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ON THE $(\mathrm{Vil}_B;\alpha)$-DIAPHONY OF THE VAN DER CORPUT SEQUENCE CONSTRUCTED IN CANTOR SYSTEMS

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In the present paper the authors consider the so-called $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony as a suitable tool to investigate sequences constructed in arbitrary Cantor systems. The definition of this kind of the diaphony is based on using Vilenkin function system and depends on two arguments – a vector α of exponential parameters and a vector γ of coordinate weights. This diaphony is used to investigate the distribution of the points of the Van der Corput sequence ω_B constructed in the same B-adic Cantor system. In this way a process of synchronization between the technique of a construction of the sequence ω_B and the tool of its studying is realized. Upper and low bounds of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B are presented. This permit us to show the influence of the exponential parameter α to the exact order of the $(\mathrm{Vil}_B;\alpha)$-diaphony of this sequence. When $\alpha = 2$ the exact order is $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ and when $\alpha > 2$ the exact order is $\mathcal{O}\left(\frac{1}{N}\right)$.

Keywords: Cantor number systems, Van der Corput sequence constructed in Cantor systems, Vilenkin function system, $(\mathrm{Vil}_B;\alpha)$-diaphony, exact orders

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1. Introduction

Let $s \geq 1$ be a fixed integer which will denote the dimension through the paper. Following Kuipers and Niederreiter [17] we will remind the concept of uniformly distributed sequence. Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in the unit cube $[0,1)^s$. Let J be an arbitrary subinterval of $[0,1)^s$ with Lebesque measure $\mu(J)$. For an arbitrary integer $N \geq 1$ let us denote $A_N(J;\xi) = \#\{n\colon 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$.

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The sequence ξ is called uniformly distributed in $[0,1)^s$ if the limit equality $\lim\limits_{N\to\infty} \frac{A_N(J;\xi)}{N} = \mu(J)$ holds for each subinterval J of $[0,1)^s$.

Some classes of complete orthonormal function systems are used as an analytical tools for studying the distribution of the points of sequences.

For an arbitrary integer k the function $e_k\colon [0,1) \to \mathbb{C}$ is defined as $e_k(x) = e^{2\pi \mathbf{i} kx}$, $x \in [0,1)$. For an arbitrary vector $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{Z}^s$ the function $e_{\mathbf{k}}\colon [0,1)^s \to \mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x}) = \prod\limits_{j=1}^{s} e_{k_j}(x_j)$, $\mathbf{x} = (x_1,\dots,x_s) \in [0,1)^s$. The set $\mathcal{T}_s = \{e_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{Z}^s,\ \mathbf{x} \in [0,1)^s\}$ is called trigonometric function system.

Let $b \geq 2$ be a fixed integer. The so-called Walsh functions in base b are defined in the following manner: For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the b-adic representations $k = \sum\limits_{i=0}^{\nu} k_i b_i$ and $x = \sum\limits_{i=0}^{\infty} x_i b^{-i-1}$, where $k_i, x_i \in \{0,1,\dots, b-1\}$, $k_\nu \neq 0$ and for infinitely many values of i we have $x_i \neq b-1$, the corresponding k-th Walsh function ${}_b\mathrm{wal}_k\colon [0,1) \to \mathbb{C}$ is defined as ${}_b\mathrm{wal}_k(x) = e^{\frac{2\pi \mathbf{i}}{b}(k_0x_0+\cdots+k_\nu x_\nu)}$.

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For an arbitrary vector $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th function of Walsh in base b is defined as ${}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}) = \prod\limits_{j=1}^{s} {}_b\mathrm{wal}_{k_j}(x_j)$, $\mathbf{x} = (x_1,\dots,x_s) \in [0,1)^s$. The set $\mathcal{W}(b) = \{{}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called the system of the Walsh functions in base b. In 1923 Walsh [24] defined the Walsh functions in base $b=2$ and in 1955 Chrestenson [6] consider the Walsh functions in arbitrary base $b \geq 2$.

In 2011 Hallekalek and Niederreiter [16] introduced the concept of the so-called b-adic function system. So, let the base b, the arbitrary integer $k \in \mathbb{N}_0$ and the real $x \in [0,1)$ be as above. Then, the corresponding k-th b-adic function ${}_b\gamma_k\colon [0,1) \to \mathbb{C}$ is defined as

$$
{}_b\gamma_k(x) = e^{2\pi \mathbf{i}\left(\frac{k_0}{b}+\frac{k_1}{b^2}+\cdots+\frac{k_\nu}{b^{\nu+1}}\right)(x_0+x_1b+x_2b^2+\cdots)}.
$$

For an arbitrary vector $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th b-adic function is defined as ${}_b\gamma_{\mathbf{k}}(\mathbf{x}) = \prod\limits_{j=1}^{s} {}_b\gamma_{k_j}(x_j)$, $\mathbf{x} = (x_1,\dots,x_s) \in [0,1)^s$. The set $\Gamma_b = \{{}_b\gamma_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{N}_0^s,$ $\mathbf{x} \in [0,1)^s\}$ is called b-adic function system.

First in 2010 Hallekalek [13] introduced the concept of the so-called $\Gamma_{\mathbf{p}}$ function system. So, let $\mathbf{p} = (p_1,\dots,p_s) \in \mathbb{N}_0^s$ be an arbitrary vector of not distinct different prime numbers. For an arbitrary vector $\mathbf{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th $\mathbf{p}$-adic function is defined as ${}_{\mathbf{p}}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod\limits_{j=1}^{s} {}_{p_j}\gamma_{k_j}(x_j)$, $\mathbf{x} = (x_1,\dots,x_s) \in [0,1)^s$. The function system $\Gamma_{\mathbf{p}}$ is defined as $\Gamma_{\mathbf{p}} = \{{}_{\mathbf{p}}\gamma_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{N}_0^s, \mathbf{x} \in [0,1)^s\}$.

Baycheva and Grozdanov [4] made a chronological survey of the diaphony as a quantitative measure for the irregularity of the distribution of sequences. Some reasons, related to the practice of the Quasi-Monte Carlo integration in weighted reproducing kernel Hilbert spaces are used, to present the different version of the diaphony. Special attention is devoted to the hybrid version of the diaphony, as quantitative measure for studying classes of hybrid sequences and nets. So, we will remind some kinds of the diaphony. In 1976 Zinterhof [25] proposed the first example of the diaphony, which today is called a classical diaphony. The concept of the classical diaphony is based on using the trigonometric function system $\mathcal{T}_s$. So, for an arbitrary integer $N \geq 1$ the diaphony $F_N(\mathcal{T}_s; \xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(\mathcal{T}_s;\xi) = \left(\sum_{\mathbf{k}\in\mathbb{Z}^s\setminus\{\mathbf{0}\}} R^{-2}(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} e_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{Z}^s$ the coefficient $R(\mathbf{k}) = \prod_{j=1}^{s} R(k_j)$ and for an arbitrary integer k the coefficient $R(k)$ is defined as

$$R(k) = \begin{cases} 1, & \text{if } k = 0, \\ |k|, & \text{if } k \neq 0. \end{cases}$$

