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PASSIVE MOTIONS OF VECTORS AND REPERS

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Румен Симеонов. ПАССИВНЫЕ ДВИЖЕНИЯ ВЕКТОРОВ И РЕПЕРОВ

Вводится понятие о пассивном движении векторов и реперов. Дается вариационная характеризация этого понятия. Указаны приложения к дифференциальной геометрии кривых, а также к аналитической механике. В частности, на этом пути естественным образом выведен интеграл С. Ковалевской о движении твердого тела около неподвижной точки.

Rumen Simeonov. PASSIVE MOTIONS OF VECTORS AND REPERS

The notion passive motion of vectors and repers is defined.
A variational characterization is given. Some applications to differential geometry of curves and to analytical mechanics are made. In particular, using this technique the S. Kowalewski integral concerning motions of a rigid body with a fixed point is derived in a natural way.

§1. DEFINITIONS AND BASIC PROPERTIES

Let V denote a threedimentional Euclidean vector space over $\mathbb{R}$, and $(a,\ b) = a \cdot b$ be the scalar product of elements $a,\ b \in V$. Further by Δ we'll denote a nonempty connected subset of $\mathbb{R}$, $C^m(\Delta;\ V)$ will denote the family of all m times ($m \geqq 0$) continuously differentiable vector-functions $a : \Delta \longrightarrow V$. We put $|a| = \sqrt{a \cdot a}$, $S^1 = S^1(V) = \{a \in V : |a| = 1\}$ and $C^m(\Delta;\ S^1) = \{a \in C^m(\Delta;\ V) : |a(t)| = 1, \forall t \in \Delta\}$.

Definition 1. Let $c \in C^m(\Delta; S^1)$, $m \geq 1$ and $a \in C^m(\Delta; V)$, $a(t) \perp c(t)$ (i. e. $a(t) \cdot c(t) = 0$), $\forall t \in \Delta$. We'll say "$a(t)$ passively follows $c(t)$ for $t \in \Delta$" iff

$$\text{(1)} \qquad \dot{a}(t) = \lambda(t)c(t), \quad \forall t \in \Delta, \text{ where } \lambda : \Delta \longrightarrow \mathbb{R}.$$

Here and farther on $()^\bullet = d/dt$. Since we suppose $a(t) \cdot c(t) = 0$, it is obvious that (1) es equivalent to the following

$$\text{(2)} \qquad \dot{a}(t) = -(a(t) \cdot \dot{c}(t))c(t), \quad \forall t \in \Delta.$$

Definition 1 is a mathematical formalization of our intuitive idea a vector $a(t)$, $a(t) \cdot c(t) = 0$ "to do no rotation around $c(t)$ when $t \in \Delta$". This is so because (1) exactly means that $\dot{a}(t)$ has no component in the plane perpendicular to $c(t)$.

Proposition 1. Let $m \geq 1$, $c \in C^m(\Delta; S^1)$. Then

a) Given any $t_0 \in \Delta$ and $a_0 \in V$, $a_0 \perp c(t_0)$, there exists unique $a \in C^m(\Delta; V)$ such that $a(t)$ passively follows $c(t)$ for $t \in \Delta$ and $a(t_0) = a_0$.

b) If a, $b \in C^m(\Delta; V)$, λ, $\mu \in R$, $r(t) = \lambda a(t) + \mu b(t)$, $t \in \Delta$, where $a(t)$, $b(t)$ passively follow $c(t)$ for $t \in \Delta$ then $r(t)$ passively follows $c(t)$ for $t \in \Delta$.

c) If a, $b \in C^m(\Delta; V)$, $a(t)$, $b(t)$ passively follow $c(t)$ for $t \in \Delta$ then $a(t) \cdot b(t) =$ const for $t \in \Delta$. Under the same assumptions it follows $|a(t)| =$ const, $|b(t)| =$ const, $\measuredangle(a(t), b(t)) =$ const, $t \in \Delta$.

