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DEFINABILITY OF JUMP CLASSES IN THE LOCAL THEORY OF THE ω-ENUMERATION DEGREES

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In the present paper we continue the study of the definability in the local substructure $\mathcal{G}_\omega$ of the ω-enumeration degrees, which was started in the work of Ganchev and Soskova [3]. We show that the class $\mathbf{I}$ of the intermediate degrees is definable in $\mathcal{G}_\omega$. As a consequence of our observations, we show that the first jump of the least ω-enumeration degree is also definable.

Keywords: Enumeration reducibility, ω-enumeration degrees, degree structures, local substructures, definability, jump classes

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1. INTRODUCTION

A major focus of research in Computability theory involves definability issues in degree structures. Considering a degree structure, natural questions arise about the definability of classes of degrees determined by the structure's jump operation. The same questions can be transferred to its local substructures as well. As an interesting special case one can ask for which natural numbers n the jump classes $\mathbf{H}_n$ and $\mathbf{L}_n$, consisting of the high$_n$ and the low$_n$ degrees respectively, are first order definable in a degree structure.

As it has been shown by Shore and Slaman in [9], the Turing jump is first order definable in the structure of the Turing degrees, $\mathcal{D}_T$, so for all natural numbers n the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first order definable in $\mathcal{D}_T$. For the local substructure

$\mathcal{G}_T$ consisting of all Turing degrees less than or equal to the first jump of the least element in $\mathcal{D}_T$, and for the substructure $\mathcal{R}$ consisting of all computably enumerable Turing degrees, Nies, Shore and Slaman [7] showed that for each natural number n the jump classes $\mathbf{H}_n$ and $\mathbf{L}_{n+1}$ are first order definable. The question whether the class $\mathbf{L}_1$ is first order definable is still open.

In the case of the structure of the enumeration degrees, $\mathcal{D}_e$, Kalimullin [5] proved that the enumeration jump is first order definable, so all of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ are first order definable as well. In the local structure $\mathcal{G}_e$ consisting of all enumeration degrees below the first jump of the least element in $\mathcal{D}_e$, we know by a recent result of Ganchev and M. Soskova [4] that the class $\mathbf{L}_1$ is definable. The problems concerning the definability of $\mathbf{H}_1$ and of the classes of the high$_n$ and low$_n$ degrees for $n \geq 2$ still resist all attempts to be solved.

Further one can consider the questions about the first order definability of the jump classes $\mathbf{H} = \bigcup \mathbf{H}_n$ of the degrees which are high$_n$ for some $n < \omega$, $\mathbf{L} = \bigcup \mathbf{L}_n$ of the degrees which are low$_n$ for some $n < \omega$ and of the class of the intermediate degrees $\mathbf{I}$. It is known that the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ are definable in $\mathcal{D}_T$. This follows from the fact that each relation on $\mathcal{D}_T$ is definable in $\mathcal{D}_T$ *if and only if* it is invariant under the automorphisms and it is induced by a degree invariant relation on 2^ω definable in Second-Order Arithmetic, see [10]. An analogous reasoning is valid for the structure $\mathcal{D}_e$, [11].

What is the situation in the local substructures? In the case of the structure of the c.e. degrees, $\mathcal{R}$, the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ are not definable. Indeed, by Solovay (see for instance [12]), the set of the indices of the c.e. sets which are intermediate is $\Pi^0_{\omega+1}$-complete, and the sets of the indices of the c.e. sets which are in $\mathbf{H}$ and $\mathbf{L}$ respectively are both $\Sigma^0_{\omega+1}$-complete and hence are not definable in First-Order Arithmetic. On the other hand, by Nies, Shore and Slaman [7], a relation on c.e. degrees invariant under the double jump[1] is definable in $\mathcal{R}$ if and only if it is definable in First-Order Arithmetic. Therefore $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{R}$. From this point one may conclude that $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{G}_T$. Indeed, following Nies, Shore and Slaman [7], a relation on degrees below $\mathbf{0}'_T$ invariant under the double jump is definable in $\mathcal{G}_T$ if and only if it is definable in First-Order Arithmetic. But the classes of the indeces of the Δ^0_2-sets having Turing degrees in $\mathbf{I}, \mathbf{H}$ or $\mathbf{L}$ respectively are not definable in First-Order Arithmetic, since otherwise adding to their definitions the condition of being c.e. (which is definable in First-Order Arithmetic) would result into definitions of the indices of the c.e. sets in $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$. So again $\mathbf{I}, \mathbf{H}$ and $\mathbf{L}$ are not definable in $\mathcal{G}_T$. Finally, let us consider $\mathcal{G}_e$. Here one can argue in a manner similar to the above by noting that $\mathcal{R}$ is isomorphic to the structure of the Π^0_1 enumeration degrees [8], and that the latter are definable in First-Order Arithmetic. Now assuming that one of the classes $\mathbf{H}, \mathbf{L}$ and $\mathbf{I}$ is definable in $\mathcal{G}_e$, one can easily show the definability of the respective class of indeces of Σ^0_2-sets in First-Order Arithmetic. So a definition in First-Order

