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INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, III

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> ... mechanica rationalis erit scientia motuum, qui ex viribus quibuscunque resultant, et virium quae ad motus quoscunque requiruntur, accurate proposita ac demonstrata
>
> Isaac Newton: *Philosophiae Naturalis Principia Mathematica* (Auctoris praefacio ad lectorem)

Иван Чобанов. ВВЕДЕНИЕ В АЛГЕБРАИЧЕСКУЮ ТЕОРИЮ СКОЛЬЗЯЩИХ ВЕКТОРОВ, III. Эта работа является третьей частью серии статьей, посвещенных алгебраической теории скользящих векторов, первые две части [1, 2] которой опубликованы в этом *Ежегоднике*. Она занимается главным образом конечными системами скользящих векторов в комплексных стандартных векторных пространствах, в частности, или в стандартных векторных пространствах над комплексными расширениями произвольных упорядоченных полях, в общем. Определения этих пространств, а также некоторые основные моменты их алгебр, приведены в вводнуя часть статьи [2]; для более подробного ознакомления в этой связи читателю рекоммендуется статья [3] или книга [4]. Замечательно, что не наблюдаются никакие существенные расхождения между реальными и комплексными случаями алгебр скользящих векторов. Теорема о ранге играет ту же самую центральную роль в теории комплексных скользящих векторов, как и в теории реальных.

Ivan Chobanov. INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, III. This article is the third part of a series of investigations on an algebraic theory of arrows or sliding vectors, the first two parts [1, 2] of which are published in this *Annual.* It is dedicated mainly to the finite systems of arrows in complex standard vector spaces or, more generally, in standard vector spaces over the complex extensions of arbitrary ordered fields. The definitions of these spaces, as well as some basic moments of their algebras, are given in the introductory part of the article [2]; for a more detailed exposition in this connection the reader is referred to the paper [3] or to the booklet [4]. It is remarkable that no essential divergences are observed between

the real and the complex cases of arrow algebras. The rank-theorem plays the same central role in the theory of complex arrows as in that of the real ones.

This article is the third part of a series of investigations on an algebraic theory of arrows or sliding vectors (*vecteurs glissants, gleitende Vektoren, скользящие векторы*), the first two parts [1, 2] of which are published in this *Annual.* The definitions and notations from [2] are systematically used in this third part, the following manner of quotation being adopted (the example is a ficticious one): notation 1, axiom 2, definition 3, proposition 4, scholium 5, and relation (6) of §7 of [2], for instance, are cited here by [2, 7 Sgn 1], [2, 7 Ax 2], [2, 7 Df 3], [2, 7 Pr 4], [2, 7 Sch 5], and [2, 7(6)] respectively. In general, the exposition of the present paper is an immediate continuation of the exposition of the second part [2] of the series in question.

This article is dedicated mainly to the theory of finite systems of arrows in real and complex standard vector spaces, in particular, and of standard vector spaces over ordered fields and over the complex extensions of such fields, in general. As it has been emphasized in [2], the definitions and the basic algebraic properties of the standard vector space V_F over the ordered field F and of the standard vector space $V_{C(F)}$ over the complex extension $C(F)$ of F may be found in the article [3], as well as in the booklet [4]. In any case, a brief account in this connection is given in the introductory paragraph (§1. Praeliminaria) of [2], where the notation V_S is used in order to reduce the real case $S = F$, as well as the complex one $S = C(F)$, to a common denominator.

In order to simplify references and to ease the exposition, some most fundamental notations from [2] are reproduced over again here. Such cases are, by the way, extremely rare: as a matter of fact, they are exceptions made in case of emergency.

Since the main, if not exclusive, domain of applications of arrows is proposed by analytical mechanics, kinematics as well as dynamics, and since the analytical mechanics deals predominantly, if not exceptionally, with finite systems of arrows, our interest will be focused on such systems in the main. In other words, we shall consider sets consisting of a finite number of arrows. Since the properties of such sets depend exclusively on the properties of the single arrows, it is quite natural to fix our attention on the latter. This has been done in [2], where the following basic definition has been adopted.

Sgn 1. W_S **sgn:** $\{(\mathbf{s}, \mathbf{m}) \in V_S^2 : \mathbf{s} \neq \mathbf{o},\ \mathbf{sm} = 0 \text{ or } \mathbf{s} = \mathbf{m} = \mathbf{o}\}$.

Df 1. The elements of W_S are called *S-arrows* or *arrows in* V_S.

Df 2. Any set

$$(1) \qquad \underset{\rightarrow}{s} = \left\{\vec{s}_\nu\right\}_{\nu=1}^{n}$$

of arrows

$$(2) \qquad \vec{s}_\nu \in W_S \qquad (\nu = 1, \dots, n)$$

is called a (*finite*) *system of S-arrows* or *of arrows in* V_S.

Sch 1. For the sake of brevity (1) is usually called a *system of arrows*, S and V_S being implied by the context.

Sgn 2. Σ_S sgn: the set of all finite systems of S-arrows.

Df 3. The vector

$$s = \sum_{\nu=1}^{n} s_\nu \tag{3}$$

provided (1) and

$$\vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, \ldots, n) \tag{4}$$

is called the basis of $\underset{\rightarrow}{s}$.

Df 4. The vector

$$m = \sum_{\nu=1}^{n} m_\nu \tag{5}$$

provided (1), (4), is called the moment of $\underset{\rightarrow}{s}$.

Sch 2. In the light of Df 3, Df 4, as well as of [2, 2 Df 2, 2 Df 3], it is clear, that the basis and the moment of a system of arrows are the sums of the bases and of the moments respectively of the particular arrows composing the system in question.

Sch 3. Whereas there is a certain *dependence* between the basis s and the moment m of a single arrow $\vec{s}$, as it is seen from Sgn 1, there is no compulsory relation between the basis (3) and the moment (5) of a system of arrows (1). In other words, the vectors s and m, defined by (3) and (5) respectively, may be *completely arbitrary*. It may be proved that for any couple (s, m) of S-vectors s and m there exists one at least (infinitely many, as a matter of fact) systems (1) of arrows (4) for which the equalities (3) and (5) respectively take place. In particular, there exist systems (1) of arrows (4), for which (3) and (5) imply $sm = 0$, as in the case of a single arrow, but there exist also systems for which sm is equal to any value in S.

Sch 4. Similarly, whereas the basis s and the moment m of a single arrow $\vec{s}$ determine it completely by virtue of the definition

$$\vec{s} = (s, m), \tag{6}$$

the basis s and the moment m of a system of arrows $\underset{\rightarrow}{s}$ by no means determine $\underset{\rightarrow}{s}$ unanimously: as it has been mentioned above, *there are infinitely many systems of arrows with the same basis and the same moment.* In such sense, the notation

$$\underset{\rightarrow}{s} = (s, m), \tag{7}$$

similar to (6), is void of meaning. It is convenient, however, to introduce the following notation.

Sgn 3: $\underset{\rightarrow}{s}(s, m)$ sgn: s and m are the basis and the moment respectively of the system of arrows $\underset{\rightarrow}{s}$.

Sch 5. As it has been manifested in [2], an extremely important attribute of an arrow $\overrightarrow{s}$ is its $\boldsymbol{r}$-moment, denoted by $\mathrm{mom}_{\mathbf{r}}\overrightarrow{s}$ and defined in the following manner. If

$$\boldsymbol{r} \in V_S, \tag{8}$$

$$\overrightarrow{s} = (\boldsymbol{s}, \boldsymbol{m}) \in W_S, \tag{9}$$

then by definition

$$\mathrm{mom}_{\mathbf{r}}\overrightarrow{s} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}. \tag{10}$$

In the same way, the notion of $\boldsymbol{r}$-moment of a system of arrows $\underset{\rightarrow}{s}$, denoted by $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$, plays an extremely important role in the theory of the systems of arrows. It is introduced by the aid of the following definition, which is complete imitation of the definition (10).

Sgn 4. $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ sgn: $\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}$ iff (8) and

$$\underset{\rightarrow}{s}(\boldsymbol{s}, \boldsymbol{m}) \in \Sigma_S \tag{11}$$

Df 5. $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ is called the *$\boldsymbol{r}$-moment* of $\underset{\rightarrow}{s}$ or the *moment of $\underset{\rightarrow}{s}$ with respect to $\boldsymbol{r}$.*

Df 6. $\boldsymbol{r}$ is called the *pole* of $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$.

Pr 1. (4), (1), (8) imply

$$\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s} = \sum_{\nu=1}^{n} \mathrm{mom}_{\mathbf{r}}\overrightarrow{s}_{\nu}. \tag{12}$$

Dm. By definition

$$\mathrm{mom}_{\mathbf{r}}\overrightarrow{s}_{\nu} = \boldsymbol{m}_{\nu} + \boldsymbol{s}_{\nu} \times \boldsymbol{r} \qquad (\nu = 1, \ldots, n). \tag{13}$$

Now (13), (3), (5), Sgn 4 imply

$$\sum_{\nu=1}^{n} \mathrm{mom}_{\mathbf{r}}\overrightarrow{s}_{\nu} = \sum_{\nu=1}^{n} \boldsymbol{m}_{\nu} + \sum_{\nu=1}^{n} \boldsymbol{s}_{\nu} \times \boldsymbol{r} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} = \mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}. \tag{14}$$

Pr 2. (11) implies $\boldsymbol{m} = \mathrm{mom}_{\mathbf{o}}\underset{\rightarrow}{s}$.

Dm. Sgn 4.

Pr 3. (11) implies: $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ is invariant with respect to $\boldsymbol{r}$ iff

$$\boldsymbol{s} = \mathbf{o}. \tag{15}$$

Dm. If (15), then Sgn 4 implies $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s} = \boldsymbol{m}$ for any (8).

Let $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ be invariant with respect to the poles (8), i.e. let

$$\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s} = \boldsymbol{n} \tag{16}$$

for any (8). Sgn 4 and (16) imply

(17) $$\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} = \boldsymbol{n} \qquad (\boldsymbol{r} \in V_S).$$

Let

(18) $$\boldsymbol{r}_\nu \in V_S \qquad (\nu = 1, 2, 3),$$

(19) $$\boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1 \neq \mathbf{o}.$$

Then (17), (18) imply

(20) $$\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{n} \qquad (\nu = 1, 2, 3)$$

whence

(21) $$\boldsymbol{s} \times (\boldsymbol{r}_\nu - \boldsymbol{r}_3) = \mathbf{o} \qquad (\nu = 1, 2).$$

The relation (19) is equivalent with

(22) $$(\boldsymbol{r}_1 - \boldsymbol{r}_3) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) \neq \mathbf{o}.$$

The system of vector equations (21) with (22), where $\boldsymbol{s}$ is unknown, has, according to [2, 1 Pr 29], the only solution (15), *q. e. d.*

Pr 4. (8), (11) imply

(23) $$\boldsymbol{s} \cdot \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = \boldsymbol{sm}.$$

Dm. Sgn 4.

Sch 6. The relation (23) displays that the scalar product $\boldsymbol{s} \cdot \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}$ is *invariant* with respect to the pole $\boldsymbol{r}$ of $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}$. This important fact is an analogue of [2, 2 Pr 5] and deserves a special attention.

Sgn 5. $\mathrm{I}(\underset{\rightarrow}{s})$ sgn: $\boldsymbol{sm}$ iff (11).

Df 7. $\mathrm{I}(\underset{\rightarrow}{s})$ is called the ***first scalar invariant*** of $\underset{\rightarrow}{s}$.

Sgn 6. $\mathrm{II}(\underset{\rightarrow}{s})$ sgn: $\frac{\boldsymbol{sm}}{\boldsymbol{s}^2}$ iff (11) and

(24) $$\boldsymbol{s} \neq \mathbf{o}.$$

Df 8. $\mathrm{II}(\underset{\rightarrow}{s})$ is called the ***second scalar invariant*** of $\underset{\rightarrow}{s}$.

Sch 7. The analogue of the notion of *directrix* of a single arrow [2, 2 Sgn 4] is played, in the theory of systems of arrows, by the so-called *axis* of a system of arrows. This notion is generated by the following considerations.

Let a system of arrows (11) be given and let the following question be put: do there exist poles (8) for which $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}$ are parallel to $\boldsymbol{s}$? (Naturally, this problem is meaningless unless (24) is satisfied.) The affirmative answer of this question is equivalent with the assumption that the vector equation with respect to $\boldsymbol{r}$:

(25) $$\boldsymbol{s} \times \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = \mathbf{o}$$

is consistent provided (11), (24). By virtue of Sgn 4, (25) is equivalent with

$$(26) \qquad \boldsymbol{s} \times (\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}) = \mathbf{o},$$

i.e. with

$$(27) \qquad (\boldsymbol{r} \times \boldsymbol{s}) \times \boldsymbol{s} = \boldsymbol{m} \times \boldsymbol{s}$$

in view of [**2**, 1 Pr 13]. Now (27), (24), and [**2**, 1 Pr 32] imply: there exists $\alpha \in S$ with

$$(28) \qquad \boldsymbol{r} \times \boldsymbol{s} = \alpha \boldsymbol{s} + \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}.$$

A scalar multiplication of (28) by $\boldsymbol{s}$ implies

$$(29) \qquad \alpha \boldsymbol{s}^2 = 0,$$

and (29), (24), [2, 1 Ax 12S] imply $\alpha = 0$; therefore (28) implies

$$(30) \qquad \boldsymbol{r} \times \boldsymbol{s} = \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}$$

In such a way, if (8) satisfies (25) or, just the same, (26), then it must be sought among the solutions $\boldsymbol{r}$ of the vector equation (30). We shall establish now that any $\boldsymbol{r}$ satisfying (30) satisfies (25) too: any solution (8) of (30) is, at the same time, a solution of our initial problem (25).

Indeed, let (8) satisfy (30). Since [**2**, 1 Pr 16] implies

$$(31) \qquad \boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s}) = \boldsymbol{s}^2 \boldsymbol{m} - (\boldsymbol{m}\boldsymbol{s})\boldsymbol{s},$$

the relation (30) is equivalent with

$$(32) \qquad \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} = \left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right) s.$$

Now (32) and Sgn 4 imply

$$(33) \qquad \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = \left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right) \boldsymbol{s},$$

and (33) implies (25).

On the other hand, (30) is the equation of a line l, if $\boldsymbol{r}$ is regarded as a fluent radius-vector. Consequently, all poles (8), for which the moments of the system of arrows $\underset{\rightarrow}{s}$ are parallel to the basis $\boldsymbol{s}$ of $\underset{\rightarrow}{s}$, are located on the line (30). This line is called the *axis* of $\underset{\rightarrow}{s}$.

Let us now formalize this conclusion.

Sgn 7. ax $\underset{\rightarrow}{s}$ sgn: l with

$$(34) \qquad \left(\boldsymbol{s}, \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}\right) \ \& \ l,$$

iff (11), (24).

Sch 8. The meaning of Sgn 7 is disclosed by the aid of Sgn 4 of §1 (p.86) of the article [5].

Df 9. ax $\underset{\to}{s}$ is called the *axis* of $\underset{\to}{s}$.

Pr 5. (11), (24), (8),

$$r \; Z \; \text{ax} \; \underset{\to}{s} \tag{35}$$

imply (33).

Dm. Sgn 7 and Sgn 1 of §4 (p. 117) of [5] imply that (35) is equivalent with (30), which on its part is equivalent with (32). Now (32) and Sgn 4 imply (33).

Sch 9. The following proposition discloses why ax $\underset{\to}{s}$ is an analogue of dir $\vec{s}$, as it has been mentioned in Sch 7.