In 1997 Hellekalek and Leeb [15] used the system $\mathcal{W}(2)$ of Walsh functions to define the so-called dyadic diaphony. In 2001 Grozdanov and Stoilova [9, 10] generalized the concept of the dyadic diaphony to the notion of the b-adic diaphony. So, for an arbitrary integer $N \geq 1$ the b-adic diaphony $F_N(\mathcal{W}(b); \xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(\mathcal{W}(b);\xi) = \left(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b\mathrm{wal}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the coefficient $\rho(\mathbf{k}) = \prod_{j=1}^{s} \rho(k_j)$ and for an arbitrary integer $k \geq 0$

$$\rho(k) = \begin{cases} 1, & \text{if } k = 0, \\ b^{-2g}, & \text{if } b^g \leq k < b^{g+1},\ g \geq 0,\ g \in \mathbb{Z}. \end{cases}$$

In 2010 Hallekalek [13] introduced the notion of the so-called $\mathbf{p}$-adic diaphony, which is based on using the system $\Gamma_{\mathbf{p}}$. So, for an arbitrary integer $N \geq 1$ the $\mathbf{p}$-adic diaphony $F_N(\Gamma_{\mathbf{p}}; \xi)$ of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is defined as

$$F_N(\Gamma_{\mathbf{p}};\xi) = \left(\frac{1}{\sigma_{\mathbf{p}} - 1} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho_{\mathbf{p}}(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathbf{p}}\gamma_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the coefficient $\rho_{\mathbf{p}}(\mathbf{k}) = \prod_{j=1}^{s} \rho_{p_j}(k_j)$ and for an arbitrary integer $k \geq 0$ and a prime p

$$\rho_p(k) = \begin{cases} 1, & \text{if } k = 0, \\ p^{-2g}, & \text{if } p^g \leq k < p^{g+1},\ g \geq 0,\ g \in \mathbb{Z}. \end{cases}$$

Here, the quantity $\sigma_{\mathbf{p}}$ is defined as $\sigma_{\mathbf{p}} = \prod_{j=1}^{s}(p_j+1)$.

Hallekalek [14] constructed the so-called hybrid function system, which is a tensor product of the trigonometric system, the system of Walsh function in base b and the b-adic function system. This function system is used to introduce the hybrid version of the diaphony.

2. The Vilenkin function system and the $(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma)$-diaphony

We will present the constructive principle of the so-called Cantor systems. They are natural generalizations of the ordinary b-adic number system. Let $B = \{b_0, b_1, b_2, \dots : b_i \geq 2 \text{ for } i \geq 0\}$ be given sequence of integers. By using the sequence B, the so-called generalized powers are defined by the next recursive equalities: $B_0 = 1$ and for $j \geq 0$ we put $B_{j+1} = B_j.b_j$. For this system we will use the name B-adic system.

An arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ in the B-adic system have representations of the form $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} \dfrac{x_i}{B_{i+1}}$, where for $i \geq 0$ $k_i, x_i \in \{0, 1, \dots, b_i - 1\}$ and $k_\nu \neq 0$. This representation of k is unique. In additional condition that for infinitely many i we have that $x_i \neq b_i - 1$ the representation of x is also unique.

Vilenkin [23] proposed new orthonormal function system defined in B-adic system. We will remind the construction of the functions of this system.

Definition 2.1. For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the B-adic representations of the form $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where for $i \geq 0$, $k_i, x_i \in \{0, 1, \dots, b_i - 1\}$, $k_\nu \neq 0$ and for infinitely many values of i we have $x_i \neq b_i - 1$, the k-th Vilenkin function ${}_B\mathrm{Vil}_k \colon [0,1) \to \mathbb{C}$ is defined as

$${}_B\mathrm{Vil}_k(x) = \prod_{i=0}^{\nu} e^{\frac{2\pi \mathbf{i}}{b_i} k_i x_i}.$$

Now, we will give the multidimensional version of the Vilenkin functions. For this purpose, for $1 \leq j \leq s$ let $B_j = \{b_0^{(j)}, b_1^{(j)}, b_2^{(j)}, \dots : b_i^{(j)} \geq 2 \text{ for } i \geq 0\}$ be given s sequences of integer numbers. Let us signify $\mathcal{B}_s = (B_1, \dots, B_s)$. The multidimensional Vilenkin functions are defined in the following manner:

Definition 2.2. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th function of Vilenkin ${}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}} \colon [0,1)^s \to \mathbb{C}$ is defined as ${}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\mathrm{Vil}_{k_j}(x_j)$, $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. The set $\mathrm{Vil}_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}) \colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ is called multidimensional Vilenkin function system.

In 1947 the function system $\mathrm{Vil}_{\mathcal{B}_s}$ was introduced by Vilenkin [23] and in 1957 independently from him this system was proposed by Price [18]. Some names are used about the system $\mathrm{Vil}_{\mathcal{B}_s}$ in the literature: both Price system, see Agaev et al. [1] and Vilenkin system, see Schipp, Wade and Simon [21]. For the system $\mathrm{Vil}_{\mathcal{B}_s}$ the name multiplicative system is also used. In this work we will use the name Vilenkin function system.

Now, we will remind the concept of the so-called $(\mathrm{Vil}_{\mathcal{B}_s}; \alpha; \gamma)$-diaphony. Baycheva and Grozdanov [2, 3] introduced the general concept of the so-called hybrid weighted diaphony. The construction of this diaphony is closely related to the worst-case error of the integration in reproducing kernel Hilbert spaces. These Hilbert spaces are characterized by two arguments. The first is a vector $\alpha = (\alpha_1, \dots, \alpha_s)$, where $\alpha_j > 1$ for $1 \le j \le s$ of exponential parameters. They determine the rate of inclining to zero of the Fourier's coefficients of the functions of this class. The second one is a vector $\gamma = (\gamma_1, \dots, \gamma_s)$, where $\gamma_1 \ge \gamma_2 \ge \dots \ge \gamma_s > 0$, of coordinate weights. They determine the dependence of the functions on their arguments. These two arguments α and γ are used to define the diaphony. In this way the worst-case error and the diaphony are connected.

On other side, the definition of the diaphony is based on using some concrete orthonormal function system. For example the definition of the hybrid weighted diaphony is based on using special kind of a hybrid function system.

Here in our work we will present very special kind of the hybrid weighted diaphony. The hybrid function system will be replaced by the system $\mathrm{Vil}_{\mathcal{B}_s}$ of the Vilenkin functions. The details are as follows: Let B be an arbitrary sequence of bases and $\{B_0, B_1, B_2, \dots\}$ be the corresponding sequence of generalized powers. For arbitrary reals $\alpha > 1$, $\gamma > 0$ and an arbitrary integer $k \ge 0$ let us define the coefficient

$$\rho(B; \alpha; \gamma; k) = \begin{cases} 1, & \text{if } k = 0, \\ \gamma / B_g^{\alpha}, & \text{if } B_g \le k \le B_{g+1} - 1,\ g \ge 0,\ g \in Z. \end{cases}$$

For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ by using the set $\mathcal{B}_s$ let us define the coefficient

$$R(\mathcal{B}_s; \alpha; \gamma; \mathbf{k}) = \prod_{j=1}^{s} \rho(B_j; \alpha_j; \gamma_j; k_j). \tag{2.1}$$

Let us define the constant

$$C(\mathcal{B}_s; \alpha; \gamma) = \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} R(\mathcal{B}_s; \alpha; \gamma; \mathbf{k}). \tag{2.2}$$

We have that $C(\mathcal{B}_s; \alpha; \gamma) = \prod_{j=1}^{s} [1 + \gamma_j \cdot \mu(B_j; \alpha_j)] - 1$, where $\mu(B; \alpha) = \sum_{g=0}^{\infty} \dfrac{b_g - 1}{B_g^{\alpha} - 1}$.

