61) are zeroes (2 Pr 24); if $\lambda\neq 0$, then Pr 19, 2 Sgn 7, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1. Now 2 Sgn 7 and Pr 21.

**Pr 32.** 2(53), (24), (34),

$$(62)\qquad \lambda_1\lambda_2\neq 0$$

imply

$$(63)\qquad \lambda_1\vec{s}_1\wedge\lambda_2\vec{s}_2.$$

**Dm.** (24) implies (10), (11) (5 Sgn 1). If 2(59), then

$$(64)\qquad \lambda_\nu\vec{s}_\nu=(\lambda_\nu s_\nu,\ \overline{\lambda}_\nu \boldsymbol{m}_\nu)\qquad (\nu=1,2)$$

(2 Sgn 7). Now (64), 1 Pr 14, 1 Pr 17, 1 Ax 8S, 1 Pr 7, (62), (10), (11) imply

$$(65)\qquad (\lambda_1 s_1)\times(\lambda_2 s_2)=(\overline{\lambda_1\lambda_2})s_1\times s_2\neq \boldsymbol{o},$$

$$(66)\qquad (\lambda_1 s_1)(\overline{\lambda_2}\boldsymbol{m}_2)+(\lambda_2 s_2)(\overline{\lambda_1}\boldsymbol{m}_1)=(\lambda_1\lambda_2)(s_1\boldsymbol{m}_2+s_2\boldsymbol{m}_1)=0,$$

and (65), (66) imply (63) (5 Sgn 1).

**Pr 33.** 2(53), (24), (34) imply

$$(67)\qquad \lambda_1\vec{s}_1+\lambda_2\vec{s}_2\qquad\text{exists.}$$

**Dm.** If $\lambda_1=0$ or $\lambda_2=0$, then (67) is implied by 2 Pr 24, Pr 10, Pr 9. If (62), then Pr 32, Pr 7.

**Sgn 4.** $L(\vec{s}_\nu)_{\nu=1}^2$ sgn: $\left\{\lambda_1\vec{s}_1+\lambda_2\vec{s}_2:\ \lambda_\nu\in S(\nu=1,2)\right\}$ Iff $\vec{s}_\nu\in W_S$ $(\nu=1,2)$, $\vec{s}_1\wedge\vec{s}_2$.

**Df 6.** $L(\vec{s}_\nu)_{\nu=1}^2$ is called the *linear span* of $\vec{s}_\nu$ $(\nu=1,2)$.

**Pr 34.** 2(53), (24),

$$(68)\qquad \vec{o}_\mu\in L(\vec{s}_\nu)_{\nu=1}^2\qquad (\mu=1,2)$$

imply

(69) $\vec{\sigma}_1 + \vec{\sigma}_2$ exists.

**Dm.** (68), Sgn 4 imply: there exist

(70) $\lambda_{\mu\nu} \in S$ $(\mu,\ \nu = 1, 2)$

with

(71) $\vec{\sigma}_\mu = \lambda_{\mu 1}\vec{s}_1 + \lambda_{\mu 2}\vec{s}_2$ $(\mu = 1, 2)$

Now (71) and 2(59) imply

(72) $\vec{\sigma}_\mu = (\lambda_{\mu 1}s_1 + \lambda_{\mu 2}s_2,\ \overline{\lambda}_{\mu 1}m_1 + \overline{\lambda}_{\mu 2}m_2)$ $(\mu = 1, 2)$.

(2 Sgn 7, Pr 6), and (72), 1 Ax 8S, 1 Pr 7, 1 Pr 14, 1 Pr 17 imply

(73) $(\lambda_{11}s_1 + \lambda_{12}s_2) \times (\lambda_{21}s_1 + \lambda_{22}s_2) = (\overline{\lambda}_{11}\overline{\lambda}_{22} - \overline{\lambda}_{12}\overline{\lambda}_{21})s_1 \times s_2,$

(74) $(\lambda_{11}s_1 + \lambda_{12}s_2)(\overline{\lambda_{21}}m_1 + \overline{\lambda_{22}}m_2) + (\lambda_{21}s_1 + \lambda_{22}s_2)(\overline{\lambda_{11}}m_1 + \overline{\lambda_{12}}m_2)$

$$= (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})(s_1m_2 + s_2m_1) = 0.$$

in view of (14) (2 Sgn 1) and (11) (5 Sgn 1).

The alternative

(75) $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \neq 0$

or

(76) $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$

now arises

If (75), then (73) and (10) (5 Sgn 1) imply

(77) $(\lambda_{11}s_1 + \lambda_{12}s_2) \times (\lambda_{21}s_1 + \lambda_{22}s_2) \neq o$

and (72), (77), (74), 5 Sgn 1 imply

(78) $\vec{\sigma}_1 \wedge \vec{\sigma}_2.$

consequently (69) (Pr 7).

If (76), then (73) implies

(79) $(\lambda_{11}s_1 + \lambda_{12}s_2) \times (\lambda_{21}s_1 + \lambda_{22}s_2) = o,$

and(79), (72) imply

(80) $\vec{\sigma}_1 | \vec{\sigma}_2$

in view of 3 Sgn 1.

The supposition

(81) $\vec{\sigma}_1 \uparrow\downarrow \vec{\sigma}_2$

is wrong. Indeed, (72) and 3 Sgn 7 imply that (81) is equivalent to

(82) $\qquad \lambda_{11}s_1 + \lambda_{12}s_2 = -(\lambda_{21}s_1 + \lambda_{22}s_2),$

(83) $\qquad \overline{\lambda_{11}}m_1 + \overline{\lambda_{12}}m_2 \neq -(\overline{\lambda_{21}}m_1 + \overline{\lambda_{22}}m_2)$

i.e. with

(84) $\qquad (\lambda_{11} + \lambda_{21})s_1 + (\lambda_{12} + \lambda_{22})s_2 = o,$

(85) $\qquad (\overline{\lambda_{11}} + \overline{\lambda_{21}})m_1 + (\overline{\lambda_{12}} + \overline{\lambda_{22}})m_2 \neq o.$

Now (84) and (10) (5 Sgn 1) imply

(86) $\qquad \lambda_{11} + \lambda_{21} = 0, \qquad \lambda_{12} + \lambda_{22} = 0,$

and (86) imply

(87) $\qquad (\overline{\lambda_{11}} + \overline{\lambda_{21}})m_1 + (\overline{\lambda_{12}} + \overline{\lambda_{22}})m_2 = 0,$

contrary to (85). In such a manner,

(88) $\qquad \vec{\sigma}_1 \uparrow\downarrow \vec{\sigma}_2$

(3 Pr 21) and (80), (88) imply (69) (Pr 7).

**Pr 35.** $L(\vec{s}_\nu)_{\nu=1}^2$ is a group with respect to the addition in $W_S$.

**Dm.** Pr 34, Pr 12, Sgn 4 with $\lambda_1 = \lambda_2 = 0$, 2 Pr 24, Pr 10, Sgn 4 with $\lambda_1$, $\lambda_2$, on the one hand, and $-\lambda_1$, $-\lambda_2$, on the other hand, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\vec{s}_\nu)_{\nu=1}^2$ instead of $V_S$).