Proof. a) The equation (2) about $a = a(t)$ is equivalent to a linear system of three scalar differential equations solved with respect to the derivatives. According to the differential equations theory, the equation (2) has unique solution $a \in C^m(\Delta; V)$, with $a(t_0) = a_0$. It remains to be noted that (2) implies

$$(d/dt)(a(t) \cdot c(t)) = \dot{a}(t) \cdot c(t) + a(t) \cdot \dot{c}(t) = -(a(t) \cdot \dot{c}(t))c(t)^2 + a(t) \cdot \dot{c}(t) = 0,$$

$$a(t) \cdot c(t) = a(t_0) \cdot c(t_0) = a_0 \cdot c(t_0) = 0, \quad \forall t \in \Delta.$$

b) This assertion automatically follows from the linearity of the equation (2), because $a(t) \perp c(t)$, $b(t) \perp c(t)$ implies $r(t) \perp c(t)$, $\forall t \in \Delta$.

c) It is sufficient to prove $a(t) \cdot b(t) =$ const, $t \in \Delta$. We calculate

$$(d/dt)(a(t) \cdot b(t)) = \dot{a}(t) \cdot b(t) + a(t) \cdot \dot{b}(t) =$$

$$-(a(t) \cdot \dot{c}(t))(c(t) \cdot b(t)) - (b(t) \cdot \dot{c}(t))(a(t) \cdot c(t)) =$$

$$-(a(t) \cdot \dot{c}(t)) \cdot 0 - (b(t) \cdot \dot{c}(t)) \cdot 0 = 0, \quad \forall t \in \Delta.$$

The proof of proposition 1 is completed.

Any ordered triple $R = (a, b, c)$, such that $\{a, b, c\}$ is an ortonormed base in V will be called strongly oriented 3-reper in V. The family of all such R will be denoted by V_3. If a, b, $c \in C^m(\Delta; V)$ and $R(t) = (a(t), b(t), c(t)) \in V_3$, $\forall t \in \Delta$ we will write $R \in C^m(\Delta; V_3)$.

Definition 2. If $m \geqq 1$, $R \in C^m(\Delta;\ V_3)$, $R(t) = (a(t),\ b(t),\ c(t))$, $a(t)$, $b(t)$ passively follow $c(t)$ for $t \in \Delta$ then we'll say "R is a passive C^m-motion of a reper in V_3 for $t \in \Delta$".

Proposition 2. Let $m \geqq 1$, $c \in C^m(\Delta;\ S^1)$, $t_0 \in \Delta$, $R_0 = (a_0,\ b_0,\ c(t_0)) \in V_3$. Then there exists an unique passive C^m-motion $R(t) \in V_3$, $t \in \Delta$ such that $R(t_0) = R_0$.

Proof. Let $a,\ b \in C^m(\Delta;\ V)$, $a(t)$, $b(t)$ passively follow $c(t)$ for $t \in \Delta$, $a(t_0) = a_0$, $b(t_0) = b_0$. According to proposition 1a) such a and b exist and are unique. We put $R(t) = (a(t),\ b(t),\ c(t))$, $t \in \Delta$ and we note that according to proposition 1c) it follows $a(t) \cdot b(t) = a_0 \cdot b_0 = 0$, $|a(t)| = |a_0| = 1$, $|b(t)| = |b_0| = 1$, $\forall t \in \Delta$. So proposition 2 is proved.

Proposition 3. Let $m \geqq 1$, $c \in C^m(\Delta;\ S^1)$, $R_i(t) = (a_i(t),\ b_i(t),\ c_i(t))$, $t \in \Delta$, $i = 1,\ 2$ be passive C^m-motions of repers R_1, $R_2 \in V_3$, $c_1(t) = c_2(t) = c(t)$, $t \in \Delta$. Then there exists an angle $\gamma = \mathrm{const}$, $t \in \Delta$ (which is uniquely determined up to an addent $2l\pi$, $l \in Z$) such that

$$
\begin{aligned}
a_2(t) &= \cos\gamma a_1(t) + \sin\gamma b_1(t), \\
b_2(t) &= -\sin\gamma a_1(t) + \cos\gamma b_1(t)
\end{aligned}
\tag{3}
$$

$\forall t \in \Delta$. Conversely, if $R_1(t) = (a_1(t),\ b_1(t),\ c_1(t))$, $t \in \Delta$ is a passive C^m-motion of a reper $R_1 \in V_3$, $m \geqq 1$, $\gamma = \mathrm{const}$, $c_2(t) = c_1(t) = c(t)$, $\forall t \in \Delta$ and $a_2(t)$, $b_2(t)$ are defined by (3) then the reper $R_2(t) = (a_2(t),\ b_2(t),\ c_2(t))$ will be a passive C^m-motion of a reper in V_3 for $t \in \Delta$.

