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LINEAR CROSS-SECTIONS AND FREDHOLM OPERATORS IN A CLASS GROUPOID C^*-ALGEBRAS

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We consider the groupoid C^*-algebra $\mathcal{T} = C^*(\mathcal{G})$, where the groupoid $\mathcal{G}$ is a Wiener-Hopf groupoid, i. e., $\mathcal{G}$ a reduction of a transformation group $\mathcal{G} = (Y \times G)|X$, and Y and X are suitable topological spaces. We give a method to construct continuous linear cross-sections using contractions in $\mathcal{G}^0$ – the unit space of $\mathcal{G}$.
We establish a criterion for an operator $T \in \mathcal{B}$ to be Fredholm.

Keywords: groupoid C^*-algebra, Fredholm operator, Wiener-Hopf groupoid, continuous cross-sections

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1. Introduction

Let G be a locally compact, second countable, unimodular group with identity e and a left Haar measure μ. We fix a solid, closed, normal subsemigroup P of G of positive measure containing the identity e of G.

For any $f \in C_c(G)$ we define the Wiener-Hopf operator with symbol f on $L^2(P)$ to be

$$W_f\xi(t) = \int_P f(ts^{-1})\xi(s)d\mu(s), \quad \xi \in L^2(P).$$

The C^*-algebra generated by $\{W_f : f \in C_c(G)\}$ is denoted by $\mathcal{T}$.

In [5, § 3.1] and [7, § 3.1] is explained how to construct a locally compact space Y such that there exist an inclusion $i : G \longrightarrow Y$ and a continuous action $G \times Y \longrightarrow Y$.

We define the space X to be the closure of $i(P)$ in Y and the groupoid $\mathcal{G}$ as a reduction [5, § 2.2.5] of the transformation group $Y \times G$ to X, i. e., $\mathcal{G} = (Y \times G)|X$. This groupoid is known as the Wiener-Hopf groupoid, associated with G and P.

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The groupoid $\mathcal{G}$ and its reduced C^*-algebra $C^*_{\mathrm{red}}(\mathcal{G})$ are important because in [5, § 3.7] and [7, § 2.4.1] is proved that the C^*-algebra $\mathcal{T}$ of Wiener-Hopf operators is isomorphic to $C^*_{\mathrm{red}}(\mathcal{G})$.

In the theory of groupoid C^*-algebras one may associate ideals with open invariant subsets of the unit space. If U is an open invariant subset of $\mathcal{G}^0$, then the set $I_U = \{f \in C_c(\mathcal{G})\colon \operatorname{supp}(f) \subset \mathcal{G}|U\}$ is a two-sided ideal in $C_c(\mathcal{G})$ and its closure in $C^*_{\mathrm{red}}(\mathcal{G})$, $\overline{I_U}$ is a closed two-sided ideal in $C^*_{\mathrm{red}}(\mathcal{G})$.

The following theorem is well known:

Theorem ([5, § 2.2.16, Prop. 2.16], [9, § 3.9, Prop. 4.5]). *The map $U \mapsto I_U$ is a one to one order preserving map from the lattice of open invariant subsets of $\mathcal{G}^0$ into the lattice of two-sided ideals of $\mathcal{T} = C^*_{\mathrm{red}}(\mathcal{G})$. For an ideal $J = I_U$ of $\mathcal{T}$, $C^*_{\mathrm{red}}(\mathcal{G})/J$ is canonically isomorphic to $C^*_{\mathrm{red}}(\mathcal{G}|F)$, where $F = \mathcal{G}^0 \setminus U$.*

Let F be a closed invariant subset of $\mathcal{G}^0 = X$. Then $U = X \setminus F$ is an open invariant subset of X.

By the above theorem we obtain the following exact sequence:

$$0 \longrightarrow J = C^*_{\mathrm{red}}(\mathcal{G}_{|U}) \xrightarrow{\ i\ } C^*_{\mathrm{red}}(\mathcal{G}) \xrightarrow{\ \gamma\ } C^*_{\mathrm{red}}(\mathcal{G}_{|F}) \longrightarrow 0 \tag{1.1}$$

In this sequence γ maps $a \in C_c(\mathcal{G})$ into its restriction to $\mathcal{G}_{|F}$. Thus $\gamma(a) \in C_c(\mathcal{G}|F)$.

Let us consider the opposite problem: when $b \in C_c(\mathcal{G}|F)$, we want to extend b to a compactly supported function $\psi(b) \in C_c(\mathcal{G})$. In this situation we obtain a continuous function $\psi\colon C_c(\mathcal{G}|F) \longrightarrow C_c(\mathcal{G})$, which may be extended to continuous linear cross-section $\psi\colon C^*_{\mathrm{red}}(\mathcal{G}|F) \longrightarrow C^*_{\mathrm{red}}(\mathcal{G})$.