Pr 6. If

$$\vec{o} \neq \vec{s} \in W_S, \tag{36}$$

$$\underset{\to}{s} = \left\{ \vec{s} \right\}, \tag{37}$$

then

$$\text{ax} \; \underset{\to}{s} = \text{dir} \; \vec{s}. \tag{38}$$

Dm. (36), (6) imply

$$s \neq o, \qquad sm = 0. \tag{39}$$

On the other hand, (37) and Df 3, Df 4 imply (11). Now (11), (39), Sgn 7 imply

$$(s, m) \;\; \& \; \text{ax} \; \underset{\to}{s}. \tag{40}$$

On the other hand, [2, 2 Sgn 4] imply

$$(s, m) \;\; \& \; \text{dir} \; \vec{s}, \tag{41}$$

and (40), (41) imply (38) by virtue of Pr 18 of §1 (p. 86) of [5].

Sch 10. The following proposition displays that the moments of a system of arrows have a minimal property for the poles located on the axis of the system.

Pr 7. (8), (11), (24), (35),

$$\bar{p} \in V_S, \tag{42}$$

$$\bar{p} \; \bar{Z} \; \text{ax} \; \underset{\to}{s} \tag{43}$$

imply

$$\left(\text{mom}_r \, \underset{\to}{s}\right)^2 < \left(\text{mom}_{\bar{p}} \, \underset{\to}{s}\right)^2. \tag{44}$$

Dm. Sgn 7 implies that (35) and (43) are equivalent with (30) and

$$(45) \qquad \overline{p} \times \boldsymbol{s} \neq \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2}$$

respectively, and Sgn 4 implies that (30) and (45) are equivalent with (33) and

$$(46) \qquad \mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} \neq \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right) \boldsymbol{s}$$

respectively. Let

$$(47) \qquad \mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} = \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right) \boldsymbol{s} + \overline{\sigma},$$

where, in view of (46),

$$(48) \qquad \overline{\sigma} \neq \mathrm{o}.$$

Now (47) and (33) imply

$$(49) \qquad \mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} = \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} + \overline{\sigma},$$

and (49), Pr 4 imply

$$(50) \qquad \boldsymbol{s}\overline{\sigma} = 0.$$

Then (33) and (50) imply

$$(51) \qquad \overline{\sigma} \cdot \mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = 0,$$

and (49), (51) imply

$$(52) \qquad \left(\mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s}\right)^2 = \left(\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}\right)^2 + \overline{\sigma}^2.$$

At last, (52) and (48) imply (44).

Sch 11. The same minimal property characterizes $\mathrm{dir}\,\overrightarrow{s}$: let us remind [2, 2 Pr 8].

Sch 12. If l is a line parallel to ax $\underset{\rightarrow}{s}$, then all moments of $\underset{\rightarrow}{s}$ with respect to poles incident with l are equal between themselves. Indeed, let

$$(53) \qquad l \parallel \text{ ax } \underset{\rightarrow}{s}.$$

Then there exists a $\boldsymbol{n} \in V_S$ with $\boldsymbol{sn} = 0$ and

$$(54) \qquad (\boldsymbol{s}, \boldsymbol{n}) \ \& \ l.$$

If (8) is incident with l, then

$$(55) \qquad \boldsymbol{r} \times \boldsymbol{s} = \boldsymbol{n}.$$

Now (55) and Sgn 4 imply

$$\text{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = \boldsymbol{m} - \boldsymbol{n}. \tag{56}$$

In such a way, the following proposition has been proved.

Pr 8. (11), (24), (54), (8), $\boldsymbol{r}$ Z l imply (56).

Sch 13. Another important property of ax $\underset{\rightarrow}{s}$ is disclosed by the following considerations. Let

$$\boldsymbol{r}_\nu \in V_S \qquad (\nu = 1, 2) \tag{57}$$

and let

$$\text{dist}(\boldsymbol{r}_1, \text{ax } \underset{\rightarrow}{s}) = \text{dist}(\boldsymbol{r}_2, \text{ax } \underset{\rightarrow}{s}), \tag{58}$$

i.e. the end-points of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are at equal distances from ax $\underset{\rightarrow}{s}$. Since

$$\text{dist}(\boldsymbol{r}_\nu, \text{ax } \underset{\rightarrow}{s}) = \frac{1}{s} \left| \boldsymbol{r}_\nu \times \boldsymbol{s} - \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s}}{\boldsymbol{s}^2} \right| \tag{59}$$

($\nu = 1, 2$), as Sgn 7 and Sgn 9 of §4 (p. 124) of [5] imply, (58) is equivalent to

$$\left(\boldsymbol{r}_1 \times \boldsymbol{s} - \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2} \right)^2 = \left(\boldsymbol{r}_2 \times \boldsymbol{s} - \frac{\boldsymbol{s} \times (\boldsymbol{m} \times \boldsymbol{s})}{\boldsymbol{s}^2} \right)^2. \tag{60}$$

Now (60) and Sgn 4 imply

$$\left(\text{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} - \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2} \right) \boldsymbol{s} \right)^2 = \left(\text{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} - \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2} \right) \boldsymbol{s} \right)^2, \tag{61}$$

and (61), Pr 4 imply

$$\left(\text{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} \right)^2 = \left(\text{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} \right)^2. \tag{62}$$

In such a manner, the following proposition has been proved.

Pr 9. (11), (24), (57), (58) imply (62).

Sch 14. The following proposition is an analogue of [2, 2 Pr 9].

Pr 10. (11), (24), (8), (42), (35) imply

$$\text{mom}_{\overline{\rho}} \underset{\rightarrow}{s} = (\boldsymbol{r} - \overline{\rho}) \times \boldsymbol{s} + \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2} \right) \boldsymbol{s}. \tag{63}$$

Dm. Sgn 4, (30).

Pr 11. (11), (57) imply

$$\text{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} - \text{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} = \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2). \tag{64}$$

Dm. Sgn 4.

Sch 15. The relation (64) is called the *connection between the moments of a system of arrows with respect to two poles*, and it plays a fundamental role in the so-called *statical-kinematical analogy*, which will be discussed briefly below. The connection (64) is a complete analogue of the *connection between the moments of an arrow with respect to two poles* [**2**, 2(19)].

Sch 16. (64) implies

$$\mathbf{s} \cdot \mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} = \mathbf{s} \cdot \mathrm{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s}. \tag{65}$$

The inference (65) from (64) is, however, trivial in the light of Pr 4.

Pr 12. (57),

$$\underset{\rightarrow}{s} \in \Sigma_S \tag{66}$$

imply

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} = (\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s}. \tag{67}$$

Dm. Pr 11.

Pr 13. (66), (57),

$$\mathbf{r}_1 \neq \mathbf{r}_2, \tag{68}$$

$$S = P \qquad \text{or} \qquad S = C(P) \tag{69}$$

imply

$$(\mathbf{r}_1 - \mathbf{r}_2)^0 \cdot \mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} = (\mathbf{r}_1 - \mathbf{r}_2)^0 \cdot \mathrm{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s}. \tag{70}$$

Dm. Pr 12, [**2**, 1 Sgn 4].

Sch 17. The relation (70) gives expression of the fact that if (68) holds and if (57) are different poles, then the projections of $\mathrm{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s}$ ($\nu = 1, 2$) on the line l connecting them, i.e. on

$$(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_2 \times \mathbf{r}_1) \mathbin{\&} l, \tag{71}$$

are equal.

Pr 14. (11), (18), (19),

$$\mathrm{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s} = \mathbf{n} \qquad (\nu = 1, 2, 3) \tag{72}$$

imply (15).

Dm. (72), Sgn 4 imply

$$\mathbf{m} + \mathbf{s} \times \mathbf{r}_\nu = \mathbf{n} \qquad (\nu = 1, 2, 3) \tag{73}$$

and (73) imply (21). Since (19) is equivalent to (22), the system of equations (21) implies (15).

Sch 18. The condition (19) implies that (18) are non-colinear, i.e. there exists no line l with $\boldsymbol{r}_\nu \; Z \; l \; (\nu = 1, 2, 3)$. In such a manner, the basis of the system $\underset{\longrightarrow}{s}$ is certainly zero if there exist three non-colinear poles, the moments of $\underset{\longrightarrow}{s}$ with respect to which are equal. The inverse statement is trivial in the light of Pr 3.

Sch 19. Before we proceed further, let us make a brief remark of an ideological character. The above exposition reveals the most important role the notion of moment of an arrow plays in the theory of arrows. It would not be far-fetched to state that both notions are genetically connected.

Indeed, an arrow $\overrightarrow{s}$ being given by (6), its $\boldsymbol{r}$-moment is determined by (10). On the other hand, (10) implies $\boldsymbol{m} = \mathrm{mom}_{\boldsymbol{o}} \overrightarrow{s}$, so that (6) may be written in the form

$$\text{(74)} \qquad \overrightarrow{s} = (\boldsymbol{s}, \mathrm{mom}_{\boldsymbol{o}} \overrightarrow{s}).$$

In other words, a special $\boldsymbol{r}$-moment of $\overrightarrow{s}$, namely its $\boldsymbol{o}$-moment, takes part in the very definition of $\overrightarrow{s}$ by means of (6). In other words, the arrow-concept is unthinkable at all without the moment-concept. On the other hand, not a step could be made in the theory of arrows without the intensive exploitation of moments. The rank-theorem, the discussion of which lies yet ahead, is a brilliant illustration of this assertion.

The general connection between arrow-concept and moment-concept being once comprehended, a problem of cardinal importance arises at once, and quite naturally at that. It concerns the logical possibilities for a formal, purely mathematical, generalization of the arrow theory.

As it has been emphasized at the very end of the introductory remarks of [**2**], discussing in his *Vortrag* [**6**] the problem of the axiomatical consolidation of the logical foundations of rational mechanics, Hilbert underlines that, in the course of this process, "auch wird der Mathematiker, wie er es in der Geometrie getan hat, nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien berücksichtigen zu haben". The real arrows (i.e. the elements of W_R) "sind der Wirklichkeit nahe kommende" mathematical entities; the complex ones are obviously not, to say nothing of the $C(F)$-arrows. And yet, maybe there are other generalizations of the arrow-concept, for instance, in multidimensional Hermitean or, at least, Euclidean spaces?

This question is answered in the negative, and the main reason for this state of affairs lies in the fact that the introduction of a fourth operation (vector multiplication) in an Euclidean or in a Hermitean space with the most economical requirement that it must satisfy the two only specific conditions [**2**, 1 Ax 13S, 1 Ax 14S] inevitably reduces the dimensions of the spaces in question to 3 (with the additional condition that this operation should not be trivial, i.e. that there must exist two at least Euclidean or Hermitean vectors, the vector product of which is different from zero). And this fourth operation is used most essentially in the definition (10) of the moment of an arrow.

It is true that there are made efforts to generalize the vector multiplication in multidimensional cases, but it is still a controversial point to what extent these generalizations may be rated as not far-fetched ones.

Sch 20. If $G \subset V_S$ and a mapping

$$(75) \qquad \mu: \quad G \longrightarrow V_S$$

is defined, then it is said that a *vector field* over G is at hand. Now, a system of arrows (11) being given, a vector field

$$(76) \qquad \mu: \quad V_S \longrightarrow V_S$$

is *eo ipso* defined, determined by the relation

$$(77) \qquad \mu(\boldsymbol{r}) \text{ sgn : } \operatorname{mom}_{\mathbf{r}} \overrightarrow{s} \qquad (\boldsymbol{r} \in V_S),$$

i.e.

$$(78) \qquad \mu(\boldsymbol{r}) = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r} \qquad (\boldsymbol{r} \in V_S),$$

by virtue of Sgn 4.

Df 10. The vector field (76) defined by (77) is called the *moment field* of $\underset{\to}{s}$.

Sch 21. Let us regard the image $\mu(V_S)$ of V_S through the mapping (78). By definition, it is a set consisting of S-vectors. Now any set of vectors has a natural characteristic, namely the maximal number of linearly independent elements of this set. Since any four standard vectors are linearly dependent, this maximal number may be 0, or 1, or 2, or 3 at most. After these explanations, the following definition may be accepted.

Sgn 8. rank $\underset{\to}{s}$ sgn: the maximal number of linearly independent elements of $\mu(V_S)$ iff (11), (77).

Df 11. rank $\underset{\to}{s}$ is called the *rank* of $\underset{\to}{s}$.

Sch 22. The rank of a system of arrows $\underset{\to}{s}$ is, as it will be seen in the sequel, a most important characteristic of $\underset{\to}{s}$. Therefore, a system of arrows $\underset{\to}{s}$ being given, the problem of the determination of rank $\underset{\to}{s}$ is a most actual one. The direct application to this end of the definition Sgn 8 is, however, a very clumsy approach to the goal.

Indeed, let us get aware of the nature of this direct application. What must one do in order to determinate rank $\underset{\to}{s}$?

To prove that rank $\underset{\to}{s} = 3$ means to find such three particular poles (18) that $\operatorname{mom}_{\mathbf{r}_\nu} \underset{\to}{s}$ $(\nu = 1, 2, 3)$ are linearly independent, i.e.

$$(79) \qquad \operatorname{mom}_{\mathbf{r}_1} \underset{\to}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\to}{s} \cdot \operatorname{mom}_{\mathbf{r}_3} \underset{\to}{s} \neq 0.$$

To prove that rank $\underset{\to}{s} = 2$ means, first, to establish that $\operatorname{mom}_{\mathbf{r}_\nu} \underset{\to}{s}$ $(\nu = 1, 2, 3)$ are linearly dependent for any three poles (18), i.e.

$$(80) \qquad \operatorname{mom}_{\mathbf{r}_1} \underset{\to}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\to}{s} \cdot \operatorname{mom}_{\mathbf{r}_3} \underset{\to}{s} = 0;$$

and second, to find such two particular poles (57), that $\operatorname{mom}_{\mathbf{r}_\nu} \underset{\to}{s}$ $(\nu = 1, 2)$ are linearly independent, i.e.

(81) $$\mathrm{mom}_{\mathbf{r}_1}\underset{\rightarrow}{s} \times \mathrm{mom}_{\mathbf{r}_2}\underset{\rightarrow}{s} \neq \mathbf{o}.$$

To prove that rank $\underset{\rightarrow}{s} = 1$ means, first, to demonstrate that $\mathrm{mom}_{\mathbf{r}_\nu}\underset{\rightarrow}{s}$ ($\nu = 1, 2$) are linearly dependent for any two poles (57), i.e.

(82) $$\mathrm{mom}_{\mathbf{r}_1}\underset{\rightarrow}{s} \times \mathrm{mom}_{\mathbf{r}_2}\underset{\rightarrow}{s} = \mathbf{o};$$

and second, to find such a particular pole (8), that $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ is linearly independent, i.e.

(83) $$\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s} \neq 0.$$

At last, to prove that rank $\underset{\rightarrow}{s} = 0$ means to establish that $\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s}$ is linearly dependent for any pole (8), i.e.

(84) $$\mathrm{mom}_{\mathbf{r}}\underset{\rightarrow}{s} = \mathbf{o}.$$

It is obvious that all these procedures cannot be assessed as a very attractive mathematical task. They certainly require some amount of inventiveness. Therefore it is quite natural to seek a certain *Schablon* that would permit to determinate rank $\underset{\rightarrow}{s}$ automatically, as the saying goes.

This *Schablon* is proposed by the so-called *theorem of the rank of a system of arrows* (or, concisely, the *rank-theorem*). It consists in the following proposition.