**Pr 36.** $L(\vec{s}_\nu)_{\nu=1}^2$ is a 2-dimensional linear space over $S$ with respect to the addition in $W_S$ and to the multiplication 2 Sgn 7 of the elements of $S$ and $W_S$.

**Dm.** As regards the addition, see Pr 35. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied (with $L(\vec{s}_\nu)_{\nu=1}^2$ instead of $V_S$).

As regards the dimension of the linear space $L(\vec{s}_\nu)_{\nu=1}^2$ over $S$, let us note, first, that there exist two linearly independent elements of $L(\vec{s}_\nu)_{\nu=1}^2$ namely $\vec{s}_1$ and $\vec{s}_2$; and, second, that any three elements of $L(\vec{s}_\nu)_{\nu=1}^2$ are linearly dependent.

Indeed, $\vec{s}_\mu \in L(\vec{s}_\nu)_{\nu=1}^2$ $(\mu = 1, 2)$ since $\vec{s}_1 = 1\vec{s}_1 + 0\vec{s}_2$, $\vec{s}_2 = 0\vec{s}_1 + 1\vec{s}_2$ (2 Pr 24, 2 Pr 26, Pr 10, Pr 9). Let now (34) and

(89) $\qquad \lambda_1\vec{s}_1 + \lambda_2\vec{s}_2 = \vec{o}$

hold. If 2(59), then (89) is equivalent to

(90) $\qquad (\lambda_1 s_1 + \lambda_2 s_2, \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2) = (o, o)$

(2 Sgn 7, Pr 6, 2 Sgn 2), whence

(91) $\qquad \lambda_1 s_1 + \lambda_2 s_2 = o.$

Now (91) and (100 (5 Sgn 1) imply $\lambda_1 = \lambda_2 = 0$, hence the linear independency of $\vec{s}_1$ and $\vec{s}_2$.

On the other hand, let

(92) $\qquad \vec{\sigma}_\mu \in L(\vec{s}_\nu)_{\nu=1}^2 \qquad (\mu = 1, 2, 3).$

Now (92) and Sgn 4 imply that there exist

(93) $\qquad \lambda_{\mu\nu} \in S \qquad (\mu = 1, 2, 3; \nu = 1, 2)$

with

(94) $\qquad \vec{\sigma}_\mu = \lambda_{\mu 1}\vec{s}_1 + \lambda_{\mu 2}\vec{s}_2 \qquad (\mu = 1, 2, 3).$

If 2(59), then (94) is equivalent to

(95) $\qquad \vec{\sigma}_\mu = (\lambda_{\mu 1}s_1 + \lambda_{\mu 2}s_2, \overline{\lambda}_{\mu 1}m_1 + \overline{\lambda}_{\mu 2}m_2) \qquad (\mu = 1, 2, 3).$

Let 1(34) be a non-zero solution of the system of equations

(96) $$\sum_{\mu=1}^{3}\alpha_\mu\lambda_{\mu 1} = 0, \qquad \sum_{\mu=1}^{3}\alpha_\mu\lambda_{\mu 2} = 0,$$

i.e. with

(97) $$\sum_{\mu=1}^{3}\alpha_\mu\overline{\alpha}_\mu \neq 0.$$

Now (96) imply

(98) $$\sum_{\mu=1}^{3}\overline{\alpha}_\mu\overline{\lambda}_{\mu 1} = 0, \qquad \sum_{\mu=1}^{3}\overline{\alpha}_\mu\overline{\lambda}_{\mu 2} = 0$$

and (95), (96), (98),

(99) $$\sum_{\mu=1}^{3}\alpha_\mu\vec{\sigma}_\mu \qquad \text{sgn}: \qquad (\alpha_1\vec{\sigma}_1 + \alpha_2\vec{\sigma}_2) + \alpha_3\vec{\sigma}_3,$$

Pr 6, 2 Sgn 2 imply

(100) $$\sum_{\mu=1}^{3}\alpha_\mu\vec{\sigma}_\mu = \vec{o}$$

with (97), i.e. the linear dependency of (92).

**Pr 37.** If

(101) $\qquad \vec{s} \in L(\vec{s}_\nu)_{\nu=1}^2.$

(102) $\qquad \vec{s} \,|\, \vec{s}_\nu \qquad (\nu = 1, 2),$

then

(103) $\qquad \vec{s} \wedge \vec{s}_\nu \qquad (\nu = 1, 2).$

**Dm.** (101), Sgn 4 imply: there exist (34) with

$$(104) \qquad \vec{s} = \lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2.$$

If 2(2), 2959), then (102), 3 Sgn 2 imply

$$(105) \qquad s \times s_\nu \neq o \qquad (\nu = 1, 2).$$

Besides, (104), 2(2), 2(59), 2 Sgn 7, Pr 6 imply

$$(106) \qquad s = \lambda_1 s_1 + \lambda_2 s_2, \qquad m = \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2.$$

now (106), 2 Sgn 1, 1 Ax 8S, 1 Pr 7, Sgn 4, 5 Sgn 1 imply

$$(107) \qquad sm_1 + s_1 m = \lambda_2(s_1 m_2 + s_2 m_1) = 0,$$

$$(108) \qquad sm_2 + s_2 m = \lambda_1(s_1 m_2 + s_2 m_1) = 0,$$

and (105), (107), (108), 5 Sgn 1 imply (103).

**Pr 38.** (68),

$$(109) \qquad \vec{\sigma}_1 | \vec{\sigma}_2$$

imply

$$(110) \qquad L(\vec{\sigma}_\nu)^2_{\nu=1} = L(\vec{s}_\nu)^2_{\nu=1}.$$

**Dm.** As in proof of Pr 34 it is proved that (73), (74) hold good. The case (76) is impossible since it implies (80) contrary to (109) (3 Pr 1). Now as in the proof of Pr 34 it is proved that (78) holds. Hence the left-hand side of (110) exists (Sgn 4). Then Pr 36 and Sgn 4.

**Pr 39.** If

$$(111) \qquad \vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2, 3),$$

$$(112) \qquad s_1 \times s_2 \cdot s_3 \neq 0,$$

$$(113) \qquad s_\mu m_\nu + s_\nu m_\mu = 0 \qquad (\mu, \nu = 1, 2, 3),$$

then there exists exactly one 1(31) with

$$(114) \qquad r \mathrm{\ Z\ dir\ } \vec{s}_\nu \qquad (\nu = 1, 2, 3),$$

namely 5(7).

**Dm.** (112) imply $s_\nu \neq o$ $(\nu = 1, 2, 3)$, whence

$$(115) \qquad \vec{s}_\nu \neq \vec{o} \qquad (\nu = 1, 2, 3)$$

(2 Pr 1). Now (115) and 2 Sgn 4 imply: dir $\vec{s}_\nu$ $(\nu = 1, 2, 3)$ exist and (114) is equivalent to

$$(116) \qquad r \times s_\nu = m_\nu \qquad (\nu = 1, 2, 3).$$

Then (112), (113), 1 Pr 30 imply that there exists exactly one 1(31) with (116), namely 5(7).