Proof. Obviously a continuous function $\gamma(t)$, $t \in \Delta$ satisfying (3) exists and is uniquely determined up to an addent $2l\pi$, $l \in Z$. According to proposition 1c) we have $\cos\gamma(t) = a_2(t) \cdot a_1(t) = \mathrm{const}$, $\sin\gamma(t) = a_2(t) \cdot b_1(t) = \mathrm{const}$, $t \in \Delta$. This proves that $\gamma(t) = \mathrm{const}$, $t \in \Delta$. The converse assertion follows immediately from proposition 1b).

§2. A VARIATIONAL CHARACTERIZATION OF THE PASSIVE MOTIONS

We denote $B = \{(\lambda,\ \mu,\ \nu) \in R^3 : \lambda^2 + \mu^2 + \nu^2 \leqq 1\}$ and consider B as a homogeneous rigid body with density 1. Let $R(t) = (\xi(t),\ \eta(t),\ \zeta(t))$, $t \in \Delta$, be an arbitrary C^m-motion of a reper $R \in V_3$, $m \geqq 1$. With each such a reper $R(t)$, $t \in \Delta$, we associate a motion of the rigid body B using the following formula

$$r(t;\ \lambda,\ \mu,\ \nu) = \lambda\xi(t) + \mu\eta(t) + \nu\zeta(t), \quad t \in \Delta.$$

The kinetic energy of this motion of B is determined by the formula

$$T_R(t) = \iiint\limits_{\lambda^2+\mu^2+\nu^2 \leqq 1} (1/2)\dot{r}^2(t;\ \lambda,\ \mu,\ \nu)\,d\lambda d\mu d\nu, \quad t \in \Delta.$$

Theorem 1. Let $m \geqq 1$, $\zeta \in C^m(\Delta;\ S^1)$, $R_0(t) = (a(t),\ b(t),\ \zeta(t))$, $t \in \Delta$, be a passive C^m-motion of a reper $R_0 \in V_3$. Let $R(t) = (\xi(t),\ \eta(t),\ \zeta(t))$, $t \in \Delta$, be an arbitrary C^1-motion of a reper $R \in V_3$ having the same third vector $\zeta(t)$, $t \in \Delta$. Then for the corresponding kinetic energies the following inequality

$$(4) \qquad T_{R_0}(t) \leqq T_R(t), \quad \forall t \in \Delta,$$

holds. In (4) equality holds for each $t \in \Delta$ if and only if $R(t)$, $t \in \Delta$, is a passive motion of a reper in V_3 too.

Proof. Since R_0 and R have one and the same third vector we can find a continuously differentiable, function $\gamma(t)$, $t \in \Delta$, for which

$$(5) \qquad \begin{aligned} \xi(t) &= \cos\gamma(t)a(t) + \sin\gamma(t)b(t), \\ \eta(t) &= -\sin\gamma(t)a(t) + \cos\gamma(t)b(t) \end{aligned}$$

holds $\forall t \in \Delta$. Now we calculate

$$\begin{aligned} T_R(t) &= \iiint\limits_{\lambda^2+\mu^2+\nu^2 \leqq 1} (1/2)(\lambda\dot\xi(t) + \mu\dot\eta(t) + \nu\dot\zeta(t))^2 d\lambda d\mu d\nu \\ &= (1/2)A(\dot\xi^2(t) + \dot\eta^2(t) + \dot\zeta^2(t)), \quad \forall t \in \Delta, \end{aligned}$$

where

$$A = \iiint \lambda^2 d\lambda d\mu d\nu > 0, \quad A = \text{const}, \quad t \in \Delta.$$