Definition 2.3. For an arbitrary integer $N \ge 1$ the weighted $(\mathrm{Vil}_{\mathcal{B}_s}; \alpha; \gamma)$-diaphony of the first N elements of the sequence $\xi = (\mathbf{x}_n)_{n \ge 0}$ of points in $[0, 1)^s$ is defined as

$$F_N(\mathrm{Vil}_{\mathcal{B}_s}; \alpha; \gamma; \xi) = \left(\frac{1}{C(\mathcal{B}_s; \alpha; \gamma)} \sum_{\mathbf{k} \in \mathbb{N}_0^s \setminus \{\mathbf{0}\}} R(\mathcal{B}_s; \alpha; \gamma; \mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s}\mathrm{Vil}_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for an arbitrary vector $\mathbf{k} \in \mathbb{N}_0^s$ the coefficient $R(\mathcal{B}_s;\alpha;\gamma;\mathbf{k})$ is defined by equality (2.1) and the constant $C(\mathcal{B}_s;\alpha;\gamma)$ by equality (2.2).

In the case when $\alpha = \mathbf{2} = (2,\dots,2)$ and $\gamma = \mathbf{1} = (1,\dots,1)$ the $(\mathrm{Vil}_{\mathcal{B}_s};\mathbf{2};\mathbf{1})$-diaphony was introduced by Grozdanov and Stoilova [11].

It is well-known fact that the sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the limit equality $\lim_{N\to\infty} F_N(\mathrm{Vil}_{\mathcal{B}_s};\alpha;\gamma;\xi) = 0$ holds for each choice of the vectors α and γ.

We note the fact that in the one-dimensional case the coordinate weight γ from Definition 2.3 is canceled. This gives us the right in the place of the notion of $(\mathrm{Vil}_B;\alpha;\gamma)$-diaphony $F_N(\mathrm{Vil}_B;\alpha;\gamma;\xi)$ of the sequence ξ to use the notion of $(\mathrm{Vil}_B;\alpha)$-diaphony and the denotation $F_N(\mathrm{Vil}_B;\alpha;\xi)$. We will follow this signification to the end of our work.

The sequence of Van der Corput is a classical example of well uniformly distributed sequence, which has a long history and many generalization related to different purposes. Bednařik et al. [5] consider the construction of this sequence, as also its multidimensional version, in Cantor systems. So, following their idea we will remind the concept of this sequence. Let B be the sequence as above.

Definition 2.4. For an arbitrary integer $n \geq 0$ which has the B-adic representation

$$n = n_m B_m + n_{m-1} B_{m-1} + \cdots + n_1 B_1 + n_0 B_0,$$

where $n_i \in \{0,1,\dots,b_i - 1\}$ for $0 \leq i \leq m$ and $n_m \neq 0$, we put

$$p_B(n) = \frac{n_0}{B_1} + \frac{n_1}{B_2} + \cdots + \frac{n_m}{B_{m+1}}.$$

The sequence $\omega_B = (p_B(n))_{n\geq 0}$ is called Van der Corput sequence constructed in the B-adic Cantor system.

Let the sequence B of bases is $B = \{b, b, \dots : b \geq 2\}$, i.e. all bases are equal to b. In this case the sequence $\omega_b = (p_b(n))_{n\geq 0}$ is obtained. If the base $b = 2$, then we find the classical Van der Corput [22] sequence $\omega_2 = (p_2(n))_{n\geq 0}$.

In 1960 Halton [12] used pairwise coprime integers $b_1,\dots,b_s$ to construct the sequence $((p_{b_1}(n),\dots,p_{b_s}(n)))_{n\geq 0}$, which is the s-dimensional version of the Van der Corput sequence.

Faure [7, 8] developed an another approach to generalize the construction of the Van der Corput sequence. He proposed to include permutations chosen either deterministically or randomly in the radical-inverse function.

In 1987 Proinov and Grozdanov [19, 20] investigated the diaphony $F_N(\mathcal{T}_s;\omega_b)$ of the Van der Corput sequence. It is shown that the classical diaphony of the sequence ω_b has an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$. In 2001 Grozdanov and Stoilova [10] showed that the b-adic diaphony $F_N(\mathcal{W}(b);\omega_b)$ has an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$.

3. Statements of the results

Now, we will present the main results of the paper. In Theorem 3.1 an upper bound of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B is presented. This bound permits us to obtain the asymptotic behaviour depending on the exponential parameter α of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B.

Theorem 3.1. *Let us assume that the sequence B of bases is limited from above, i.e. there exists a constant M such that for each $i \geq 0$ we have $b_i \leq M$. Let $N \geq 1$ be an arbitrary integer which in the B-adic system has a representation of the form*

$$N = a_1 B_{\nu_1} + a_2 B_{\nu_2} + \cdots + a_t B_{\nu_t},$$

where $\nu_1 > \nu_2 > \cdots > \nu_t \geq 0$ and $a_i \in \{1, 2, \ldots, b_i - 1\}$ for $1 \leq i \leq t$. Let the exponential parameter $\alpha \geq 2$. Then, the following holds:

(i) (*an upper bound*) *The $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B satisfies the inequality*

$$[N \cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 \leq \frac{1}{\mu(B;\alpha)} \left[\left(\frac{2^{\alpha+2}}{2^\alpha - 2} \right)^2 M^4 - \frac{M^{2\alpha} - M^{\alpha+1} + M^\alpha - 1}{(M^\alpha - 1)(M^\alpha - M)} \right] \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha};$$

(ii) (*an asymptotic behaviour*) *The $(\mathrm{Vil}_B;\alpha;\gamma)$-diaphony of the sequence ω_B has the following asymptotic behaviour:*

(ii_1) *If $\alpha = 2$, then* $F_N(\mathrm{Vil}_B;\alpha;\omega_B) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$;

(ii_2) *If $\alpha > 2$, then* $F_N(\mathrm{Vil}_B;\alpha;\omega_B) \in \mathcal{O}\left(\frac{1}{N}\right)$.

Let us note the fact that the quantity $\sum\limits_{i=1}^{t} B_{\nu_i}^{2-\alpha}$ gives the orders of the $(\mathrm{Vil}_B;\alpha;\gamma)$-diaphony of the sequence ω_B. The main sense of this quantity is that it shows the influence of the exponential parameter α to these orders. This result shows the importance of the parameter α to the orders of the considered diaphony. *The authors think that this is the priority of using the parameter α to obtain the wide spectrum of the orders of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B.*

We also note the fact that Grozdanov and Stoilova [11] obtain the order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ of the B-adic diaphony of the sequence ω_B. But in this result missing the idea for the exponential parameter an only this order is obtained.

With a purpose to prove the exactness of the obtained in Theorem 3.1 orders, in Theorem 3.2 a lower bound of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B is presented.