**Pr 40.** (111) — (113) imply

$$(117) \qquad \vec{s}_\mu \wedge \vec{s}_\nu \qquad (\mu, \nu = 1, 2, 3;\ \mu \neq \nu).$$

**Dm.** 5 Sgn 1.

**Pr 41.** (111) — (113),

$$(118) \qquad \lambda_\nu \in S \qquad (\nu = 1, 2, 3),$$

$$(119) \qquad \lambda_1\lambda_2\lambda_3 \neq 0$$

imply

$$(120) \qquad \lambda_\mu \vec{s}_\mu \wedge \lambda_\nu \vec{s}_\nu \qquad (\mu, \nu = 1, 2, 3;\ \mu \neq \nu).$$

**Dm.** Pr 40, Pr 32.

**Pr 42.** (111) — (113), (118), (119) imply

$$(121) \qquad \lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2 \wedge \lambda_3 \vec{s}_3.$$

**Dm.** Pr 40, Pr 33 imply (67). Besides, (111), 2 Sgn 7, Pr 6 imply

$$(122) \qquad \lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2 = (\lambda_1 s_1 + \lambda_2 s_2,\ \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2).$$

On the other hand, (111) and 2 Sgn 7 imply

$$(123) \qquad \lambda_3 \vec{s}_3 = (\lambda_3 s_3,\ \overline{\lambda}_3 m_3).$$

Now (122), (123) imply

$$(124) \qquad (\lambda_1 s_1 + \lambda_2 s_2) \times (\lambda_3 s_3) = \overline{\lambda}_3(\overline{\lambda}_1(s_1 \times s_3) + \overline{\lambda}_2(s_2 \times s_3))$$

(1 Pr 14, 1 Pr 17), whence

$$(125) \qquad (\lambda_1 s_1 + \lambda_2 s_2) \times (\lambda_3 s_3) \neq \mathbf{o}.$$

Indeed, otherwise (124), (119) imply

$$(126) \qquad \overline{\lambda}_1(s_1 \times s_3) + \overline{\lambda}_2(s_2 \times s_3) = \mathbf{o}$$

and scalar multiplication of (126) with $s_1$ implies

$$(127) \qquad \lambda_2(s_1 \cdot s_2 \times s_3) = 0$$

by virtue of 1 Pr 7, contrary to (119), (112) in view of 1 Pr 8. On the other hand, 1 Pr 7, (113) imply

(128) $$(\lambda_1 s_1 + h_2 s_2)m_3 + s_3(\overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2)$$

$$= \lambda_1(s_1 m_3 + s_3 m_1) + \lambda_2(s_2 m_3 + s_3 m_2) = 0,$$

and (122), (123), (125), 5 Sgn 1 imply (121).

**Pr 43.** (111) — (113), (118) imply

(129) $$(\lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2) + \lambda_2 \vec{s}_3 \quad \text{exists.}$$

**Dm.** If $\lambda_1\lambda_2\lambda_3 = 0$, then (129) is implied by 2 Pr 24, Pr 10, Pr 9. if (119) then Pr 33, Pr 42, Pr 7.

**Sgn 5.** $\vec{s}_1 + \vec{s}_2 + \vec{s}_3$ sgn: $(\vec{s}_1 + \vec{s}_2) + \vec{s}_3$ if (31).

**Sgn 6.** $L(\vec{s}_\nu)_{\nu=1}^3$ sgn: $\{\lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2 + \lambda_3 \vec{s}_3\colon\ \lambda_\nu \in S\ (\nu = 1, 2, 3)\}$ iff (111) — (113).

**Df 7.** $L(\vec{s}_\nu)_{\nu=1}^3$ is called the *linear span* of $\vec{s}_\nu$ $(\nu = 1, 2, 3)$.

**Pr 44.** (111) — (113),

(130) $$\vec{\sigma}_\mu \in L(\vec{s}_\nu)_{\nu=1}^3 \qquad (\mu = 1, 2)$$

imply (69).

**Dm.** (130), Sgn 6 imply: there exist

(131) $$\lambda_{\mu\nu} \in S \qquad (\mu = 1, 2;\ \nu = 1, 2, 3)$$

with

(132) $$\vec{\sigma}_\mu = \lambda_{\mu 1} \vec{s}_1 + \lambda_{\mu 2} \vec{s}_2 + \lambda_{\mu 3} \vec{s}_3 \qquad (\mu = 1, 2).$$

Now (132) and (111) imply

(133) $$\vec{\sigma}_\mu = (\lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3,\ \overline{\lambda}_{\mu 1} m_1 + \overline{\lambda}_{\mu 2} m_2 + \overline{\lambda}_{\mu 3} m_3) (\mu = 1, 2)$$

(2 Sgn 7, Pr 6), and (133), 1 Ax 8S, 1 Pr 7, 1 Pr 14, 1 Pr 17 imply

(134) $$(\lambda_{11} s_1 + \lambda_{12} s_2 + \lambda_{13} s_3) \times (\lambda_{21} s_1 + \lambda_{22} s_2 + \lambda_{23} s_3)$$

$$= (\overline{\lambda}_{11}\overline{\lambda}_{22} - \overline{\lambda}_{12}\overline{\lambda}_{21}) s_1 \times s_2 + (\overline{\lambda}_{12}\overline{\lambda}_{23} - \overline{\lambda}_{13}\overline{\lambda}_{22}) s_2 \times s_3 + (\overline{\lambda}_{13}\overline{\lambda}_{21} - \overline{\lambda}_{11}\overline{\lambda}_{23}) s_3 \times s_1$$

and

(135) $$(\lambda_{11} s_1 + \lambda_{12} s_2 + \lambda_{13} s_3)(\overline{\lambda}_{21} m_1 + \overline{\lambda}_{22} m_2 + \overline{\lambda}_{23} m_3)$$

$$+(\lambda_{21} s_1 + \lambda_{22} s_2 + \lambda_{23} s_3)(\overline{\lambda}_{11} m_1 + \overline{\lambda}_{12} m_2 + \overline{\lambda}_{13} m_3)$$

$$= (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})(s_1 m_2 + s_2 m_1) + (\lambda_{12}\lambda_{23} + \lambda_{13}\lambda_{22})(s_2 m_3 + s_3 m_2)$$

$$+(\lambda_{13}\lambda_{21} + \lambda_{11}\lambda_{23})(s_3 m_1 + s_1 m_3) = 0$$

in view of

(136) $$s_\nu m_\nu = 0 \qquad (\nu = 1, 2, 3)$$

(2 Sgn 1) and (113) (Sgn 6).

The alternative

(137) $$|\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}|^2 + |\lambda_{12}\lambda_{21} - \lambda_{13}\lambda_{22}|^2 + |\lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{23}|^2 \neq 0$$

or

(138) $$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21} = 0, \qquad \lambda_{12}\lambda_{21}-\lambda_{13}\lambda_{22} = 0 \qquad \lambda_{13}\lambda_{21}-\lambda_{11}\lambda_{23} = 0$$

now arises.

If (137), then (1340 implies

(139) $$(\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3) \times (\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3) \neq o.$$

Indeed, the right-hand side of (134) represents a linear combination of Gibbs' vectors $\overrightarrow{s}_\nu^{-1}$ ($\nu = 1, 2, 3$) defined by 1(23) provided 1(24); they exist by virtues of (112) (Sgn 6): in view of 1 Pr 20 and 1 Pr 12 such a combination may be zerro if, and only if, all coefficients are zeroes, and this is not the case if (137) holds. Now (132), (139), (135), 5 Sgn 1 imply (78), consequently (69) (Pr 7).