Using (5) we find

$$\begin{aligned} \dot\xi(t) &= \dot\gamma(t)\eta(t) + \cos\gamma(t)\dot a(t) + \sin\gamma(t)\dot b(t), \\ \dot\eta(t) &= -\dot\gamma(t)\xi(t) - \sin\gamma(t)\dot a(t) + \cos\gamma(t)\dot b(t), \quad t \in \Delta. \end{aligned}$$

Since $a(t)$ and $b(t)$ passively follow $\zeta(t)$ we have $\dot a(t) \perp \xi(t)$, $\dot a(t) \perp \eta(t)$, $\dot b(t) \perp \xi(t)$, $\dot b(t) \perp \eta(t)$ and consequently

$$\begin{aligned} \dot\xi^2(t) &= \dot\gamma^2(t) + (\cos\gamma(t)\dot a(t) + \sin\gamma(t)\dot b(t))^2, \\ \dot\eta^2(t) &= \dot\gamma^2(t) + (-\sin\gamma(t)\dot a(t) + \cos\gamma(t)\dot b(t))^2, \\ \dot\xi^2(t) + \dot\eta^2(t) &= 2\dot\gamma^2(t) + \dot a^2(t) + \dot b^2(t), \\ T_R(t) &= A\dot\gamma^2(t) + T_{R_0}(t), \quad \forall t \in \Delta. \end{aligned}$$

This equality proves (4) and it is clear that we have equality in (4) if and only if $\dot\gamma(t) = 0$, $\forall t \in \Delta$, i. e. iff $\gamma(t)$ =const, $t \in \Delta$. According to proposition 3, the last means that $R(t)$, $t \in \Delta$, is a passive motion of a reper in V_3. Thus Theorem 1 is proved.

§3. CURVATURE, TORSION AND PASSIVE VECTORS

Suppose in V is introduced an orientation (one of two possible) and let $\times$ denote the corresponding vector product operation. Let $r \in C^3(\Delta;\ V)$ be regular, i. e. $\dot{r}(t) \times \ddot{r}(t) \neq 0$, $\forall t \in \Delta$. As it is common in differential geometry of curves we put $\tau,\ \nu,\ \beta \in C^1(\Delta;\ V)$, $\tau(t) = r\cdot(t)/|\dot{r}(t)|$, $\beta(t) = (\dot{r}(t) \times \ddot{r}(t))/|\dot{r}(t) \times \ddot{r}(t)|$, $\nu(t) = \beta(t) \times \tau(t)$, $\forall t \in \Delta$. In such a case r is called regular curve in V; $\tau,\ \nu,\ \beta$ are called tangent, main normal, binormal unit vectors of r. The plane containing the end of the vector $r(t)$ and collinear to the vectors $\tau(t)$, $\nu(t)$ is called tangent plane; the plane containing the end of the vector $r(t)$ and collinear to the vectors $\nu(t)$, $\beta(t)$ is called normal plane.

Let $t_0 \in \Delta$ and $R_\tau(t) = (a(t),\ b(t),\ \tau(t)) \in V_3$, $t \in \Delta$, be a passive motion of a reper in V_3, determined by $R_\tau(t_0) = (\nu(t_0),\ \beta(t_0),\ \tau(t_0))$. According to proposition 3 such a reper exists and is unique. By virtue of Theorem 1 this reper has the property to rotate itself about $\tau(t)$, $t \in \Delta$, with a minimal kinetic energy. We involve the angle $\gamma(t)$, $t \in \Delta$, $\gamma \in C^0(\Delta;\ R)$ determined by $\gamma(t_0) = 0$ and

$$
\begin{aligned}
\nu(t) &= \cos\gamma(t)a(t) + \sin\gamma(t)b(t), \\
\beta(t) &= -\sin\gamma(t)a(t) + \cos\gamma(t)b(t), \quad \forall t \in \Delta.
\end{aligned}
\tag{6}
$$

Since $r \in C^3$ it is easy to see that $\gamma \in C^1(\Delta;\ R)$. In an analogical way we introduce a passive reper $R_\beta(t) = (p(t),\ q(t),\ \beta(t))$, $t \in \Delta$, $R_\beta(t_0) = (\tau(t_0),\ \nu(t_0),\ \beta(t_0))$, and the corresponding angle $\theta \in C^1(\Delta;\ R)$, $\theta(t_0) = 0$,

$$
\begin{aligned}
\tau(t) &= \cos\theta(t)p(t) + \sin\theta(t)q(t), \\
\nu(t) &= -\sin\theta(t)p(t) + \cos\theta(t)q(t), \quad \forall t \in \Delta.
\end{aligned}
\tag{7}
$$

Now it is natural to give the following.