Theorem 3.2. *Let us assume that the sequence B of bases is limited from above, i.e. there exists a constant M such that for each $i \geq 0$ we have $b_i \leq M$. Let $N \geq 1$ be an arbitrary integer with the B-adic representation of the form*

$$N = 101\ldots101,$$

where the number of the ones is exactly r and $r \geq 2$. Let the exponential parameter $\alpha \geq 2$. Then, the following holds:

(i) (*a lower bound*) *For infinitely many values of N of the above form the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B satisfies the inequality*

$$[N \cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 > \frac{1}{M^6\mu(B;\alpha)}\sum_{h=0}^{r-2} B_{2h+1}^{2-\alpha};$$

(ii) (*an asymptotic inclusions*) *The $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B has the following asymptotic inclusions:*

(ii$_1$) *If $\alpha = 2$, then $F_N(\mathrm{Vil}_B;\alpha;\omega_B) \in \Omega\left(\frac{\sqrt{\log N}}{N}\right)$;*

(ii$_2$) *If $\alpha > 2$, then $F_N(\mathrm{Vil}_B;\alpha;\xi_B) \in \Omega\left(\frac{1}{N}\right)$.*

We note the fact that the quantity $\sum_{h=0}^{r-2} B_{2h+1}^{2-\alpha}$, which is related to the special form of N, again gives us the dependence of the exact orders of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B on the exponential parameter α. In this way the exactness of the orders $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ and $\mathcal{O}\left(\frac{1}{N}\right)$ is proved.

4. Preliminary statements

To prove the main results of the paper we need to present some preliminary statements, related to the exact value of the trigonometric sum of the sequence ω_B with respect to the functions of the Vilenkin system.

Lemma 4.1. *Let $\omega_B = (p_B(n))_{n\geq 0}$ be the sequence of Van der Corput constructed in the B-adic system. Let $k \geq 1$ be an arbitrary integer with the B-adic representation*

$$k = k_1 B_{\alpha_1} + k_2 B_{\alpha_2} + \cdots + k_p B_{\alpha_p},$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p \geq 0$ and $k_j \in \{1, 2, \ldots, b_{\alpha_j} - 1\}$ for $1 \leq j \leq p$. Let N be an arbitrary integer with the B-adic representation

$$N = a_1 B_{\nu_1} + a_2 B_{\nu_2} + \cdots + a_t B_{\nu_t},$$

where $\nu_1 > \nu_2 > \cdots > \nu_t \geq 0$ and $a_j \in \{1, 2, \ldots, b_j - 1\}$ for $1 \leq j \leq t$. Then, the trigonometric sum of the sequence ω_B with respect to the functions of the Vilenkin system satisfies the equalities

$$\left|\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n))\right|$$

$$= \begin{cases} 0, & \text{if } \nu_t > \alpha_p, \\ \left|\sum_{h=0}^{a_t-1} e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}} k_p h}\right| B_{\nu_t}, & \text{if } \nu_t = \alpha_p, \\ \sum_{j=s+1}^{t} a_j B_{\nu_j}, & \text{if there is some } s, \\ & 1 < s \leq t-1,\ \nu_s > \alpha_p > \nu_{s+1}, \\ \left|\sum_{h=0}^{a_s-1} e^{\frac{2\pi \mathrm{i}}{b_{\nu_s}} k_p h} \cdot B_{\nu_s} + e^{\frac{2\pi \mathrm{i}}{b_{\nu_s}} k_p a_s} \sum_{j=s+1}^{t} a_j B_{\nu_j}\right|, & \text{if there is some } s, \\ & 1 \leq s \leq t-1,\ \nu_s = \alpha_p > \nu_{s+1}, \\ N, & \text{if } \alpha_p > \nu_1; \end{cases}$$

Proof. For arbitrary integers $\alpha, \nu \geq 0$ let us define the function

$$\delta_{B_\alpha}(\nu) = \begin{cases} 1, & \text{if } \alpha \geq \nu, \\ 0, & \text{if } \alpha < \nu. \end{cases}$$

First of all we will prove an useful equality. Let $\nu \geq 0$ and $P \equiv 0 \pmod{B_\nu}$ be arbitrary and fixed integers. Then, for each integer k as in the condition of the Lemma, the equality holds

$$\left|\sum_{n=P}^{P+B_\nu-1} {}_B\mathrm{Vil}_k(p_B(n))\right| = B_\nu \cdot \delta_{B_{\alpha_p}}(\nu). \tag{4.1}$$

Really, let an arbitrary integer n, $P \leq n \leq P + B_\nu - 1$, have the B-adic representation $n = n_q n_{q-1} \ldots n_\nu n_{\nu-1} n_{\nu-2} \ldots n_1 n_0$, where $n_i \in \{0, 1, \ldots, b_i - 1\}$ for $0 \leq i \leq q$. Here n_i, $0 \leq i \leq \nu - 1$, are variable digits and n_i, $\nu \leq j \leq q$, are fixed digits. Then, we have that $p_B(n) = 0.n_0 n_1 \ldots n_{\nu-1} n_\nu \ldots n_q$ and hence

$$\sum_{n=P}^{P+B_\nu-1} {}_B\mathrm{Vil}_k(p_B(n)) = \sum_{n_0=0}^{b_0-1} \sum_{n_1=0}^{b_1-1} \cdots \sum_{n_{\nu-1}=0}^{b_{\nu-1}-1} e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}} k_p n_{\alpha_p}} \cdot e^{\frac{2\pi \mathrm{i}}{b_{\alpha_{p-1}}} k_{p-1} n_{\alpha_{p-1}}} \cdots e^{\frac{2\pi \mathrm{i}}{b_{\alpha_1}} k_1 n_{\alpha_1}}. \tag{4.2}$$

Let us assume that $\alpha_p \leq \nu - 1$. Then, the corresponding sum $\sum_{n_{\alpha_p}=0}^{b_{\alpha_p}-1} e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}} k_p n_{\alpha_p}} = 0$ and from equality (4.2) we obtain that $\sum_{n=P}^{P+B_\nu-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0$.

Let us assume that $\alpha_p \geq \nu$. Then, from equality (4.2) we obtain that

$$\left|\sum_{n=P}^{P+B_\nu-1} {}_B\mathrm{Vil}_k(p_B(n))\right|$$
$$= \left|e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}}k_p n_{\alpha_p}} \cdot e^{\frac{2\pi \mathrm{i}}{b_{\alpha_{p-1}}}k_{p-1} n_{\alpha_{p-1}}} \cdots e^{\frac{2\pi \mathrm{i}}{b_{\alpha_1}}k_1 n_{\alpha_1}}\right| \cdot \sum_{n_0=0}^{b_0-1} 1 \cdots \sum_{n_{\nu-1}=0}^{b_{\nu-1}-1} 1 = B_\nu.$$

According to the defined function $\delta_{B_\alpha}(\nu)$ the above two results can be written as

$$\left|\sum_{n=P}^{P+B_\nu-1} {}_B\mathrm{Vil}_k(p_B(n))\right| = B_\nu \cdot \delta_{B_{\alpha_p}}(\nu).$$

Now, we can prove the statements of the Lemma. For this purpose let us introduce the significations

$$N_0 = 0,$$
$$N_1 = a_1 B_{\nu_1},$$
$$N_2 = a_1 B_{\nu_1} + a_2 B_{\nu_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$N_t = a_1 B_{\nu_1} + a_2 B_{\nu_2} + \cdots + a_t B_{\nu_t}, \text{ so } N_t = N.$$

Then, for each integer $k \geq 0$ we have that

$$\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) = \sum_{j=0}^{t-1} \sum_{h=0}^{a_{j+1}-1} \sum_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)). \tag{4.3}$$

I. Let us assume that $\nu_t > \alpha_p$. From equality (4.1) for each integers $0 \leq j \leq t-1$ and $0 \leq h \leq a_{j+1}-1$ we have that $\sum\limits_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0$ and from (4.3) we obtain that $\sum\limits_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0$.