[1]A n-ary relation R on degrees is invariant under the double jump if and only if whenever $R(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ and $\mathbf{x}''_1 = \mathbf{y}''_1, \ldots, \mathbf{x}''_n = \mathbf{y}''_n$, it is also true that $R(\mathbf{y}_1, \ldots, \mathbf{y}_n)$.

Arithmetic of the corresponding class of c.e. sets is obtained, once again leading to a contradiction.

In this paper we investigate the question about the definability of the classes $\mathbf{I}$, $\mathbf{H}$ and $\mathbf{L}$ in the local theory of the structure of the ω-enumeration degrees, $\mathcal{D}_\omega$, which is a proper extension of $\mathcal{D}_e$.

The structure of the ω-enumeration degrees was introduced by Soskov [14] and further studied in a sequence of works by Soskov, M. Soskova and Ganchev [15,17,3]. Unlike the structures of the Turing degrees and of the enumeration degrees, $\mathcal{D}_\omega$ is based on a reducibility relation between sequences of sets of natural numbers. To be more precise, a sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ is said to be *ω-enumeration reducible* to a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ if and only if $J_\mathcal{B} \subseteq J_\mathcal{A}$, where for any sequence $\mathcal{X} = \{X_k\}_{k<\omega}$, $J_\mathcal{X}$ denotes the jump class

$$J_\mathcal{X} = \{\mathbf{d}_T(Y) | \ X_k \text{ is c.e. in } Y^{(k)} \text{ uniformly in } k\}.$$

The jump $\mathcal{A}'$ of a sequence $\mathcal{A}$ is defined [15] so that the class $J_{\mathcal{A}'}$ consists exactly of the jumps of the Turing degrees in $J_\mathcal{A}$, i.e. so that $J_{\mathcal{A}'} = J'_\mathcal{A}$. The jump operator on sequences is monotone and thus induces a jump operation $'$ in $\mathcal{D}_\omega$. Like the jump operation in $\mathcal{D}_T$, the range of the jump operation in $\mathcal{D}_\omega$ is exactly the cone above the first jump of the least element $\mathbf{0}_\omega$. In other words, a general jump inversion theorem is valid for $\mathcal{D}_\omega$. Moreover, even a stronger statement turns out to be true, namely, for every ω-enumeration degree $\mathbf{a}$ above $\mathbf{0}'_\omega$ there is a least degree with jump equals to $\mathbf{a}$. This property is neither true for $\mathcal{D}_T$ nor for $\mathcal{D}_e$.

The strong jump inversion theorem makes the structure $\mathcal{D}_\omega$ worth studying, since using it one may consider a natural copy of the structure $\mathcal{D}_e$ definable in $\mathcal{D}_\omega$ augmented by the jump operation. Moreover, the automorphism groups of $\mathcal{D}_e$ and ${\mathcal{D}_\omega}'$ (i.e. the structure of ω-enumeration degrees augmented with jump operation) are isomorphic.