The main purpose of this paper is to give a method how to construct continuous linear cross-sections $\psi\colon C^*_{\mathrm{red}}(\mathcal{G}|F) \longrightarrow C^*_{\mathrm{red}}(\mathcal{G})$.

This paper is organized as follows: In Section 2 the most interesting case ($\mathcal{K} \subset \mathcal{T}$) is considered and we give a necessary and sufficient conditions for an operator in $\mathcal{T}$ to be Fredholm. In the Section 3 we give a method how to construct a continuous linear cross-section in Wiener-Hopf groupoid algebras using contractions in the unit space of $\mathcal{G}$, and we give some examples.

2. When $\mathcal{K} \subset C^*(\mathcal{G})$

The most interesting case is when $\mathcal{T}$ contains $\mathcal{K}$ – the ideal of compact operators on $L^2(P)$.

Sufficient conditions ($P \cap P^{-1} = \{e\}$ and X to be a regular compactification of P) are given in [5, § 3.7.2]. We recall that X is called a regular compactification of P if $i(P)$ is open in X and the embedding of P in X is a homeomorphism of P to $i(P)$. Later, Sheu [10, Theorem 1] proved that if X is not a regular compactification of P, then $\mathcal{T}$ is not of type I and contains no nontrivial compact operators.

The algebras, discussed in [7] and [5] satisfy those conditions.

Let X be a regular compactification of P. Then $U = i(P)$ is an open and invariant subset of $X = \mathcal{G}^0$ and the above exact sequence (1.1) is

$$0 \longrightarrow \mathcal{K} \xrightarrow{i} C^*(\mathcal{G}) \xrightarrow{\gamma} C^*(\mathcal{G})/\mathcal{K} = C^*(\mathcal{G}_{|F}) \longrightarrow 0.$$

This short exact sequence gives a criterion for operator $T \in \mathcal{T}$ to be Fredholm.

Theorem 2.1. *An operator $T \in \mathcal{B}$ is Fredholm if and only if $\gamma(T)$ is invertible in $C^*(\mathcal{G}_{|F})$.*

Proof. This theorem is a corollary of well known statement, called as Theorem of Atkinson in [6, Theorem 1.4.16] and as Theorem of Nikolskii in [4, Ch. 3, § 3, Theorem 19]. □

Remark 2.2. *If $a \in C^*(\mathcal{G})$, then $a - \psi\gamma(a) \in \mathcal{K}$, because $\gamma(a - \psi\gamma(a)) = 0$ and the exactness of the sequence.*

So a is Fredholm iff $\psi\gamma(a)$ is Fredholm, and a and $\psi\gamma(a)$ have a same Fredholm index.

3. A linear cross-section in $C^*(\mathcal{G})$, generated by contractions in the unit space of the groupoid $\mathcal{G}$

It is natural to ask how to define a continuous linear cross-section.

In the case of groupoid C^* algebras of Wiener-Hopf groupoids we may define continuous linear cross-sections using contractions in $X = \mathcal{G}^0$ – the unit space of $\mathcal{G}$.

Let F be a closed and invariant subset of $X = \mathcal{G}^0$ and let $\lambda\colon X \longrightarrow F$ be a continuous contraction (i. e. $\lambda(x) = x$, for all $x \in F$).

Theorem 3.1. *In the above notations, the map*

$$\psi(b)(x, n) = b(\lambda(x), n), \quad b \in C_c(\mathcal{G}_{|F})$$

is a continuous cross- section.

Example 3.1. Let $G = \mathbb{Z}$, and $P = \mathbb{Z}_+ = \{0, 1, 2, \dots, n, \dots\}$ be the subsemigroup of the natural numbers. Define $Y = \mathbb{Z} \cup \{\infty\}$. There is an obvious embedding $i\colon G \hookrightarrow Y$ and let G acts as translations on the points of $\mathbb{Z}$, and let ∞ remain fixed. Put $X = \mathrm{clos}(i(P))$. Define the groupoid $\mathcal{G} = (Y \times G)|X$.

The orbits in $\mathcal{G}^0 = X$ are $\mathbb{Z}_+$ and ∞. The isotropy group of ∞ is $\mathbb{Z}$, while $\mathcal{G}|\mathbb{Z}_+$ is principal and transitive. It is isomorphic to the trivial groupoid on $\mathbb{Z}_+$ under the map $(x, n) \mapsto \big(r(x, n), d(x, n)\big) = (x, x + n)$. So by [5, § 2.7.1] we conclude that $C^*(\mathcal{G}|\mathbb{Z}_+) \cong \mathcal{K}$, the ideal of the compact operators. By [5, Prop. 2.16] the quotient $C^*(\mathcal{G})/\mathcal{K}$ is isomorphic to $C^*(\mathbb{Z}) = C(T)$, since $\mathcal{G}|\{\infty\}$ is topologically isomorphic to $\mathbb{Z}$.