II. Let us assume that $\nu_t = \alpha_p$. According to equality (4.1) for arbitrary integers $0 \leq j \leq t-2$ and $0 \leq h \leq a_{j+1}-1$ we have that $\sum\limits_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0.$

Let the integer h such that $0 \leq h \leq a_t - 1$ be fixed. Let an arbitrary integer n such that $N_{t-1}+h\cdot B_{\nu_t} \leq n \leq N_{t-1}+(h+1)\cdot B_{\nu_t}-1$ have the B-adic representation $n = n_\mu n_{\mu-1} \ldots n_{\nu_t+1} h n_{\nu_t-1} \ldots n_1 n_0$. Hence from (4.3) we obtain that

$$\left|\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n))\right| = \left|e^{\frac{2\pi \mathrm{i}}{b_{\alpha_{p-1}}}k_{p-1}\cdot n_{\alpha_{p-1}}} \cdots e^{\frac{2\pi \mathrm{i}}{b_{\alpha_1}}k_1\cdot n_{\alpha_1}}\right|$$
$$\times \left|\sum_{h=0}^{a_t-1} e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}}k_p\cdot h}\right| \cdot \sum_{n=N_{t-1}+h\cdot B_{\nu_t}}^{N_{t-1}+(h+1)\cdot B_{\nu_t}-1} 1 = \left|\sum_{h=0}^{a_t-1} e^{\frac{2\pi \mathrm{i}}{b_{\alpha_p}}k_p\cdot h}\right| \cdot B_{\nu_t}.$$

III. Let us assume that there is some s, $1 < s \le t-1$, such that $\nu_s > \alpha_p > \nu_{s+1}$. For each fixed integer j, $0 \le j \le s-1$, we have that $\nu_{j+1} > \alpha_p$. Then, from (4.1) for each fixed h, $0 \le h \le a_{j+1}-1$, the equality holds $\sum\limits_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0$.

For each fixed integer j, $s \le j \le t-1$, the inequality $\alpha_p > \nu_{j+1}$ holds. Let the integer h, $0 \le h \le a_{j+1}-1$, be fixed. Then, an arbitrary integer n such that $N_j + h \cdot B_{\nu_{j+1}} \le n \le N_j + (h+1)\cdot B_{\nu_{j+1}} - 1$ has the B-adic representation of the form

$$n = n_\mu \dots n_{\alpha_1} \dots n_{\alpha_2} \dots n_{\alpha_p} \dots n_{\nu_{j+1}+1} h n_{\nu_{j+1}-1} \dots n_1 n_0,$$

where $n_{\nu_{j+1}-1}, \dots, n_1, n_0$ are variable digits and $n_{\nu_{j+1}+1}, \dots, n_\mu$ are fixed digits. Hence we obtain that ${}_B\mathrm{Vil}_k(p_B(n)) = \prod\limits_{\beta=1}^{p} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_\beta}} k_\beta n_{\alpha_\beta}}$.

Then, from (4.3) and the above suppositions we obtain that

$$\left|\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n))\right| = \left|\prod_{\beta=1}^{p} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_\beta}} k_\beta n_{\alpha_\beta}}\right| \cdot \sum_{j=s}^{t-1} \sum_{h=0}^{a_{j+1}-1} B_{\nu_{j+1}} = \sum_{j=s}^{t-1} a_{j+1} B_{\nu_{j+1}}$$
$$= \sum_{j=s+1}^{t} a_j B_{\nu_j}.$$

IV. Let us assume that there is some s, $1 \le s \le t-1$, such that $\nu_s = \alpha_p > \nu_{s+1}$. According to equality (4.3) we will use the presentation

$$\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) = \sum_{j=0}^{s-2} \sum_{h=0}^{a_{j+1}-1} \sum_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n))$$
$$+ \sum_{h=0}^{a_s-1} \sum_{n=N_{s-1}+h\cdot B_{\nu_s}}^{N_{s-1}+(h+1)\cdot B_{\nu_s}-1} {}_B\mathrm{Vil}_k(p_B(n)) + \sum_{j=s}^{t-1} \sum_{h=0}^{a_{j+1}-1} \sum_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)). \tag{4.4}$$

For each j, $0 \le j \le s-2$, the inequality $\alpha_p < \nu_{j+1}$ holds. Then, from (4.1) for each fixed integer h, $0 \le h \le a_{j+1}-1$, the equality $\sum\limits_{n=N_j+h\cdot B_{\nu_{j+1}}}^{N_j+(h+1)\cdot B_{\nu_{j+1}}-1} {}_B\mathrm{Vil}_k(p_B(n)) = 0$ holds.

It is obvious that N_{s-1} has the B-adic representation of the form

$$N_{s-1} = a_1 0 \dots 0 a_2 0 \dots 0 \dots a_{s-1} \underbrace{00\dots0}_{\nu_{s-1}},$$

where for $1 \le q \le s-1$ the digit a_q stays on the ν_q-th position. Let h, $0 \le h \le a_s - 1$, be a fixed integer. Then, an arbitrary integer n such that $N_{s-1} + h \cdot B_{\nu_s} \le n \le N_{s-1} + (h+1) B_{\nu_s} - 1$ has the B-adic representation of the form

$$n = a_1 0 \dots 0 a_2 0 \dots 0 \dots a_{s-1} 0 \dots 0 h n_{\nu_s - 1} n_{\nu_s - 2} \dots n_1 n_0,$$

hence $p_B(n) = 0.n_0 n_1 \dots n_{\nu_s - 1} h 0 \dots 0 a_{s-1} \dots 0 \dots 0 \dots a_1$ and let us signify

$$p_B(n) = 0.n_0 n_1 \dots n_{\nu_s-1} h n_{\nu_s+1} n_{\nu_s+2} \dots n_{\nu_1}. \tag{4.5}$$

Then, we have that ${}_B\mathrm{Vil}_k(p_B(n)) = e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p h} \prod_{r=1}^{p-1} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_r}} k_r n_{\alpha_r}}$.

For each integer j, $s \le j \le t-1$, the number N_j has the B-adic representation of the form $N_j = a_1 0 \dots 0 a_2 0 \dots 0 \dots a_{s-1} 0 \dots 0 \dots a_j \underbrace{00\dots0}_{\nu_j}$, where for $1 \le q \le j$ the digit a_q stays on the ν_q-th position. Let the index h, $0 \le h \le a_{j+1} - 1$, be fixed. Then, an arbitrary integer n, $N_j + h \cdot B_{\nu_{j+1}} \le n \le N_j + (h+1) \cdot B_{\nu_{j+1}} - 1$, has the B-adic representation of the form

$$n = a_1 0 \dots 0 a_2 0 \dots 0 \dots a_{s-1} 0 \dots 0 \dots a_j 0 \dots 0 h n_{\nu_{j+1}-1} n_{\nu_{j+1}-2} \dots n_1 n_0.$$

Hence we have that $p_B(n) = 0.n_0 n_1 \dots n_{\nu_{j+1}-1} h 0 \dots 0 \dots a_j 0 \dots 0 \dots a_1$ and let us signify

$$p_B(n) = 0.n_0 n_1 \dots n_{\nu_{j+1}-1} h n_{\nu_{j+1}+1} n_{\nu_{j+1}+2} \dots n_{\nu_1}. \tag{4.6}$$

Then, we obtain that ${}_B\mathrm{Vil}_k(p_B(n)) = e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p a_s} \cdot \prod_{r=1}^{p-1} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_r}} k_r n_{\alpha_r}}$.