Pr 15 (the *rank-theorem*). If (11), then

(85) $$\mathrm{rank}\,\underset{\rightarrow}{s} = 0$$

iff

(86) $$\boldsymbol{s} = \mathbf{o}, \qquad \boldsymbol{m} = \mathbf{o};$$

(87) $$\mathrm{rank}\,\underset{\rightarrow}{s} = 1$$

iff

(88) $$\boldsymbol{s} = \mathbf{o}, \qquad \boldsymbol{m} \neq \mathbf{o};$$

(89) $$\mathrm{rank}\,\underset{\rightarrow}{s} = 2$$

iff

(90) $$\boldsymbol{s} \neq \mathbf{o}, \qquad \boldsymbol{sm} = \mathbf{o};$$

(91) $$\mathrm{rank}\,\underset{\rightarrow}{s} = 3$$

iff

(92) $$\boldsymbol{sm} = 0.$$

Sch 23. The following scheme proposes an abbreviated formulation of the rank-theorem:

$$\text{(93)} \qquad \operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \text{iff} \quad \begin{cases} \boldsymbol{s} = \mathbf{o}, \quad \boldsymbol{m} = \mathbf{o}, \\ \boldsymbol{s} = \mathbf{o}, \quad \boldsymbol{m} \neq \mathbf{o}, \\ \boldsymbol{s} \neq \mathbf{o}, \quad \boldsymbol{sm} = 0, \\ \qquad \boldsymbol{sm} \neq 0. \end{cases}$$

Sch 24. Two different proofs of the rank-theorem (93) will be given below. Before proceeding to these, let us get aware of the meaning of Pr 15.

Let a system (1) of arrows (4) be given. Then its basis $\boldsymbol{s}$ and its moment $\boldsymbol{m}$ are trivially determined by (3) and (5) respectively. Now, in order to determine rank $\underset{\rightarrow}{s}$, one must simply check up which of the cases (86), or (88), or (90), or (92) is at hand. This is why the rank-theorem has been called a *Schablon* in Sch 22. Where did the difficulties in the determination of rank $\underset{\rightarrow}{s}$, described in Sch 22, vanish? The answer is: these difficulties are overcome, once for ever, in the proof of Pr 15.

First proof of the rank-theorem. *Sufficiency.* Let (86) hold. Then (8) and Sgn 4 imply (84), whence (85) (Sgn 8).

Let (88) hold. Then (86) and Sgn 4 imply (83). On the other hand, (57) and Pr 3 imply (82), whence (87) (Sgn 8).

Let (90) hold. If (8), (35), then the second relation (90) and (30) (Sgn 7) imply

$$\text{(94)} \qquad \boldsymbol{r} \times \boldsymbol{s} = \boldsymbol{m}.$$

Let

$$\text{(95)} \qquad \boldsymbol{s}_\nu \in V_S \qquad (\nu = 1, 2),$$

$$\text{(96)} \qquad \boldsymbol{s}_1 \times \boldsymbol{s}_2 \cdot \boldsymbol{s}_3 \neq 0,$$

and let by definition

$$\text{(97)} \qquad \boldsymbol{r}_\nu = \boldsymbol{r} + \boldsymbol{s}_\nu \qquad (\nu = 1, 2, 3).$$

Then (97), Sgn 4, (94) imply

$$\text{(98)} \qquad \operatorname{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s} = \boldsymbol{m} + \boldsymbol{s} \times (\boldsymbol{r} + \boldsymbol{s}_\nu) = \boldsymbol{s} \times \boldsymbol{s}_\nu \qquad (\nu = 1, 2),$$

whence

$$\text{(99)} \qquad \operatorname{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} = (\boldsymbol{s} \times \boldsymbol{s}_1) \times (\boldsymbol{s} \times \boldsymbol{s}_2),$$

i.e.

$$\text{(100)} \qquad \operatorname{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} = (\boldsymbol{s} \times \boldsymbol{r}_1 \cdot \boldsymbol{s}_2)\boldsymbol{s}.$$

Now (100), (96), and the first relation (90) imply (81). On the other hand, if (18), then Sgn 4 and the second relation (90) imply

(101) $$\operatorname{mom}_{\mathbf{r}_1} \underset{\longrightarrow}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\longrightarrow}{s} \cdot \operatorname{mom}_{\mathbf{r}_3} \underset{\longrightarrow}{s}$$

$$= ((\boldsymbol{r}_2\boldsymbol{m} - \boldsymbol{r}_1\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1 \cdot \boldsymbol{r}_2)\boldsymbol{s})(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_3).$$

Now (101) and the second relation (90) imply (80), whence (89) (Sgn 8).

At last, let (92) hold. If (8), (35), then (30) holds. Let (95), (96),

(102) $$\boldsymbol{s}_3 = \boldsymbol{s},$$

and let by definition

(103) $$\boldsymbol{r}_\nu = \boldsymbol{r} + \boldsymbol{s}_\nu \qquad (\nu = 1, 2, 3).$$

Then (103), Sgn 4, (30) imply

(104) $$\operatorname{mom}_{\mathbf{r}_\nu} \underset{\longrightarrow}{s} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{m} + \boldsymbol{s} \times (\boldsymbol{r} \times \boldsymbol{s}_\nu) = \left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_\nu$$

$$(\nu = 1, 2, 3),$$

and (104), (102) imply

(105) $$\operatorname{mom}_{\mathbf{r}_1} \underset{\longrightarrow}{s} \times \operatorname{mom}_{\mathbf{r}_2} \underset{\longrightarrow}{s} \cdot \operatorname{mom}_{\mathbf{r}_3} \underset{\longrightarrow}{s}$$

$$= \left(\left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_1\right) \times \left(\left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_2\right) \cdot \left(\left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right)\boldsymbol{s}\right)$$

$$= \left(\frac{\boldsymbol{m}\boldsymbol{s}}{\boldsymbol{s}^2}\right)(\boldsymbol{s} \times \boldsymbol{s}_1 \cdot \boldsymbol{s}_2)\boldsymbol{s}^2.$$

Now (105), (92), (96) imply (79), whence (91) (Sgn 8).

Necessity. The sufficiency of the conditions (86), (88), (90), (92) for (85), (87), (89), (91) respectively once proved, the necessity of these conditions is trivial, since they are mutually inconsistent.

Second proof of the rank-theorem. *Necessity.* Let (85) hold. Then Sgn 8 implies (84) for any (8) whence

(106) $$\boldsymbol{m} = \mathbf{o}$$

(Pr 2) and (15) (Pr 3). Now (15), (106) imply (86).

Let (87) hold. Then Sgn 8 implies that there exists (42) with

(107) $$\operatorname{mom}_{\bar{\rho}} \underset{\longrightarrow}{s} \neq \mathbf{o}$$

and that (57) imply (82). If

(108) $$\bar{\mu} = \operatorname{mom}_{\bar{\rho}} \underset{\longrightarrow}{s},$$

then (82) implies

(109) $$\operatorname{mom}_{\mathbf{r}} \underset{\longrightarrow}{s} \times \bar{\mu} = \mathbf{o}$$

for any (8), and (109), Pr 2 imply

(110) $\boldsymbol{m} \times \overline{\mu} = \mathbf{o}.$

On the other hand, (109) and Sgn 4 imply

(111) $(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}) \times \overline{\mu} = \mathbf{o},$

and (110), (111) imply

(112) $(\boldsymbol{s} \times \boldsymbol{r}) \times \overline{\mu} = \mathbf{o}$

for any (8). If (24), then there exist (57) with

(113) $(\boldsymbol{s} \times \boldsymbol{r}_1) \times (\boldsymbol{s} \times \boldsymbol{r}_2) \neq \mathbf{o}.$

Since (112) holds for any (8),

(114) $(\boldsymbol{s} \times \boldsymbol{r}_\nu) \times \overline{\mu} = \mathbf{o}$ $(\nu = 1, 2),$

and (113), (114) imply

(115) $\overline{\mu} = \mathbf{o},$

contrary to (107). This consideration is due to the supposition that (24) holds. Therefore (15). Now Pr 3, Pr 2 imply $\boldsymbol{m} = \overline{\mu}$, whence

(116) $\boldsymbol{m} \neq \mathbf{o},$

and (15), (116) imply (88).

Let (89) hold. Then Sgn 8 implies that there exist $\overline{\rho}_\nu \in V_S$ $(\nu = 1, 2)$ with

(117) $\mathrm{mom}_{\overline{\rho}_1} \underset{\longrightarrow}{S} \times \mathrm{mom}_{\overline{\rho}_2} \underset{\longrightarrow}{S} \neq \mathbf{o},$

and that (18) imply (80). If (15), then (64) implies

(118) $\mathrm{mom}_{\overline{\rho}_1} \underset{\longrightarrow}{S} = \mathrm{mom}_{\overline{\rho}_2} \underset{\longrightarrow}{S}$

contrary to (117), whence (24). On the other hand, if

(119) $\boldsymbol{m}_\nu = \mathrm{mom}_{\overline{\rho}_\nu} \underset{\longrightarrow}{S}$ $(\nu = 1, 2).$

Then (80), Pr 2, (8) imply

(120) $\boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \boldsymbol{m} = 0,$

(121) $\boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \mathrm{mom}_{\mathbf{r}} \underset{\longrightarrow}{S} = 0.$

Now (121), Sgn 4 imply

(122) $\boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot (\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}) = 0,$

and (120), (122) imply

$$(123) \qquad \boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \boldsymbol{s} \times \boldsymbol{r} = 0$$

for any (8). Let (18),

$$(124) \qquad \boldsymbol{r}_1 \times \boldsymbol{r}_2 \cdot \boldsymbol{r}_3 \neq 0$$

hold. Then (123) implies

$$(125) \qquad (\boldsymbol{m}_1 \times \boldsymbol{m}_2) \times \boldsymbol{s} \cdot \boldsymbol{r}_\nu = 0 \qquad (\nu = 1, 2, 3)$$

and(124), (125) imply

$$(126) \qquad (\boldsymbol{m}_1 \times \boldsymbol{m}_2) \times \boldsymbol{s} = \mathbf{o},$$

i.e.

$$(127) \qquad (\boldsymbol{m}_1 \boldsymbol{s})\boldsymbol{m}_2 - (\boldsymbol{m}_2 \boldsymbol{s})\boldsymbol{m}_1 = 0.$$

Now (127), (118), (119) imply

$$(128) \qquad \boldsymbol{m}_1 \boldsymbol{s} = 0,$$

and (128), Pr 4 imply

$$(129) \qquad \boldsymbol{s}\boldsymbol{m} = 0.$$

Then (24), (129) imply (90).

Let (91) hold. Then there exist (18) with (79). Let

$$(130) \qquad \boldsymbol{m}_\nu = \mathrm{mom}_{\mathrm{r}_\nu} \underset{\rightarrow}{s} \qquad (\nu = 1, 2, 3).$$

Now (130), (79), Pr 2, Pr 4 imply

$$(131) \qquad \boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \boldsymbol{m}_3 \neq 0,$$

$$(132) \qquad \boldsymbol{s}\boldsymbol{m} = \boldsymbol{s}\boldsymbol{m}_\nu \qquad (\nu = 1, 2, 3).$$

If (129), then (131), (132) imply (15), and (15), (130), Sgn 4 imply

$$(133) \qquad \boldsymbol{m}_\nu = \boldsymbol{m} \qquad (\nu = 1, 2, 3),$$

contrary to (131). Hence (92).

Sufficiency. The necessity of the conditions (86), (88), (90), (92) for (85), (87), (89), (91) respectively being, in such a manner, proved, the sufficiency of these conditions is verified trivially, since they are mutually inconsistent.

Sch 25. The first proof of the rank-theorem may be assessed as simpler than the second one.

Pr 16. (18), (66),

$$(134) \qquad \boldsymbol{m}_\nu = \mathrm{mom}_{\mathrm{r}_\nu} \underset{\rightarrow}{s} \qquad (\nu = 1, 2, 3)$$

imply

$$(135) \qquad (\boldsymbol{r}_\nu - \boldsymbol{r}_{\nu+1})(\boldsymbol{m}_\nu - \boldsymbol{m}_{\nu+1}) = 0 \qquad (\nu = 1, 2, 3)$$

provided

$$(136) \qquad \boldsymbol{r}_4 = \boldsymbol{r}_1, \qquad \boldsymbol{m}_4 = \boldsymbol{m}_1.$$

Dm. (134), (11), Sgn 4 imply

$$(137) \qquad \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{m}_\nu \qquad (\nu = 1, 2, 3),$$

i.e.

$$(138) \qquad \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_{\nu+1} = \boldsymbol{m}_{\nu+1} \qquad (\nu = 1, 2, 3)$$

provided (136). Now (137), (138) imply

$$(139) \qquad \boldsymbol{s} \times (\boldsymbol{r}_\nu - \boldsymbol{r}_{\nu+1}) = \boldsymbol{m}_\nu - \boldsymbol{m}_{\nu+1} \qquad (\nu = 1, 2, 3)$$

and (139) imply (135).

Sch 26. In such a manner, the conditions (135) provided (136) are necessary for the consistency of the relations (134) or, just the same, of (137). As a matter of fact, (134) imply seemingly much more burdensome necessary conditions than (135), namely

$$(140) \qquad (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu) = 0$$

$$(\mu, \nu, \sigma, \tau = 1, 2, 3).$$

Indeed, (137) imply

$$(141) \qquad \boldsymbol{s} \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) = \boldsymbol{m}_\mu - \boldsymbol{m}_\nu (\mu, \nu = 1, 2, 3),$$

$$(142) \qquad \boldsymbol{s} \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) = \boldsymbol{m}_\sigma - \boldsymbol{m}_\tau (\sigma, \tau = 1, 2, 3),$$

and (141), (142) imply

$$\begin{aligned}(143) \qquad & (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu) \\ &= (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \cdot \boldsymbol{s} \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) \cdot \boldsymbol{s} \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \\ &= \boldsymbol{s} \cdot (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) + \boldsymbol{s} \cdot (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)\end{aligned}$$

$(\mu, \nu, \sigma, \tau = 1, 2, 3)$, whence (140).

It will be seen, however, that all the relations (140) are simple corollaries from (135), rather than mutually independent.