We note that the element of $\mathcal{T}$, defined with $S(x, n) = \delta_1(n)\chi_X(x)\chi_X(x + 1)$ is an isometry and generates $\mathcal{T}$. So, $\mathcal{T}$ is isomorphic to the C^*-algebra, generated by one isometry and here we gave a new proof of Theorem 1 and Theorem 2 of [1].

In this example we put $\lambda(y) = \infty$, for all $y \in Y$ and we obtain $\psi(b(x, n)) = b(\infty, n)$. This cross-section is equal to the cross-section, given in [2, Theorem 1].

There is an analogue of this formula, which defines continuous linear cross-section in the case when F is an union of finite number of closed and invariant subsets of X.

Suppose that $F_1, F_2, \ldots, F_n$ are closed and invariant subsets of X and $F = \bigcup_{i=1}^{n} F_i$. For $\sigma \subset \{1, 2, \ldots, n\}$, define rank$(\sigma)$ to be the number of the elements of σ and denote $F_\sigma = \bigcap_{i \in \sigma} F_i$. Let $\lambda_\sigma \colon X \longrightarrow F_\sigma$ be continuous contractions, such that $\lambda_{\sigma \cup \tau} = \lambda_\sigma \circ \lambda_\tau$ for all $\sigma, \tau \subset \{1, 2, \ldots, n\}$.

Theorem 3.2. *In the above notations, the map ψ given by the formula*

$$\psi(b)(x,n) = \sum_{\varnothing \neq \sigma \subset \{1,2,\ldots,n\}} (-1)^{\mathrm{rank}(\sigma)+1} b(\lambda_\sigma(x), n), b \in C_c\big(\mathcal{G}_{|F}\big)$$

is a continuous cross-section.

Proof. Let us choose $x \in F$. We may think that $x \in F_1$. In this case we have $\lambda_1(x) = x$. We have to prove that $\psi(b)(x,n) = b(x,n)$, i. e. ψ is an extension.

We will show that an appropriate grouping of the terms of the right hand sum annihilate each other, and there will remain only one term, namely $b(\lambda_1(x), n) = b(x,n)$.

Let $\sigma \subset \{1, 2, \ldots, n\}$. In the case when $1 \notin \sigma$, we choose $\rho = \sigma \cup \{1\}$. We have $\mathrm{rank}(\rho) = \mathrm{rank}(\sigma) + 1$ and $\lambda_\rho(x) = \lambda_{\sigma \cup \{1\}} = \lambda_\sigma \circ \lambda_1(x) = \lambda_\sigma(x)$. So the terms, corresponding to σ and ρ have equal values and opposite signs and therefore annihilate.

In the case when $1 \in \sigma$, we choose $\tau = \sigma \setminus \{1\}$. Again the terms, corresponding to σ and τ have equal values and opposite signs and therefore annihilate. Thus we see, that only one term stays on the right hand side, namely $b(\lambda_1(x), n) = b(x,n)$. So the map ψ is a continuous cross-section. □

Example 3.2. Let $G = \mathbb{Z}^2$, and $P = (\mathbb{Z}_+)^2$ be the subsemigroup of the integer valued points in the first quarterplane. Define $Y = \big(\mathbb{Z} \cup \{\infty\}\big)^2$. $\mathcal{F}$ is the C^*-algebra of Toeplitz operators on the quarterplane, investigated in [3].

There is an obvious embedding $i \colon G \hookrightarrow Y$. Let G act as translations on the i-th coordinate y_i of $y \in Y$ when y_i is a finite number, and let ∞ remain fixed. Put $X = \mathrm{clos}(i(P)) = \big(\mathbb{Z}_+ \cup \{\infty\}\big)^n$. Define the groupoid $\mathcal{G} = (Y \times G)|X$. For $\sigma \subset \{1,2\}$ define

$$F_\sigma = \{x \in X \colon x_j = \infty \text{ for } j \in \sigma\}.$$

Orbits in X are

$$\begin{aligned} F_{\{1\}} &= \{x = (\infty, x_2) \colon x_2 \in \mathbb{Z}_+ \cup \{\infty\}\}, \\ F_{\{2\}} &= \{x = (x_1, \infty) \colon x_1 \in \mathbb{Z}_+ \cup \{\infty\}\}, \\ F_{\{1,2\}} &= \{x = (\infty, \infty)\}. \end{aligned}$$

Define the contractions

$$\lambda_1 \colon (x_1, x_2) \mapsto (\infty, x_2),$$
$$\lambda_2 \colon (x_1, x_2) \mapsto (x_1, \infty),$$
$$\lambda_{1,2} \colon (x_1, x_2) \mapsto (\infty, \infty).$$

The cross-section ψ in this example is given by the formula

$$\psi b(x_1, x_2) = b(\infty, x_2) + b(x_1, \infty) - b(\infty, \infty).$$

This cross-section is equal to the cross-section given in [8, Prop. 2.2].

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