ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 103

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 103

TOPOLOGY AND DIFFERENTIAL STRUCTURE ON DESCARTES FOLIUM

CONSTANTIN UDRISTE, ADRIAN CONSTANTINESCU, STELUTA PRICOPIE

The Descartes Folium carries families of group laws, defined entirely in terms of algebraic operations over the base field. To highlight these structures and their properties, we introduced and studied new topologies and differential structures on Descartes Folium.

Keywords: Descartes Folium, differential structure, topologies.

2000 Math. Subject Classification: 14H45, 14L10, 14A10

1. DESCARTES FOLIUM

Let $\mathbb{K}$ be a field with $\operatorname{char}\mathbb{K} \neq 3$. Our theory (see also [2], [3], [8]) refers to the *Descartes Folium* which is a non-smooth curve

$$DF : x^3 + y^3 - 3axy = 0 \subset \mathbb{A}^2_{\mathbb{K}},\ a \in \mathbb{K} \setminus \{0\}$$

and to its *projective closure* defined by homogenization, i.e.,

$$\overline{DF} : x^3 + y^3 - 3axyz = 0 \subset \mathbb{P}^2_{\mathbb{K}}$$

and called the *projective Descartes Folium*, too. DF, resp $\overline{DF}$, has only a non-smooth point, namely $O = (0,0) \in DF \subset \overline{DF}$.

Suppose now $\mathbb{K}=\mathbb{R}$. We have a bijective map p:

$$\begin{array}{cccc} DF & \left(x=\frac{3at}{1+t^3},\, y=\frac{3at^2}{1+t^3}\right) & (x,y)\in DF\setminus\{(0,0)\} & (0,0) \\ \wr\Big\uparrow p & \Big\uparrow & \Big\downarrow & \Big\downarrow \\ \mathbb{R}\setminus\{-1\} & t & t=\frac{y}{x} & t=0 \end{array}$$

On the other hand, the function $t\to\tau=t+1$, with the inverse $t=\tau-1\xleftarrow{\alpha}\tau$, establishes a bijection

$$\mathbb{R}\setminus\{-1\}\underset{\sim}{\xleftarrow{\alpha}}\mathbb{R}\setminus\{0\}.$$

It appears the bijective map $p\alpha$:

$$\begin{array}{cccc} DF & \left(x=\frac{3a(\tau-1)}{1+(\tau-1)^3},\, y=\frac{3a(\tau-1)^2}{1+(\tau-1)^3}\right) & (x,y)\in DF\setminus\{(0,0)\} & (0,0) \\ \wr\Big\uparrow p\,\alpha & \Big\uparrow & \Big\downarrow & \Big\downarrow \\ R\setminus\{0\} & \tau & \tau=\frac{y}{x}+1 & 1 \end{array}$$

It follows that the natural group law $\cdot$ on $R\setminus\{0\}$ transfers to a group composition law $\circ$ on DF defined as follows:

$$(p\alpha)(\tau)\circ(p\alpha)(\tau')\stackrel{def}{=}(p\alpha)(\tau\tau').$$

Consequently, we have an isomorphism of groups:

$$(DF,\circ)\simeq(\mathbb{R}\setminus\{0\},\cdot)$$

$$(p\,\alpha)(\tau)\leftarrow\tau.$$

Remarks. (i) For each $t\in\mathbb{R}\setminus\{-1,0\}$, we can observe that $p(t)\in DF\setminus\{0\}$ is the intersection point in $\mathbb{A}^2_{\mathbb{R}}=\mathbb{R}^2$, of DF with the affine straight line $y-tx=0$, different of $O=(0,0)$. In this way, p admits a geometric definition.

(ii) p (and $p\alpha$) can be related to the normalization morphism of an algebraic curve ([4]). Namely, p can be uniquely extended as algebraic map to a (non-bijective) algebraic map $\bar{p}$:

$$\begin{array}{ccc} \overline{DF} & (x=3at,\, y=3at^2,\, z=1+t^3) & (x,y,z)\in\overline{DF}\setminus\{O\} \\ \bar{p}\Big\uparrow & \Big\uparrow & \Big\downarrow \\ P^1_{\mathbb{R}}=A^1_{\mathbb{R}}\cup\{\infty\} & t\in A^1_{\mathbb{R}} & t=\frac{y}{x}\ , \end{array}$$

where we indicated the definition of $\bar{p}$ and of a partial inverse of $\bar{p}$.