As a matter of fact, the number of different among themselves relations (140) is 6, rather than 81, as one might think at a first glance. Indeed, let by definition

$$(144) \qquad L(\mu, \nu, \sigma, \tau) = (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu)$$

($\mu, \nu, \sigma, \tau = 1, 2, 3$). Now obviously

$$L(\mu, \mu, \sigma, \tau) = 0, \tag{145}$$
$$L(\mu, \nu, \sigma, \sigma) = 0 \tag{146}$$

($\mu, \nu, \sigma, \tau = 1, 2, 3$). On the other hand,

$$L(\nu, \mu, \sigma, \tau) = -L(\mu, \nu, \sigma, \tau), \tag{147}$$
$$L(\mu, \nu, \tau, \sigma) = -L(\mu, \nu, \sigma, \tau), \tag{148}$$
$$L(\tau, \sigma, \mu, \nu) = L(\mu, \nu, \sigma, \tau) \tag{149}$$

($\mu, \nu, \sigma, \tau = 1, 2, 3$), etc. In such a manner, the relations (140) reduce to the following ones: (135) provided (136) and

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_1 - \boldsymbol{m}_3) + (\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{m}_1 - \boldsymbol{m}_2) = 0, \tag{150}$$
$$(\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_2 - \boldsymbol{m}_1) + (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_2 - \boldsymbol{m}_3) = 0, \tag{151}$$
$$(\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_3 - \boldsymbol{m}_2) + (\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_3 - \boldsymbol{m}_1) = 0. \tag{152}$$

Now (135) imply

$$\begin{aligned} &(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_1 - \boldsymbol{m}_3) + (\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{m}_1 - \boldsymbol{m}_2) \\ &= (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_1 - \boldsymbol{m}_2 + \boldsymbol{m}_2 - \boldsymbol{m}_3) + (\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{m}_1 - \boldsymbol{m}_3 + \boldsymbol{m}_3 - \boldsymbol{m}_2) \\ &\quad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_1 - \boldsymbol{m}_2) + (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_2 - \boldsymbol{m}_3) \\ &\quad +(\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{m}_1 - \boldsymbol{m}_3) + (\boldsymbol{r}_1 - \boldsymbol{r}_3)(\boldsymbol{m}_3 - \boldsymbol{m}_2) \\ &\quad = (\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_2 - \boldsymbol{m}_3) = 0, \end{aligned} \tag{153}$$

i.e. (150). Similarly, (135) imply

$$\begin{aligned} &(\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_2 - \boldsymbol{m}_1) + (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_2 - \boldsymbol{m}_3) \\ &= (\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_2 - \boldsymbol{m}_3 + \boldsymbol{m}_3 - \boldsymbol{m}_1) + (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_2 - \boldsymbol{m}_1 + \boldsymbol{m}_1 - \boldsymbol{m}_3) \\ &\quad = (\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_2 - \boldsymbol{m}_3) + (\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_3 - \boldsymbol{m}_1) \\ &\quad +(\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_2 - \boldsymbol{m}_1) + (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_1 - \boldsymbol{m}_3) \\ &\quad = (\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_1 - \boldsymbol{m}_3) = 0, \end{aligned} \tag{154}$$

i.e. (151). At last, (135) imply

$$\begin{aligned} &(\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_3 - \boldsymbol{m}_2) + (\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_3 - \boldsymbol{m}_1) \\ &= (\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_3 - \boldsymbol{m}_1 + \boldsymbol{m}_1 - \boldsymbol{m}_2) + (\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_3 - \boldsymbol{m}_2 + \boldsymbol{m}_2 - \boldsymbol{m}_1) \\ &\quad (\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_3 - \boldsymbol{m}_1) + (\boldsymbol{r}_3 - \boldsymbol{r}_1)(\boldsymbol{m}_1 - \boldsymbol{m}_2) \\ &\quad +(\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_3 - \boldsymbol{m}_2) + (\boldsymbol{r}_3 - \boldsymbol{r}_2)(\boldsymbol{m}_2 - \boldsymbol{m}_1) \\ &\quad = (\boldsymbol{r}_2 - \boldsymbol{r}_1)(\boldsymbol{m}_1 - \boldsymbol{m}_2) = 0, \end{aligned} \tag{155}$$

i.e. (152).

Now the following question quite naturally arises: are the conditions (135) also sufficient for the consistency of the relations (134). In other words, (18) and

$$(156) \qquad \boldsymbol{m}_\nu \in V_S \qquad (\nu = 1, 2, 3)$$

with (135) being given, does there exist a system of arrows (66) satisfying (234)? Or otherwise, does there exist, under the same conditions, a solution $\boldsymbol{s}$, $\boldsymbol{m}$ of the system of vector equations (137). As the following proposition displays, this question is answered in the affirmative, provided (19) holds.

Pr 17. (18), (19), (156), (135) provided (136) imply: there exist infinitely many systems of arrows (11) with (134), namely all these for which

$$(157) \qquad \boldsymbol{s} = \frac{1}{2}\sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times \cdot \boldsymbol{b}_\nu$$

provided

$$(158) \qquad \boldsymbol{a}_1 = \boldsymbol{r}_1 - \boldsymbol{r}_2, \qquad \boldsymbol{a}_2 = \boldsymbol{r}_2 - \boldsymbol{r}_3,$$

$$(159) \qquad \boldsymbol{b}_1 = \boldsymbol{m}_1 - \boldsymbol{m}_2, \qquad \boldsymbol{b}_2 = \boldsymbol{m}_2 - \boldsymbol{m}_3,$$

$$(160) \qquad \boldsymbol{a}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1,$$

$$(161) \qquad \boldsymbol{b}_3 = (\boldsymbol{b}_1 \cdot \boldsymbol{a}_2 \times \boldsymbol{a}_1)\boldsymbol{a}_1^{-1} + (\boldsymbol{b}_2 \cdot \boldsymbol{a}_2 \times \boldsymbol{a}_1)\boldsymbol{a}_2^{-1}.$$

and

$$(162) \qquad \boldsymbol{m} = \boldsymbol{m}_1 + \boldsymbol{r}_1 \times \boldsymbol{s}.$$

Dm. Since (134) is equivalent with (137), a solution $\boldsymbol{s}$, $\boldsymbol{m}$ of this system of vector equations at the hypothesis (135) provided (136) must be found. The system (137) implies

$$(163) \qquad \boldsymbol{s} \times \boldsymbol{a}_\nu = \boldsymbol{b}_\nu \qquad (\nu = 1, 2)$$

provided (158), (159). As it is well-known [**2**, 1 Pr 28], necessary conditions for the consistency of (163) are

$$(164) \qquad \boldsymbol{a}_\mu \boldsymbol{b}_\nu + \boldsymbol{a}_\nu \boldsymbol{b}_\mu = 0 \qquad (\mu, \nu = 1, 2).$$

Now the definitions (158), (159) imply that (164) are equivalent to

$$(165) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_1 - \boldsymbol{m}_2) = 0, \qquad (\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_2 - \boldsymbol{m}_3) = 0$$

and

$$(166) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m}_2 - \boldsymbol{m}_3) + (\boldsymbol{r}_2 - \boldsymbol{r}_3)(\boldsymbol{m}_1 - \boldsymbol{m}_2) = 0.$$

As regards the conditions (165), they are satisfied by virtue of the hypothesis (135); as regards the condition (166), it is satisfied by virtue of (151), which is a corollary from (135), as proved in Sch 26.

On the other hand, the conditions (164) are sufficient for the consistency of (163), provided

$$(167) \qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq \boldsymbol{o}$$

[**2**, 1 Pr 31]: if (167), (164) hold, then the system (163) has exactly one solution $\boldsymbol{s}$, namely (157), provided

$$(168) \qquad \boldsymbol{a}_3 = \boldsymbol{a}_1 \times \boldsymbol{a}_2$$

and (161). Now (158) imply that (167) is equivalent to

$$(169) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times (\boldsymbol{r}_2 - \boldsymbol{r}_3) \neq \mathbf{o}$$

by virtue of (19), and that (168) is equivalent to (160).

The basis $\boldsymbol{s}$ of $\underset{\rightarrow}{s}$ being once determined in such a manner, the moment $\boldsymbol{m}$ is defined by (162). Let

$$(170) \qquad \boldsymbol{p} = \boldsymbol{m}_2 + \boldsymbol{r}_2 \times \boldsymbol{s}.$$

Now (162), (170) imply

$$(171) \qquad \boldsymbol{m} - \boldsymbol{p} = \boldsymbol{m}_1 - \boldsymbol{m}_2 + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}.$$

Since $\boldsymbol{s}$ nullifies the right-hand side of (171), the last equation implies $\boldsymbol{p} = \boldsymbol{m}$ and (170) implies

$$(172) \qquad \boldsymbol{m} = \boldsymbol{m}_2 + \boldsymbol{r}_2 \times \boldsymbol{s}.$$

Similarly, let

$$(173) \qquad \boldsymbol{q} = \boldsymbol{m}_3 + \boldsymbol{r}_3 \times \boldsymbol{s}.$$

Now (170), (173) imply

$$(174) \qquad \boldsymbol{p} - \boldsymbol{q} = \boldsymbol{m}_2 - \boldsymbol{m}_3 + (\boldsymbol{r}_2 - \boldsymbol{r}_3) \times \boldsymbol{s}.$$

Since $\boldsymbol{s}$ annulates the right-hand side of (174), the last equation implies $\boldsymbol{q} = \boldsymbol{p} = \boldsymbol{m}$, and (173) implies

$$(175) \qquad \boldsymbol{m} = \boldsymbol{m}_3 + \boldsymbol{r}_3 \times \boldsymbol{s}.$$

In such a manner, the proposition is proved.

Sch 27. Pr 17 being proved, the following problem arises: determine the rank of the systems of arrows $\underset{\rightarrow}{s}$, satisfying (137), (18) with (19) and (156) with (135) being given.

According to Pr 3, Pr 15 and Pr 17, the following relations hold

$$(176) \qquad \operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 0 & \text{iff} \quad \boldsymbol{m}_\nu = \mathbf{o} \qquad (\nu = 1, 2, 3), \\ 1 & \text{iff} \quad \boldsymbol{m}_1 = \boldsymbol{m}_2 = \boldsymbol{m}_3 \neq \mathbf{o}. \end{cases}$$

as regards the other cases, they demand a more detailed discussion.

According to Pr 4, Pr 15 and Pr 17,

$$(177) \qquad \operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 2 \\ 3 \end{cases} \text{iff} \begin{cases} \boldsymbol{s} \neq \mathbf{o}, \quad \boldsymbol{s}\boldsymbol{m}_1 = 0, \\ \qquad \boldsymbol{s}\boldsymbol{m}_1 \neq 0. \end{cases}$$

Now (157) and the definition [2, 1(23) provided 1(24)] of the reciprocal vectors $\boldsymbol{a}_\nu^{-1}$ $(\nu = 1, 2, 3)$ imply

$$(178) \qquad 2(\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3)\boldsymbol{s} = \sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}) \times \boldsymbol{b}_\nu.$$

provided

$$(179) \qquad \boldsymbol{a}_{\nu+3} = \boldsymbol{a}_\nu \qquad (\nu = 1, 2),$$

and (178) implies (24) iff

$$(180) \qquad \sum_{\nu=1}^{3}((\boldsymbol{a}_{\nu+1}\boldsymbol{b}_\nu)\boldsymbol{a}_{\nu+2} - (\boldsymbol{a}_{\nu+2}\boldsymbol{b}_\nu)\boldsymbol{a}_{\nu+1}) \neq \mathbf{o}.$$

On the other hand, by virtue of (179) the relation (180) may be written in the form

$$(181) \qquad \sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+2}\boldsymbol{b}_{\nu+1} - \boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})\boldsymbol{a}_\nu \neq \mathbf{o}$$

provided

$$(182) \qquad \boldsymbol{b}_{\nu+3} = \boldsymbol{b}_\nu \qquad (\nu = 1, 2),$$

and (181), in view of

$$(183) \qquad \boldsymbol{a}_\mu\boldsymbol{b}_\nu + \boldsymbol{a}_\nu\boldsymbol{b}_\mu = 0 \qquad (\mu, \nu = 1, 2, 3)$$

is equivalent to

$$(184) \qquad \sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})\boldsymbol{a}_\nu \neq \mathbf{o}.$$

Because of

$$(185) \qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \neq 0,$$

the condition (184) is equivalent to

$$(186) \qquad \sum_{\nu=1}^{3}(\boldsymbol{a}_\nu\boldsymbol{b}_{\nu+1})(\boldsymbol{b}_{\nu+1}\boldsymbol{a}_\nu) \neq 0.$$

As regards the condition

$$(187) \qquad \boldsymbol{s}\boldsymbol{m}_1 = 0,$$

the above considerations display that it is equivalent to

$$(188) \qquad \sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})(\boldsymbol{a}_\nu\boldsymbol{m}_1) = 0,$$

and

(189) $\quad sm_1 \neq 0$

is equivalent to

(190) $$\sum_{\nu=1}^{3}(a_{\nu+1}b_{\nu+2})(a_\nu m_1) \neq 0.$$

Summing up, we may formulate the following proposition.

Pr 18. (18), (19), (156), (135) provided (136) imply: if (66) satisfies (134), then

(191) $$\operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \text{iff} \quad \begin{cases} m_\nu = o \qquad (\nu = 1, 2, 3), \\ m_1 = m_2 = m_3 \neq o, \\ (184), (188), \\ (190) \end{cases}$$

provided (158) — (161), (179), (182).

Sch 28. As it has been emphasized in Sch 4, although the basis s and the moment m of a system of arrows $\underset{\rightarrow}{s}$ are extremely important characteristics of $\underset{\rightarrow}{s}$ (see, for instance, the role of s and m in the rank-theorem Pr 15), they do not determine $\underset{\rightarrow}{s}$ completely, so that the notation (7) is void of sense: there are infinitely many systems of arrows that have the same basis s and the same moment m for any $(s, m) \in V_S^2$. And yet, in the applications of the theory of arrows to analytical mechanics at least, dynamics as well as statics, the role of these characteristics of the systems of arrows is, to the highest degree, a predestinating one. The reason is concealed in the fact that, both in statics and dynamics, the systems of arrows available in the mechanical problem in the capacity of forces (both active and passive) acting on mass-points and rigid bodies, are authoritative by means of their bases and moments rather than by themselves as individual mechanical entities. Putting it more specific, a system of arrows (66) is of interest to statics and dynamics inasmuch as its basis s and its moment m are concerned, rather than the particular arrows entering into its composition. Figuratively speaking, one may say that for analytical statics and analytical dynamics the definition of (66) by (11) is sufficient, and the definition of (66) by (1) provided (2) is not necessary (inasmuch as the determination of the particular reactions of the geometrical constraints imposed on the mass-points or on the rigid bodies is neglected). To put it another way, both the statical and the dynamical effects of a system of forces (arrows) $\underset{\rightarrow}{s}$ remain unaffected if $\underset{\rightarrow}{s}$ is replaced by a system of forces $\underset{\rightarrow}{\sigma}$ having the same basis and the same moment as $\underset{\rightarrow}{s}$.

The causality of all these circumstances is rooted in the fact that a system of forces acting on a mass-point or a rigid body is presented in the basis equations of analytical statics and analytical dynamics by its basis and its moment namely, and not by the particular arrow-components which this system involves. This question is discussed in some details in the following three scholiums.

Sch 29. Analytical statics and analytical dynamics are concerned with the mechanical behaviour of mass-points and rigid bodies subjected to certain *geometrical constraints* and to the action of *active forces* (wholly determined in the conditions of the statical or dynamical problem in question) and of *passive forces* or *reactions of the constraints.* More precisely, analytical statics is concerned with the *equilibrium* of the said mass-points and rigid bodies, and analytical dynamics with their *motion.*

In order to fix the ideas, let a rigid body B be given, subjected to certain geometrical constraints, restricting to l its degrees of freedom, and let

$$q_\lambda \in R \qquad (\lambda = 1, \ldots, l) \tag{192}$$

be mutually independent parameters of B. Let B be under the action of the active forces (arrows)

$$\vec{F}_\mu = (\boldsymbol{F}_\mu, \boldsymbol{M}_\mu) \qquad (\mu = 1, \ldots, m), \tag{193}$$

and let the geometrical constraints generate (by virtue of special statical or dynamical axioms) the passive forces (reactions of the constraints)

$$\vec{R}_\nu = (\boldsymbol{R}_\nu, \boldsymbol{N}_\nu) \qquad (\nu = 1, \ldots, n). \tag{194}$$

For the sake of brevity let by definition

$$\vec{F}_{m+\nu} = \vec{R}_\nu \qquad (\nu = 1, \ldots, n), \tag{195}$$

i.e.

$$\boldsymbol{F}_{m+\nu} = \boldsymbol{R}_\nu, \qquad \boldsymbol{M}_{m+\nu} = \boldsymbol{N}_\nu \qquad (\nu = 1, \ldots, n) \tag{196}$$

provided

$$\vec{F}_{m+\nu} = (\boldsymbol{F}_{m+\nu}, \boldsymbol{M}_{m+\nu}) \qquad (\nu = 1, \ldots, n), \tag{197}$$

and let

$$\boldsymbol{F} = \sum_{\nu=1}^{m+n} \boldsymbol{F}_\nu, \qquad \boldsymbol{M} = \sum_{\nu=1}^{m+n} \boldsymbol{M}_\nu. \tag{198}$$

If

$$\underset{\rightarrow}{F} = \left\{ \vec{F}_\nu \right\}_{\nu=1}^{m+n} \tag{199}$$

denotes the system of all forces (both active and passive) acting on B, then obviously $\boldsymbol{F}$ and $\boldsymbol{M}$ are the basis and the moment of $\underset{\rightarrow}{F}$ respectively.