If (138), then (134) implies

(140) $$(\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3) \times (\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3) = o,$$

and (140), (132) imply (80) in view of 3 Sgn 1.

The supposition (81) is wrong. Indeed, (132) and 3 Sgn 7 imply that (81) is equivalent to

(141) $$\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3 = -(\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3),$$

(142) $$\overline{\lambda}_{11}m_1 + \overline{\lambda}_{12}m_2 + \overline{\lambda}_{13}m_3 \neq -(\overline{\lambda}_{21}m_1 + \overline{\lambda}_{22}m_2 + \overline{\lambda}_{23}m_3),$$

i.e. to

(143) $$(\lambda_{11} + \lambda_{21})s_1 + (\lambda_{12} + \lambda_{22})s_2 + (\lambda_{13} + \lambda_{23})s_3 = o,$$

(144) $$(\overline{\lambda}_{11} + \overline{\lambda}_{21})m_1 + (\overline{\lambda}_{12} + \overline{\lambda}_{22})m_2 + (\overline{\lambda}_{13} + \overline{\lambda}_{23})m_3 \neq o.$$

Now (143) and (112), 1 Pr 12 imply

(145) $$\lambda_{11} + \lambda_{21} = 0, \qquad \lambda_{12} + \lambda_{22} = 0, \qquad \lambda_{13} + \lambda_{23} = 0,$$

and (145) imply

(146) $$(\overline{\lambda}_{11} + \overline{\lambda}_{21})m_1 + (\overline{\lambda}_{12} + \overline{\lambda}_{22})m_2 + (\overline{\lambda}_{13} + \overline{\lambda}_{23})m_3 = o.$$

contrary to (144). In such a manner, (88) holds (3 Pr 21) and (80), (88) imply (69) (Pr 7).

**Pr 45.** $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ is a group with respect to the addition in $W_S$.

**Dm.** Pr 44, Sgn 6 with $\lambda_\nu = 0$ ($\nu = 1, 2, 3$), 2 Pr 24, Pr 10, Sgn 6 with $\lambda_1$, $\lambda_2$, $\lambda_3$, on the one hand, and $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, respectively, on the other hand, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ instead of $V_S$).

**Pr 46.** $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ is a 3-dimensional linear space over $S$ with respect to the addition in $W_S$ and to the multiplication 2Sgn 7 of the elements of $S$ and $W_S$.

**Dm.** As regards the addition, see Pr 45. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied with $L(\vec{s}_\nu)_{\nu=1}^3$ instead of $V_S$).

As regards the dimension of the linear space $L(\vec{s}_\nu)_{\nu=1}^3$ over $S$, let us note, first, that there exist three linearly independent elements of $L(\vec{s}_\nu)_{\nu=1}^3$, namely $\vec{s}_\nu$ ($\nu = 1, 2, 3$); and second, that any four elements of $L(\vec{s}_\nu)_{\nu=1}^3$ are linearly dependent.

Indeed, $\vec{s}_\mu \in L(\vec{s}_\nu)_{\nu=1}^3$ ($\mu = 1, 2, 3$), since $\vec{s}_1 = 1\vec{s}_1 + 0\vec{s}_2 + 0\vec{s}_3$, $\vec{s}_2 = 0\vec{s}_1 + 1\vec{s}_2 + 0\vec{s}_3$, $\vec{s}_3 = 0\vec{s}_1 + 0\vec{s}_2 + 1\vec{s}_3$ (2 Pr 24, 2 Pr 26, Pr 10, Pr 9, Sgn 6). Let now (118) and

(147) $$\lambda_1\vec{s}_1 + \lambda_2\vec{s}_2 + \lambda_3\vec{s}_3 = \vec{o}$$

hold. If (111), then (147) is equivalent to

(148) $$(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3,\ \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2 + \overline{\lambda}_3 m_3) = (o,\ o)$$

(2 Sgn 7, Pr 6, 2 Sgn 2, Sgn 5), whence

(149) $$\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 = o.$$

Now (149) and (112) imply $\lambda_\nu = 0$ ($\nu = 1, 2, 3$) (1 Pr 12), hence the linear independency of $\vec{s}_\nu$ ($\nu = 1, 2, 3$).

On the other hand, let

(150) $$\vec{\sigma}_\mu \in L(\vec{s}_\nu)_{\nu=1}^3 \qquad (\mu = 1, 2, 3, 4).$$

Now (150) and Sgn 6 imply that there exist

(151) $$\lambda_{\mu\nu} \in S \qquad (\mu = 1, 2, 3, 4;\ \nu = 1, 2, 3)$$

with

(152) $$\vec{\sigma}_\mu = \lambda_{\mu 1}\vec{s}_1 + \lambda_{\mu 2}\vec{s}_2 + \lambda_{\mu 3}\vec{s}_3 \qquad (\mu = 1, 2, 3, 4).$$

If (111), then (152) is equivalent to

(153) $$\vec{\sigma}_\mu = (\lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3,\ \overline{\lambda}_{\mu 1} m_1 + \overline{\lambda}_{\mu 2} m_2 + \overline{\lambda}_{\mu 3} m_3)$$
$$(\mu = 1, 2, 3, 4).$$

Let

(154) $$\alpha_\mu \in S \qquad (\mu = 1, 2, 3, 4)$$

be a non-zero solution of the system of equations

(155) $$\sum_{\mu=1}^{4} \alpha_\mu \lambda_{\mu\nu} = 0 \qquad (\nu = 1, 2, 3),$$

i.e.

(156) $$\sum_{\mu=1}^{4} \alpha_\mu \overline{\alpha}_\mu \neq 0.$$

Now (155) imply

(157) $$\sum_{\mu=1}^{4} \overline{\alpha}_\mu \overline{\lambda}_\mu = 0 \qquad (\nu = 1, 2, 3)$$

and (153), (155), (157) imply

(158) $$\sum_{\mu=1}^{4} \alpha_\mu \overrightarrow{\sigma}_\mu = \overrightarrow{o}$$

with (156), i.e. the linear dependency of (150).

**Pr 47.** If

(159) $$\overrightarrow{s} \in L(\overrightarrow{s})_{\nu=1}^{3},$$

(160) $$\overrightarrow{s} \,|\, \overrightarrow{s}_\nu \qquad (\nu = 1, 2, 3),$$

then

(161) $$\overrightarrow{s} \wedge \overrightarrow{s}_\nu \qquad (\nu = 1, 2, 3).$$

**Dm.** (159), Sgn 6 imply: there exist (118) with

(162) $$\overrightarrow{s} = \lambda_1 \overrightarrow{s}_1 + \lambda_2 \overrightarrow{s}_2 + \lambda_3 \overrightarrow{s}_3.$$

If 2(2), 2(59), then (160), 3 Sgn 2 imply

(163) $$\boldsymbol{s} \times \boldsymbol{s}_\nu \neq \boldsymbol{o} \qquad (\nu = 1, 2, 3).$$