The jump operation gives rise to the local substructure $\mathcal{G}_\omega$ consisting of all ω-enumeration degrees below $\mathbf{0}'_\omega$. Thanks to the strong jump inversion, $\mathcal{G}_\omega$ contains a class of remarkable degrees having no analogue in either $\mathcal{R}$, $\mathcal{G}_T$ or $\mathcal{G}_e$. These degrees are denoted by $\mathbf{o}_n$, $n < \omega$, and are defined so that $\mathbf{o}_n$ is the least degree whose n-th jump is equal to the $(n+1)$-th jump of $\mathbf{0}_\omega$. In other words, $\mathbf{o}_n$ is the least high$_n$ degree. The degrees $\mathbf{o}_n$ turn out be also connected to low$_n$ degrees. Indeed, a degree in $\mathcal{G}_\omega$ is low$_n$ *if and only if* it forms a minimal pair with $\mathbf{o}_n$.

Each one of the degrees $\mathbf{o}_n$ turns out to be definable in $\mathcal{G}_\omega$, [3], and hence so are the classes $\mathbf{H}_n$ and $\mathbf{L}_n$, for $n \in \omega$. The definition in $\mathcal{G}_\omega$ of $\mathbf{o}_n$ given by Ganchev and M. Soskova [3] is based on the notion of Kalimullin pairs, or more simply $\mathcal{K}$-pairs — a notion first introduced and studied by Kalimullin in the context of the enumeration degrees. For an arbitrary partial order $\mathcal{D} = (\mathbf{D}, \leq)$ a pair $\{\mathbf{a}, \mathbf{b}\}$ is called a $\mathcal{K}$-pair *if and only if*

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})$$

holds for every $\mathbf{x} \in \mathcal{D}$.

The $\mathcal{K}$-pairs in $\mathcal{G}_\omega$ can be separated into two disjoint classes. The first class consists of the $\mathcal{K}$-pairs formed by two almost zero degrees (a degree in $\mathcal{G}_\omega$ is called *almost zero* if and only if it is bellow each $\mathbf{o}_n$). The other class contains the $\mathcal{K}$-pairs *inherited* from $\mathcal{D}_e[\mathbf{0}_e^{(n)}, \mathbf{0}_e^{(n+1)}]$ for some natural number n. The degrees $\mathbf{o}_n$ are strongly connected with the inherited $\mathcal{K}$-pairs. In fact the degree $\mathbf{o}_n$ is the greatest degree which is the least upper bound of an inherited $\mathcal{K}$-pair, which cannot be cupped above $\mathbf{o}_{n-1}$ by a degree less then $\mathbf{o}_{n-1}$. On the other hand if a $\mathcal{K}$-pair is not inherited, then it is bounded by every $\mathbf{o}_{n-1}$ so that we can relax the condition on the $\mathcal{K}$-pairs to be inherited in the above characterisation. Since $\mathbf{o}_0$ is the top element in $\mathcal{G}_\omega$, we can define each of the degrees $\mathbf{o}_n$ inductively in $\mathcal{G}_\omega$.

In this paper we continue the study of the connections between the degrees $\mathbf{o}_n$ and the $\mathcal{K}$-pairs. Our aim is to prove the following theorem.

Theorem 1. *The classes* $\mathbf{H}, \mathbf{L}$ *and* $\mathbf{I}$ *are first order definable in the local substructure* $\mathcal{G}_\omega$ *of the* ω*-enumeration degrees.*

To obtain the above mentioned definability result it suffices to prove that the set $\mathfrak{O} = \{\mathbf{o}_n | n < \omega\}$ is definable in $\mathcal{G}_\omega$. Indeed, we obviously have that

$$\mathbf{x} \in \mathbf{H} \iff (\exists n)[\mathbf{x} \in \mathbf{H}_n] \iff (\exists n)[\mathbf{o}_n \leq_\omega \mathbf{x}] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{o} \leq_\omega \mathbf{x}].$$

Similarly

$$\mathbf{x} \in \mathbf{L} \iff (\exists n)[\mathbf{x} \in \mathbf{L}_n] \iff (\exists n)[\mathbf{o}_n \wedge \mathbf{x} = \mathbf{0}_\omega] \iff (\exists \mathbf{o} \in \mathfrak{O})[\mathbf{o} \wedge \mathbf{x} = \mathbf{0}_\omega].$$

