ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 109

ANNUAL OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 109

SUBRECURSIVE INCOMPARABILITY OF THE GRAPHS OF STANDARD AND DUAL BAIRE SEQUENCES

IVAN GEORGIEV

Our main question of interest is the existence or the non-existence of a subrecursive reduction between different representations of the irrational numbers. For any representation, considered as a total function, we consider the characteristic function of its graph. The graph is computably equivalent to the function itself, but not subrecursively equivalent. In some cases, the graph of a representation is subrecursively equivalent to an already known representation, but in other cases it is a new representation. In the present paper we undertake a systematic study of the graphs of standard and dual Baire sequences. By combining our new results with the previously known results on the graph of the continued fraction, we obtain a total of eight new subrecursive degrees, which lie strictly between the Dedekind cut and the continued fraction.

Keywords: irrational number representations, graphs of representations, Baire sequences, subrecursive classes, computable analysis

2020 Mathematics Subject Classification: 03D78, 03D20, 03D15, 03F60

1. Introduction

Let us consider some popular representations of the irrational numbers. Any irrational $\alpha \in (0,1)$ has a unique expansion in base 10 and we denote by E_{10} the enumeration of its decimal digits. A Cauchy sequence for α is a function $C\colon \mathbb{N} \to \mathbb{Q}$, such that $|C(n) - \alpha| < (n+1)^{-1}$. We want to know if it possible to compare the complexity of these two representations and we ask the question: *is it possible to convert one of these representations into the other without using unbounded search*? We will call such conversions *subrecursive*. The answer is yes for converting from E_{10} to C, because we can take $C(n) = \mathtt{0}.E_{10}(1)E_{10}(2)\dots E_{10}(n)$. But now let us assume we have access to C and we would like to compute E_{10}. The first decimal digit $E_{10}(1)$ is the unique natural number d, such that $\frac{d}{10} < \alpha < \frac{d+1}{10}$. In order to

DOI: 10.60063/GSU.FMI.109.41-55

find d, we need to compute a proper n, such that $C(n)$ is close enough to α so that we can decide the two inequalities. To find this n we need unbounded search. For example, suppose $\mathtt{0.1} = C(0) = C(1) = \cdots$. Since α is irrational, we know that there exists n, such that $C(n)$ is far enough from $\mathtt{0.1}$, but as long as $C(n) = \mathtt{0.1}$ we cannot tell if $E_{10}(1)$ is $\mathtt{0}$ or $\mathtt{1}$. Of course, we could potentially have the same problem with all the other decimal digits. Indeed, it turns out that there does not exist a subrecursive reduction from C to E_{10}. So we shall say that C is strictly subrecursive in E_{10}.

Consider also the Dedekind cut $D\colon \mathbb{Q} \to \{0, 1\}$, such that $D(q) = 0$ if and only if $q < \alpha$. It is possible to subrecursively convert D to E_{10} (and in fact to E_b for any base b). We can test if d is the right decimal digit by asking two questions to D. In this way, we can successively compute all decimal digits of α. No unbounded search is needed, because $d < 10$. In the other direction, assume we have access to E_{10}. For some rational numbers q we can tell whether $q < \alpha$ without unbounded search. For example, if $q = \mathtt{0.3}$ we need to check only if $E_{10}(1) < \mathtt{3}$ to compute $D(q)$. The situation is the same for all q, which have finite decimal expansion. But now consider $q = \frac{1}{3}$. Then we might have $\mathtt{3} = E_{10}(1) = E_{10}(2) = \cdots$. Since α is irrational, there will be some n, such that $E_{10}(n) \neq 3$ which allows to compute $D(\frac{1}{3})$, but we need unbounded search to find this n. Indeed, it can be formally shown that no subrecusive reduction exists from E_{10} do D, therefore E_{10} is strictly subrecursive in D.

More formally, we say that the representation R_1 is subrecursive in the representation R_2, if there exists a subrecursive reduction, which given as oracle any R_2-representation of an irrational $\alpha \in (0, 1)$, produces a R_1-representation of α.

The study of the structure of representations of irrationals with respect to subrecursive reduciblity was formally initiated in [6, 7] and since then it has been actively further investigated in [1–4, 8].

In the last of these papers [1], the following topic was posed: For a representation R, viewed as a function, we consider its graph $\mathcal{G}(R)$, which we may also regard as a representation. In general, R is not subrecursive in $\mathcal{G}(R)$ and we are mostly interested in the case when $\mathcal{G}(R)$ is not subrecursively equivalent to any of the known representations. Such is the case with the graph of the continued fraction, as shown in [1]. In the present paper, we shall see that the graphs of the dual and the standard Baire sequences are also in a similar very interesting and intricate position with respect to the already studied representations.

2. Some facts on subrecursive classes

In our computational model we will use total functions of several arguments over the natural numbers $\mathbb{N}$. We will also freely use other discrete domains, such as the rational numbers $\mathbb{Q}$ and the finite strings $\{\mathtt{L}, \mathtt{R}\}^{\star}$, which are assumed to be properly coded with natural numbers. We also assume a monotonic coding is fixed of finite sequences of natural numbers.

For a function f, we denote by $\mathcal{G}(f)$ the graph of f, considered as a relation, that is $\mathcal{G}(f)(x,y)$ has value *true* if $f(x)=y$ and value *false* if $f(x)\neq y$.

Given two functions f, g, we denote $f \leq_S g$ and we say that f *is subrecursive* in g, if there exists an algorithm, which given input x computes the value $f(x)$, where the algorithm is allowed to invoke $g(y)$ for any y already computed, but *it is not allowed to use unbounded search.*

For two functions f, g, let us denote by $\langle f,g\rangle$ the function, defined with

$$\langle f,g\rangle(x,y)=\langle f(x),g(y)\rangle.$$

It is easy to see that $\leq_S$ is a preorder on the set of all functions and $\langle f,g\rangle$ is the least upper bound of f and g with respect to $\leq_S$.

We denote:

$$f \equiv_S g \text{ if } f \leq_S g \text{ and } g \leq_S f; \quad f <_S g \text{ if } f \leq_S g \text{ and } g \not\leq_S f.$$

Of course, $\equiv_S$ is an equivalence relation on the set of functions and we will call its equivalence classes *subrecursive degrees.*

One easily verifies that $\mathcal{G}(\langle f,g\rangle) \equiv_S \langle \mathcal{G}(f),\mathcal{G}(g)\rangle$.

For a function $f\colon \mathbb{N}\to\mathbb{N}$, we denote by $f^{\Sigma}\colon \mathbb{N}\to\mathbb{N}$ its bounded sum $f^{\Sigma}(y)=\sum_{x=0}^{y} f(x)$.

Lemma 2.1. *For any function $f\colon \mathbb{N}\to\mathbb{N}$ we have*

(a) $\mathcal{G}(f)\leq_S f$;

(b) $\mathcal{G}(f^{\Sigma})\leq_S \mathcal{G}(f)$.