D e f i n i t i o n 3. The angle $\gamma(t)$, $t \in \Delta$, is called turning angle (reading from the moment $t = t_0$) of the normal plane. The angle $\theta(t)$, $t \in \Delta$, is called turning angle (reading from the moment $t = t_0$) of the tangent plane.

T h e o r e m 2. For each regular curve $r \in C^3(\Delta;\ V)$ the curvature $k(t)$ and the torsion $\sigma(t)$ can be expressed in the following way

$$
k(t) = \dot{\theta}(t)|\dot{r}(t)|^{-1}, \quad \sigma(t) = \dot{\gamma}(t)|\dot{r}(t)|^{-1}, \quad \forall t \in \Delta. \tag{8}
$$

The following (Frenet's) formulas

$$
\dot{\tau}(t) = \dot{\theta}(t)\nu(t), \tag{9}
$$

$$
\dot{\nu}(t) = -\dot{\theta}(t)\tau(t) + \dot{\gamma}(t)\beta(t), \tag{10}
$$

$$
\dot{\beta}(t) = -\dot{\gamma}(t)\nu(t), \tag{11}
$$

hold $\forall t \in \Delta$.

Proof. First we'll prove (9), (10), (11). From (7) and (6) it follows

$$\dot{r}(t) = \dot{\theta}(t)\nu(t) + \cos\theta(t)\dot{p}(t) + \sin\theta(t)\dot{q}(t),$$

$$\dot{\beta}(t) = -\dot{\gamma}(t)\nu(t) - \sin\gamma(t)\dot{a}(t) + \cos\gamma(t)\dot{b}(t).$$

Since $\tau(t) = \lambda(t)\dot{r}(t)$, $\lambda(t) = |\dot{r}(t)|^{-1}$, $\beta(t) = \mu(t)(\dot{r}(t)\times\ddot{r}(t))$, $\mu(t) = |\dot{r}(t)\times\ddot{r}(t)|^{-1}$ $\dot{\tau}(t)\cdot\beta(t) = (\dot{\lambda}(t)\dot{r}(t)+\lambda(t)\ddot{r}(t))\cdot\mu(t)(\dot{r}(t)\times\ddot{r}(t)) = 0$, $\forall t\in\Delta$, and $p(t)$, $q(t)$ passivel follow $\beta(t)$ for $t\in\Delta$ it is evident that

$$\cos\theta(t)\dot{p}(t) + \sin\theta(t)\dot{q}(t) = h(t)\beta(t) = (\dot{\tau}(t)\cdot\beta(t))\beta(t) = 0.$$

Thus (9) is proved. By the same manner we have

$$-\sin\gamma(t)\dot{a}(t) + \cos\gamma(t)\dot{b}(t) = g(t)\tau(t) = (\dot{\beta}(t)\cdot\tau(t))\tau(t) = (-\beta(t)\cdot\dot{\tau}(t))\tau(t) = 0$$

and (11) is proved. Since $\nu = \beta\times\tau$ we obtain

$$\dot{\nu}(t) = \dot{\beta}(t)\times\tau(t) + \beta(t)\times\dot{\tau}(t) = -\dot{\gamma}(t)(-\beta(t)) + \dot{\theta}(t)(-\tau(t))$$

which proves (10).

It is known from differential geometry [1, p. 44] that $\dot{\tau}(t)\cdot\nu(t) = |\dot{r}(t)|k(t$ and $\dot{\beta}(t)\cdot\nu(t) = -\sigma(t)|\dot{r}(t)|$, $\forall t\in\Delta$. This, (9) and (11) prove (8). Theorem 2 i proved.