So how do we define $\mathfrak{O}$? As we stated above, each $\mathbf{o}_n$ is the least upper bound of an inherited $\mathcal{K}$-pair. Then our first goal is to define the set of the inherited $\mathcal{K}$-pairs in $\mathcal{G}_\omega$. We achieve this using a result by Kent and Sorbi [6]. Namely, we show that a $\mathcal{K}$-pair is inherited *if and only if* each of its elements bounds a non-splittable degree. So we concentrate only on least upper bounds of inherited $\mathcal{K}$-pairs. First we show that for each $\mathbf{o}_n$ and for each inherited $\mathcal{K}$-pair, the elements of the $\mathcal{K}$-pair are either bellow $\mathbf{o}_n$ or are incomparable with $\mathbf{o}_n$. Then a result by Ganchev and M. Soskova [3] allows us to show that this necessary condition is also sufficient, so that we obtain the desired definition of $\mathfrak{O}$.

Moreover, we shall extend our observations for the $\mathcal{K}$-pairs in $\mathcal{G}_\omega$ and characterise the $\mathcal{K}$-pairs in $\mathcal{D}_\omega$. We shall see that the $\mathcal{K}$-pairs in $\mathcal{D}_\omega$ either consists only of *a.z.* degrees, or are inherited just like in the case of $\mathcal{G}_\omega$. But the inherited $\mathcal{K}$-pairs are always below $\mathbf{0}'_\omega$. So, knowing how to distinguish (in $\mathcal{D}_\omega$) the inherited $\mathcal{K}$-pairs from the others and using the fact that $\mathbf{0}'_\omega$ can be represented as a least upper bound of an inherited $\mathcal{K}$-pair, we conclude that $\mathbf{0}'_\omega$ is the greatest degree which is least upper bound of an inherited $\mathcal{K}$-pair. Thus we have

Theorem 2. *The first jump of the least element* $\mathbf{0}_\omega$ *is first order definable in* $\mathcal{D}_\omega$.

2. PRELIMINARIES

We denote the set of natural numbers by ω. If not stated otherwise, a, b, $c, \dots$ stand for natural numbers, A, B, $C, \dots$ for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}, \dots$ for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}, \dots$ for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

The notation $A \oplus B$ stands for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

We assume that the reader is familiar with the notion of enumeration reducibility, $\leq_e$, and with the structure of the enumeration degrees (for an introduction on the enumeration reducibilities and the respective degree structure we refer the reader to [1, 13]).

For a natural number e and a set $A \subseteq \omega$, we denote by W_e^A the domain of the partial function computed by the oracle Turing machine with index e and using A as an oracle.

Intuitively, a set A is enumeration reducible (*e-reducible*) to a set B, if there is an effective algorithm transforming each enumeration of B into an enumeration of A. More formally, $A \leq_e B$ *if and only if* there is a natural number i, such that for every enumeration f of B, the function $\{i\}^f$ is an enumeration of A. It turns out that $A \leq_e B$ *if and only if* there is a c.e. set W, such that

$$x \in A \iff (\exists u)[\langle x, u\rangle \in W \;\&\; D_u \subseteq B], \tag{2.1}$$

where $\langle x, u\rangle$ denotes the code of the pair of natural numbers (x, u) under some fixed encoding, and D_u is the finite set with canonical index u. Usually this is taken as the formal definition of the enumeration reducibility. If the set W in (2.1) has index i, we say that A is e-reducible to B via W_i, and we shall write $A = W_i(B)$.

The relation $\leq_e$ is a preorder on the powerset $\mathcal{P}(\omega)$ of the natural numbers and induces a nontrivial equivalence relation $\equiv_e$. The equivalence classes under $\equiv_e$ are called enumeration degrees. The enumeration degree which contains the set A is denoted by $\mathbf{d}_e(A)$. The set of all enumeration degrees is denoted by $\mathbf{D}_e$. The enumeration reducibility between sets induces a partial order $\leq_e$ on $\mathbf{D}_e$ by

$$\mathbf{d}_e(A) \leq_e \mathbf{d}_e(B) \iff A \leq_e B.$$

We denote by $\mathcal{D}_e$ the partially ordered set $(\mathbf{D}_e, \leq_e)$. The least element of $\mathcal{D}_e$ is the enumeration degree $\mathbf{0}_e$ of $\emptyset$. Also, the enumeration degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_e$ is an upper semilattice with least element.