Proof. In order to check, given (x,y), whether $f(x)=y$, we can just compute $f(x)$ and then compare the result with y. This simple observation shows (a).

For (b) we have

$$z=f^{\Sigma}(y) \iff \exists u\left(\forall x\leq y[u_x=f(x)]\ \&\ z=\sum_{x=0}^{y} u_x\right)$$

and the code of the sequence u can be bounded by the code of the sequence $z,z,\ldots,z$ of length $y+1$. □

We will need the notion of a subrecursive class in order to give precise estimates for the complexity of functions.

Definition 2.2. A non-empty set $\mathcal{S}$ of functions will be called *a subrecursive class* if:

1. $\mathcal{S}$ is contained in an efficiently enumerable class, which means that there exists a computable function $U\colon \mathbb{N}^2\to\mathbb{N}$, such that for any $f\in\mathcal{S}$ there exists e with $f(x_1,x_2,\ldots,x_n)=U(e,\langle x_1,x_2,\ldots,x_n\rangle)$;

2. if $f \leq_S g$ and $g \in \mathcal{S}$, then $f \in \mathcal{S}$.

For our purposes we may safely assume that $\mathcal{S}$ contains all primitive recursive functions.

Informally, we may say that $f \in \mathcal{S}$ implies that f is simple and accordingly $f \notin \mathcal{S}$ implies that f is complex.

We will need two important tools for working with subrecursive classes.

Lemma 2.3. *For any subrecursive class $\mathcal{S}$, there exists $f \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $f \notin \mathcal{S}$ and $\mathcal{G}(f) \in \mathcal{S}$.*

For a proof, see [3, Sect. 2, 3]. We have $f \not\leq_S \mathcal{G}(f)$ and thus by Lemma 2.1(a) we obtain $\mathcal{G}(f) <_S f$. So we may say that f is complex, but its graph $\mathcal{G}(f)$ is simple.

Lemma 2.4. *For any subrecursive class $\mathcal{S}$, there exists $s \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $\mathcal{G}(s) \notin \mathcal{S}$ and $\mathcal{G}(s^\Sigma) \in \mathcal{S}$.*

For a proof, see [1, Theorem 1] (we can add 1 to produce a function s with non-zero values). Applying Lemma 2.1(b), we obtain $\mathcal{G}(s^\Sigma) <_S \mathcal{G}(s)$. Thus we may say that the graph $\mathcal{G}(s)$ of s is complex, but the graph $\mathcal{G}(s^\Sigma)$ of its bounded sum s^Σ is simple. Observe that we have $s \equiv_S s^\Sigma$, but $\mathcal{G}(s) \not\equiv_S \mathcal{G}(s^\Sigma)$. This shows that the graph $\mathcal{G}$ is not a degree-theoretic operation, that is two functions from the same subrecursive degree may have graphs, which belong to different subrecursive degrees.

3. Representations of irrational numbers

A *Farey pair* is a pair $(\frac{a}{b}, \frac{c}{d})$ of rational numbers, such that $bc - ad = 1$. We will only consider Farey pairs with $\frac{a}{b}, \frac{c}{d} \in [0, 1]$. It will also be convenient to consider a Farey pair as an open interval in $\mathbb{R}$.

All Farey pairs can be generated using an infinite binary tree (called *the Farey pair tree*) in the following way: For $\tau \in \{\mathtt{L}, \mathtt{R}\}^\star$ we denote $I[\tau] = (\frac{a}{b}, \frac{c}{d})$, where $I[\epsilon] = (\frac{0}{1}, \frac{1}{1})$ and $I[\tau\mathtt{L}] = (\frac{a}{b}, \frac{a+c}{b+d})$, $I[\tau\mathtt{R}] = (\frac{a+c}{b+d}, \frac{c}{d})$.

The fraction $\frac{a+c}{b+d}$ is called *the mediant* of $\frac{a}{b}$ and $\frac{c}{d}$. An easy computation shows that if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

For any rational $q \in (0, 1)$ there exists a unique $\tau \in \{\mathtt{L}, \mathtt{R}\}^\star$, such that q is the mediant of the endpoints of $I[\tau]$.

For more details on Farey pairs, see [11, Chapter 6].

Let $\alpha \in (0, 1)$ be an irrational number.

On each fixed level, the intervals in the Farey pair tree determine a partition of the irrationals in the interval $(0, 1)$, therefore α belongs to exactly one of these intervals. Moreover, the length of the intervals converges to 0 as the level goes to infinity. This justifies the following

Definition 3.1. *The Hurwitz characteristic* $H^\alpha \colon \mathbb{N} \to \{\mathtt{L}, \mathtt{R}\}$ is the unique infinite string over $\{\mathtt{L}, \mathtt{R}\}$, such that $\alpha \in I[\tau]$ for any finite prefix τ of H^α.

Note that the infinite strings H^α, which represent irrationals $\alpha \in (0,1)$ are exactly those, which contain infinitely many occurrences of both $\mathtt{L}$ and $\mathtt{R}$.

Lemma 3.2. *Let H^α be the Hurwitz characteristic of α. Then the Hurwitz characteristic $H^{1-\alpha}$ is obtained from H^α by exchanging the symbols $\mathtt{L}$ and $\mathtt{R}$.*

Proof. For any finite string $\tau \in \{\mathtt{L},\mathtt{R}\}^\star$, let τ' be obtained from τ by exchanging $\mathtt{L}$ and $\mathtt{R}$. We will prove that for all $\tau \in \{\mathtt{L},\mathtt{R}\}^\star$:

$$[\tau] = \left(\frac{a}{b}, \frac{c}{d}\right) \implies I[\tau'] = \left(\frac{d-c}{d}, \frac{b-a}{b}\right) \tag{*}$$

For $\tau = \epsilon$, we have $\tau' = \epsilon$ and $I[\tau] = (\frac{0}{1}, \frac{1}{1}) = (\frac{1-1}{1}, \frac{1-0}{1}) = I[\tau']$. Suppose $I[\tau] = \left(\frac{a}{b}, \frac{c}{d}\right)$ and $I[\tau'] = \left(\frac{d-c}{d}, \frac{b-a}{b}\right)$. Then $I[\tau\mathtt{L}] = \left(\frac{a}{b}, \frac{a+c}{b+d}\right)$ and

$$I[(\tau\mathtt{L})'] = I[\tau'\mathtt{R}] = \left(\frac{d-c+b-a}{d+b}, \frac{b-a}{b}\right) = \left(\frac{(b+d)-(a+c)}{b+d}, \frac{b-a}{b}\right),$$

therefore (*) is true for $\tau\mathtt{L}$. Similarly, $I[\tau\mathtt{R}] = \left(\frac{a+c}{b+d}, \frac{c}{d}\right)$ and

$$I[(\tau\mathtt{R})'] = I[\tau'\mathtt{L}] = \left(\frac{d-c}{d}, \frac{d-c+b-a}{d+b}\right) = \left(\frac{d-c}{d}, \frac{(b+d)-(a+c)}{b+d}\right),$$

therefore (*) is true for $\tau\mathtt{R}$. So by induction on τ, (*) is true for all $\tau \in \{\mathtt{L},\mathtt{R}\}^\star$.