In any statical and dynamical problem all active forces (193) are completely determined functions

$$\vec{F}_\mu = \vec{F}_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{200}$$

($\mu = 1, \ldots, m$) of the parameters (192) of B, of their velocities (derivatives with respect to the time t)

$$\dot{q}_\lambda \in R \qquad (\lambda = 1, \ldots, l), \tag{201}$$

and possibly of the time t itself. (As a matter of fact, tradionally in statics the active forces (193) do not depend explicitely on the velocities (201) and on t.) In other words, the bases and the moments of (193) are completely determined functions

$$F_\mu = F_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{202}$$

and

$$M_\mu = M_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{203}$$

($\mu = 1, \ldots, m$) respectively of (192), (201) and t.

As regards the reactions of the geometrical constraints (194), the situation is a quite different one: they are unknown quantities the determination of which is in store for statics and dynamics, in the process of the solving of the corresponding mechanical problem. Now, any of the reactions (194) is generated by a single geometrical constraint imposed on B, so that the number of the passive forces is equal to the number of these geometrical constraints, i.e. to n. Let

$$c_\nu \in V \qquad (\nu = 1, \ldots, n) \tag{204}$$

be the radius-vector of the corresponding point of contact of the rigid body B with the ν-th geometrical constraint. Then (204) are wholly determined functions

$$c_\nu = c_\nu(q_1, \ldots, q_l; t) \qquad (\nu = 1, \ldots, n) \tag{205}$$

of the parameters (192) of B and possibly of t. According to a kinetical (statical as well as dynamical) axiom, the directrix of the ν-th reaction $\vec{R}_\nu$ is incident with the point of contact c_ν of the geometrical constraint generating $\vec{R}_\nu$ ($\nu = 1, \ldots, n$), i.e.

$$c_\nu \times R_\nu = N_\nu \qquad (\nu = 1, \ldots, n). \tag{206}$$

In such a manner, it is enough to know the bases R_ν of the reactions $\vec{R}_\nu$ in order to know $\vec{R}_\nu$ themselves ($\nu = 1, \ldots, n$). In other words, these namely bases R_ν ($\nu = 1, \ldots, n$) are unknown quantities in the statical or dynamical problem under consideration, as well as the parameters (192) of B. In the case of a dynamical problem, initial values

$$q_{\lambda_0} = q_\lambda(0) \qquad (\lambda = 1, \ldots, l) \tag{207}$$

and

$$\dot{q}_{\lambda_0} = \dot{q}_\lambda(0) \qquad (\lambda = 1, \ldots, l) \tag{208}$$

are prescribed to (192) and (201) respectively, and functions

$$q_\lambda = q_\lambda(t) \qquad (\lambda = 1, \ldots, l) \tag{209}$$

of the time t are sought.

Sch 30. This information once grasped, let us proceed to the formulation of a statical problem concerning a rigid body B.

The fundamental problem of rigid body statics may be described in the following manner.

Let a rigid body B be given, subjected to n geometrical constraints with points of contact (204) which generate the reactions (194), and let B be under the action of the active forces (193). Find the equilibrium of B and the reactions of the constraints.

In this formulation the term *equilibrium* has been used. It is a fundamental statical notion ant it requires a special definition. This definition is given in the following manner.

It is said that a rigid body is *in equilibrium* under the action of a system of forces when the rank of this system iz zero.

Under the notations introduced in Sch 29, the rigid body will be in equilibrium under the action of the active forces (193) and of the reactions of the constraints (194) if, and only if, the following conditions are satisfied:

$$\boldsymbol{F} = \boldsymbol{O}, \qquad \boldsymbol{M} = \boldsymbol{O}. \tag{210}$$

In such a manner, it is seen, that the necessary and sufficient condition (210) for the equilibrium of the rigid body is expressed exclusively by the aid of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system of forces (199) acting on the rigid body. Let us, however, discuss this formulation somewhat closer.

The equations (210) may be written in the form

$$\sum_{\mu=1}^{m} \boldsymbol{F}_\mu + \sum_{\nu=1}^{n} \boldsymbol{R}_\nu = \boldsymbol{O} \tag{211}$$

and

$$\sum_{\mu=1}^{m} \boldsymbol{M}_\mu + \sum_{\nu=1}^{n} \boldsymbol{c}_\nu \times \boldsymbol{R}_\nu = \boldsymbol{O} \tag{212}$$

respectively, by virtue of (198), (196), and (206). In such a manner, the fundamental problem of rigid body statics is reduced mathematically to the vector equations (211), (212), where the unknown quantities are the parameters (192) of the rigid body B and the reactions $\boldsymbol{R}_\nu$ ($\nu = 1, \ldots, n$) of the constraints. In this general formulation the statical problem is indeterminate: as a rule, the number of the unknown quantities is much greater than the number of the equations (211), (212) (two vectors equations, equivalent to six scalar ones) available for their determination. In order to make this problem mathematically wholly determinate one, various additional hypotheses about the mechanical nature of the geometrical constraints are made, which reduce the number of the unknown quantities to six. (A classical for analytical statics hypothesis of this kind, possessing a sound physical motivation at that, consists in the assumption that the geometrical constraints are *smooth*, i.e. that they generate reactions perpendicular to the corresponding constraints.)

It is clear that the equations (211), (212) include unknown quantities of two kinds: first, the parameters (192) of B; and, second, the reactions $\boldsymbol{R}_\nu$ $(\nu = 1, \ldots, n)$.

With a view to a better understanding of the essence of the problem, let us suppose that $Oxyz$ is an orthonormal right-hand orientated Cartesian system of reference, the origin O of which coincides with the zero-pole (i.e. the pole, with respect to which the moments $\boldsymbol{M}_\mu$ and $\boldsymbol{N}_\nu$ $(\mu = 1, \ldots, m;\ \nu = 1, \ldots, n)$ of (193) and (194) respectively are taken). If $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ denote the vectores of the axes Ox, Oy, Oz respectively, then these conditions imply

(213) $$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \qquad \boldsymbol{ij} = \boldsymbol{jk} = \boldsymbol{ki} = 0$$

and

(214) $$\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}.$$

Let by definition

(215) $$\boldsymbol{F}_\mu = F_{\mu x}\boldsymbol{i} + F_{\mu y}\boldsymbol{j} + F_{\mu z}\boldsymbol{k} \qquad (\mu = 1, \ldots, m),$$

(216) $$\boldsymbol{M}_\mu = M_{\mu x}\boldsymbol{i} + M_{\mu y}\boldsymbol{j} + M_{\mu z}\boldsymbol{k} \qquad (\mu = 1, \ldots, m),$$

(217) $$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n),$$

(218) $$\boldsymbol{c}_\nu = c_{\nu x}\boldsymbol{i} + c_{\nu y}\boldsymbol{j} + c_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n).$$

At these notations the vector equation (211) is equivalent to the following three scalar equations

(219) $$\sum_{\mu=1}^{m} F_{\mu x} + \sum_{\nu=1}^{n} R_{\nu x} = 0,$$

(220) $$\sum_{\mu=1}^{m} F_{\mu y} + \sum_{\nu=1}^{n} R_{\nu y} = 0,$$

(221) $$\sum_{\mu=1}^{m} F_{\mu z} + \sum_{\nu=1}^{n} R_{\nu z} = 0,$$

and the vector equation (212) is equivalent to the following three scalar equations

(222) $$\sum_{\mu=1}^{m} M_{\mu x} + \sum_{\nu=1}^{n}(c_{\nu y}R_{\nu z} - c_{\nu z}R_{\nu y}) = 0,$$

(223) $$\sum_{\mu=1}^{m} M_{\mu y} + \sum_{\nu=1}^{n}(c_{\nu z}R_{\nu x} - c_{\nu x}R_{\nu z}) = 0,$$

(224) $$\sum_{\mu=1}^{m} M_{\mu z} + \sum_{\nu=1}^{n}(c_{\nu x}R_{\nu y} - c_{\nu y}R_{\nu x}) = 0.$$

In such a way, the statical problem is reduced to the determination, by means of the equations (219) — (224), of the equilibrium values (if any) of the parameters (192) of B and of the equilibrium reactions (if any) of the constraints

$$R_{\nu x},\ R_{\nu y},\ R_{\nu z} \qquad (\nu = 1, \ldots, n). \tag{225}$$

The expression "if any" used above means "if such exist". Now some explanations in this connection are more than indispensable.

Any statical problem consists, as a matter of fact, of three subproblems:

1. Does equilibrium exist?

2. If yes, then which is the equilibrium position of the rigid body?

3. If yes, then which are the reactions that realize it?

The first question is equivalent with the problem of existence of a solution of the statical problem. As in any mathematical problem, in a statical problem the existence of a solution is a fact that must be proved *ad hoc*, rather than a presumptive postulate.

As a matter of fact, the statical problem under consideration has a solution if, and only if, the system of equations (219) — (224) is consistent. And it is consistent if, and only if, there exist such quantities (192) and (225) that satisfy it. Tautological though these statements seem, they emphasize the importance of the existence problem in statics, which in mechanics in general not infrequent is found to be neglected.

The second question is equivalent with the problem of determination of the equilibrium values (i.e. those satisfying the system of equations (219) — (224)) of the parameters (192) of the rigid body.

The third problem is equivalent to the problem of determination of the equilibrium values (i.e. those satisfying the system of equations (219) — (224)) of the components (225) of the reactions of the geometrical constraints.

And that is that. In such, and in such only, sense the statement in Sch 28 must be understood, namely that "a system of arrows (66) is of interest to statics and dynamics, inasmuch as its basis $\boldsymbol{s}$ and its moment $\boldsymbol{m}$ are concerned, rather than the particular arrows entering into its composition": the conditions of equilibrium (210) are expressed by means of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system (199) of all forces (passive as well as active) acting on the rigid body, and not by means of the particular arrows this system contains.

Sch 31. An analogous situation is observed when dealing with a dynamical problem concerning a rigid body.

The fundamental problem of rigid body dynamics may be formulated in the following manner.

Let a rigid body B be given, subjected to n geometrical constraints with points of contact (204) which generate the reactions (194), and let B be under the action of the active forces (193). Initial conditions (207), (208) being prescribed to the parameters (192) of B and to their velocities (201), find the motion of B and the reactions of the constraints.

In this formulation the term *motion* has been used. It is a fundamental mechanical notion and it requires a special definition. Since this definition is a rather intricate one, we shall not reproduce it here and, making a long story short, we shall substitute a mathematical surrogate for it.

To this end, let us first introduce two fundamental dynamical attributes for any rigid body.

If P is an arbitrary point of the rigid body B, $\boldsymbol{r} = \boldsymbol{OP}$, $v = d\boldsymbol{r}/dt$ (the derivative being taken with respect to the system of reference $Oxyz$), and dm denotes a "mass-element" of B (all these terms being duly defined), then

$$\boldsymbol{K} = \int \boldsymbol{v}dm \tag{226}$$

and

$$\boldsymbol{L} = \int \boldsymbol{r} \times \boldsymbol{v}dm \tag{227}$$

are by definition the *momentum* and the *moment of momentum* (*kinetical moment*) of B respectively with regard to $Oxyz$, the integrals in (226) and (227) being taken over the part of the space occupied by the rigid body.

One of the greatest discoveries in all the history not only of rational mechanics, but of the whole mathematical physics as well, was made by Euler in 1775, and is reflected in his article [7]. In accordance with it, there exists one at least system of reference (*inertial system of reference*) that, all derivatives being taken with respect to it, for any rigid body B and for any system of forces $\underset{\rightarrow}{F}$ (passive as well as active) acting on B, the derivative with respect to the time t of the momentum and of the moment of momentum of B are equal to the basis and the moment of $\underset{\rightarrow}{F}$ respectively, both moments being taken with respect to the origin of $Oxyz$.

These laws or principles of Euler are called the *first* and the *second Eulerian dynamical axioms* respectively (or *Euler's laws* or *principles of momentum* and *of moment of momentum* of a rigid body respectively).

The mathematical formulation of both Eulerian dynamical axioms read

$$\frac{d}{dt}\int \boldsymbol{v}dm = \boldsymbol{F} \tag{228}$$

and

$$\frac{d}{dt}\int \boldsymbol{r} \times \boldsymbol{v}dm = \boldsymbol{M} \tag{229}$$

respectively provided (193) — (199).

These preliminaries settled, it is said that a rigid body is *moving* or is *in motion* under the action of the system of forces $\underset{\rightarrow}{F}(\boldsymbol{F}, \boldsymbol{M})$ when (228) and (229) hold good.

In such a manner, it is seen that the necessary and sufficient conditions (228), (229) for the motion of the rigid body are expressed exclusively by the aid of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system of forces (199) acting on the rigid body. Let us, however, discuss this formulation somewhat closer.

The equations (228) and (229) may be written in the form

$$\frac{d}{dt}\int v dm = \sum_{\mu=1}^{m} F_\mu + \sum_{\nu=1}^{n} R_\nu \tag{230}$$

and

$$\frac{d}{dt}\int r \times v dm = \sum_{\mu=1}^{m} M_\mu + \sum_{\nu=1}^{n} c_\nu \times R_\nu \tag{231}$$

respectively, by virtue of (198), (196), and (206). In such a manner, the fundamental problem of rigid body dynamics is reduced mathematically to the vector equations (230), (231), where the unknown quantities are the parameters (192) of the rigid body B as functions (209) of the time t and the reactions R_ν $(\nu = 1, \ldots, n)$ of the constraints. As in the statical case, in this general formulation the dynamical problem is indeterminate: as a rule, the number of the unknown quantities is much greater than the number of the equations (230), (231) (two vector equations equivalent to six scalar ones) available for their determination. In order to make this problem mathematically a whole determinate one, in dynamics, as well as in statics, various additional hypotheses about the mechanical nature of the geometrical constraints are made, which reduce the number of the unknown quantities to six. (As in statics, the most popular, traditional, and even classical hypothesis of this kind is the *postulate of smooth geometrical constraints.*)

It is clear that the equations (230), (231) include unknown quantities of two kinds: first, the parameters (192) of B, their velocities (201), and their accelerations

$$\ddot{q}_\lambda \in R \qquad (\lambda = 1, \ldots, l) \tag{232}$$

and, second, the reactions R_ν $(\nu = 1, \ldots, n)$. In other words, the equations (230), (231) are, as the saying goes, of a heterogeneous type: being differential equations of second order with respect to the time t with regard to the unknown functions (209), they are, in the same time, linear algebraic equations with regard to the unknown reactions of the costraints R_ν $(\nu = 1, \ldots, n)$.