We would like to mention that if $t = s$ is natural parameter, i. e. if $|\dot{r}(t)| = 1$ $\forall t\in\Delta$ then (8) has the very simple form

(8′) $$k(s) = \theta'(s),\ \sigma(s) = \gamma'(s),\quad \forall s\in\Delta.$$

§4. PASSIVE VECTORS AND THE S. KOWALEWSKI INTEGRAL

Let us consider the classical problem concerned the motion of a rigid bod having a fixed point O. Let $R(t) = (\xi(t), \eta(t), \zeta(t))$, $t\in\Delta$, be a reper immobill attached to the body and such that $O\xi$, $O\eta$, $O\zeta$ are main inertial axes for the poin O. Let A, B, C be the corresponding inertial moments, and let

$$P = -Mg(a_{31}(t)\xi(t) + a_{32}(t)\eta(t) + a_{33}(t)\zeta(t)) = -Mgk,$$

$$k = a_{31}(t)\xi(t) + a_{32}(t)\eta(t) + a_{33}(t)\zeta(t),$$

$$\dot{k}(t) = 0,\qquad t\in\Delta,\qquad |k| = 1$$

be the weight-force. The force P is applied at the mass center G of the body bein in consideration. Let us suppose that $OG = \xi_G\xi(t)+\eta_G\eta(t)+\zeta_G\zeta(t)$, $\eta_G = \zeta_G = 0$ $\xi_G = \text{const}$, $\forall t\in\Delta$. As it is wellknown [2, p. 513], the Euler equation for thi problem has the form

(12) $$\begin{aligned} A\dot{\omega}_\xi &= (B-C)\omega_\eta\omega_\zeta, \\ B\dot{\omega}_\eta &= (C-A)\omega_\zeta\omega_\xi + Mg\xi_G a_{33}, \\ C\dot{\omega}_\zeta &= (A-B)\omega_\xi\omega_\eta - Mg\xi_G a_{32}, \\ \dot{a}_{31} &= \omega_\zeta a_{32} - \omega_\eta a_{33}, \\ \dot{a}_{32} &= \omega_\xi a_{33} - \omega_\zeta a_{31}, \\ \dot{a}_{33} &= \omega_\eta a_{31} - \omega_\xi a_{32}. \end{aligned}$$

Let us remember also the following identities

$$(13)\qquad \begin{aligned}\dot\xi(t) &= \omega_\zeta(t)\eta(t)-\omega_\eta(t)\zeta(t),\\ \dot\eta(t) &= -\omega_\zeta(t)\xi(t)+\omega_\xi(t)\zeta(t),\\ \dot\zeta(t) &= \omega_\eta(t)\xi(t)-\omega_\xi(t)\eta(t),\end{aligned}$$

true for $t\in\Delta$.

According to Proposition 1c) for any vector $b(t)=b_1(t)\xi(t)+b_2(t)\eta(t)$, b_1, b_2 — scalars, which is in the plane $O\xi\eta$ and passively follows the vector $\zeta(t)$ we have $|b(t)|=\text{const}$, $t\in\Delta$, i. e. $b_1^2(t)+b_2^2(t)=\text{const}$, $t\in\Delta$. It becomes clear that if b_1 and b_2 are polynomials of the unknown ω_ξ, ω_η, ..., a_{31}, a_{32}, ... then $I=b_1^2+b_2^2$ will be a first integral of system (12). Following the S. Kowalewski idea let us involve the isomorphism $\langle\,\rangle: C\to O\xi\eta$, $\langle\lambda+\mu i\rangle=\lambda\xi+\mu\eta$. In the plane $O\xi\eta$ we have the vectors $\omega_{\xi\eta}=\omega_\xi\xi+\omega_\eta\eta$ and $a=a_{31}\xi+a_{32}\eta$. It is clear that a suitable way to find a vector b with coefficients b_1; b_2 being polynomials of ω_ξ, ω_η, a_{31}, a_{32} is to put $b=\langle b_1+b_2i\rangle$, $b_1+b_2i=f(z,\,m)$, $z=\omega_\xi+\omega_\eta i$, $m=a_{31}+a_{32}i$, where $f(z,\,m)$ is a polynomial of the complex variables z, m. We could begin with $f=m$, $f=z$, and so on. Very soon we come to the possibility $f(z,\,m)=z^2-m$.