Now by definition, $\alpha \in I[\tau]$ for any finite prefix τ of H^α and the claim implies that $(1-\alpha) \in I[\tau']$ for all such τ. Therefore, τ' is a prefix of $H^{1-\alpha}$ and the lemma follows. □

Definition 3.3. Let $\alpha \in (0,1)$ be irrational with Hurwitz characteristic H^α.
We can write $H^\alpha = \mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\ldots\mathtt{L}^{A(n)}\mathtt{R}\ldots$ for a unique function $A\colon \mathbb{N} \to \mathbb{N}$, which will be called *the dual Baire sequence* of α.
Similarly, $H^\alpha = \mathtt{R}^{B(0)}\mathtt{L}\mathtt{R}^{B(1)}\mathtt{L}\ldots\mathtt{R}^{B(n)}\mathtt{L}\ldots$ for a unique function $B\colon \mathbb{N} \to \mathbb{N}$, which will be called *the standard Baire sequence* of α.

Dual and standard Baire sequences are introduced in [8], where it is shown that they are subrecursively equivalent to the complete left and right best approximations and to the general sum approximations from below and from above.

Observe the symmetry between the dual and the standard Baire sequences A and B, which follows from Lemma 3.2: $A^\alpha = B^{1-\alpha}$ and $B^\alpha = A^{1-\alpha}$.

Let $D^\alpha\colon \mathbb{Q} \to \{0,1\}$ be *the Dedekind cut* of α, $D^\alpha(q) = 0 \Leftrightarrow q < \alpha$.

We sketch a proof that $H^\alpha \equiv_S D^\alpha$, for more details see [9, 10].

Given input n, we can compute $H^\alpha(n)$ by successively constructing the corresponding intervals. We decide whether to go left or right by asking for $D^\alpha(m)$, where m is the mediant of the current interval. This algorithm shows that $H^\alpha \le_S D^\alpha$ (no unbounded search is used).

Given $q \in \mathbb{Q} \cap (0,1)$, we compute the level s of its first occurrence as an endpoint of an interval in the Farey pair tree (an upper bound for s is the sum of the numerator and the denominator of q). Then using H^α we construct the interval (a_s, b_s) on level s, which contains α. Since $q \notin (a_s, b_s)$, we have $D^\alpha(q) = 0$ if $q \leq a_s$ and $D^\alpha(q) = 1$ if $q \geq b_s$. So we have a subrecursive reduction, which shows $D^\alpha \leq_S H^\alpha$.

It is easy to see that $D^\alpha \equiv_S D^{1-\alpha}$, therefore $H^\alpha \equiv_S H^{1-\alpha}$ for all α (of course, this also follows from Lemma 3.2).

Definition 3.4. The unique sequence $c\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that

$$\alpha = 0 + \cfrac{1}{c(0) + \cfrac{1}{c(1) + \cfrac{1}{c(2) + \ddots}}}$$

will be called *the continued fraction* of α.

We also denote $c = [\,]^\alpha$ and sometimes, for convenience, we use the standard notation $\alpha = [0;\ c(0),\ c(1),\ \ldots,\ c(k),\ \ldots]$.

The next lemma shows that, similarly to the Dedekind cut and the Hurwitz characteristic, the subrecursive degrees of the continued fractions of α and $1-\alpha$ coincide and the same is true for their graphs.

Lemma 3.5. *For all irrational $\alpha \in (0,1)$:*

$$[\,]^\alpha \ \equiv_S\ [\,]^{1-\alpha} \quad \textit{and} \quad \mathcal{G}([\,]^\alpha)\ \equiv_S\ \mathcal{G}([\,]^{1-\alpha}).$$

Proof. We will use the following elementary algebraic fact:

$$1 - \alpha = \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{\alpha} - 1}}.$$

Let $\alpha = [0;\ c(0),\ c(1),\ \ldots,\ c(k),\ \ldots]$. In the case $c(0) \neq 1$, we have $1 - \alpha = [0;\ 1,\ c(0) - 1,\ c(1),\ c(2),\ \ldots,\ c(k),\ \ldots]$. In the case $c(0) = 1$, we have $1 - \alpha = [0;\ c(1) + 1,\ c(2),\ \ldots,\ c(k),\ \ldots]$. Therefore, $[\,]^{1-\alpha}$ can be obtained from $[\,]^\alpha$ by the following definition by cases:

$$[\,]^{1-\alpha}(n) = \begin{cases} 1 & \text{if } n = 0 \ \&\ [\,]^\alpha(0) \neq 1, \\ [\,]^\alpha(0) \dot{-} 1 & \text{if } n = 1 \ \&\ [\,]^\alpha(0) \neq 1, \\ [\,]^\alpha(n \dot{-} 1) & \text{if } n > 1 \ \&\ [\,]^\alpha(0) \neq 1, \\ [\,]^\alpha(1) + 1 & \text{if } n = 0 \ \&\ [\,]^\alpha(0) = 1, \\ [\,]^\alpha(n+1) & \text{if } n > 0 \ \&\ [\,]^\alpha(0) = 1. \end{cases}$$

It trivially follows that $[\,]^{1-\alpha} \leq_S [\,]^\alpha$ and $\mathcal{G}([\,]^{1-\alpha}) \leq_S \mathcal{G}([\,]^\alpha)$. By applying the same argument for $1 - \alpha$, the proof is finished. $\square$

Now we present a key observation that links the Hurwitz characteristic and the continued fraction. It is an old result of Hurwitz (also cited in a modified form in [10]) and we include a proof for completeness.

Lemma 3.6 (Hurwitz)**.** *For any irrational $\alpha \in (0,1)$, its Hurwitz characteristic and its continued fraction are connected by the equality*

$$H^\alpha = \mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\mathtt{L}^{c(2)}\mathtt{R}^{c(3)}\dots \, . \tag{3.1}$$

Proof. We will need some standard facts about continued fractions (see [5]). We have $\alpha = \lim\limits_{n\to\infty}\dfrac{p_n}{q_n}$, where $\dfrac{p_n}{q_n} = [0;\ c(0),\ c(1),\ \dots\ c(n-1)]$ is the finite part of the continued fraction of α up to the term $c(n-1)$ and p, q satisfy the equalities:

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_{n+2} = c(n+1)\cdot p_{n+1} + p_n,\\ q_0 &= 1, \quad q_1 = c(0), \quad q_{n+2} = c(n+1)\cdot q_{n+1} + q_n, \end{aligned}$$

for any $n \in \mathbb{N}$. The sequence $\dfrac{p_n}{q_n}$ has the following behavior:

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \alpha < \cdots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Since H^α is unique, we need to prove that α lies in each of the intervals, determined by the right-hand side of (3.1).