As in the statical case, $Oxyz$ being an inertial system of reference with (213), (214), let by definition the relations (215) — (218) hold. Then the vector equation (230) is equivalent to the following three scalar equations:

$$\frac{d}{dt}\int \dot{x} dm = \sum_{\mu=1}^{m} F_{\mu x} + \sum_{\nu=1}^{n} R_{\nu x}, \tag{233}$$

$$\frac{d}{dt}\int \dot{y} dm = \sum_{\mu=1}^{m} F_{\mu y} + \sum_{\nu=1}^{n} R_{\nu y}, \tag{234}$$

$$\frac{d}{dt}\int \dot{z} dm = \sum_{\mu=1}^{m} F_{\mu z} + \sum_{\nu=1}^{n} R_{\nu z} \tag{235}$$

provided

$$r = xi + yj + zk \tag{236}$$

and the vector equation (231) is equivalent with the following three scalar equations:

$$\text{(237)} \qquad \frac{d}{dt}\int (y\dot{z} - z\dot{y})dm = \sum_{\mu=1}^{m} M_{\mu x} + \sum_{\nu=1}^{n}(c_{\nu y}R_{\nu z} - c_{\nu z}R_{\nu y}),$$

$$\text{(238)} \qquad \frac{d}{dt}\int (z\dot{x} - x\dot{z})dm = \sum_{\mu=1}^{m} M_{\mu y} + \sum_{\nu=1}^{n}(c_{\nu z}R_{\nu x} - c_{\nu x}R_{\nu z}),$$

$$\text{(239)} \qquad \frac{d}{dt}\int (x\dot{y} - y\dot{x})dm = \sum_{\mu=1}^{m} M_{\mu z} + \sum_{\nu=1}^{n}(c_{\nu x}R_{\nu y} - c_{\nu y}R_{\nu x}).$$

In such a way, the dynamical problem is reduced to the determination, by means of the equations (233) — (235) and (237) — (239) (or by the aid of other systems of differential equations, mathematically equivalent to (233) — (235) and (237) — (239), but technically much more convenient; dynamical equations, containing such characteristic for a rigid body quantities, as its moments of inertia and moments of deviation, have been proposed by Euler and are called today the *Eulerian dynamical equations*), of those functions (209) (if any) that govern the motion of rigid body and of the reactions of the constraints (225) (if any) that produce this motion (along with the active forces (193) acting on the rigid body).

The expression "if any" used above means "if such exists". As in statics, some explanations in this connection are more than unavoidable.

As a matter of fact, any dynamical problem consists of three subproblems:

1. Does motion exist?

2. If yes, then which is the motion of the rigid body?

3. If yes, then which are the reactions that realize it?

The first question is equivalent to the problem of existence of a solution of the dynamical problem. As in any mathematical problem, *in a dynamical problem the existence of a solution is not an a priori datum*, but a fact that needs an *ad hoc* demonstration. As a matter of fact, the dynamical problem under consideration has a solution if, and only if, the system of equations (233) — (235) and (237) — (239) is consistent.

This point needs a closer elucidation. Most of the authors of mechanical writings do not investigate at all the problem of existence of solutions of statical and dynamical problems. One of the reasons for such a mathematical behaviour is purely psychological. Solution of a statical or dynamical problem means, for these authors, rest or motion of the rigid body involved, and the existence of rest and motion is regarded by them as something self-apparent, self-evident, and self-explanatory.

This is not so. As Brelot [8] notes:

"On sait que lorsqu'on schématise selon l'habitude des problèmes méchaniques concrete physiquement possibles, on obtient parfois des problèmes mathématiques impossibles ... Exemples avec le frottement (Painlevé), exemple connu de la barre pesante diamétrale glissant sans frottement dans une sphère" (p. 7).

Writing "des problèmes mathèmatiques impossibles" this author has in mind mechanical problems without solutions. In this connection the reader may see, for instance, our article [9].

Another reason that explains the traditional carelessness of most authors of mechanical writings as regards the existence of dynamical solutions is rooted in the fact that, in their majority, they solve dynamical problems by the aid of the *Lagrangean dynamical equations*

$$(240)\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = 0 \qquad (\lambda = 1, \ldots, l),$$

rather than by means of the Eulerian dynamical axioms (228), (229). Now the equations (240) do not contain the unknown reactions of the costraints (194): these reactions are expelled from the mechanical paradise by the aid of the *postulate of ideal constraints*

$$(241)\qquad \sum_{\nu=1}^{n} \boldsymbol{R}_\nu d\dot{c}_\nu = 0,$$

and this act of heroism is regarded by the adherents of Lagrangean dynamical tradition as the chief achievement of their Teacher. The reactions of the constraints are, however, the mathematical factor that may deprive a dynamical problem of its solution. In such a manner, the Lagrangean dynamical equations (240), faced with the existence problem, may be compared with daltonists faced with a colour-table. Moreover, they are pointblank blind when confronted with this problem. It is needless to underline that the Lagrangeanists are as helpless in front of the third of the above questions (which are the reactions of the constraints?) as they are in front of the first of them: the only thing the Lagrangean dynamical equations (240) can do is to describe mathematically the motion *if any*.

The second question is equivalent to the solving of the equations (240) which are neither more nor less than the projections of the Eulerian dynamical axioms (228), (229) on appropriate axes: these axes are chosen in such a manner that the unknown reactions of the constraints (194) vanish when (228), (229) are projected on them. In such a way the functions (209) are determined, satisfying the initial conditions (207), (208).

The third question is equivalent to the problem of determination of the reactions (194) of the constraints. Since they take part linearly in the equations (230), (231), this determination, provided the functions (209) are known, offers no difficulties.

And that is that. In such, and in such only, sense the statement in Sch 28 must be understood, namely that "the systems of arrows available in the mechanical problem in the capacity of forces ... are authoritative by means of their bases and moments rather than by themselves as individual mechanical entities": the conditions of motion (228), (229) are expressed by means of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system (199) of all forces (passive as well as active) acting on the rigid body, and not by means of the particular arrows this system contains.

Sch 32. Analytical mechanics is sometimes described as the mathematics of the equilibria and motions of mass-points and rigid bodies, and of the forces that generate these equilibria and motions and are generated by them.

This description is clear in the light of the above explanations in connection with the fundamental problems of statics and dynamics. Not less clear is also that the theory of arrows is predestinated to play an important role in the problems of analytical mechanics.

Still the first generations of mechanicians have become well aware of this fact. Not a few experimental and intellectual work has been dedicated to the doctrine of forces, aiming at the elucidation of various aspects of the force-concept from all angles. In order to realize that it is sufficient to remember the amount of painstaking work accomplished by mathematicians and physicists in connection with the problem of the *parallelogram of forces*.

This problem is a most remarkable one. Let B be a *physical* rigid body and let P_1 and P_2 be its points. Let Q_1 and Q_2 be small fixed pulleys and let w_1 and w_2 be weights fastened to the ends of two inextensible flexible strings, the other ends of which are fastened to P_1 and P_2 respectively, and which are passed through Q_1 and Q_2 respectively. If B is in equilibrium under the action of its weight w_3 and of the weights w_1 and w_2, then Q_1, Q_2, and the mass-center G of B lie in a vertical plane, as a simple experiment displays.

Let $\boldsymbol{e}_\nu$ be the unit vectors of the vectors $\boldsymbol{P_\nu Q_\nu}$ $(\nu = 1, 2)$ respectively and let $\boldsymbol{e}_3$ be the unit vector of the downwards directed vertical. Let by definition $\boldsymbol{s}_\nu = w_\nu \boldsymbol{e}_\nu$ $(\nu = 1, 2, 3)$. If O is any fixed point, chosen for the zero-pole, let by definition $\boldsymbol{m}_\nu = \boldsymbol{OP}_\nu \times \boldsymbol{s}_\nu$ $(\nu = 1, 2)$, $\boldsymbol{m}_3 = \boldsymbol{OG} \times \boldsymbol{s}_3$. Then obviously

$$\overrightarrow{s}_\nu = (\boldsymbol{s}_\nu, \boldsymbol{m}_\nu) \in W_S \qquad (\nu = 1, 2, 3), \tag{242}$$

and the rigid body B is in equilibrium under the action of the forces (242) iff

$$\sum_{\nu=1}^{3} \boldsymbol{s}_\nu = \boldsymbol{o}, \qquad \sum_{\nu=1}^{3} \boldsymbol{m}_\nu = \boldsymbol{o}. \tag{243}$$

Only a genius could guess that the action of the forces $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$ could be replaced, without disturbing the equilibrium of B, by a single force $\overrightarrow{s}$, the basis $\boldsymbol{s}$ of which is equal to the sum $\boldsymbol{s}_1 + \boldsymbol{s}_2$ of the bases of $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$, and the directrix of which is passing through G. The name of this genius is unknown. It is also an enigma where, when, and who was the first to confirm this brilliant conjecture by an experiment fated to become classical even in the primary course in physics today. It is easily seen that

$$\overrightarrow{s} = (\boldsymbol{s}_1 + \boldsymbol{s}_2, \boldsymbol{m}_1 + \boldsymbol{m}_2), \tag{244}$$

i.e. that

$$\overrightarrow{s} = \overrightarrow{s}_1 + \overrightarrow{s}_2. \tag{245}$$

This story is instructive to the highest degree.

In the first place, here lies the germ of the idea of *reduction of a system of forces.*

In the second place, here lies the germ of the notion of *equivalent systems of forces.*

In the third place, here lies the germ of the concept of *sum of two forces.*

In the fourth place, here lies the germ of the idea of *elementary statical operations.*

In the fifth place, this proces has provoked the mathematicians to solve the problem of the *parallelogram of forces* and to initiate, in such a manner, a scientific politics in mechanics aimed at its mathematification — a tendency half-finished as yet.

For what reason did the ancient mechanician invent the intricate mechanism with the pulleys described above? The answer is obvious: simplicity. Two arrows are something more complex than a single one. Now this simple discovery may be generalized. Anyone knows from his school-days how this device is accomodated to the addition of two arrows with parallel directrices, etc. The common of all these processes is the simplification of a system of arrows by diminishing the number of its elements.

It must not be left unnoticed that replacing two arrows by their sum in a system of arrows $\underset{\rightarrow}{s}$ one does not change the basis and the moment of $\underset{\rightarrow}{s}$. In the due course of time the mechanicians became concious of the fact that there exist four elementary operations, by means of which any system of forces may be transformed into the simplest possible form; at that it preserves unchanged its basis and its moment. This mathematical process has been called the reduction of the system of forces in question.

All these facts, initially established in the statical case, have been inductively transferred from it to the dynamical one. The fact, that a reduced system of forces has the same, dynamical as well as statical, effect as the starting system, has been realized long before the definition (210) of equilibrium has been formulated in its complete generality, and out and away earlier than the Eulerian dynamical axioms (228), (229) have been proclaimed.

After these explications the aim of which is to give a physical motivation and a heuristic background for the following considerations, we shall proceed now to the mathematical formalization of the circumstances described,

Sgn 9. $\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_2}$ sgn:

(246) $$\boldsymbol{s}_1 = \boldsymbol{s}_2, \qquad \boldsymbol{m}_1 = \boldsymbol{m}_2$$

iff

(247) $$\underset{\rightarrow}{s_\nu}(\boldsymbol{s}_\nu, \boldsymbol{m}_\nu) \in \Sigma_S \qquad (\nu = 1, 2).$$

Df 12. $\underset{\rightarrow}{s_1}$ is called equivalent to $\underset{\rightarrow}{s_2}$ if

(248) $$\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_2}.$$

Pr 19. (66) implies $\underset{\rightarrow}{s} \sim \underset{\rightarrow}{s}$.
Dm. Sgn 9.
Pr 20. If

(249) $$\underset{\rightarrow}{s_\nu} \in \Sigma_S \qquad (\nu = 1, 2),$$

(248), then $\underset{\rightarrow}{s_2} \sim \underset{\rightarrow}{s_1}$.
Dm. Sgn 9.

Pr 21. If

$$\underrightarrow{s_\nu} \in \Sigma_S \qquad (\nu = 1, 2, 3), \tag{250}$$

(248), $\underrightarrow{s_2} \sim \underrightarrow{s_3}$, then $\underrightarrow{s_1} \sim \underrightarrow{s_3}$.

Dm. Sgn 9.

Pr 22. The relation $\sim$ in Σ_S defined by Sgn 9 is an equivalence relation in Σ_S.

Dm. Pr 19 — Pr 21.

Df 13. Any equivalence class in Σ_S, generated by the relation $\sim$ in Σ_S, is called an S-action or an action in V_S.

Sgn 10. A_S sgn: the set of all actions in V_S.

Sch 33. In the light of the explanation given in Sch 29 — Sch 31 one could say that namely the actions in V, alias the elements of A_R, rather than the particular systems of arrows in V, alias the elements of Σ_R, are that determine the mechanical (dynamical as well as statical) behaviour of a rigid body in analytical mechanics, the term behaviour meaning equilibrium or motion.

Sch 34. $\underrightarrow{s}$ being a system of arrows, the qualification of $\underrightarrow{s}$ as simple or as complicated is deprived of the possibilities of an objective judgement, save, maybe, by means of the number of its elements. And yet, immediately below four types of systems of arrows are proposed, any of them with its particular designation, in the capacity of the simplest possible kinds of systems.

Sgn 11. $\underrightarrow{o}$ sgn: $\{\vec{o}\}$.

Df 14. $\underrightarrow{o}$ is called the *zero-system.*

Df 15. $\{\vec{s}\}$ is called a *monosystem* iff $\vec{o} \neq \vec{s} \in W_S$.

Df 16. $\{\vec{s}_\nu\}_{\nu=1}^2$ is a called a *dipole* iff

$$\vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2), \tag{251}$$

$$s_1 + s_2 = o, \qquad m_1 + m_2 \neq o. \tag{252}$$

Df 17. $\{\vec{s}_\nu\}_{\nu=1}^2$ is called a *bisystem* iff (251),

$$s_1 \times s_2 \neq o, \qquad s_1 m_2 + s_2 m_1 \neq 0. \tag{253}$$

Sgn 12. $\Sigma_\nu (\nu = 0, 1, 2, 3)$ sgn: $\underrightarrow{o}$ iff $\nu = 0$; the set of all dipoles iff $\nu = 1$; the set of all monosystems iff $\nu = 2$; the set of all bisystems iff $\nu = 3$.

Sgn 13. Σ sgn: the union of $\Sigma_\nu (\nu = 0, 1, 2, 3)$.

Pr 23. $\{\vec{s}_\nu\}_{\nu=1}^2 \in \Sigma_1$ implies $\vec{s}_1 \uparrow\downarrow \vec{s}_2$.

Dm. Sgn 12, Df 16, [2, 3 Sgn 7].

Pr 24. $\{\vec{s}_\nu\}_{\nu=1}^2 \in \Sigma_3$ implies $\vec{s}_1 \otimes \vec{s}_2$.

Dm. Sgn 12, Df 17 [2, 5 Sgn 3].

Pr 25. (66) implies: there exists $\underrightarrow{\sigma} \in \Sigma$ with

$$\underrightarrow{s} \sim \underrightarrow{\sigma}. \tag{254}$$

Dm. If (11), then one exactly of the cases (86), (88), (90), and (92) is possible.

If (86), then $\underrightarrow{\sigma} = \underrightarrow{0}$ satisfies (254) (Sgn 11, [2, 2 Sgn 2], Sgn 9, Sgn 12, Sgn 13).

If (88), let $\bar{\rho}, \bar{\sigma} \in V_S$, $\bar{\rho} \times \bar{\sigma} = m$, $\vec{s}_1 = (-\bar{\sigma}, o)$, $\vec{s}_2 = (\bar{\sigma}, m)$. Then $m \neq o$ implies $\bar{\sigma} \neq o$. Besides, obviously $\bar{\sigma} m = 0$. Hence

$$(255) \qquad \vec{s}_\nu \in W_S \qquad (\nu = 1, 2)$$

(Sgn 1). If

$$(256) \qquad \underset{\rightarrow}{\sigma} = \{\vec{s}_\nu\}_{\nu=1}^{2},$$

then $\underset{\rightarrow}{\sigma} \in \Sigma_1$ (Df 16, Sgn 12) and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sgn 9, Sgn 13).

If (90), let by definition (9) hold (Sgn 1). If $\underset{\rightarrow}{\sigma} = \{\vec{s}\}$, then $\underset{\rightarrow}{\sigma} \in \Sigma_2$ (Df 15, Sgn 12) and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sgn 9, Sgn 13).