Besides, (162), 2(2), 2(59), 2 Sgn 7, Pr 6 imply

(164) $$\boldsymbol{s} = \lambda_1 \boldsymbol{s}_1 + \lambda_2 \boldsymbol{s}_2 + \lambda_3 \boldsymbol{s}_3, \qquad \boldsymbol{m} = \overline{\lambda}_1 \boldsymbol{m}_1 + \overline{\lambda}_2 \boldsymbol{m}_2 + \overline{\lambda}_3 \boldsymbol{m}_3.$$

Now (164), 2 Sgn 1, 1 Ax 8S, 1 Pr 7, Sgn 6, 5 Sgn 1 imply

(165) $$\boldsymbol{s}\boldsymbol{m}_1 + \boldsymbol{s}_1\boldsymbol{m} = \lambda_2(\boldsymbol{s}_1\boldsymbol{m}_2 + \boldsymbol{s}_2\boldsymbol{m}_1) + \lambda_3(\boldsymbol{s}_1\boldsymbol{m}_3 + \boldsymbol{s}_3\boldsymbol{m}_1) = 0,$$

(166) $$\boldsymbol{s}\boldsymbol{m}_2 + \boldsymbol{s}_2\boldsymbol{m} = \lambda_1(\boldsymbol{s}_1\boldsymbol{m}_2 + \boldsymbol{s}_2\boldsymbol{m}_1) + \lambda_3(\boldsymbol{s}_2\boldsymbol{m}_3 + \boldsymbol{s}_3\boldsymbol{m}_2) = 0,$$

(167) $$\boldsymbol{s}\boldsymbol{m}_3 + \boldsymbol{s}_3\boldsymbol{m} = \lambda_1(\boldsymbol{s}_1\boldsymbol{m}_3 + \boldsymbol{s}_3\boldsymbol{m}_1) + \lambda_3(\boldsymbol{s}_3\boldsymbol{m}_3 + \boldsymbol{s}_3\boldsymbol{m}_2) = 0,$$

and(163), (165) — (167), 5 Sgn 1 imply (161).

**Pr 48.** If

(168) $$\overrightarrow{\sigma}_\mu \in L(\overrightarrow{s})_{\nu=1}^{3} \qquad (\mu = 1, 2, 3),$$

(169) $\quad \vec{\sigma}_\mu = (\bar{\sigma}_\mu, n_\mu) \qquad (\mu = 1, 2, 3),$

(170) $\quad \bar{\sigma}_1 \times \bar{\sigma}_2 \cdot \bar{\sigma}_3 \neq 0,$

then

(171) $\quad L(\vec{\sigma})^3_{\nu=1} = L(\vec{s})^3_{\nu=1}.$

**Dm.** (168), Sgn 6 imply: there exist

(172) $\quad \lambda_{\mu\nu} \in S \qquad (\mu, \nu = 1, 2, 3)$

with

(173) $\quad \vec{\sigma}_\mu = \lambda_{\mu 1} \vec{s}_1 + \lambda_{\mu 2} \vec{s}_2 + \lambda_{\mu 3} \vec{s}_3 \qquad (\mu = 1, 2, 3).$

If (111), then (169) and (173) imply

(174) $\quad \bar{\sigma}_\mu = \lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3 \qquad (\mu = 1, 2, 3),$

(175) $\quad n_\mu = \bar{\lambda}_{\mu 1} m_1 + \bar{\lambda}_{\mu 2} m_2 + \bar{\lambda}_{\mu 3} m_3 \qquad (\mu = 1, 2, 3)$

and (174), (175), 1 Ax 10S, 1 Ax 8S, 1 Pr 7, 2 Sgn 1, Sgn 6 imply

(176) $\quad \bar{\sigma}_\mu n_\nu + \bar{\sigma}_\nu n_\mu = 0 \qquad (\mu, \nu = 1, 2, 3).$

Now (170) and (176) imply that the left-hand side of (171) exists (Sgn 6). Then Pr 46.

**Sgn 7.** $\vec{s}_1 \vec{s}_2$ sgn: $s_1 s_2$ iff (20).

**Df 8.** $\vec{s}_1 \vec{s}_2$ is called the *scalar product* of $\vec{s}_1$ and $\vec{s}_2$.

**Sgn 8.** $\vec{s}^2$ sgn: $\vec{s}\vec{s}$ iff 2(1).

**Df 9.** $\vec{s}^2$ is called the *scalar square* of $\vec{s}$.

**Pr 49.** 2(53) imply $\vec{s}_1 \vec{s}_2 = \overline{\vec{s}_2 \vec{s}_1}$.
**Dm.** Sgn 7, 1 Ax 8S.

**Pr 50.** 2(49), 2(53) imply $(\lambda \vec{s}_1) \vec{s}_2 = \lambda (\vec{s}_1 \vec{s}_2)$.
**Dm.** Sgn 7, 2 Sgn 7, 1 Ax 9S.

**Pr 51.** (31), (22) imply $(\vec{s}_1 + \vec{s}_2) \vec{s}_3 = \vec{s}_1 \vec{s}_3 + \vec{s}_2 \vec{s}_3$.
**Dm.** Sgn 7, Pr 6, 1 Ax 10S.

**Pr 52.** 2(1) implies $\vec{s}^2 \geqq 0$.
**Dm.** Sgn 8, Sgn 7, 1 Ax 11S.

**Pr 53.** 2(1), $\vec{s}^2 = 0$ imply $\vec{s} = \vec{o}$.
**Dm.** Sgn 8, Sgn 7, 1 Ax 12S, 2 Sgn 1.

**Pr 54.** $L(\vec{s})^3_{\nu=1}$ is a 3-dimensional Hermitean space over $S$ with respect to the addition in $W_S$, to the multiplication 2 Sgn 7 of the elements of $S$ and $W_S$, and to the scalar multiplication Sgn 7 of the elements of $W_S$.
**Dm.** Pr 45, Pr 46, Pr 49 — Pr 53.

**Sch 3.** The following considerations will be useful in the sequel.

Let the following problem be solved. If $\vec{s}_\nu = (s_\nu, m_\nu)$ $\nu = 1, 2)$ be given intersecting arrows, then find a third arrow $\vec{s} = (s_3, m_3)$ such that, first, $s_3 = s_1 \times s_2$; and, second, that there exists a $r \in V_S$ with $r$ Z dir $\vec{s}_\nu$ $(\nu = 1, 2, 3)$, i.e. $r \times s_\nu = m_\nu$ $(\nu = 1, 2, 3)$. (The directrices of $\vec{s}_\nu$ exist, since $\vec{s}_\nu \neq \vec{o}$ $(\nu = 1, 2, 3)$, as 5 Sgn 1 and the definition of $\vec{s}_3$ imply.) in other words, conditions are sought for the consistency of the system of vector equations $r \times s_\nu = m_\nu$ $(\nu = 1, 2, 3)$, provided $s_3 = s_1 \times s_2$. According to 1 Pr 30, it is necessary and sufficient to this end that

$$s_\mu m_\nu + s_\nu m_\mu = 0 \qquad (\mu, \nu = 1, 2, 3). \tag{177}$$

hold, i.e. that

$$m_3 s_1 = -m_1 s_3, \qquad m_3 s_2 = -m_2 s_3, \qquad m_3 s_3 = 0 \tag{178}$$

are satisfied (1 Ax 8S, 2 Sgn 1). The system (178) is equivalent to

$$m_3 s_1 = m_1 \cdot s_2 \times s_1, \qquad m_3 s_2 = m_2 \cdot s_2 \times s_1, \qquad m_3 s \cdot s_1 \times s_2 = 0 \tag{179}$$