We have $\alpha < \dfrac{p_1}{q_1} = \dfrac{1}{c(0)}$, therefore $\alpha \in I[\mathtt{L}^i] = (\frac{0}{1}, \frac{1}{i+1})$ for each $i < c(0)$. Notice also that $\left(\dfrac{p_0}{q_0}, \dfrac{p_1}{q_1}\right) = I[\mathtt{L}^{c(0)-1}]$.

Suppose inductively that for some even n, $I[\mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\dots\mathtt{L}^{c(n)}]$ coincides with $\left(\dfrac{p_n}{q_n}, \dfrac{p_{n+1}}{q_{n+1}}\right)$. Consider $I[\tau_i] = I[\mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\dots\mathtt{L}^{c(n)}\mathtt{R}^i]$ for $i = 0,1,\dots,c(n+1)$. Clearly, $I[\tau_i] = \left(\dfrac{p_n + i\cdot p_{n+1}}{q_n + i\cdot q_{n+1}}, \dfrac{p_{n+1}}{q_{n+1}}\right)$. The left endpoints can be computed by taking mediants successively with the fixed right endpoint. Therefore, the left endpoints form an increasing sequence for $i = 0,1,\dots,c(n+1)$. Its last element is the left endpoint of $I[\tau_{c(n+1)}] = \left(\dfrac{p_n + c(n+1)\cdot p_{n+1}}{q_n + c(n+1)\cdot q_{n+1}}, \dfrac{p_{n+1}}{q_{n+1}}\right) = \left(\dfrac{p_{n+2}}{q_{n+2}}, \dfrac{p_{n+1}}{q_{n+1}}\right)$. In particular, all these left endpoints lie to the left of α and $\alpha < \dfrac{p_{n+1}}{q_{n+1}}$, that is $\alpha \in I[\tau_i]$ for all $i = 0,1,\dots,c(n+1)$.

The other case when for an odd n, $I[\mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\dots\mathtt{L}^{c(n-1)}\mathtt{R}^{c(n)}]$ coincides with $\left(\dfrac{p_{n+1}}{q_{n+1}}, \dfrac{p_n}{q_n}\right)$ is completely symmetric. $\square$

It turns out that the subrecursive degree of the continued fraction is the least upper bound of the subrecursive degrees of the dual and the standard Baire sequences. A proof of this result can be found in [8] using contractors or in [6] using trace functions and general sum approximations (contractors and trace functions are subrecursively equivalent to continued fractions). Here we will give a direct proof, using equality (3.1).

Proposition 3.7. *For all irrational* $\alpha \in (0,1)$: $[\,]^\alpha \equiv_S \langle A^\alpha, B^\alpha\rangle$.

Proof. Let us fix α and omit the superscripts for brevity. First we provide algorithm for $A \le_S [\,]$.

Input: natural number n. **Output:** $A(n) \in \mathbb{N}$.

1. Compute $\tau = \mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\mathtt{L}^{c(2)}\dots\mathtt{R}^{c(2n+1)}$, where $c(i) = [\,](i)$.
2. Let τ' be the shortest prefix of τ, containing $n+1$ occurrences of $\mathtt{R}$.
3. Write τ' in the form $\mathtt{L}^{a_0}\mathtt{R}\mathtt{L}^{a_1}\mathtt{R}\dots\mathtt{L}^{a_n}\mathtt{R}$.
4. Return the result a_n.

End of algorithm.

Observe that τ is chosen to contain at least $n+1$ occurrences of $\mathtt{R}$, therefore τ' is well defined. Moreover, τ' is a prefix of H^α, because τ' is a prefix of τ, which in turn is a prefix of H^α by equality (3.1). Therefore, $a_i = A(i)$ for all $i \le n$ and the algorithm is correct.

A symmetric algorithm shows that $B \le_S [\,]$. We take

$$\tau = \mathtt{L}^{c(0)-1}\mathtt{R}^{c(1)}\mathtt{L}^{c(2)}\dots\mathtt{R}^{c(2n+1)}\mathtt{L}^{c(2n+2)}$$

and in 2., 3. we exchange $\mathtt{L}$ and $\mathtt{R}$. We can also use Lemma 3.5 to obtain $B^\alpha = A^{1-\alpha} \le_S [\,]^{1-\alpha} \equiv_S [\,]^\alpha$.

Now let us show that $[\,] \le_S \langle A, B\rangle$. We will use primitive recursion (as well as A and B as oracles).

Input: natural number n. **Output:** $[\,](n) \in \mathbb{N} \setminus \{0\}$.

1. If $n = 0$, return output $A(0) + 1$.

2. Assume that $n > 0$ is even and that $[\,](0), \dots, [\,](n-1)$ have been computed. Then compute $r = [\,](1) + [\,](3) + \dots + [\,](n-1)$ and return output $A(r)$.

3. Assume that $n > 0$ is odd and that $[\,](0), \dots, [\,](n-1)$ have been computed. Then compute $l = [\,](0) - 1 + [\,](2) + \dots + [\,](n-1)$ and return output $B(l)$.

End of algorithm.

The algorithm gives a correct output for $n = 0$, because H^α begins with $A(0)$ symbols $\mathtt{L}$ by the definition of A and also with $[\,](0) - 1$ symbols $\mathtt{L}$ by equality (3.1). Assume that $n > 0$ is even and that $[\,](0), \dots, [\,](n-1)$ have been computed correctly. The Hurwitz characteristic H^α begins with the prefix $\tau = \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\dots\mathtt{L}^{[\,](n-2)}\mathtt{R}^{[\,](n-1)}$. We want to compute $[\,](n)$, which on one hand, by equality (3.1), is the number of $\mathtt{L}$-s that come after τ in H^α. On the other hand, the number of occurrences of $\mathtt{R}$ in τ is the number r, computed in Step 2. Therefore, we can write $\tau = \mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\dots\mathtt{L}^{A(r-1)}\mathtt{R}$ (note that τ ends in $\mathtt{R}$). By the definition of A, the next symbols in H^α after τ are $\mathtt{L}^{A(r)}\mathtt{R}$. But this entails that $A(r)$ symbols $\mathtt{L}$ follow τ in H^α, therefore the output is indeed correct. The case when $n > 0$ is odd is justified in a completely symmetric way. $\square$

Observe that the proposition can be used to show that $[\,]^\alpha \equiv_S \langle A^\alpha, B^\alpha\rangle = \langle B^{1-\alpha}, A^{1-\alpha}\rangle \equiv_S [\,]^{1-\alpha}$. But as we shall see the situation with the graphs $\mathcal{G}([\,])$, $\mathcal{G}(A)$, $\mathcal{G}(B)$ is a lot more complicated. We know from [1] that $\mathcal{G}([\,])$ is subrecursively incomparable with A and B (note that in [1] the subrecursive degrees of A and B are represented by the complete left and right best approximations L and R, respectively). At the end of the next section, we will be able to see the exact position of the subrecursive degrees of the three graphs $\mathcal{G}([\,])$, $\mathcal{G}(A)$, $\mathcal{G}(B)$ relative to the subrecursive degrees of D, A, B, $[\,]$.