If we pass from the base field $\mathbb{R}$ to its algebraic closure $\overline{\mathbb{R}}=\mathbb{C}\supset\mathbb{R}$, then DF, $\overline{DF}$, $\bar{p}$, α are all defined over $\mathbb{C}$, by the same equations and formulae. According to [4], one verifies that the pair $(\mathbb{P}^1_{\mathbb{C}},\bar{p})$ is a normalization of the algebraic $\mathbb{C}$-variety $\overline{DF}$. Recall that a normalization pair is uniquely determined up to an isomorphism of algebraic $\mathbb{C}$-varieties.

We have a similar situation with the pair $(\mathbb{P}^1_{\mathbb{C}}, \bar{p}\alpha)$, which is also a normalization of $\overline{DF}$ over $\mathbb{C}$. Based on this natural relation of p and $p\alpha$, with the normalizations of $\overline{DF}$ over $\mathbb{C}$, we will say shortly in the following that p and $p\alpha$ are also "normalization maps".

So the algebraic maps p and $p\alpha$ are natural and then the group composition law $\circ$ on DF is still natural. But the group $(DF, \circ)$ *is not* a Lie group over $\mathbb{R}$ because DF is not a topological manifold w.r.t. the induced real topology of $\mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2 \supset DF$ (If $U \subset DF$ is a small connected open subset w.r.t. this topology, containing $O = (0,0)$, then $U \setminus \{O\}$ has at least 4 connected components).

2. BRANCHES OF DESCARTES FOLIUM

Suppose $\mathbb{K} = \mathbb{R}$. Let us consider the branches of the non-smooth point $O = (0,0)$ of DF as follows: (i) the "South branch" $S = p(-1,1)$; (ii) the "West branch" $W = p(1,\infty) \cup \{O\} \cup p(-\infty,-1)$.

We have $S \cap W = \{O\}$ and $S \cup W = DF \setminus \{V\}$, where $V = p(1) = \left(\frac{3}{2}, \frac{3}{2}\right)$ is the "vertex" of DF. The branches S and W are symmetric w.r.t. the first bisector $x - y = 0$ of $\mathbb{A}^2_{\mathbb{R}}$. This means that applying the symmetry σ w.r.t. the bisector $x - y = 0$, given by

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{R}} & \overset{\sigma}{\underset{\sim}{\longrightarrow}} & \mathbb{A}^2_{\mathbb{R}} \\ (x,y) & \longrightarrow & (y,x), \end{array}$$

we have $\sigma(DF) = DF$ and the branches S and W interchange by σ (i.e., $\sigma(S) = W$ and $\sigma(W) = S$).

By the parametrization p, the point $O = p(0)$ is reached on the branch S and it is not reached on the branch W.

Let us consider the parametrization $p' = \sigma p$ of DF. Then, by using the interchange of S and W by σ, it follows that $W = p'(-1,1)$ and $S = p'(1,\infty) \cup \{O\} \cup p'(-\infty,-1)$. By the parametrization $p' = \sigma p$, the point $\{O\}$ is reached only on the branch W (not on the branch S).

It is easy to see that the pair $((\mathbb{A}^1_{\mathbb{R}} \setminus \{-1\}) \cup \{\infty\}, p' = \sigma p)$ is also related to a normalization of $\overline{DF}$ over $\mathbb{C}$, as p and $p\alpha$. We will say shortly, in the following, that p' is also a "normalization map".