(1 Pr 13). according to 1 Pr 26, the only solution $m_3$ of the system (179) of vector equations is

$$m_3 = (m_1 \cdot s_2 \times s_1) s_1^{-1} + (m_2 \cdot s_2 \times s_1) s_2^{-1}. \tag{180}$$

i.e.

$$m_3 = \frac{(m_1 \cdot s_2 \times s_1)(s_2 \times (s_1 \times s_2)) + (m_2 \cdot s_2 \times s_1)(s_1 \times (s_2) \times s_1)}{(s_1 \times s_2)^2}. \tag{181}$$

Now (181) and 1 Pr 14, 1 Pr 15, 1 Pr 8 imply

$$m_3 = \frac{s_1 \times s_2 2}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2) s_1 + (s_1 \times s_2 \cdot m_1) s_2). \tag{182}$$

These conclusions give rise to the following definitions.

**Sgn 9.** $\vec{s}_1 \times \vec{s}_2$ sgn: $\vec{o}$ iff 2(53), 3(1), 3(8).

**Sgn 10.** $\vec{s}_1 \times \vec{s}_2$ sgn: $(s, m)$ with

$$s = s_1 \times s_2, \tag{183}$$

$$m = \frac{s_1 \times s_2 2}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2) s_1 + (s_1 \times s_2 \cdot m_1) s_2) \tag{184}$$

iff (20), 5(1).

**Df 10.** $\vec{s}_1 \times \vec{s}_2$ is called the *vector product* of $\vec{s}_1$ and $\vec{s}_2$.

**Pr 55.** 2(53) imply: $\vec{s}_1 \times \vec{s}_2$ exists iff $\vec{s}_1 + \vec{s}_2$ exists.

**Dm.** Sgn 9, Sgn 10, Pr 7.

**Pr 56.** (130) imply: $\vec{\sigma}_1 \times \vec{\sigma}_2$ exists.

**Dm.** Pr 55, Pr 44.

**Pr 57.** (31) imply

(185) $$\vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1,$$

provided the vector product exist.

**Dm.** Two cases are possible:

(186) $$\vec{s}_\nu = \vec{o} \qquad (1 \leqq \nu \leqq 3)$$

or

(187) $$\vec{s}_\nu \neq \vec{o} \qquad (\nu = 1, 2, 3).$$

If (186), then both sides of (185) are zeroes (2 Sgn 2, Sgn 9, 3 Pr 5, 3 Pr 26, Sgn 7).

If (187), then the following subcases are possible:

(188) $$\vec{s}_1 \mid \vec{s}_2$$

or

(189) $$\vec{s}_2 \mid \vec{s}_3$$

or

(190) $$\vec{s}_\nu \wedge \vec{s}_{\nu+1} \qquad (\nu = 1, 2).$$

Let (111) hold.

If (188), then $s_1 \times s_2 = \mathbf{o}$ (3 Sgn 1) and (187) imply $s_1 = \lambda s_2$ ($\lambda \in S$). Besides, the left-hand side of (185) is zero because of Sgn 9, Pr 54. Now if (189), then the right-hand side of (185) is zero too by the same reasons, hence (185) holds. If $\vec{s}_2 \,\bar{|}\, \vec{s}_3$, then $\vec{s}_2 \wedge \vec{s}_3$ by the assumption that $\vec{s}_2 \times \vec{s}_3$ exists (Sgn 9, Sgn 10), and Sgn 10 implies $\vec{s}_2 \times \vec{s}_3 = (s_2 \times s_3, m)$ with an appropriate $m \in V_S$. Hence $\vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = s_2 \times s_3 \cdot (\lambda s_2) = 0$ (Sgn 7, 1 Pr 7), i.e. (185) holds again.

In the same way it is proved that (185) holds in the subcase (189) of (187).

Let now (190) hold. Then Sgn 10 implies

(191) $$\vec{s}_1 \times \vec{s}_2 = (s_1 \times s_2, p), \qquad \vec{s}_2 \times \vec{s}_3 = (s_2 \times s_3, q)$$

with appropriate $p, q \in V_S$. Now (191) and Sgn 7 imply

(192) $$\vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = s_1 \times s_2 \cdot s_3, \qquad \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = s_2 \times s_3 \cdot s_1$$

and the validity of (185) is a direct corollary of (192) and 1 Ax 13S.

**Pr 58.** (111) — (113), 5(7), $s \in V_S$,

(193) $$\vec{s} = (s, r \times s)$$

imply

(194) $$\vec{s} \in L(\vec{s}_\nu)_{\nu=1}^3$$

**Dm.** (111) — (113), 5(7) imply (116) and $s \in V_S$, (112) imply

$$(195)\qquad s = \sum_{\nu=1}^{3}(ss_\nu^{-1})s_\nu$$

(1 Pr 30). Now (195) implies

$$(196)\qquad r \times s = \sum_{\nu=1}^{3}(s_\nu^{-1}s)r \times s_\nu$$

(1 Pr 15, 1 Pr 13, 1 Pr 17, 1 Ax 8S) and (196), (116) imply

$$(197)\qquad r \times s = \sum_{\nu=1}^{3}(s_\nu^{-1}s)m_\nu.$$

Then (193), (195), (197), (111), 2 Sgn 7 imply (162) with

$$(198)\qquad \lambda_\nu = ss_\nu^{-1} \qquad (\nu = 1, 2, 3),$$

whence (194) (Sgn 6).

**Pr 59.** (111) — (113), 5(7), (194), 2(2) imply (193).

**Dm.** Sgn 6 implies: there exist (118) with (162), and (162), 2(1), (111) imply (164). The first relation (164) implies (198) (1 Pr 24). Now (198), (116), and the second relation (164) imply

$$(199)\qquad m = \sum_{\nu=1}^{3}(s_\nu^{-1}s)r \times s_\nu = r \times \sum_{\nu=1}^{3}(ss_\nu^{-1})s_\nu = r \times s$$

in view of 1 Pr 15, 1 Pr 13, 1 Pr 17, 1 Ax 8S.

**Pr 60.** (111) — (113), (130) imply

$$(200)\qquad \vec{\sigma}_1 \times \vec{\sigma}_2 \in L(\vec{s}_\nu)_{\nu=1}^{3}.$$

**Dm.** The existence of $\vec{\sigma}_1 \times \vec{\sigma}_2$ is proved in Pr 56. Let

$$(201)\qquad \vec{\sigma}_\nu = (\bar{\sigma}_\nu, n_\nu) \qquad (\nu = 1, 2).$$

Then (201) and Pr 59 imply

$$(202)\qquad n_\nu = r \times \bar{\sigma}_\nu \qquad (\nu = 1, 2),$$

$r$ being defined by 5(7).