T h e o r e m 3. The vector $b=\langle z^2-m\rangle$, where $z=\omega_\xi+\omega_\eta i$, $m=a_{31}+a_{32}i$, passively follows ζ when $t\in\Delta$, for $\omega_\xi(t_0)$, $\omega_\eta(t_0)$, $\omega_\zeta(t_0)$, $a_{33}(t_0)$ — arbitrary initial values, if and only if $A=B=2C$ and $Mg\xi_G=C$.

Proof. We have $z^2-m=\omega_\xi^2-\omega_\eta^2-a_{31}+(2\omega_\xi\omega_\eta-a_{32})i$, $b=b_1\xi+b_2\eta$,

$$(14)\qquad b_1=\omega_\xi^2-\omega_\eta^2-a_{31},\qquad b_2=2\omega_\xi\omega_\eta-a_{32}.$$

In accordance with Definition 1, b will follow passively if and only if $\dot b(t)\cdot\xi(t)=0$ and $\dot b(t)\cdot\eta(t)=0$, $\forall t\in\Delta$. Using (13) we obtain

$$\dot b(t)=\dot b_1(t)\xi(t)+\dot b_2(t)\eta(t)+b_1(t)\dot\xi(t)+b_2(t)\dot\eta(t),$$

$$\dot b(t)\cdot\xi(t)=\dot b_1(t)-b_2(t)\omega_\zeta(t),$$

$$\dot b(t)\cdot\eta(t)=\dot b_2(t)+b_1(t)\omega_\zeta(t)$$

for each $t\in\Delta$. According to (14) we have

$$\dot b_1-b_2\omega_\zeta=2\omega_\xi\dot\omega_\xi-2\omega_\eta\dot\omega_\eta-\dot a_{31}-(2\omega_\xi\omega_\eta-a_{32})\omega_\zeta,$$

$$\dot b_2+b_1\omega_\zeta=2\dot\omega_\xi\omega_\eta+2\omega_\xi\dot\omega_\eta-\dot a_{32}+(\omega_\xi^2+\omega_\eta^2-a_{31})\omega_\zeta.$$

Now from (12) follows

$$\dot b_1-b_2\omega_\zeta=2\left(\frac{B-C}{A}-1-\frac{C-A}{B}\right)\omega_\xi\omega_\eta\omega_\zeta+\left(-2Mg\frac{\xi_G}{B}+1\right)\omega_\eta a_{33},$$

$$\dot b_2+b_1\omega_\zeta=\left(2\frac{B-C}{A}-1\right)\omega_\eta^2\omega_\zeta+\left(2\frac{C-A}{B}+1\right)\omega_\xi^2\omega_\eta$$

$$+\left(2Mg\frac{\xi_G}{B}-1\right)\omega_\xi a_{33}.$$

Consequently $\dot{b}_1 - b_2\omega_\zeta = 0$ and $\dot{b}_2 + b_1\omega_\zeta = 0$, $\forall t \in \Delta$ and for arbitrary ω_ξ, ω_η, ω_ζ, a_{33} if and only if $2Mg\xi_G = B$ and

$$\frac{B-C}{A} - 1 - \frac{C-A}{B} = 0, \quad 2\frac{B-C}{A} - 1 = 0, \quad 2\frac{C-A}{B} + 1 = 0.$$

Solving the last system we find the unique possibility $A = B = 2C = 2Mg\xi_G$. With this Theorem 3 is proved.

If we change the time t by $\tau = \lambda t$, where $\lambda = (Mg\xi_G/C)^{1/2}$then in (12) on the place of $Mg\xi_G$ will appear C. In such a way we see that the condition $Mg\xi_G = C$ is not essential.

C o r o l l a r y. If $\eta_G = \zeta_G = 0$ and the change $\tau = \lambda t$, $\lambda = (Mg\xi_G/C)^{1/2}$ is already made then

$$I = (\omega_\xi^2 - \omega_\eta^2 - a_{31})^2 + (2\omega_\xi\omega_\eta - a_{32})^2$$

will be a first integral of the system (12) if and only if $A = B = 2C$.

After all above we have seen how the S. Kowalewski integral could be derived constructing an appropriate passive vector. Something more — the conditions $A = B = 2C$ appear as some necessary conditions for realizing such a construction.

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