3. SOME TOPOLOGIES ON AFFINE DESCARTES FOLIUM

Concerning the topological properties of the map p, we have the following

Proposition 1. *Suppose $\mathbb{K} = \mathbb{R}$ and $DF \subset \mathbb{A}^2_{\mathbb{R}}$ endowed with the topology τ induced by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Then*

(i) the bijective map $p : \mathbb{R} \setminus \{-1\} \overset{\sim}{\longrightarrow} DF$ is continuous but not a homeomorphism;

(ii) $p|_{\mathbb{R}\setminus\{0,-1\}} : \mathbb{R} \setminus \{0, -1\} \xrightarrow{\sim} DF \setminus \{O\}$ *is a homeomorphism.*

Similar properties hold for the map $p' = \sigma p : \mathbb{R} \setminus \{-1\} \xrightarrow{\sim} DF$.

Proof. (i) Suppose, by contrary, that $p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$ is a homeomorphism. Then $p\alpha : \mathbb{R} \setminus \{0\} \xrightarrow{\sim} DF$ is also a homeomorphism. Since $(\mathbb{R} \setminus \{0\}, +)$ is a topological group and $p\alpha$ is a group isomorphism onto $(DF, \circ)$, it follows easy that $(DF, \circ)$ is a topological group, which is not possible.

An alternative proof based on the different connectivity properties of $\mathbb{R} \setminus \{0\}$ and DF can be done.

(ii) The inverse map

$$\begin{array}{ccc} \mathbb{R} \setminus \{0, -1\} & \overset{p^{-1}}{\underset{}{\xleftarrow{\sim}}} & DF \setminus \{O\} \\ t = \frac{y}{x} & \longleftarrow & (x, y) \end{array}$$

is also continuous. □

In the previous Proposition we have worked with the topology τ on DF which is induced on $DF \subset \mathbb{A}^2_{\mathbb{R}}$ by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Now let us change the topology τ on DF with the topology τ_S (resp. τ_W) defined as follows:

Definition 1. *τ_S (resp. τ_W) is the image on DF of the real topology of $\mathbb{R}\setminus\{-1\}$ by the bijective map $p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$ (resp. by $p' = \sigma p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$).*

Hence the new topology τ_S (resp. τ_W) on DF is obtained by carrying the real topology of an open subset of DF by the normalization map p (resp. p'). It follows that the topology τ_S (resp. τ_W) is separated, paracompact and locally compact, and with countable basis, as well as the fact that $S = p(-1, 1)$ (resp. $W = p'(-1, 1)$) is open in DF w.r.t. τ_S (resp. τ_W). Moreover, the topological space (DF, τ_S) (resp. (DF, τ_W)) has two connected components.

3.1. SOME PROPERTIES OF TOPOLOGY τ_S (RESP. τ_W)

Theorem 1. *(i) τ_S (resp. τ_W) is a finer topology than τ (i.e., $\tau_S, \tau_W \succ \tau$).*

(ii) The induced topology $\tau_S|_{DF\setminus\{O\}}$ ($\tau_W|_{DF\setminus\{O\}}$) on $DF \setminus \{O\} \subset \mathbb{A}^2_{\mathbb{R}}$ is that induced on $DF \setminus \{O\}$ by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Equivalently,

$$\tau_S|_{DF\setminus\{O\}} = \tau_{DF\setminus\{O\}} \ \left(\text{resp.}\, \tau_W|_{DF\setminus\{O\}} = \tau_{DF\setminus\{O\}}\right).$$

(iii) If $\{U_n\}_{n\in\mathbb{N}}$ is a fundamental system of open neighborhoods of O in $\mathbb{A}^2_{\mathbb{R}}$, with respect to the real topology, then $\{U_n \cap S\}_{n\in\mathbb{N}}$ (resp. $\{U_n \cap W\}_{n\in\mathbb{N}}$) is a fundamental system of open neighborhoods of $O \in DF$, in DF, with respect to the topology τ_S (resp. τ_W).

(iv)

$$\tau \cup \{U \cap S \,|\, U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}$$
$$\big(\text{resp. } \tau \cup \{U \cap W \,|\, U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}\big)$$

is a basis for the topology τ_S (resp. τ_W). Moreover, for each $V \in \tau_S$ (resp. τ_W),

$$V = (U' \cap DF) \cup (U \cap S) \;\; (\text{resp. } V = (U' \cap DF) \cup (U \cap W)),$$

with $U', U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open subsets.