If (92), let $s_1 \in V_S$,

$$(257) \qquad s_1 \times s \neq o, \qquad s_1 m = s m$$

and let by definition

$$(258) \qquad \vec{s}_\nu = (s_\nu, m_\nu) \qquad (\nu = 1, 2)$$

provided

$$(259) \qquad m_1 = o, \qquad s_2 = s - s_1, \qquad m_2 = m.$$

Then (257) — (259) imply (255) (Sgn 1). If (256), then $\underset{\rightarrow}{\sigma} \in \Sigma_3$ (Df 17, Sgn 12), since (257), (259), (92) imply

$$(260) \qquad s_1 \times s_2 = s_1 \times s \neq o, \qquad s_1 m_2 + s_2 m_1 = s_1 m \neq 0,$$

and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sign 9, Sgn 13).

The following proposition is an immediate corollary from Pr 25 and from the rank-theorem.

Pr 26. (66) implies: a necessary and sufficient condition for the existence of a

$$(261) \qquad \underset{\rightarrow}{\sigma} \in \Sigma_\nu \qquad (\nu = 0, 1, 2, 3)$$

with (254) is

$$(262) \qquad \text{rank}\, \underset{\rightarrow}{s} = \nu \qquad (\nu = 0, 1, 2, 3)$$

respectively.

Sch 35. Naturally, the ancient mechanicians worked, technically at least, in a quite different way. Not disposing with an algebraic definition of the arrow concept and with all the technical facilitations this definition proposes, they were compelled to work synthetic-geometrically and invented, with an eye on the reduction of the systems of forces, the so-called *elementary statical operations.* We give here a non-formal description of these operations.

Let $\underset{\rightarrow}{s}$ be a system of forces and let two of its elements $\vec{s}_\nu \in \underset{\rightarrow}{s} (\nu = 1, 2)$ possess a sum $\vec{s}_1 + \vec{s}_2$. Let $\underset{\rightarrow}{\sigma}$ be a system of forces, defined in the following manner: $\underset{\rightarrow}{\sigma}$ consists of all elements of $\underset{\rightarrow}{s}$ except for $\vec{s}_1$ and $\vec{s}_2$; besides, $\vec{s}_1 + \vec{s}_2 \in \underset{\rightarrow}{\sigma}$. Then it is said that $\underset{\rightarrow}{\sigma}$ is obtained from $\underset{\rightarrow}{s}$ by means of the *first elementary statical operation*, and the transition from $\underset{\rightarrow}{s}$ to $\underset{\rightarrow}{\sigma}$ or, alias, the substitution of

$\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *first elementary statical operation.*

Let $\underrightarrow{s}$ be a system of forces and let one of its elements $\vec{s} \in \underrightarrow{s}$ be decomposed in two addends $\vec{s}_\nu(\nu = 1, 2)$, i.e. $\vec{s} = \vec{s}_1 + \vec{s}_2$. Let $\underrightarrow{\sigma}$ be a system of forces defined in the following manner: $\underrightarrow{\sigma}$ consists of all elements of $\underrightarrow{s}$ except for $\vec{s}$; besides, $\vec{s}_\nu \in \underrightarrow{\sigma}(\nu = 1, 2)$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *second elementary statical operation*, and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *second elementary statical operation.*

Let $\underrightarrow{s}$ be a system of forces including the zero-arrow, i.e. $\vec{o} \in \underrightarrow{s}$. Let $\underrightarrow{\sigma}$ be a system of forces defined in the following manner: $\underrightarrow{\sigma}$ consists of all elements of $\underrightarrow{s}$ except for $\vec{o}$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *third elementary statical operation*, and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *third elementary statical operation.*

Let $\underrightarrow{s}$ be any system of forces and let the system of forces $\underrightarrow{\sigma}$ be defined in the following manner: $\underrightarrow{\sigma}$ consists of all the elements of $\underrightarrow{s}$ and, moreover, of the zero-arrow $\vec{o}$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *fourth elementary statical operation* and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *fourth elemehtary statical operation.*

It is trivially seen that, if $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of any of the described four elementary statical operations, then $\underrightarrow{s} \sim \underrightarrow{\sigma}$. In other words, *any of the four elementary statical operations preserves the basis and the moment of the initial system of forces.*

It is self-evident that the desultory application of the four elementary statical operations on a particular system of forces $\underrightarrow{s}$ leads to a dead-end. It turn out, however, that by the aid of purposeful applications of these operations on any system of forces $\underrightarrow{s}$ the latter may be reduced to a zero-system, or to a dipole, or to a mono-system, or, at last, to a bi-system, in other words, to some of the systems of $\Sigma_\nu(0 \leqq \nu \leqq 3)$ [**10**, p. 115—122]. This fact reveals the meaning of the term reduction of a system of arrows.

A system of arrows $\underrightarrow{s}$ being given, the process of discovering, by means of the elementary statical operations, a system $\underrightarrow{\sigma} \in \Sigma_\nu(0 \leqq \nu \leqq 3)$ with $\underrightarrow{s} \sim \underrightarrow{\sigma}$, is called a *reduction* of $\underrightarrow{s}$.

N. B. 1. This is obviously a non-mathematical description of the term reduction.

N. B. 2. In the above description it is said "a reduction" and not "the reduction". The reason for that is the fact that a reduction of a system of arrows $\underrightarrow{s}$ is not an univocally determined mathematical phenomenon: if (11) and $\boldsymbol{s} = \mathbf{o}$, $\boldsymbol{m} \neq \mathbf{o}$, then there exist two at least different $\underrightarrow{\sigma_\nu} \in \Sigma_1$ with $\underrightarrow{s} \sim \underrightarrow{\sigma_\nu}(\nu = 1, 2)$. Similarly, if (11) and $\boldsymbol{sm} \neq 0$, then there exist two at least different $\underrightarrow{\sigma_\nu} \in \Sigma_3$ with

$\underset{\rightarrow}{s} \sim \underset{\rightarrow}{\sigma_\nu} (\nu = 1, 2)$.

Sch 36. The indefiniteness of the simplest dipoles or bisystems equivalent with a given system of arrows $\underset{\rightarrow}{s}$ in the cases rank $\underset{\rightarrow}{s} = 1$ and rank $\underset{\rightarrow}{s} = 3$ respectively leads to the formulation of various problems aimed at the clarification of the nature and degree of this indefiniteness. Some of these problems are discussed immediately below.

Sch 37. The following question quite naturally arises. A system of arrows (66) with (91), as well as (57), being given, do there exist (255) with

$$\text{(263)} \qquad \boldsymbol{r}_\nu \ \mathrm{Z} \ \mathrm{dis}\, \overrightarrow{s}_\nu \qquad (\nu = 1, 2)$$

and

$$\text{(264)} \qquad \underset{\rightarrow}{s} \sim \{\overrightarrow{s}_\nu\}_{\nu=1}^{2}.$$

Pr 26 implies that a necessary and sufficient condition for (264) is

$$\text{(265)} \qquad \{\overrightarrow{s}_\nu\}_{\nu=1}^{2} \in \Sigma_3,$$

and Sgn 12 implies that (265) is equivalent with (253) provided (258). In other words, the problem is equivalent with the following one.

If (57) and

$$\text{(266)} \qquad (\boldsymbol{s}, \boldsymbol{m}) \in V_S^2,$$

(92), then do there exist

$$\text{(267)} \qquad (\boldsymbol{s}_\nu, \boldsymbol{m}_\nu) \in V_S^2 \qquad (\nu = 1, 2)$$

with (253),

$$\text{(268)} \qquad \boldsymbol{s}_\nu \boldsymbol{m}_\nu = 0 \qquad (\nu = 1, 2),$$

$$\text{(269)} \qquad \boldsymbol{s}_1 + \boldsymbol{s}_2 = \boldsymbol{s}, \qquad \boldsymbol{m}_1 + \boldsymbol{m}_2 = \boldsymbol{m},$$

$$\text{(270)} \qquad \boldsymbol{r}_\nu \times \boldsymbol{s}_\nu = \boldsymbol{m}_\nu \qquad (\nu = 1, 2).$$

In order to answer this question, let us first note that necessarily (68) must hold. Indeed, the supposition $\boldsymbol{r}_1 = \boldsymbol{r}_2 = \boldsymbol{r}$, along with (270), implies

$$\text{(271)} \qquad \boldsymbol{r} \times \boldsymbol{s}_\nu = \boldsymbol{m}_\nu \qquad (\nu = 1, 2),$$

contrary to the second relation (253) [**2**, 1 Pr 28].

Let us suppose that the above question is answered in the affirmative, and let by definition

$$\begin{aligned} &\text{(272)} \qquad & \boldsymbol{m}_1 &= \boldsymbol{r}_1 \times \boldsymbol{s}_1, \\ &\text{(273)} \qquad & \boldsymbol{s}_2 &= \boldsymbol{s} - \boldsymbol{s}_1, \\ &\text{(274)} \qquad & \boldsymbol{m}_2 &= \boldsymbol{r}_2 \times (\boldsymbol{s} - \boldsymbol{s}_1). \end{aligned}$$

In other words, the left-hand sides of (272) — (274) are certainly known if $\boldsymbol{s}_1$ is known, and the problem is reduced to the determination of $\boldsymbol{s}_1$ namely.

Such a $\boldsymbol{s}_1$ does not necessarily exist. Indeed, (272), (274), and the second relation (269) imply

(275) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}_1 = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2,$$

and (275) implies that the condition

(276) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) = 0$$

is necessary for the affirmative answer of the above question. Let us discuss this problem somewhat closer.

Sgn 4 implies that (275) is equivalent with

(277) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}_1 = \mathrm{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s},$$

and (276) is equivalent to

(278) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s} = 0.$$

On the other hand , (278) and (67) imply

(279) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} = 0,$$

and (279), Sgn 4 imply

(280) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1) = 0.$$

The relations (276) and (280) are, however, not mutually independent. Indeed, the supposition

(281) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1) \neq 0$$

leads, along with (275), to the contradiction

(282) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \neq 0.$$

The conditions (68) and (276) display that the equation (275) with respect to $\boldsymbol{s}_1$ is consistent. As it is well known, (275) implies

(283) $$\boldsymbol{s}_1 = \lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \frac{(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} \qquad (\lambda \in S),$$

and (283) implies

(284) $$\boldsymbol{s}_1 \times \boldsymbol{s} = \overline{\lambda}((\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}) + \frac{(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} \times \boldsymbol{s}.$$

Two cases are possible:

(285) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s} = \mathbf{o}.$$

or

(286) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s} \neq \mathbf{o}.$$

If (285), then (68), (92) imply

(287) $$\boldsymbol{s} = \mu(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (0 \neq \mu \in S),$$

and (284), (285), (287), (276) imply

$$(288) \qquad s_1 \times s = \overline{\mu}(m + s \times r_2).$$

The supposition

$$(289) \qquad m + s \times r_2 = o$$

is wrong, since it implies $sm = 0$ contrary to (92). In such a manner, if (185), then (287), (288) imply

$$(290) \qquad s_1 \times s \neq o.$$

If (286), then (284) and

$$(291) \qquad s_1 \times s = o$$

imply

$$(292) \qquad \lambda((r_1 - r_2) \times s)^2 = (r_1 - r_2) \times s \cdot s \times \frac{(m + s \times r_2) \times (r_1 - r_2)}{(r_1 - r_2)^2}.$$

Let by definition

$$(293) \qquad A = (r_1 - r_2) \times s \cdot s \times ((m + s \times r_2) \times (r_1 - r_2)).$$

Then

$$(294) \qquad \begin{aligned} A &= (s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s) \\ &- ((m + s \times r_2) \times (r_1 - r_2) \cdot (r_1 - r_2))s^2, \end{aligned}$$

i.e.

$$(295) \qquad A = (s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)$$

and (292), (293), (295) imply

$$(296) \qquad \lambda = \frac{(s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)}{(r_1 - r_2)^2((r_1 - r_2) \times s)^2}.$$

In other words, (291) is equivalent to (296).

In such a manner, (290) holds good for any $\lambda \in S$ if (285) and for any

$$(297) \qquad \lambda \neq \frac{(s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)}{(r_1 - r_2)^2((r_1 - r_2) \times s)^2}$$

if (286).

It will be now proved that (283) with any $\lambda \in S$ if (285) and with any $\lambda \in S$ satisfying (297) if (286), along with (272) — (274), propose a solution of the problem under consideration.

Indeed, (273) implies the first relation (269). On the other hand, (272) and (274) imply

$$(298) \qquad m_1 + m_2 = (r_1 - r_2) \times s + r_2 \times s.$$

Now (298) and (275) imply the second relation (269).

Besides, (272) and (274), (273) imply (270).

As regards the first relation (253), the definition (273) implies that it is equivalent to (290) and, by hypothesis, is satisfied for any $\lambda \in S$ in the case (285) and for any $\lambda \in S$ satisfying (287) in the case (286).

At last, the left-hand side of the relation (253) is equal to

$$(299) \qquad \boldsymbol{s}_1 \cdot \boldsymbol{r}_2 \times (\boldsymbol{s} - \boldsymbol{s}_1) + (\boldsymbol{s} - \boldsymbol{s}_1) \cdot \boldsymbol{r}_1 \times \boldsymbol{s}_1 = \boldsymbol{s}_1 \cdot \boldsymbol{r}_2 \times \boldsymbol{s} + \boldsymbol{s} \cdot \boldsymbol{r}_1 \times \boldsymbol{s}_1 = \boldsymbol{s}_1 \cdot \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2).$$

On the other hand, (283) implies

$$(300) \qquad \boldsymbol{s}_1 \cdot \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) = \frac{(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} = \boldsymbol{s}\boldsymbol{m} \neq 0$$

by virtue of (276) and (92), i.e. the second relation (253) is satisfied for any $\lambda \in S$. *Q. e. d.*

There do not exist solutions of the problem under consideration other than the ones described above, since the latter are consistent with all the necessary conditions for the existence of a solution, being their corollaries, as a matter of fact.

In such a way, the following proposition has been proved.

Pr 27. If $\boldsymbol{r}_\nu \in V_S (\nu = 1, 2)$ and $\underset{\rightarrow}{s}(\boldsymbol{s}, \boldsymbol{m})$ is a given system of arrows with rank $\underset{\rightarrow}{s} = 3$, then (68) and (276) are necessary conditions for the existence of

$$(301) \qquad \vec{s}_\nu = (\boldsymbol{s}_\nu, \boldsymbol{m}_\nu) \in W_S \qquad (\nu = 1, 2)$$

satisfying (263) and (264). These conditions are sufficient at the same time: if they are satisfied, then (301) defined by (283) with an arbitrary $\lambda \in S$ in the case (285) or with any $\lambda \in S$ satisfying (297) in the case (286) and by (272) — (274) are the only arrows satisfying (263) and (264).

Sch 38. The second problem may by formulated in the following manner. A system of arrows (66) with (91) being given, let (42) be any point and l with

$$(302) \qquad (\boldsymbol{a}, \boldsymbol{b}) \& l \qquad (\boldsymbol{a} \neq \mathbf{o}, \ \boldsymbol{ab} = 0)$$

a line. The question now arises whether there exist arrows (255) with (264) and

$$(303) \qquad \bar{p} \ \mathrm{Z} \ \mathrm{dir} \ \vec{s}_1,$$

$$(304) \qquad l = \mathrm{dir} \ \vec{s}_2.$$

The relations (303), (304) are equvalent to

$$(305) \qquad \bar{p} \times \boldsymbol{s}_1 = \boldsymbol{m}_1,$$

$$(306) \qquad \boldsymbol{s}_2 = \lambda \boldsymbol{a}, \qquad \boldsymbol{m}_2 = \bar{\lambda} \boldsymbol{b} \qquad (\lambda \in S)$$

respectively, by virtue of (302); and (264) is equivalent to (269). Now (305), (306), (269) imply

$$(307) \qquad \boldsymbol{s}_1 + \lambda \boldsymbol{a} = \boldsymbol{s}, \qquad \bar{p} \times \boldsymbol{s}_1 + \bar{\lambda} \boldsymbol{b} = \boldsymbol{m},$$

and (307) imply

$$(308)\qquad \bar{p}\times \boldsymbol{s}-\bar{\lambda}(\bar{p}\times \boldsymbol{a})=\boldsymbol{m}-\bar{\lambda}\boldsymbol{b}.$$

Then (308), (302) imply

$$(309)\qquad (\bar{p}\times \boldsymbol{s}-\boldsymbol{m})\boldsymbol{a}=0.$$

In other words, (309) is a necessary condition for the affirmative answer of the problem under consideration.