By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

The enumeration jump A'_e of A is defined by $A'_e = \{x \mid x \in W_x(A)\}^+$. The jump operation preserves the enumeration reducibility, hence we can define $\mathbf{d}_e(A)' = \mathbf{d}_e(A')$. Since $A <_e A'$, then we have $\mathbf{a} <_e \mathbf{a}'$ for every enumeration degree $\mathbf{a}$. The jump operator is uniform, i.e. there exists a recursive function j such that for every sets A and B, if $A = W_e(B)$ then $A' = W_{j(e)}(B')$.

The jump operation gives rise to the local substructure $\mathcal{G}_e$, consisting of all degrees bellow $\mathbf{0}'_e$ – the jump of the least enumeration degree. Cooper [1] has proved that $\mathcal{G}_e$ is exactly the collection of all Σ^0_2 enumeration degrees.

Finally we need the following definition, which we shall use in order to characterise ω-enumeration reducibility. Given a sequence $\mathcal{A} \in S_\omega$ we define the *jump sequence* $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ as the sequence $\{P_k(\mathcal{A})\}_{k<\omega}$ such that:

1. $P_0(\mathcal{A}) = A_0$;
2. $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$.

3. THE ω-ENUMERATION DEGREES

Soskov [14] introduced the structure of the ω-enumeration degrees $\mathcal{D}_\omega$ in the following way. For every sequence $\mathcal{A} \in \mathcal{S}_\omega$, we define its jump class $J_{\mathcal{A}}$ to be the set:

$$J_{\mathcal{A}} = \{\mathbf{d}_T(X) \mid A_k \text{ is c.e. in } X^{(k)} \text{ uniformly in } k\}. \tag{3.1}$$

We set

$$\mathcal{A} \leq_\omega \mathcal{B} \iff J_{\mathcal{B}} \subseteq J_{\mathcal{A}}.$$

Clearly $\leq_\omega$ is a reflexive and transitive relation, and the relation $\equiv_\omega$ defined by

$$\mathcal{A} \equiv_\omega \mathcal{B} \iff \mathcal{A} \leq_\omega \mathcal{B} \;\&\; \mathcal{B} \leq_\omega \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-enumeration degrees. In particular, the equivalence class $\mathbf{d}_\omega(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$ is called the ω-enumeration degree of $\mathcal{A}$. The relation $\leq_\omega$ defined by

$$\mathbf{a} \leq_\omega \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_\omega \mathcal{B})$$

is a partial order on the set of all ω-enumeration degrees $\mathbf{D}_\omega$. By $\mathcal{D}_\omega$ we shall denote the structure $(\mathbf{D}_\omega, \leq_\omega)$. The ω-enumeration degree $\mathbf{0}_\omega$ of the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_\omega$. Further, the ω-enumeration degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \mathbf{d}_\omega(\mathcal{A})$ and $\mathbf{b} = \mathbf{d}_\omega(\mathcal{B})$. Thus $\mathcal{D}_\omega$ is an upper semi-lattice with least element.

An explicit characterisation of the ω-enumeration reducibility is derived in [16]. According to it, $\mathcal{A} \leq_\omega \mathcal{B} \iff A_n \leq_e P_n(\mathcal{B})$ uniformly in n. More formally,

$\mathcal{A} \leq_\omega \mathcal{B}$ *if and only if* there is a computable function f, such that for every natural number k, $A_k = W_{f(k)}(P_k(\mathcal{B}))$. From here, one can show that each sequence is ω-enumeration equivalent with its jump sequence, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A}). \tag{3.2}$$