(v) Let

$$\pi : DF \to \mathbb{R}, \pi(x,y) = \begin{cases} \frac{y}{x} & if \quad (x,y) \neq O \\ 0 & if \quad (x,y) = O \end{cases}$$
$$\left(\text{resp. } \pi' : DF \to \mathbb{R}, \pi'(x,y) = \begin{cases} \frac{x}{y} & if \quad (x,y) \neq O \\ 0 & if \quad (x,y) = O \end{cases} \right).$$

Then τ_S (resp. τ_W) is the weakest topology on DF such that π (resp. π') is continuous ($\mathbb{R}$ endowed with the real topology).

(vi) $\{O\} \subset W$ (resp. $\{O\} \subset S$) is a connected component of the subspace W (resp. S) w.r.t. the topology τ_S (resp. τ_W). Moreover

$$W = p(1, \infty) \cup \{O\} \cup p(-\infty, -1)$$
$$(\text{resp. } S = p'(1, \infty) \cup \{O\} \cup p'(-\infty, -1))$$

is the representation of W (resp. S) as the union of its connected components w.r.t. τ_S (resp. τ_W). On the other hand S (resp. W) is connected w.r.t. τ_S (resp. τ_W).

Proof. Properties (i), (ii) and (v) are direct consequences of the definition of τ_S (resp. τ_W) and of the fact that the maps

$$p, p' : \mathbb{R} \setminus \{-1\} \xrightarrow{\sim} DF$$

(having $\pi, \pi' : DF \to \mathbb{R} \setminus \{-1\}$ as inverse maps) are continuous and

$$p, p' : \mathbb{R} \setminus \{0, -1\} \xrightarrow{\sim} DF \setminus \{O\}$$

are homeomorphisms, where DF (resp. $DF \setminus \{O\}$) above is endowed with the topology τ (resp. $\tau|_{DF\setminus\{O\}}$).

For property (iii), let us point out firstly that $U \cap S$ (resp. $U \cap W$) is an open subset of DF w.r.t. τ_S (resp. τ_W), in particular an open neighborhood of the point $O \in DF$ w.r.t. τ_S (resp. τ_W), if $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ is an open subset w.r.t. τ_S (resp. τ_W), resp. an open neighborhood of the point O in $\mathbb{A}^2_{\mathbb{R}}$. In fact, S (resp. W) is open in DF w.r.t. τ_S (resp. τ_W) and $U \cap DF \in \tau \subseteq \tau_S$ (resp. $U \cap DF \in \tau_W$) and so $U \cap S = (U \cap DF) \cap S \in \tau_S$ (resp. τ_W).

To end the proof of (iii) it suffices to resume to the topology τ_S and to show that for an open neighborhood V of O in DF w.r.t τ_S, there exists an open neighborhood

U of O in $\mathbb{A}^2_{\mathbb{R}}$ such that $V \supseteq U \cap S$. Indeed, we can reduce the situation to the case $V = p((-\delta, \delta))$, with $0 < \delta < 1$, because always for such V we have $V \supseteq p((-\delta, \delta))$, with $0 < \delta < 1$, and $p((-\delta, \delta))$ is an open neighborhood of O in DF w.r.t. τ_S.

For $t \in p((-\delta, \delta))$, with $0 < \delta < 1$, we have $|t| < \delta < 1$ and from the relation $x = \frac{3at}{1+t^3}$, where $t = \pi(x, y)$, with $(x, y) \in S$, it follows $3at = (1 + t^3)x$ and

$$3|a|\,|t| \leq |1 + t^3|\,|x| \leq (1 + |t|^3)|x| < (1 + \delta^3)|x| < 2|x|.$$

Hence $|t| < \frac{2}{3|a|}\,|x|$. If we consider ϵ, with $0 < \epsilon < \delta$, and

$$U = \{(x, y) \in \mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2 \, |x| < \frac{3|a|}{2}\,\epsilon\},$$

then $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ is open w.r.t. the standard real topology and we have $|t| < \epsilon$, for each $(x, y) \in U \cap S$, i.e., $|\pi(x, y)| < \epsilon$, for $(x, y) \in U \cap S$. Therefore $\pi(U \cap S) \subset (-\epsilon, \epsilon) \subset (-\delta, \delta)$. Since $\pi = p^{-1}$, we have then

$$U \cap S = p\pi(U \cap S) \subset p(-\epsilon, \epsilon) \subset p(-\delta, \delta) = V.$$

For property (iv), recall firstly that

$$\tau = \{U \cap DF \mid U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}.$$

Also, we resume to the topology τ_S. Then the family

$$\tau \cup \{U \cap S \mid U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}$$

is closed w.r.t. the finite intersections.