4. Main results

Theorem 4.1. *Let $\alpha \in (0,1)$ be irrational and A^Σ, B^Σ be the bounded sums of its dual and standard Baire sequence A, B. Then $\mathcal{G}(A^\Sigma) \equiv_S D^\alpha \equiv_S \mathcal{G}(B^\Sigma)$.*

Proof. First we prove $H^\alpha \leq_S \mathcal{G}(A^\Sigma)$. For each given $n \in \mathbb{N}$, the prefix

$$\mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\ldots\mathtt{L}^{A(n)}\mathtt{R}$$

of H^α contains at least $n+1$ symbols. Therefore we can compute $H^\alpha(n)$ by the following algorithm.

Input: natural number n. **Output:** $H^\alpha(n) \in \{\mathtt{L}, \mathtt{R}\}$.

Test whether: there exists $k \leq n$, such that $A^\Sigma(k) = n \dot{-} k$.

Return $\mathtt{R}$ if the test succeeds and $\mathtt{L}$, otherwise.

End of algorithm.

Let us check that the algorithm is correct. If the test succeeds for some $k \leq n$, then $n = k + A^\Sigma(k) = k + \sum_{i=0}^{k} A(i)$. Therefore, n is equal to length of the prefix $\mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\ldots\mathtt{L}^{A(k-1)}\mathtt{R}\mathtt{L}^{A(k)}$ of H^α and $H^\alpha(n)$ is the symbol immediately after this prefix, which is $\mathtt{R}$. Conversely, if $H^\alpha(n) = \mathtt{R}$ and this is the k-th occurrence of $\mathtt{R}$ in H^α, then the test succeeds with the value $k-1$. The algorithm is clearly subrecursive in $\mathcal{G}(A^\Sigma)$, so we have proved $H^\alpha \leq_S \mathcal{G}(A^\Sigma)$.

Now we prove $\mathcal{G}(A^\Sigma) \leq_S H^\alpha$.

Input: natural numbers k, n. **Output:** $A^\Sigma(k) = n$ (*true* or *false*).

1. Compute $\tau = H^\alpha(0)H^\alpha(1)\ldots H^\alpha(n+k)$.
2. If the last symbol of τ is $\mathtt{L}$, return *false*.
3. If τ does not contain $k+1$ occurrences of $\mathtt{R}$, return *false*.
4. Otherwise, compute the number m of occurrences of $\mathtt{L}$ in τ.
5. If $m = n$, return *true*, else return *false*.

End of algorithm.

For correctness: first, assume $A^\Sigma(k) = n$. Since τ is the unique prefix of H^α having length $n+k+1$, we have $\tau = \mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\ldots\mathtt{L}^{A(k)}\mathtt{R}$. Therefore, the output *false* in 2. and 3. is correct. Moreover, the number of occurrences of $\mathtt{L}$ in τ is $m = A^\Sigma(k) = n$, therefore the output *false* in 5. is also correct. Second, suppose we output *true* in 5. so that $m = n$. Since we have reached 4., τ must have the same form as above. Therefore, $m = A^\Sigma(k)$ and since $m = n$, the output *true* is correct. This proves $\mathcal{G}(A^\Sigma) \leq_S H^\alpha$.

Now we can consider $1-\alpha$: we have $B^\alpha = A^{1-\alpha}$ and the proof shows that $\mathcal{G}(B^\Sigma) \equiv_S H^{1-\alpha}$. Since $D^\alpha \equiv_S H^\alpha \equiv_S H^{1-\alpha}$, the theorem follows. $\square$

Observe that all results on representations that we have considered so far are positive, in the sense that one representation is subrecursive in another and the reduction works (uniformly) for all irrational $\alpha \in (0,1)$. In the next corollary we present some negative results. To prove the claim that one representation is not subrecursive in another, we construct a specific irrational number α for which the claim holds.

Corollary 4.2. *$D^\alpha <_S \mathcal{G}(A^\alpha) <_S A^\alpha$ and $D^\alpha <_S \mathcal{G}(B^\alpha) <_S B^\alpha$.*

Proof. Let us take the irrational number α with dual Baire sequence $A = s$, where s is the function from Lemma 2.4 (note that A attains non-zero values at infinitely many arguments, because $A \notin \mathcal{S}$). On one hand, by the choice of A, $\mathcal{G}(A^\alpha) \not\leq_S \mathcal{G}(A^\Sigma)$, because $\mathcal{G}(A^\Sigma) \in \mathcal{S}$ and $\mathcal{G}(A^\alpha) \notin \mathcal{S}$. On the other hand, by Lemma 2.1(b) we have $\mathcal{G}(A^\Sigma) \leq_S \mathcal{G}(A^\alpha)$ for any irrational α. Therefore, $\mathcal{G}(A^\Sigma) <_S \mathcal{G}(A^\alpha)$. The theorem gives $\mathcal{G}(A^\Sigma) \equiv_S D^\alpha$, thus $D^\alpha <_S \mathcal{G}(A^\alpha)$.

Similarly, let us take the irrational α with dual Baire sequence $A = f$, where f is the function from Lemma 2.3. Then $A^\alpha \not\leq_S \mathcal{G}(A^\alpha)$, because $\mathcal{G}(A^\alpha) \in \mathcal{S}$ and $A^\alpha \notin \mathcal{S}$. Moreover, by Lemma 2.1(a) $\mathcal{G}(A^\alpha) \leq_S A^\alpha$ for any irrational α. We obtain $\mathcal{G}(A^\alpha) <_S A^\alpha$.

The proof of the second part is analogous using standard Baire sequences. It also follows from the first part by considering $1-\alpha$. $\square$

The corollary shows that the subrecursive degrees of $\mathcal{G}(A^\alpha)$ and $\mathcal{G}(B^\alpha)$ are different from the subrecursive degrees of any of the considered representations. Our main motivation is to compare them with the degree of the graph of the continued fraction $\mathcal{G}([\,]^\alpha)$.

Theorem 4.3. *The following hold: $\mathcal{G}(A^\alpha) \leq_S \mathcal{G}([\,]^\alpha)$, $\mathcal{G}(A^\alpha) \not\leq_S B^\alpha$ and $\mathcal{G}(B^\alpha) \leq_S \mathcal{G}([\,]^\alpha)$, $\mathcal{G}(B^\alpha) \not\leq_S A^\alpha$.*

Proof. Again we omit the superscripts. First we show $\mathcal{G}(A) \leq_S \mathcal{G}([\,])$ with a primitive recursive algorithm (using the graph of $[\,]$ as oracle).