We note the important fact that for $\nu_s + 1 \le q \le \nu_1$ the digits n_q in the presentations (4.5) and (4.6) are equal.

Hence from the presentation (4.4) and the above assumptions we obtain that

$$\left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right| = \left| \prod_{r=1}^{p-1} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_r}} k_r n_{\alpha_r}} \right| \cdot \left| \sum_{h=0}^{a_s-1} e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p h} B_{\nu_s} + e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p a_s} \sum_{j=s}^{t-1} a_{j+1} B_{\nu_{j+1}} \right|$$
$$= \left| \sum_{h=0}^{a_s-1} e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p h} B_{\nu_s} + e^{\frac{2\pi \mathbf{i}}{b_{\nu_s}} k_p a_s} \sum_{j=s+1}^{t} a_j B_{\nu_j} \right|.$$

V. Let us assume that $\alpha_p > \nu_1$. An arbitrary integer n, $0 \le n \le N-1$, has the B-adic representation $n = n_{\nu_1} n_{\nu_1 - 1} \dots n_{\nu_2} \dots n_{\nu_t} n_{\nu_t - 1} \dots n_1 n_0$. This means that on the positions biggest that ν_1 the digits of n are equal to zero and hence ${}_B\mathrm{Vil}_k(p_B(n)) = \prod_{\beta=0}^{p} e^{\frac{2\pi \mathbf{i}}{b_{\alpha_\beta}} k_\beta \cdot 0} = 1$. This gives us that $\sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) = N$. □

By using the equalities presented in Lemma 4.1 it is easy to prove the following result.

Corollary 4.1. *Let $\omega_B = (p_B(n))_{n \ge 0}$ be the Van der Corput sequence constructed in the B-adic system. Let k and N be as in the condition of Lemma 4.1. Then, the trigonometric sum of the sequence ω_B with respect to the functions of the Vilenkin system satisfies the inequality*

$$\left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right| \le \sum_{i=1}^{t} a_i B_{\nu_i} \delta_{B_{\alpha_p}}(\nu_i).$$

5. Proof of Theorem 3.1

(i) According to Definition 2.3 the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B satisfies the equality

$$[N\cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2=\frac{1}{\mu(B;\alpha)}\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\sum_{k=B_g}^{B_{g+1}-1}\left|\sum_{n=0}^{N-1}{}_B\mathrm{Vil}_k(p_B(n))\right|^2. \tag{5.1}$$

For an arbitrary integer $g\ge 0$ let us introduce the set

$$B(g)=\{k\colon k=k_gB_g,\ k_g\in\{1,2,\dots,b_g-1\}\}.$$

For arbitrary integers $g\ge 1$ and q such that $0\le q\le g-1$ let us introduce the set

$$A(g;q)=\{k\colon k=k_gB_g+k_{g-1}B_{g-1}+\cdots+k_{q+1}B_{q+1}+k_qB_q,k_g\in\{1,2,\dots,b_g-1\},$$
$$k_q\in\{1,2,\dots,b_q-1\}\text{ for }q+1\le j\le g-1,\ k_j\in\{0,1,\dots,b_j-1\}\}.$$

The cardinalities $|B(g)|=b_g-1$ and $|A(g;q)|=(b_g-1)(b_{g-1}\dots b_{q+1})(b_q-1)$ hold. In this way from the equality (5.1) we obtain that

$$[N\cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2=\frac{1}{\mu(B;\alpha)}\left[\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\sum_{k\in B(g)}\left|\sum_{n=0}^{N-1}{}_B\mathrm{Vil}_k(p_B(n))\right|^2\right.$$
$$\left.+\sum_{g=1}^{\infty}\frac{1}{B_g^{\alpha}}\sum_{q=0}^{g-1}\sum_{k\in A(g;q)}\left|\sum_{n=0}^{N-1}{}_B\mathrm{Vil}_k(p_B(n))\right|^2\right]=\frac{1}{\mu(B;\alpha)}(\Sigma_1+\Sigma_2). \tag{5.2}$$

Now, we will obtain upper bounds of the sums Σ_1 and Σ_2. According to the statement of Corollary 4.1 we consecutively obtain the next results:

$$\Sigma_1=\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\sum_{k\in B(g)}\left|\sum_{n=0}^{N-1}{}_B\mathrm{Vil}_k(P_B(n))\right|^2$$
$$\le\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\sum_{k\in B(g)}\left[2\sum_{i=1}^{t}\sum_{j=1}^{i}a_ia_jB_{\nu_i}B_{\nu_j}\delta_{B_g}(\nu_i)\delta_{B_g}(\nu_j)-\sum_{i=1}^{t}a_i^2B_{\nu_i}^2\delta_{B_g}(\nu_i)\right]$$
$$=2\sum_{i=1}^{t}a_iB_{\nu_i}\sum_{j=1}^{i}a_jB_{\nu_j}\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\delta_{B_g}(\nu_i)\delta_{B_g}(\nu_j)\sum_{k\in B(g)}1$$
$$-\sum_{i=1}^{t}a_i^2B_{\nu_i}^2\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\delta_{B_g}(\nu_i)\sum_{k\in B(g)}1$$
$$\le 2(M-1)^3\sum_{i=1}^{t}B_{\nu_i}\sum_{j=1}^{i}B_{\nu_j}\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\delta_{B_g}(\nu_i)\delta_{B_g}(\nu_j)-\sum_{i=1}^{t}B_{\nu_i}^2\sum_{g=0}^{\infty}\frac{1}{B_g^{\alpha}}\delta_{B_g}(\nu_i). \tag{5.3}$$

The condition $j \le i$ shows that $\nu_j \ge \nu_i$. If we put the condition $g \ge \nu_j$ we will have that $g \ge \nu_i$ and $\delta_{B_g}(\nu_i)\cdot\delta_{B_g}(\nu_j) = 1$. By analogy in the second sum of (5.3) we put the condition $g \ge \nu_i$ and we will have that $\delta_{B_g}(\nu_i) = 1$. In this way from (5.3) we obtain

$$\begin{aligned}
\Sigma_1 &\le 2(M-1)^3 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j}^{\infty} \frac{1}{B_g^\alpha} - \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i}^{\infty} \frac{1}{B_g^\alpha} \\
&\le 2(M-1)^3 \frac{2^\alpha}{2^\alpha-1} \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} \frac{1}{B_{\nu_j}^{\alpha-1}} - \frac{M^\alpha}{M^\alpha-1} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} \\
&\le 2(M-1)^3 \frac{2^\alpha}{2^\alpha-1} \sum_{i=1}^{t} B_{\nu_i} \sum_{j=\nu_i}^{\infty} \frac{1}{B_j^{\alpha-1}} - \frac{M^\alpha}{M^\alpha-1} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} \\
&\le 2(M-1)^3 \frac{2^\alpha}{2^\alpha-1} \frac{2^\alpha}{2^\alpha-2} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} - \frac{M^\alpha}{M^\alpha-1} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} \\
&= \left[2(M-1)^3 \frac{2^{2\alpha}}{(2^\alpha-1)(2^\alpha-2)} - \frac{M^\alpha}{M^\alpha-1}\right] \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}. \qquad (5.4)
\end{aligned}$$