Let $V \subseteq DF$ be an open subset w.r.t. τ_S. If $O \in V$, then $V \supseteq U \cap S$ with $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open and $O \in U$, according to (iii) and its proof. If $P \in V$, $P \neq O$, then

$$V \supseteq V \cap (DF \setminus \{O\}) \in \tau$$

according to (ii) and $P \in V \cap (DF \setminus \{O\})$. It follows that

$$V = (V \cap (DF \setminus \{O\}) \cup (U \cap S),$$

where $V \cap (DF \setminus \{O\}) \in \tau$ (hence $V \cap (DF \setminus \{O\}) = U' \cap DF$ with $U' \subseteq \mathbb{A}^2_{\mathbb{R}}$ open) and $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open such that $O \in U$. The proof of (iv) is achieved.

For property (vi), we use the fact that $S \cap W = \{O\}$ and then for an open neighborhood $U \subset \mathbb{A}^2_{\mathbb{R}}$ of O w.r.t. the real topology of $\mathbb{A}^n_{\mathbb{R}}$, $U \cap S$ (resp. $U \cap W$) is an open neighborhood of $O \in DF$ w.r.t. τ_S (resp. τ_W) and $(U \cap S) \cap W = \{O\}$ (resp. $(U \cap W) \cap S = \{O\}$). Hence $\{O\}$ is open in W (resp. in S) w.r.t. τ_S (resp. τ_W) and so it is a connected component of W (resp. S), because $\{O\}$ is also closed in W (resp. S) w.r.t the separated topology τ_S (resp. τ_W). The connectivity of $p(1, \infty)$, $p(-\infty, -1)$, $S = p(-1, 1)$ (resp. $p'(1, \infty)$, $p'(-\infty, -1)$, $W = p'(-1, 1)$) w.r.t. τ_S (resp. τ_W) is clear because p (resp. p') is a homeomorphism. $\square$

Comment (ii) in conjunction to (iii), as well as (iv), determine completely the topology τ_S (resp. τ_W) by means of the real topology of the ambient space $\mathbb{A}^2_{\mathbb{R}}$, DF and its branch S (resp. W).

3.2. DIFFERENTIAL STRUCTURES ON AFFINE DESCARTES FOLIUM

On the topological space (DF, τ_S) (resp. (DF, τ_W)) we can introduce a structure $\mathcal{A}_S$ (resp $\mathcal{A}_W$) of smooth *differential manifold* by means of the simple atlas $\{(DF, \pi)\}$, (resp. $\{(DF, \pi')\}$ having only one chart, where

$$\pi : DF \xrightarrow{\sim} \mathbb{R} \setminus \{-1\} \subset \mathbb{R} \;\; (\text{resp. } \pi' : DF \xrightarrow{\sim} \mathbb{R} \setminus \{-1\} \subset \mathbb{R})$$

is the bijective map defined above, i.e.,

$$\pi : DF \to \mathbb{R}, \;\; \pi(x, y) = \begin{cases} \frac{y}{x} & if \quad (x, y) \neq O \\ 0 & if \quad (x, y) = O \end{cases}$$

$$\left(\text{resp. } \pi' : DF \to \mathbb{R}, \;\; \pi'(x, y) = \begin{cases} \frac{x}{y} & if \quad (x, y) \neq O \\ 0 & if \quad (x, y) = O \end{cases}\right).$$

Recall that the inverse of the map π (resp. π') is the map p (resp. $p' = \sigma p$) and all are continuous, hence homeomorphisms. In this way,

$$DF \;\; \underset{p}{\overset{\pi}{\rightleftarrows}} \;\; \mathbb{R} \setminus \{-1\} \;\; \left(\text{resp.} DF \;\; \underset{p'}{\overset{\pi'}{\rightleftarrows}} \;\; \mathbb{R} \setminus \{-1\}\right)$$

become diffeomorphisms of differentiable manifolds.