Input: natural numbers k, n. **Output:** $A(k) = n$ (*true* or *false*).

1. If $k = 0$, return *true* if $n+1 = [\,](0)$ and *false* if $n+1 \neq [\,](0)$.

2. If $k > 0$, suppose we have determined the truth values in the sequence $A(0) = 0, \ldots, A(k-1) = 0$. Compute l = the number of *true* values in this sequence and l' = the number of *true* values that come at the end of the sequence.

3. If $l = k$, let $p = l$. If $l \neq k$, let $p = l' + 1$.

4. If $[\,](0) = 1$, let $m = (2k \dot{-} 2l) + 1$. If $[\,](0) \neq 1$, let $m = (2k \dot{-} 2l) \dot{-} 1$.

5. If $[\,](m) \neq p$, then return *true* if $n = 0$ and *false* if $n > 0$.

6. If $[\,](m) = p$, then return *true* if $n = [\,](m+1)$ and *false* if $n \neq [\,](m+1)$.

End of algorithm.

For $k = 0$ the algorithm gives correct answer, because $[\,](0) = A(0) + 1$. Suppose $k > 0$ and consider $\tau = \mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\dots\mathtt{L}^{A(k-1)}\mathtt{R}$. The values of p and m are computed in such a way that

$$\tau = \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m-1)}\mathtt{R}^{p}. \qquad (**)$$

We show this by induction on k. First, consider the base case $k = 1$. If $A(0) = 0$, that is $[\,](0) = 1$, we have $l = l' = 1$. Following 3. and 4. we have $p = l = 1$, $m = (2k \dot{-} 2l) + 1 = 1$. If $A(0) \neq 0$, that is $[\,](0) \neq 1$, we have $l = l' = 0$, so by 3. and 4. $p = l' + 1 = 1$, $m = (2k \dot{-} 2l) \dot{-} 1 = 1$. In both cases, we obtain $p = m = 1$ and $\tau = \mathtt{L}^{A(0)}\mathtt{R} = \mathtt{L}^{[\,](0)-1}\mathtt{R}^{p}$, therefore (**) is true. Now let us assume that m and p have been computed correctly for some $k \geq 1$, so that we have (**) for τ. Consider $\tau_1 = \tau\mathtt{L}^{A(k)}\mathtt{R}$ and let l_1, l'_1, p_1, m_1 be the corresponding new values of l, l', p, m.

Case $A(k) = 0$. We have $l_1 = l + 1$, $l'_1 = l' + 1$. If $l_1 = k + 1$, then $p_1 = l_1 = l + 1 = p + 1$. If $l_1 \neq k + 1$, then $p_1 = l'_1 + 1 = l' + 2 = p + 1$. Therefore, $p_1 = p + 1$ and also by 4. $m_1 = m$ (this is true in both cases, since $2(k+1) \dot{-} 2l_1 = 2k \dot{-} 2l$). We obtain (**) is true for τ_1:

$$\begin{aligned}\tau_1 = \tau\mathtt{L}^{A(k)}\mathtt{R} = \tau\mathtt{R} &= \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m-1)}\mathtt{R}^{p+1}\\ &= \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m_1-1)}\mathtt{R}^{p_1}.\end{aligned}$$

Case $A(k) \neq 0$. We have $l_1 = l$, $l'_1 = 0$ and $p_1 = l'_1 + 1 = 1$. By 4. we obtain $m_1 = m + 2$ (in both cases). We have:

$$\begin{aligned}\tau_1 = \tau\mathtt{L}^{A(k)}\mathtt{R} = \tau\mathtt{R} &= \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m-1)}\mathtt{R}^{p}\mathtt{L}^{A(k)}\mathtt{R}\\ &= \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m-1)}\mathtt{R}^{[\,](m)}\mathtt{L}^{[\,](m+1)}\mathtt{R}\\ &= \mathtt{L}^{[\,](0)-1}\mathtt{R}^{[\,](1)}\mathtt{L}^{[\,](2)}\mathtt{R}^{[\,](3)}\dots\mathtt{L}^{[\,](m_1-1)}\mathtt{R}^{p_1}.\end{aligned}$$

Indeed, $p = [\,](m)$ and $A(k) = [\,](m+1)$, because $p > 0$, $A(k) > 0$ and τ_1 is a prefix of H^α, so that the length of each portion of consecutive $\mathtt{L}$-s and $\mathtt{R}$-s is uniquely determined by α. We obtained that (**) is true for τ_1 in this case as well.

Now assuming (**), let us have $[\,](m) \neq p$. Then the next symbol after τ in the Hurwitz characteristic is $\mathtt{R}$. It follows that $A(k) = 0$ and the output in 5. is correct. Finally, let us have $[\,](m) = p$. Then $\mathtt{L}^{[\,](m+1)}\mathtt{R}$ comes immediately after τ in the Hurwitz characteristic. It follows that $A(k) = [\,](m+1)$ and thus the output in 6. is also correct.

We succeeded in proving $\mathcal{G}(A) \leq_S \mathcal{G}([\,])$.

For the next part we consider the irrational α with dual Baire sequence $A = s$, where s is the function from Lemma 2.4. Then $\mathcal{G}(A^\Sigma) \in \mathcal{S}$ and by Theorem 4.1, $D^\alpha \in \mathcal{S}$, which also implies $H^\alpha \in \mathcal{S}$. Now we show that $B^\alpha \in \mathcal{S}$ by subrecursively reducing B^α to H^α. The important observation is that $H^\alpha = \mathtt{L}^{A(0)}\mathtt{R}\mathtt{L}^{A(1)}\mathtt{R}\mathtt{L}^{A(2)}\mathtt{R}\dots$ and $A(x) > 0$ for all x.

Given input n: 1. We compute $\tau = H^\alpha(0)H^\alpha(1)\dots H^\alpha(2n)$. 2. We take the shortest prefix τ' of τ containing $n+1$ occurrences of $\mathtt{L}$. 3. We represent τ' in the form $\mathtt{R}^{b_0}\mathtt{L}\mathtt{R}^{b_1}\mathtt{L}\dots\mathtt{R}^{b_n}\mathtt{L}$ and return output b_n.

The algorithm correctly computes $B(n)$. The only thing that is not obvious is that τ' exists. Assume that τ contains $\leq n$ occurrences of $\mathtt{L}$. Since the length of τ is $2n+1$ it must contain $\geq n+1$ occurrences of $\mathtt{R}$. But the choice of α guarantees that there are no consecutive $\mathtt{R}$ symbols in H^α. So immediately before any occurrence of $\mathtt{R}$ in τ, there is an occurrence of $\mathtt{L}$. But this means that τ contains $\geq n+1$ occurrences of $\mathtt{L}$, which is a contradiction.