Another necessary condition is that

$$(310)\qquad \bar{p}\times \boldsymbol{s}\neq \boldsymbol{m}.$$

Indeed,

$$(311)\qquad \bar{p}\times \boldsymbol{s}=\boldsymbol{m}$$

and Sgn 4 imply

$$(312)\qquad \mathrm{mom}_{\bar{p}}\underset{\rightarrow}{s}=\mathrm{o}.$$

Now (312) and Pr 4 imply $\boldsymbol{sm}=0$ contrary to (92) (Pr 15).

A third necessary condition for the affirmative answer of the question under consideration reads

$$(313)\qquad \bar{p}\times \boldsymbol{a}-\boldsymbol{b}\neq \mathrm{o}.$$

Indeed, otherwise (302), (304) imply

$$(314)\qquad \bar{p}\ \mathrm{Z}\ \mathrm{dir}\ \vec{s}_2,$$

i.e.

$$(315)\qquad \bar{p}\times \boldsymbol{s}_2=\boldsymbol{m}_2.$$

Now (305) and (315) contradict (253).

Let now (309), (310), and (313) be satisfied. The relations (307) imply

$$(316)\qquad \bar{p}\times(\boldsymbol{s}-\lambda\boldsymbol{a})+\bar{\lambda}\boldsymbol{b}=\boldsymbol{m},$$

i.e.

$$(317)\qquad \bar{\lambda}(\boldsymbol{b}-\bar{p}\times\boldsymbol{a})=\boldsymbol{m}+\boldsymbol{s}\times\bar{p},$$

and (317) implies

$$(318)\qquad \lambda\,(\boldsymbol{s}(\boldsymbol{b}-\bar{p}\times\boldsymbol{a}))=\boldsymbol{sm}.$$

The right-hand side of (318) is different from zero by virtue of (92). Therefore necessarily

$$(319)\qquad (\bar{p}\times\boldsymbol{a}-\boldsymbol{b})\boldsymbol{s}\neq 0,$$

and (318), (319) imply

$$\lambda = \frac{sm}{s(a \times \overline{p} + b)}. \tag{320}$$

Let us note that the necessary condition (319) is obviously stronger than the condition (313). Now (320) and the first relation (307) imply

$$s_1 = s - \left(\frac{sm}{s(a \times \overline{p} + b)}\right) a. \tag{321}$$

In such a manner, all unknown quantities are determined, provided the conditions of the problem are consistent. Now it remains to be proved that these quantities propose a solution of the problem.

First of all, (321) and the first relation (306) with (320) imply the first relation (269).

In the second place, (305) and the second relation (306) with (320), along with (308) and (321) imply

$$\begin{aligned} m_1 + m_2 &= \overline{p} \times s_1 + \overline{\lambda} b = \overline{p} \times s - \overline{\lambda}(\overline{p} \times a) + \overline{\lambda} b \\ &= m - \overline{\lambda} b + \overline{\lambda}(\overline{p} \times a) - \overline{\lambda}(\overline{p} \times a) + \overline{\lambda} b = m, \end{aligned} \tag{322}$$

i.e. the second relation (269).

In the third place, (321) with (320) and the first relation (306) imply

$$s_1 \times s_2 = s \times (\lambda a) = \overline{\lambda}(s \times a). \tag{323}$$

Now (320) and (92) imply $\overline{\lambda} \neq 0$. In other words, according to (323), the first relation (253) is equivalent to

$$s \times a \neq o, \tag{324}$$

and it is clear that the necessary condition (324) must be also hypothesized.

At last, (321) with (320) and (305), (306), (302) imply

$$\begin{aligned} s_1 m_2 + s_2 m_1 &= (s - \lambda a)(\lambda b) + (\lambda a)(\overline{p} \times (s - \lambda a)) \\ &= \lambda(sb) + \lambda(a \cdot \overline{p} \times s) = \lambda(sb + s \cdot a \times \overline{p}) = \lambda(s(b + a \times \overline{p})) \neq 0 \end{aligned} \tag{325}$$

by virtue of $\lambda \neq 0$ and (319).

In such a way, the following proposition is proved.

Pr 28. If $\underset{\rightarrow}{s}(s, m)$ is a given system of arrows with rank $\underset{\rightarrow}{s} = 3$, $\overline{p} \in V_S$ and the line l is defined by means of (302), then (309), (310), (319), and (324) are necessary conditions for the existence of arrows (301) with (264) and (303), (304). These conditions are sufficient too: if they are satisfied, then the only solution is proposed by (321) and (305), (306) with (320).

Sch 39. The last problem of this kind we shall now discuss is the following one. Let a system of arrows (11) with (91) be given and let

$$(a_\nu, b_\nu) \ \& \ l_\nu \qquad (\nu = 1, 2) \tag{326}$$

be two lines, i.e.

$$(327)\qquad \boldsymbol{a}_\nu \neq \mathbf{o}, \qquad \boldsymbol{a}_\nu \boldsymbol{b}_\nu = 0 \qquad (\nu = 1, 2).$$

Now we ask: do there exist two arrows (301) with (264) and

$$(328)\qquad \operatorname{dir} \overrightarrow{s}_\nu = l_\nu \qquad (\nu = 1, 2).$$

By virtue of (326), the relations (328) are equivalent to

$$(329)\qquad \boldsymbol{s}_\nu = \lambda_\nu \boldsymbol{a}_\nu, \qquad \boldsymbol{m}_\nu = \overline{\lambda}_\nu \boldsymbol{b}_\nu \qquad (\nu = 1, 2)$$

with appropriate

$$(330)\qquad \lambda_\nu \in S \qquad (\nu = 1, 2).$$

In view of (329), the relation (264) is equivalent to

$$(331)\qquad \lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 = \boldsymbol{s}, \qquad \overline{\lambda}_1 \boldsymbol{b}_1 + \overline{\lambda}_2 \boldsymbol{b}_2 = \boldsymbol{m}.$$

Pr 26, Sgn 12, Df 17, and (264) imply (253). Hence (329) imply

$$(332)\qquad \lambda_\nu \neq 0 \qquad (\nu = 1, 2),$$

$$(333)\qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq \mathbf{o}, \qquad \boldsymbol{a}_1 \boldsymbol{b}_2 + \boldsymbol{a}_2 \boldsymbol{b}_1 \neq 0.$$

The first relation (331) implies

$$(334)\qquad \lambda_1 \boldsymbol{a}_1^2 + \lambda_2 (\boldsymbol{a}_2 \boldsymbol{a}_1) = \boldsymbol{s}\boldsymbol{a}_1, \qquad \lambda_1 (\boldsymbol{a}_1 \boldsymbol{a}_2) + \lambda_2 \boldsymbol{a}_2^2 = \boldsymbol{s}\boldsymbol{a}_2,$$

and the first relation (333) implies

$$(335)\qquad \boldsymbol{a}_1^2 \boldsymbol{a}_2^2 - (\boldsymbol{a}_1 \boldsymbol{a}_2)(\boldsymbol{a}_2 \boldsymbol{a}_1) \neq 0.$$

Now (334), (335) imply

$$(336)\qquad \lambda_1 = \frac{\begin{vmatrix} \boldsymbol{s}\boldsymbol{a}_1 & \boldsymbol{a}_2 \boldsymbol{a}_1 \\ \boldsymbol{s}\boldsymbol{a}_2 & \boldsymbol{a}_2^2 \end{vmatrix}}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}, \qquad \lambda_2 = \frac{\begin{vmatrix} \boldsymbol{a}_1^2 & \boldsymbol{s}\boldsymbol{a}_1 \\ \boldsymbol{a}_1 \boldsymbol{a}_2 & \boldsymbol{s}\boldsymbol{a}_2 \end{vmatrix}}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}.$$

On the other hand, (331) implies that the condition

$$(337)\qquad \boldsymbol{s} \cdot \boldsymbol{a}_1 \times \boldsymbol{a}_2 = 0$$

is necessary for the consistency of the problem under consideration. Now (337) and the first relation (333) imply

$$(338)\qquad \boldsymbol{s} = (\boldsymbol{s}\boldsymbol{a}_1^{-1})\boldsymbol{a}_1 + (\boldsymbol{s}\boldsymbol{a}_2^{-1})\boldsymbol{a}_2,$$

where

$$(339)\qquad \boldsymbol{a}_1^{-1} = \frac{\boldsymbol{a}_2 \times (\boldsymbol{a}_1 \times \boldsymbol{a}_2)}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}, \qquad \boldsymbol{a}_2^{-1} = \frac{(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \times \boldsymbol{a}_1}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2},$$

and (339) imply

$$(340)\qquad (a_1 \times a_2)^2 s a_1^{-1} = a_1 \times a_2 \cdot s \times a_2 = (s a_1) a_2^2 - (a_2 a_1)(s a_2),$$
$$(341)\qquad (a_1 \times a_2)^2 s a_2^{-1} = a_1 \times a_2 \cdot a_1 \times s = a_1^2 (s a_2) - (s a_1)(a_1 a_2).$$

Now (340), (341), and (336) imply

$$(342)\qquad \lambda_\nu = s a_\nu^{-1} \qquad (\nu = 1, 2)$$

and (342), (331), (338), and the first relation (329) imply the first relation (269).

The second relation (269) is equivalent to the second relation (331), i.e. with

$$(343)\qquad \begin{vmatrix} a_1 s & a_1 a_2 \\ a_2 s & a_2^2 \end{vmatrix} b_1 + \begin{vmatrix} a_1^2 & a_1 s \\ a_2 a_1 & a_2 s \end{vmatrix} b_2 = (a_1 \times a_2)^2 m$$

by virtue of (336). In other words, the condition (343) is necessary for the affirmative answer of the question under consideration. As regards the conditions (253), they are satisfied in view of (329), (332), (333).

In such a manner, the following proposition is proved.

Pr 29. If $\underset{\rightarrow}{s}(s, m)$ is a given system of arrows with rank = 3 and the lines l_ν ($\nu = 1, 2$) are defined by means of (326) with (327), then the conditions (333), (337), and (343) are necessary for the existence of arrows (301) with (264) and (328). These conditions are sufficient too: if they are satisfied, then the only solution of the problem is proposed by (329) with (336).

Sch 40. As it has been promised, we shall conclude our exposition with some remarks in connection with the *statical-kinematical analogy*. Under this term the following mathematical phenomenon is understood.

A rigid body B being given and a denoting any of its points, there exists exactly one function $\overline{\omega}$ of the time t with the following property: a point r belongs to B if, and only if, the condition

$$(344)\qquad \frac{d}{dt}(r - a) = \overline{\omega} \times (r - a)$$

is satisfied for any t.

This discovery is due to Euler, and $\overline{\omega}$ is called the *instantaneous angular velocity* of B. Euler's theorem (344) implies

$$(345)\qquad \frac{dr_1}{dt} - \frac{dr_2}{dt} = \overline{\omega} \times (r_1 - r_2)$$

for any two points r_ν ($\nu = 1, 2$) of B.

There is a formal analogy between (345) and the connection (64) between the moments of a system of arrows with respect to two poles r_ν ($\nu = 1, 2$). Indeed, it is sufficient to substitute $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ for dr_ν/dt ($\nu = 1, 2$) respectively and s for $\overline{\omega}$ in (345) in order to obtain (64). Inversely, it is sufficient to substitute dr_ν/dt for $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ ($\nu = 1, 2$) respectively and $\overline{\omega}$ for s in (64) in order to obtain (345). This fact has far reaching consequences.

It turns out that for any theorem concerning systems of arrows there corresponds a true theorem concerning rigid body kinematics. At that, there is no need to prove the kinematical theorem: it is certainly true, provided the arrow-theorem is proved. Furthermore, to any construction in the theory of the systems of arrows there corresponds a meaningful kinematical construction concerning rigid bodies. We shall enter in no details in this connection, and we shall confine our exposition to two examples only.

Let $\underset{\rightarrow}{\omega}(\overline{\omega}, \frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega})$ be a system of arrows *intrinsic* to the rigid body B by virtue of the statical-kinematical analogy. The kinematical analogue of the axis of a system of arrows (11) with (24) (Sgn 7) is the *helicoidal axis* of the rigid body B provided $\overline{\omega} \neq \mathbf{o}$, i.e. ax $\underset{\rightarrow}{\omega}$, in other words the line l defined by

$$(346) \qquad \left(\overline{\omega}, \frac{\overline{\omega} \times \left(\frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega}\right) \times \overline{\omega}}{\overline{\omega}^2}\right) \;\&\; l.$$

The helicoidal axis (346) of B has important kinematical interpretations.

The second example is connected with the rank-theorem (Pr 15). Its kinematical analogue reads:

Pr 30. $\underset{\rightarrow}{\omega}(\overline{\omega}, \frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega})$ denoting a system of arrows intrinsic for the rigid body B by virtue of the statical-kinematical analogy, the motion of B in the moment of time t is a rest, a translation, a rotation, or a most general helicoidal movement according to the cases rank $\underset{\rightarrow}{\omega} = 0, 1, 2$, or 3 respectively.

As mentioned, we shall enter in no more details here in connection with the statical-kinematical analogy. A special study will be dedicated to this theme in due time.

LITERATURE

1. I. Hristova, R. Stoyanova, I. Chobanov. Introduction to an algebraic theory of arrows. I. — Год. Соф. унив., Фак. мат. мех., **71**, 1976/1977, ч. I, 5–53.
2. I. Chobanov. Introduction to an algebraic theory of arrows. II. — Год. Соф. унив., Фак. мат. инф., **82**, 1988, кн. 1 — Математика, 59–111.
3. I. Chobanov. Complex standard vector spaces. — Год. Соф. унив., Фак. мат. мех., **75**, 1981, кн. 2 — Механика, 3–26.
4. И. Чобанов. Векторна алгебра. Исторически и съвременни аспекти. С., 1982.
5. I. Chobanov. Analytic geometries in complex standard vector spaces. — Год. Соф. унив., Фак. мат. мех., **77**, 1983, кн. 1 — Математика, 73–138.
6. D. Hilbert. Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreßzu Paris, 1900. Aus den Nachrichten der K. Ges. der Wiss. zu Göttingen, Math.-phys. Klasse, 1900, Heft 3. Mit Zusätzen des Verfassers. — Arch. d. Math. u. Phys., III. Reiche, **1**, 1901, 44 — 63, 213 — 237.
7. L. Euler. Nova methodus motum corporum rigidorum determinandi. — Novi Comm. Acad. Sci. Petrop., **20**, 1775, 208 — 238.
8. M. Brelot. Les Principes Mathématiques de la Mécanique Classique. Grenoble — Paris, 1945.
9. G. Chobanov, I. Chobanov. Sapienti sat. — Год. Соф. унив., Фак. мат. мех., **77**, 1983, кн. 2 — Механика, 21 — 25.
10. Б. Долапчиев. Аналитична механика. Книга първа: Векторно смятане. Кинематика. Статика. С., 1958.

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