In particular, $p\alpha : \mathbb{R} \setminus \{0\} \xrightarrow{\sim} DF$ is then also a diffeomorphism, where DF is endowed with the topology τ_S and the atlas $\{(DF, \pi)\}$. Since

$$p\alpha : (\mathbb{R} \setminus \{0\}, \cdot) \xrightarrow{\sim} (DF, \circ)$$

is a group isomorphism, it follows directly

Theorem 2. *(i) $(DF, \circ)$ is a Lie group over $\mathbb{R}$ (in particular a topological group), where DF is endowed with the topology τ_S and the differential manifold structure given by the atlas $\{(DF, \pi)\}$.*

(ii)

$$p\alpha : (\mathbb{R} \setminus \{0\}, \cdot) \xrightarrow{\sim} (DF, \circ)$$

is then an isomorphism of Lie groups over $\mathbb{R}$ (in particular an isomorphism of topological groups).

4. COMMENTS

The original ideas regarding the group laws on elliptic curves and algebraic (Lie) groups are found in [1], [4]-[7], [10],[11].

In this paper, and also in our papers (see also [2], [3], [8]), the Descartes Folium is just one example of (irreducible) projective plane algebraic cubic curve with singularity (node), which support group laws. Some of these structures require to introduce and to exploit new topologies and some differential structures. One of our aims is to extend the study to the family of cubic curves

$$C_{a,b} : x^3 + y^3 - 3axy = b,$$

which bifurcates in smooth curves and non-smooth ones (as Descartes Folium). This could help us to understand whether the Cryptography on Descartes Folium is simpler than that on Elliptic curves.

5. REFERENCES

[1] Borel, A.: *Linear Algebraic Groups*, Princeton, 1969.

[2] Constantinescu, A., Udriste, C., Pricopie, S.: Genome of Descartes Folium via normalization, manuscript.

[3] Constantinescu, A., Udriste, C., Pricopie, S.: Classic and special Lie groups strucrures on some plane cubic curves with singularities. I. *ROMAI J.*, **10**, no. 2, 2014, 75–88.

[4] Hartshorne, R.: *Algebraic Geometry*, Springer-Verlag, 1977.

[5] Grothendieck, A.: *Elements de Geometrie Algebrique*, IHES Sci. Publ. Math., I–IV, 1960–1967 (EGA).

[6] Grothendieck, A., Demazure, M.: Schemas en Groupes, Seminaire de Geometrie algebrique du Bois Marie 1962/1964 (SGA 3).

[7] Merzliakov, Yu. I.: *Rational groups*, Nauka, Moscow, 1980, (in Russian).

[8] Pricopie, S., Udriste, C.: Multiplicative group law on the Folium of Descartes, *Balkan J. Geom. Appl.*, **18**, no. 1, 2013, 57–73.

[9] Radu, N.: *Local Rings. I*, Romanian Academy Publishing House, 1968, (in Romanian).

[10] Silverman, J. H.: *The Arithmetic of Elliptic Curves*, Springer-Verlag, 1985.

[11] The Stacks Project, version 75c9e41, Varieties, Columbia University, 2014. Available at: `http://stacks.math.columbia.edu/download/varieties.pdf`

Received on November 5, 2015

Constantin Udriste, Steluta Pricopie
University Politehnica of Bucharest
Faculty of Applied Sciences
Department of Mathematics and Informatics
Splaiul Independentei 313, RO-060042, Bucharest
ROMANIA
e-mail: udriste@mathem.pub.ro

Adrian Constantinescu
Institute of Mathematics "Simion Stoilov" of Romannian Academy
Calea Grivitei 21, P.O. BOX 1-764, Bucharest 010702
ROMANIA
e-mail: Adrian.Constantinescu@imar.ro, adrian.constantinescu67@yahoo.com