Two cases are now possible:

$$(203)\qquad \bar{\sigma}_1 \times \bar{\sigma}_2 = o$$

or

$$(204)\qquad \bar{\sigma}_1 \times \bar{\sigma}_2 \neq o.$$

If (203), then $\vec{\sigma}_1 \mid \vec{\sigma}_2$ (3 Sgn 1). The supposition $\vec{\sigma}_1 \uparrow\downarrow \vec{\sigma}_2$ is wrong. Indeed, together with (201) it implies

(205) $\qquad \overline{\sigma}_1 + \overline{\sigma}_2 = \mathbf{o}, \qquad \boldsymbol{n}_1 + \boldsymbol{n}_2 \neq \mathbf{o}$

(3 Sgn 7). Now the first relation (205) and (202) imply $\boldsymbol{n}_1 + \boldsymbol{n}_2 = \boldsymbol{r} \times (\overline{\sigma}_1 + \overline{\sigma}_2) = \boldsymbol{r} \times \mathbf{o} = \mathbf{o}$, contrary to the second relation (205). In such a manner, $\vec{\sigma}_1 \uparrow\downarrow \vec{\sigma}_2$, whence $\vec{\sigma}_1 \times \vec{\sigma}_2 = \vec{o}$ (Sgn 9), i.e. (200) holds good in the case (203) in view of Pr 45.

If (204), then $\vec{\sigma}_1 \wedge \vec{\sigma}_2$ (5 Sgn 1), since (202) imply

(206) $$\overline{\sigma}_1\boldsymbol{n}_2 + \overline{\sigma}_2\boldsymbol{n}_1 = \overline{\sigma}_1 \cdot \boldsymbol{r} \times \overline{\sigma}_2 + \overline{\sigma}_2 \cdot \boldsymbol{r} \times \overline{\sigma}_1 = \overline{\boldsymbol{r} \times \overline{\sigma}_2 \cdot \overline{\sigma}_1} + \overline{\boldsymbol{r} \times \overline{\sigma}_1 \cdot \overline{\sigma}_2}$$
$$= \boldsymbol{r} \cdot \overline{\sigma}_2 \times \overline{\sigma}_1 + \boldsymbol{r} \cdot \overline{\sigma}_1 \times \overline{\sigma}_2 = \boldsymbol{r} \cdot (\overline{\sigma}_2 \times \overline{\sigma}_1 + \overline{\sigma}_1 \times \overline{\sigma}_2) = \boldsymbol{r}\mathbf{o} = 0$$

(1 Ax 8S, 1 Pr 8, 1 Pr 13). Now (201), (204), Sgn 10 imply

(207) $$\vec{\sigma}_1 \times \vec{\sigma}_2 = (\overline{\sigma}_1 \times \overline{\sigma}_2, \boldsymbol{n})$$

with

(208) $$\boldsymbol{n} = \frac{\overline{\sigma}_1 \times \overline{\sigma}_2}{(\overline{\sigma}_1 \times \overline{\sigma}_2)^2} \times ((\overline{\sigma}_2 \times \overline{\sigma}_1 \cdot \boldsymbol{n}_2)\overline{\sigma}_1 + (\overline{\sigma}_1 \times \overline{\sigma}_2 \cdot \boldsymbol{n}_1)\overline{\sigma}_2).$$

On the other hand, (201), (202) and 2 Sgn 4 imply

(209) $$\boldsymbol{r}\ \mathsf{Z}\ \mathrm{dir}\ \vec{\sigma}_\nu \qquad (\nu = 1, 2)$$

Now $\vec{\sigma}_1 \wedge \vec{\sigma}_2$, (209), 5 Pr 31 imply

(210) $$\boldsymbol{r} = \frac{1}{2}\sum_{\nu=1}^{3} \overline{\sigma}_\nu{}^{-1} \times \boldsymbol{n}_\nu$$

provided

(211) $$\overline{\sigma}_3 = \overline{\sigma}_1 \times \overline{\sigma}_2,$$

(212) $$\boldsymbol{n}_3 = (\boldsymbol{n}_1 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)\overline{\sigma}_1{}^{-1} + (\boldsymbol{n}_2 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)\overline{\sigma}_2{}^{-1}.$$

At that, (210) satisfies

(213) $$\boldsymbol{r}\ \mathsf{Z}\ \mathrm{dir}(\overline{\sigma}_3, \boldsymbol{n}_3),$$

i.e.

(214) $$\boldsymbol{r} \times \overline{\sigma}_3 = \boldsymbol{n}_3.$$

Now (211), (212) imply

(215) $$\boldsymbol{n}_3 = \frac{(\boldsymbol{n}_1 \cdot \overline{\sigma}_2 \times \overline{\sigma}_3)(\overline{\sigma}_2 \times (\overline{\sigma}_1 \times \overline{\sigma}_2)) + (\boldsymbol{n}_2 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)((\overline{\sigma}_1 \times \overline{\sigma}_2) \times \overline{\sigma}_1)}{(\overline{\sigma}_1 \times \overline{\sigma})^2},$$

i.e.

(216) $$\boldsymbol{n}_3 = \frac{\overline{\sigma}_1 \times \overline{\sigma}_2}{(\overline{\sigma}_1 \times \overline{\sigma}_2)^2} \times ((\overline{\sigma}_2 \times \overline{\sigma}_1 \cdot \boldsymbol{n}_2)\overline{\sigma}_1 + (\overline{\sigma}_1 \times \overline{\sigma}_2 \cdot \boldsymbol{n}_1)\overline{\sigma}_2),$$

and (208), (216) imply

$$(217)\qquad \boldsymbol{n} = \boldsymbol{n}_3.$$

Now (211), (214), (217) imply

$$(218)\qquad \boldsymbol{r} \times (\bar{\sigma}_1 \times \bar{\sigma}_2) = \boldsymbol{n},$$

and (207), (218), Pr 58 imply (200) in the case (204).

**Sch 4.** The following remark may be useful in the capacity of an economizer of technical and intellectual work.

As it is immediately seen, two different definitions are given of the vector product $\vec{s}_1 \times \vec{s}_2$ (when it exists) of two arrows $\vec{s}_1$ and $\vec{s}_2$, namely Sgn 9 and Sgn 10, in accordance with the mutual disposition of $\vec{s}_1$ and $\vec{s}_2$: whether $\vec{s}_1 \mid \vec{s}_2$ or $\vec{s}_1 \wedge \vec{s}_2$. This distinction may be avoided by virtue of Pr 58 and Pr 59. Indeed, if $\boldsymbol{r}$ denotes the intersecting point of the directrices of the arrows $\vec{s}_1$ and $\vec{s}_2$ in the case $\vec{s}_1 \wedge \vec{s}_2$ (i.e. $\boldsymbol{r}$ is defined by 5(7) provided 5(8), 5(9), see 5 Pr 31), then $\vec{s}_1 \times \vec{s}_2$ may be defined by means of the relation

$$(219)\qquad \vec{s}_1 \times \vec{s}_2 \text{ sgn} : (\boldsymbol{s}_1 \times \boldsymbol{s}_2, \boldsymbol{r} \times (\boldsymbol{s}_1 \times \boldsymbol{s}_2)),$$

as it has been shown in the proof of Pr 60. Now the same relation (219) may be used in the case $\vec{s}_1 \mid \vec{s}_2$ too, $\boldsymbol{r}$ denoting in this latter case an arbitrary vector. Indeed, if $\vec{s}_1 \mid \vec{s}_2$, then $\vec{s}_1 \times \vec{s}_2 = \vec{o}$ according to Sgn 9; the right-hand side of (219) is, however, also equal to $\vec{o}$, since $\boldsymbol{s}_1 \times \boldsymbol{s}_2 = \boldsymbol{o}$, in view of 3Sgn 1.