Now we have $B^\alpha \in \mathcal{S}$ and by the choice of α, $\mathcal{G}(A^\alpha) \notin \mathcal{S}$. We conclude that $\mathcal{G}(A^\alpha) \not\leq_S B^\alpha$.

The other half $\mathcal{G}(B^\alpha) \leq_S \mathcal{G}([\ ]^\alpha)$, $\mathcal{G}(B^\alpha) \not\leq_S A^\alpha$ of the theorem follows by considering $1-\alpha$. □

Corollary 4.4. *$\mathcal{G}(A^\alpha) <_S \mathcal{G}([\ ])$, $\mathcal{G}(B^\alpha) <_S \mathcal{G}([\ ])$ and $\mathcal{G}(A^\alpha)$, $\mathcal{G}(B^\alpha)$ are subrecursively incomparable.*

Proof. Assume that $\mathcal{G}(A^\alpha) \leq_S \mathcal{G}(B^\alpha)$. By Lemma 2.1(a), $\mathcal{G}(B^\alpha) \leq_S B^\alpha$, therefore $\mathcal{G}(A^\alpha) \leq_S B^\alpha$, which contradicts the theorem. It follows that $\mathcal{G}(A^\alpha) \not\leq_S \mathcal{G}(B^\alpha)$. Symmetrically, $\mathcal{G}(B^\alpha) \not\leq_S \mathcal{G}(A^\alpha)$, thus $\mathcal{G}(A^\alpha)$ and $\mathcal{G}(B^\alpha)$ are subrecursively incomparable.

For the first part, we have $\mathcal{G}(A^\alpha) \leq_S \mathcal{G}([\ ])$ and $\mathcal{G}(B^\alpha) \leq_S \mathcal{G}([\ ])$ by the theorem. If $\mathcal{G}([\ ]) \leq_S \mathcal{G}(A^\alpha)$ or $\mathcal{G}([\ ]) \leq_S \mathcal{G}(B^\alpha)$, it would follow that $\mathcal{G}(B^\alpha) \leq_S \mathcal{G}(A^\alpha)$ or $\mathcal{G}(A^\alpha) \leq_S \mathcal{G}(B^\alpha)$, which as we saw is impossible. So indeed we have $\mathcal{G}(A^\alpha) <_S \mathcal{G}([\ ])$ and $\mathcal{G}(B^\alpha) <_S \mathcal{G}([\ ])$. □

Theorem 4.5. *$\mathcal{G}([\ ]^\alpha) \not\leq_S \langle B^\alpha, \mathcal{G}(A^\alpha)\rangle$ and $\mathcal{G}([\ ]^\alpha) \not\leq_S \langle A^\alpha, \mathcal{G}(B^\alpha)\rangle$.*

Proof. Let s be the function from Lemma 2.4, $t = s^\Sigma$ and let us take α with Hurwitz characteristic

$$H^\alpha = \mathtt{L}^{t(0)}\mathtt{R}^{s(0)}\mathtt{L}^{t(1)}\mathtt{R}^{s(1)}\mathtt{L}^{t(2)}\mathtt{R}^{s(2)}\ldots,$$

that is $\alpha = [0;\ t(0)+1,\ s(0),\ t(1),\ s(1),\ t(2),\ s(2),\ \ldots]$. By the choice of s, we have $\mathcal{G}(s) \notin \mathcal{S}$. It follows that $\mathcal{G}([\ ]^\alpha) \notin \mathcal{S}$, because $s(k) = n \iff [\ ]^\alpha(2k+1) = n$ for all k, n.

Now the dual and the standard Baire sequences A^α and B^α have the following form:

$$A^\alpha = t(0),\ \underbrace{0,\ 0,\ \ldots,\ 0}_{s(0)-1},\ t(1),\underbrace{0,\ 0,\ \ldots,\ 0}_{s(1)-1},\ t(2),\ \ldots,$$

$$B^\alpha = \underbrace{0,\ 0,\ \ldots,\ 0}_{t(0)},\ s(0),\ \underbrace{0,\ 0,\ \ldots,\ 0}_{t(1)-1},\ s(1),\underbrace{0,\ 0,\ \ldots,\ 0}_{t(2)-1},\ s(2),\ \ldots.$$

More precisely, $A^\alpha(0) = t(0)$, $A^\alpha(s(0)) = t(1)$, $A^\alpha(s(0)+s(1)) = t(2)$ and so on, $A^\alpha(t(x)) = t(x+1)$ for all x, $A^\alpha(k) = 0$ for $k \notin \mathit{Range}(t) \cup \{0\}$.

Similarly, $B^\alpha(t(0)) = s(0)$, $B^\alpha(t(0)+t(1)) = s(1)$, $B^\alpha(t(0)+t(1)+t(2)) = s(2)$ and so on, $B^\alpha(t^\Sigma(x)) = s(x)$ for all x, $B^\alpha(k) = 0$ for $k \notin \mathit{Range}(t^\Sigma)$.

First we show that $\mathcal{G}(A) \in \mathcal{S}$. We will use that $\mathcal{G}(t) = \mathcal{G}(s^\Sigma) \in \mathcal{S}$ by the choice of s.

Input: natural numbers k, n. **Output:** $A^\alpha(k) = n$ (*true* or *false*).

1. If $k = 0$, return *true* if $n = t(0)$ and *false* if $n \neq t(0)$.

2. If $k > 0$, search for $x < k$, such that $k = t(x)$. If the search is successful, go to 3. If not, proceed to 4.

3. Return *true* if $n = t(x+1)$ and *false* if $n \neq t(x+1)$.

4. Return *true* if $n = 0$ and *false* if $n \neq 0$.

End of algorithm.

The algorithm is correct due to the above equalities for A^α. Note that if $k = t(x)$ for some x, then $x < s^\Sigma(x) = t(x) = k$ and the search will be successful.

Now we show that $B \in \mathcal{S}$. We will again use $\mathcal{G}(t)$ and also $\mathcal{G}(t^\Sigma)$, which belongs to $\mathcal{S}$ thanks to Lemma 2.1(b).

Input: natural number k. **Output:** $B^\alpha(k) \in \mathbb{N}$.

1. Search for $x < k$, such that $k = t^\Sigma(x)$. If the search is not successful, return output 0.

2. If $x = 0$, return output k.

3. If $x > 0$, search for $y < k$ and $z < k$ with $y = t(x)$ and $z = t(x \dot{-} 1)$. Return output $y \dot{-} z$.

End of algorithm.

The correctness of the outputs follows from the above equalities for B^α. Indeed, if the search for x is not successful, $k \notin Range(t^\Sigma)$ and $B^\alpha(k) = 0$. Suppose the search for x is successful. For $x = 0$, we have $s(x) = s(0) = t(0) = t^\Sigma(0) = k$. For $x > 0$, we have $t(x-1) < t(x) < t^\Sigma(x) = k$. Therefore, the search for y and z is successful and $s(x) = t(x) \dot{-} t(x-1) = y \dot{-} z$.