We will use again the facts that the equalities $\delta_{B_g}(\nu_i)\delta_{B_g}(\nu_j) = 1$ and $\delta_{B_g}(\nu_i) = 1$ hold for $j \le i$ and $g \ge \nu_j$. In this way, for the sum Σ_2 we consecutively obtain the next results:

$$\begin{aligned}
\Sigma_2 &= \sum_{g=1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=0}^{g-1} \sum_{k\in A(g;q)} \left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right|^2 \\
&\le \sum_{g=1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=0}^{g-1} \sum_{k\in A(g;q)} \left[2\sum_{i=1}^{t}\sum_{j=1}^{i} a_i a_j B_{\nu_i} B_{\nu_j} \delta_{B_q}(\nu_i)\delta_{B_q}(\nu_j) - \sum_{i=1}^{t} a_i^2 B_{\nu_i}^2 \delta_{B_q}(\nu_i) \right] \\
&= 2\sum_{i=1}^{t} a_i B_{\nu_i} \sum_{j=1}^{i} a_j B_{\nu_j} \sum_{g=1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=0}^{g-1} \sum_{k\in A(g;q)} \delta_{B_q}(\nu_i)\delta_{B_q}(\nu_j) \\
&\qquad - \sum_{i=1}^{t} a_i^2 B_{\nu_i}^2 \sum_{g=1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=0}^{g-1} \sum_{k\in A(g;q)} \delta_{B_q}(\nu_i) \\
&\le 2(M-1)^2 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=\nu_j}^{g-1} \sum_{k\in A(g;q)} 1 \\
&\qquad - \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=\nu_i}^{g-1} \sum_{k\in A(g;q)} 1 \\
&\le 2(M-1)^2 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^\alpha} \sum_{q=\nu_j}^{g-1} (b_g-1)(b_{g-1}\dots b_{q+1})(b_q-1)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha}} \sum_{q=\nu_i}^{g-1} (b_g - 1)(b_{g-1} \dots b_{q+1})(b_q - 1) \\
&\le 2(M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^{\alpha}} \sum_{q=\nu_j}^{g-1} (b_{g-1} \dots b_{q+1}) \\
&\qquad - \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha}} \sum_{q=\nu_i}^{g-1} (b_{g-1} \dots b_{q+1}) \\
&\le \frac{2(M-1)^4}{2} \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^{\alpha}} \sum_{q=\nu_j}^{g-1} \frac{B_g}{B_q} - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha}} \sum_{q=\nu_i}^{g-1} \frac{B_g}{B_q} \\
&= (M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \sum_{q=\nu_j}^{g-1} \frac{1}{B_q} - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \sum_{q=\nu_i}^{g-1} \frac{1}{B_q} \\
&\le (M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \sum_{q=\nu_j}^{\infty} \frac{1}{B_q} - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i}^2 \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \cdot \frac{1}{B_{\nu_i}} \\
&= (M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} B_{\nu_j} \sum_{g=\nu_j+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \cdot \frac{1}{B_{\nu_j}} \left(1 + \frac{1}{b_{\nu_j}} + \frac{1}{b_{\nu_j} b_{\nu_j+1}} + \cdots \right) \\
&\qquad - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i} \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \\
&\le 2(M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} \sum_{g=\nu_j}^{\infty} \frac{1}{B_g^{\alpha-1}} - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i} \sum_{g=\nu_i+1}^{\infty} \frac{1}{B_g^{\alpha-1}} \\
&= 2(M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} \frac{1}{B_{\nu_j}^{\alpha-1}} \left(1 + \frac{1}{b_{\nu_j}^{\alpha-1}} + \frac{1}{(b_{\nu_j} b_{\nu_j+1})^{\alpha-1}} + \cdots \right) \\
&\qquad - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i} \frac{1}{B_{\nu_i+1}^{\alpha-1}} \left(1 + \frac{1}{b_{\nu_i+1}^{\alpha-1}} + \frac{1}{(b_{\nu_i+1} b_{\nu_i+2})^{\alpha-1}} + \cdots \right) \\
&\le 2(M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} \frac{1}{B_{\nu_j}^{\alpha-1}} \left(1 + \frac{1}{2^{\alpha-1}} + \frac{1}{(2^{\alpha-1})^2} + \cdots \right) \\
&\qquad - \frac{1}{M} \sum_{i=1}^{t} B_{\nu_i} \cdot \frac{1}{B_{\nu_i}^{\alpha-1} \cdot b_{\nu_i+1}^{\alpha-1}} \left(1 + \frac{1}{M^{\alpha-1}} + \frac{1}{(M^{\alpha-1})^2} + \cdots \right) \\
&\le \frac{2^{\alpha+1}}{2^{\alpha} - 2} (M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=1}^{i} \frac{1}{B_{\nu_j}^{\alpha-1}} - \frac{1}{M^{\alpha} - M} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} \\
&\le \frac{2^{\alpha+1}}{2^{\alpha} - 2} (M-1)^4 \sum_{i=1}^{t} B_{\nu_i} \sum_{j=\nu_i}^{\infty} \frac{1}{B_j^{\alpha-1}} - \frac{1}{M^{\alpha} - M} \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}
\end{aligned}
$$

$$\le \left(\frac{2^{\alpha+1}}{2^\alpha-2}\right)^2 (M-1)^4 \sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} - \frac{1}{M^\alpha - M}\sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}$$
$$= \left[\left(\frac{2^{\alpha+1}}{2^\alpha-2}\right)^2 (M-1)^4 - \frac{1}{M^\alpha - M}\right]\sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}. \quad (5.5)$$

From (5.2), (5.4) and (5.5) we obtain that

$$[N\cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 \le C(B;\alpha;M)\sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}, \quad (5.6)$$

where $C(B;\alpha;M) = \frac{1}{\mu(B;\alpha)}\left[\left(\frac{2^{\alpha+2}}{2^\alpha-2}\right)^2 M^4 - \frac{M^{2\alpha}-M^{\alpha+1}+M^\alpha-1}{(M^\alpha-1)(M^\alpha-M)}\right]$. The part (i) of the Theorem is proved.

(ii) Now, by using the statement (i) of the Theorem we are able to show the asymptotic behaviour of the $(\mathrm{Vil}_B;\alpha)$-diaphony of the sequence ω_B.

(ii_1) Let us assume that $\alpha = 2$. Then, from (5.6) we obtain the inequality

$$[N\cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 \le C(B;\alpha;M)\cdot t. \quad (5.7)$$

From the conditions $\nu_1 > \nu_2 > \cdots > \nu_t \ge 0$ we consecutively obtain that $\nu_t \ge 0$, $\nu_{t-1}\ge 1$, $\nu_{t-2}\ge 2, \dots, \nu_1 \ge t-1$. From the B-adic representation of N we have that $N \ge 2^{\nu_1}+2^{\nu_2}+\cdots+2^{\nu_t} > 2^{t-1}+2^{t-2}+\cdots+2^1+2^0 = 2^t-1$ and obtain that $t < \frac{\log(N+1)}{\log 2}$. Hence from (5.7) we obtain that

$$F_N(\mathrm{Vil}_B;\alpha;\omega_B) \le \sqrt{\frac{C(B;\alpha;M)}{\log 2}}\cdot\frac{\sqrt{\log(N+1)}}{N}.$$

The last inequality gives us that $F_N(\mathrm{Vil}_B;\alpha;\omega_B)\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$.