To summarize $\vec{s}_1 \times \vec{s}_2$ may be defined by means of (219) with 5(7) if $\vec{s}_1 \wedge \vec{s}_2$ and with any $\boldsymbol{r}$ if $\vec{s}_1 \mid \vec{s}_2$.

**Sch 5.** In order to manifest the effectiveness of the "new definition" (219), let us prove, by its aid, the relation (185): we know, from the proof of Pr 57, that its direct deduction of the basis of Sgn 9 and Sgn 10 is a rather complicated one.

And so, let (111) hold. Then, according to (219),

$$(220)\qquad \vec{s}_1 \times \vec{s}_2 = (\boldsymbol{s}_1 \times \boldsymbol{s}_2, \boldsymbol{r}_1 \times (\boldsymbol{s}_1 \times \boldsymbol{s}_2)),$$

$$(221)\qquad \vec{s}_2 \times \vec{s}_3 = (\boldsymbol{s}_2 \times \boldsymbol{s}_3, \boldsymbol{r}_2 \times (\boldsymbol{s}_2 \times \boldsymbol{s}_3)).$$

At that, $\boldsymbol{r}_1$ is any vector if $\vec{s}_1 \mid \vec{s}_2$ and it is defined by the conditions $\boldsymbol{r}_1 \text{ Z dir } \vec{s}_\nu$ $(\nu = 1, 2)$ if $\vec{s}_1 \wedge \vec{s}_2$; similarly, $\boldsymbol{r}_2$ is any vector if $\vec{s}_2 \mid \vec{s}_3$, and it is defined by the conditions $\boldsymbol{r}_2 \text{ Z dir } \vec{s}_\nu$ $(\nu = 2, 3)$ if $\vec{s}_2 \wedge \vec{s}_3$.

Now (220), (221), (111), and Sgn 7 imply

$$(222)\qquad \vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = (\boldsymbol{s}_1 \times \boldsymbol{s}_2, \boldsymbol{r}_1 \times (\boldsymbol{s}_1 \times \boldsymbol{s}_2)) \cdot (\boldsymbol{s}_3, \boldsymbol{m}_3) = \boldsymbol{s}_1 \times \boldsymbol{s}_2 \cdot \boldsymbol{s}_3,$$

$$(223)\qquad \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = (\boldsymbol{s}_2 \times \boldsymbol{s}_3, \boldsymbol{r}_2 \times (\boldsymbol{s}_2 \times \boldsymbol{s}_3)) \cdot (\boldsymbol{s}_1, \boldsymbol{m}_1) = \boldsymbol{s}_2 \times \boldsymbol{s}_3 \cdot \boldsymbol{s}_1,$$

and (185) is implied by (222), (223), and 1 Ax 13S.

Pr 57 has been proved directly above in order to afford an opportunity for a parallel between the two approaches. Some abridgements in the proofs of Pr 44, Pr 46, etc. are also posible, on the basis of Pr 58 and Pr 59.

**Pr 61.** (111) — (113), (168) imply

(224) $(\vec{\sigma}_1 \times \vec{\sigma}_2) \times \vec{\sigma}_3 = (\vec{\sigma}_1 \vec{\sigma}_3)\vec{\sigma}_2 - (\vec{\sigma}_2 \vec{\sigma}_3)\vec{\sigma}_1.$

**Dm.** Let

(225) $\vec{\sigma}_\nu = (\bar{\sigma}_\nu, \boldsymbol{r} \times \bar{\sigma}_\nu)$ $(\nu = 1, 2, 3),$

$\boldsymbol{r}$ being defined by 5(7) (Pr 59). Then, according to Sch 4, the relations

(226) $\vec{\sigma}_1 \times \vec{\sigma}_2 = (\bar{\sigma}_1 \times \bar{\sigma}_2, \boldsymbol{r} \times (\bar{\sigma}_1 \times \bar{\sigma}_2)),$

(227) $(\vec{\sigma}_1 \times \vec{\sigma}_2) \times \vec{\sigma}_3 = ((\bar{\sigma}_1 \times \bar{\sigma}_2) \times \bar{\sigma}_3, \boldsymbol{r} \times ((\bar{\sigma}_1 \times \bar{\sigma}_2) \times \bar{\sigma}_3)),$

hold good. At that, the left-hand sides of (226), (227) exist by virtue of Pr 56.

On the other hand, (225) and Sgn 7, 2 Sgn 7, 1 Pr 17, Pr 21 imply

(228)
$$\begin{aligned}(\vec{\sigma}_1 \vec{\sigma}_3)\vec{\sigma}_2 - (\vec{\sigma}_2 \vec{\sigma}_3)\vec{\sigma}_1 &\\ = (\bar{\sigma}_1\bar{\sigma}_3)(\bar{\sigma}_2, \boldsymbol{r} \times \bar{\sigma}_2) - (\bar{\sigma}_2\bar{\sigma}_3)(\bar{\sigma}_1, \boldsymbol{r} \times \bar{\sigma}_1) &\\ = ((\bar{\sigma}_1\bar{\sigma}_3)\bar{\sigma}_2, \boldsymbol{r} \times ((\bar{\sigma}_1\bar{\sigma}_3)\bar{\sigma}_2)) - ((\bar{\sigma}_2\bar{\sigma}_3)\bar{\sigma}_1, \boldsymbol{r} \times ((\bar{\sigma}_2\bar{\sigma}_3)\bar{\sigma}_1)) &\\ = ((\bar{\sigma}_1\bar{\sigma}_3)\bar{\sigma}_2 - (\bar{\sigma}_2\bar{\sigma}_3)\bar{\sigma}_1, \boldsymbol{r} \times ((\bar{\sigma}_1\bar{\sigma}_3)\bar{\sigma}_2) - (\bar{\sigma}_2\bar{\sigma}_3)\bar{\sigma}_1)) .\end{aligned}$$

Now (224) is a direct corollary from (227), (228), and 1 Ax 14S.

**Pr 62.** $L(\vec{s}_\nu)_{\nu=1}^3$ is a standard vector space over $S$ with respect to the addition in $W_S$, to the multiplication 2 Sgn 7 of the elements of $S$ and $W_S$, to the scalar multiplication Sgn 7 of the elements of $W_S$, and to the vector multiplication Sgn 9, Sgn 10 in $W_S$.

**Dm.** Pr 45, Pr 46, Pr 54, Pr 56, Pr 57, Pr 61, and $\vec{s}_1 \times \vec{s}_2 \neq \vec{o}$ (Sgn 10).

**Sch 6.** A comparison of the present paper with the article [1] at once displays the complete analogy between the real and the complex algebras of arrows.

Further developments exposed in [1] will not be extended here for the complex case. They concern mainly the associativity of addition of arrows, as well as some facts about incidence of poles and arrows and metrical relations (distances, etc.). Some questions about incidence of arrows and planes are also omitted here, since they concern mainly systems of arrows.

In the third part of this series of articles dedicated to the algebraic theory of arrows finite systems of arrows will be discussed.

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