We succeeded in proving: $\mathcal{G}([\,]^\alpha) \notin \mathcal{S}$, $\mathcal{G}(A^\alpha) \in \mathcal{S}$ and $B^\alpha \in \mathcal{S}$, so we can conclude $\mathcal{G}([\,]^\alpha) \not\leq_S \langle B^\alpha, \mathcal{G}(A^\alpha)\rangle$.

The other half of the theorem follows symmetrically. □

Corollary 4.6. *The subrecursive degrees of* $\mathcal{G}([\,]^\alpha)$, $\langle A^\alpha, \mathcal{G}(B^\alpha)\rangle$, $\langle B^\alpha, \mathcal{G}(A^\alpha)\rangle$ *are pairwise subrecursively incomparable.*

Proof. We must further prove that $\langle B^\alpha, \mathcal{G}(A^\alpha)\rangle \not\leq_S \mathcal{G}([\,]^\alpha)$ and symmetrically $\langle A^\alpha, \mathcal{G}(B^\alpha)\rangle \not\leq_S \mathcal{G}([\,]^\alpha)$. These follow easily from [1, Theorem 4], where it is shown that $A^\alpha \not\leq_S \langle B^\alpha, \mathcal{G}([\,]^\alpha)\rangle$ and also $B^\alpha \not\leq_S \langle A^\alpha, \mathcal{G}([\,]^\alpha)\rangle$. If $\langle B^\alpha, \mathcal{G}(A^\alpha)\rangle \leq_S \langle A^\alpha, \mathcal{G}(B^\alpha)\rangle$, then $B^\alpha \leq_S \langle A^\alpha, \mathcal{G}(B^\alpha)\rangle \leq_S \langle A^\alpha, \mathcal{G}([\,]^\alpha)\rangle$, which contradicts [1, Theorem 4]. Of course, $\langle A^\alpha, \mathcal{G}(B^\alpha)\rangle \leq_S \langle B^\alpha, \mathcal{G}(A^\alpha)\rangle$ leads to a symmetric contradiction. □

Corollary 4.7. $\langle \mathcal{G}(A^\alpha), \mathcal{G}(B^\alpha)\rangle <_S \mathcal{G}([\,]^\alpha)$.

Proof. We know that $\mathcal{G}(A^\alpha) \leq_S \mathcal{G}([\,]^\alpha)$ and $\mathcal{G}(B^\alpha) \leq_S \mathcal{G}([\,]^\alpha)$, therefore

$$\langle \mathcal{G}(A^\alpha), \mathcal{G}(B^\alpha)\rangle \leq_S \mathcal{G}([\,]^\alpha).$$

If $\mathcal{G}([\,]^\alpha) \leq_S \langle \mathcal{G}(A^\alpha), \mathcal{G}(B^\alpha)\rangle$, then we also have $\mathcal{G}([\,]^\alpha) \leq_S \langle B^\alpha, \mathcal{G}(A^\alpha)\rangle$, which contradicts the theorem. □

We are ready to give the final picture, which shows the partial ordering of all of the considered subrecursive degrees (Figure 1). The points represent the corresponding subrecursive degrees of representations. For any two degrees, represented by two points, the lowest reachable point which lies above them, represents the least upper bound of the two degrees.

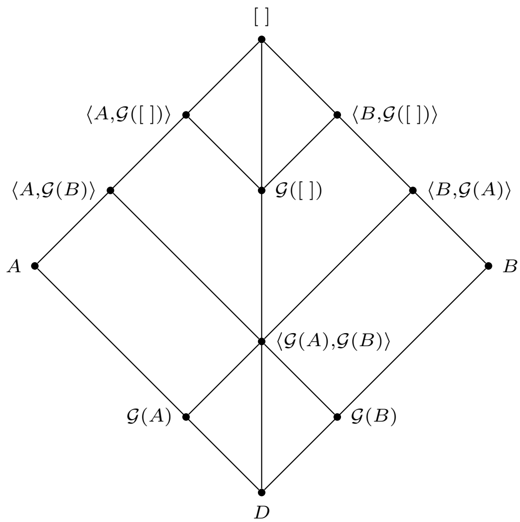

Figure 1. Subrecursive degrees between D and $[\,]$.

Acknowledgements

This work is supported by the Sofia University Science Fund through contract SU-FNI 80-10-210/22.05.2023.

References

[1] I. Georgiev, Subrecursive graphs of representations of irrational numbers, in: Unity of Logic and Computation, CiE 2023, ed. by G. Della Vedova, B. Dundua, S. Lempp and F. Manea, Lecture Notes in Computer Science, 13967, 2023, 154–165.

[2] I. Georgiev, Dedekind cuts and long strings of zeros in base expansions, in: Connecting with Computability, CiE 2021, ed. by L. De Mol, A. Weiermann, F. Manea and D. Fernández-Duque, Lecture Notes in Computer Science, 12813, 2021, 248–259.

[3] I. Georgiev, L. Kristiansen and F. Stephan, Computable irrational numbers with representations of surprising complexity, Ann. Pure Appl. Log. 172(2) (2021) 102893.

[4] I. Georgiev, L. Kristiansen and F. Stephan, On general sum approximations of irrational numbers, in: Sailing Routes in the World of Computation, CiE 2018, ed. by F. Manea, R. Miller and D. Nowotka, Lecture Notes in Computer Science, 10936, 2018, 194–203.

[5] A. Ya. Khintchine, Continued Fractions, P. Noordhoff Ltd, Groningen, The Netherlands, 1963.

[6] L. Kristiansen, On subrecursive representability of irrational numbers, Computability 6 (2017) 249–276.

[7] L. Kristiansen, On subrecursive representability of irrational numbers, part II, Computability 8 (2019) 43–65.

[8] L. Kristiansen, On subrecursive representation of irrational numbers: Contractors and Baire sequences, in: Connecting with Computability, CiE 2021, ed. by L. De Mol, A. Weiermann, F. Manea and D. Fernández-Duque, Lecture Notes in Computer Science, 12813, 2021, 308–317.

[9] L. Kristiansen and J. G. Simonsen, On the complexity of conversion between classic real number representations, in: Beyond the Horizon of Computability, CiE 2020, ed. by M. Anselmo, G. Della Vedova, F. Manea and A. Pauly, Lecture Notes in Computer Science, 12098, 2020, 75–86.

[10] R. S. Lehman, On primitive recursive real numbers, Fundamenta Mathematica 49(2) (1961) 105–118.

[11] I. Niven, H. S. Zuckerman and H. L. Montgomery, An Introduction to the Theory of Numbers, Fifth Edition, John Wiley & Sons Inc, New York, USA, 1991.

Received on November 7, 2023
Accepted on November 17, 2023

Ivan Georgiev

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: `ivandg@fmi.uni-sofia.bg`