(ii_2) The condition $\alpha > 2$ permits us to obtain an upper bound of the sum $\sum_{i=1}^{t} B_{\nu_i}^{2-\alpha}$. So, the following inequalities holds

$$\sum_{i=1}^{t} B_{\nu_i}^{2-\alpha} = \sum_{i=1}^{t}\frac{1}{B_{\nu_i}^{\alpha-2}} < \frac{1}{B_0^{\alpha-2}}+\frac{1}{B_1^{\alpha-2}}+\frac{1}{B_2^{\alpha-2}}+\cdots \le 1+\frac{1}{2^{\alpha-2}}+\frac{1}{(2^{\alpha-2})^2}+\cdots$$
$$= \frac{2^\alpha}{2^\alpha-4}.$$

From (5.6) and the above result we obtain that

$$F_N(\mathrm{Vil}_B;\alpha;\omega_B) \le \sqrt{C(B;\alpha;M)\cdot\frac{2^\alpha}{2^\alpha-4}}\cdot\frac{1}{N},$$

which gives us that $F_N(\mathrm{Vil}_B;\alpha;\omega_B)\in\mathcal{O}\left(\frac{1}{N}\right)$. Theorem 3.1 is finally proved.

6. Proof of Theorem 3.2

Let the integer N be as in the condition of Theorem 3.2, so we have that

$$N = B_{2r-2} + B_{2r-4} + \cdots + B_2 + B_0. \tag{6.1}$$

We will use the general concept of the B-adic representation of N exposed in the condition of Lemma 4.1. Especially the representation of N of the form (6.1) gives us that $\nu_1 = 2(r-1)$, $\nu_2 = 2(r-2)$, $\ldots$, $\nu_r = 2(r-r) = 0$. For $0 \le g \le 2r-2$ let us introduce the set $B(g) = \{k\colon k = k_g B_g, k_g \in \{1, 2, \ldots, b_g - 1\}\}$. Then, from Definition 2.3 the low bound holds

$$[N \cdot F_N(\mathrm{Vil}_B; \alpha; \omega_B)]^2 \ge \frac{1}{\mu(B;\alpha)} \sum_{g=0}^{2r-2} B_g^{-\alpha} \sum_{k \in B(g)} \left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right|^2 .$$

In the first sum of the above expression we will realize a summation only on the odd subscripts g. So, let $g = 2h+1$, where $h = 0, 1, 2, \ldots, r-1$. Then, we obtain that

$$[N \cdot F_N(\mathrm{Vil}_B; \alpha; \omega_B)]^2 \ge \frac{1}{\mu(B;\alpha)} \sum_{h=0}^{r-2} B_{2h+1}^{-\alpha} \sum_{k \in B(2h+1)} \left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right|^2 . \tag{6.2}$$

Now, for each integer $k \in B(2h+1)$ we will obtain a low bound of the trigonometric sum $\left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right|$. An arbitrary integer $k \in B(2h+1)$ has the B-adic representation $k = k_{2h+1} B_{2h+1}$, i.e. we have that $\alpha_p = 2h+1$. The presentation (6.1) of N shows us that there is some s, $1 \le s \le t-1$, $\nu_s > \alpha_p > \nu_{s+1}$. In our case $s = r-h$. Hence the third case of Lemma 4.1 is realized. From this statement we have that

$$\begin{aligned} \left| \sum_{n=0}^{N-1} {}_B\mathrm{Vil}_k(p_B(n)) \right| &= \sum_{j=s+1}^{t} a_j B_{\nu_j} = \sum_{j=r-h+1}^{r} B_{2(r-j)} > B_{2(h-1)} \\ &= \frac{B_{2h+1}}{b_{2h-2} \cdot b_{2h-1} \cdot b_{2h}} \ge \frac{1}{M^3} \cdot B_{2h+1}. \end{aligned}$$

We put the above inequality in (6.2) and obtain

$$\begin{aligned} [N \cdot F_N(\mathrm{Vil}_B; \alpha; \omega_B)]^2 &> \frac{1}{\mu(B;\alpha)} \sum_{h=0}^{r-2} B_{2h+1}^{-\alpha} \sum_{k \in B(2h+1)} \left(\frac{1}{M^3} \cdot B_{2h+1} \right)^2 \\ &= \frac{1}{M^6 \cdot \mu(B;\alpha)} \sum_{h=0}^{r-2} B_{2h+1}^{2-\alpha} \cdot \sum_{k \in B(2h+1)} 1 \ge \frac{1}{M^6 \cdot \mu(B;\alpha)} \sum_{h=0}^{r-2} B_{2h+1}^{2-\alpha} \end{aligned} \tag{6.3}$$

and the part (i) of the Theorem is proved.

(ii$_1$) Let us assume $\alpha = 2$. Then, from (6.3) we obtain that

$$[N \cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 > \frac{1}{M^6\mu(B;\alpha)}(r-1) \geq \frac{r}{2M^6\mu(B;\alpha)}.$$

From the presentation of N of the form (6.1) the inequality holds

$$N \leq M^0 + M^2 + \cdots + M^{2r-2} = \frac{M^{2r}-1}{M-1},$$

whence we find that $r \geq \frac{\log N}{2\log M}$. Finally, we obtain that

$$F_N(\mathrm{Vil}_B;\alpha;\omega_B) > \frac{1}{2M^3\sqrt{\mu(B;\alpha)\log M}} \cdot \frac{\sqrt{\log N}}{N},$$

which gives us that $F_N(\mathrm{Vil}_B;\alpha;\omega_B) \in \Omega\left(\frac{\sqrt{\log N}}{N}\right)$.

(ii2) Let us assume that $\alpha > 2$. Then, from the inequality (6.3) we obtain that

$$\begin{aligned}
[N \cdot F_N(\mathrm{Vil}_B;\alpha;\omega_B)]^2 &> \frac{1}{M^6\mu(B;\alpha)}\sum_{h=0}^{r-2} B_{2h+1}^{2-\alpha} \\
&= \frac{1}{M^6\mu(B;\alpha)}(B_1^{2-\alpha} + B_3^{2-\alpha} + \cdots + B_{2r-3}^{2-\alpha}) \\
&> \frac{1}{M^6\mu(B;\alpha)}B_1^{2-\alpha} = \frac{1}{M^6\mu(B;\alpha)} \cdot \frac{1}{b_0^{\alpha-2}} \\
&\geq \frac{1}{M^{\alpha-2} \cdot M^6\mu(B;\alpha)} = \frac{1}{M^{4+\alpha}\mu(B;\alpha)}.
\end{aligned}$$

From the above inequality we obtain that

$$F_N(\mathrm{Vil}_B;\alpha;\omega_B) > \frac{1}{\sqrt{M^{4+\alpha}\mu(B;\alpha)}} \cdot \frac{1}{N},$$

which gives us that $F_N(\mathrm{Vil}_B;\alpha;\omega_B) \in \Omega\left(\frac{1}{N}\right)$. In this way Theorem 3.2 is finally proved.

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