Further, for the sake of convenience, for sequences $\mathcal{A}, \mathcal{B} \in S_\omega$ we shall write $\mathcal{A} \leq_e \mathcal{B}$ if and only if for each $k < \omega, A_k \leq_e B_k$ uniformly in k. So $\mathcal{A} \leq_\omega \mathcal{B} \iff \mathcal{A} \leq_e \mathcal{P}(\mathcal{B})$. Note that there exist only countably many computable functions, so that there could be only countably many sequences ω-enumeration reducible to a given sequence. In particular every ω-enumeration degree cannot contain more than countably many sequences and hence there are continuum many ω-enumeration degrees.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \ldots, \emptyset, \ldots)$. From the definition of $\leq_\omega$ and the uniformity of the jump operation, we have that for every sets A and B,

$$A \uparrow \omega \leq_\omega B \uparrow \omega \iff A \leq_e B. \tag{3.3}$$

The last equivalence means, that the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_\omega$, defined by, $\kappa(\mathbf{x}) = \mathbf{d}_\omega(X \uparrow \omega)$, where X is an arbitrary set in $\mathbf{x}$, is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$. Further, the so defined embedding κ preserves the least element and the binary least upper bound operation. We shall denote the range of κ with $\mathbf{D}_1$.

4. THE JUMP OPERATOR

Following the lines of Soskov and Ganchev [15], the ω-enumeration jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots).$$

This operator is defined so that if $\mathcal{A}'$ is the jump of $\mathcal{A}$, then the jump class $J_{\mathcal{A}'}$ of $\mathcal{A}'$ contains exactly the jumps of the degrees in the jump class $J_{\mathcal{A}}$ of $\mathcal{A}$. Note also, that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_\omega \{P_{k+1}(\mathcal{A})\}$.

The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_\omega \mathcal{A}'$ and $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$. This allows to define a jump operation on the ω-enumeration degrees by setting

$$\mathbf{a}' = \mathbf{d}_\omega(\mathcal{A}'),$$

where $\mathcal{A}$ is an arbitrary sequence in $\mathbf{a}$. Clearly, $\mathbf{a} <_\omega \mathbf{a}'$ and $\mathbf{a} \leq_\omega \mathbf{b} \Rightarrow \mathbf{a}' \leq_\omega \mathbf{b}'$.

Also the jump operation on ω-enumeration degrees agrees with the jump operation on the enumeration degrees, i.e. we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathcal{D}_e.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \dots) \equiv_\omega \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{4.1}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the ω-enumeration degree $\mathbf{a}$.

The jump operator on $\mathcal{D}_\omega$ preserves the greatest lower bound, i.e. for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{D}_\omega$,

$$\mathbf{x} \wedge \mathbf{y} = \mathbf{c} \Rightarrow \mathbf{x}' \wedge \mathbf{y}' = \mathbf{c}'.[2] \tag{4.2}$$

Further, Soskov and Ganchev [15] showed that for every natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then there is a least ω-enumeration degree $\mathbf{x}$ above $\mathbf{a}$ with $\mathbf{x}^{(n)} = \mathbf{b}$. We denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_k, \dots), \tag{4.3}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

From here it follows that for every given $\mathbf{a} \in \mathbf{D}_\omega$ and $n < \omega$, the operation $\mathbf{I}^n_{\mathbf{a}}$ is monotone. Further, its range is a downwards closed subset of the upper cone with least element $\mathbf{a}$. In fact, even a stronger property holds: if $\mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ are such that $\mathbf{a} \le_\omega \mathbf{x}$, $\mathbf{a}^{(n)} \le_\omega \mathbf{b}$ and $\mathbf{x} \le_\omega \mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$, then $\mathbf{x}$ is equal to $\mathbf{I}^n_{\mathbf{a}}(\mathbf{x}^{(n)})$. The above property can be easily verified by simple relativisation of claim ($I2$) of Lemma 1 in [3].

It what follows, when $\mathbf{a} = \mathbf{0}_\omega$, we shall write $\mathbf{I}^n$ instead of $\mathbf{I}^n_{\mathbf{0}_\omega}$. Finally, we provide a property of the jump inversion operation, a proof of which can be found in [3].

$$(\mathbf{x} \vee \mathbf{I}^n(\mathbf{a}))^{(n)} = \mathbf{x}^{(n)} \vee \mathbf{a}. \tag{4.4}$$

5. THE LOCAL THEORY AND THE $\mathbf{o}_n$ DEGREES

The structure of the degrees lying beneath the first jump of the least element is usually referred to as the local structure of a degree structure. In the case of the ω-enumeration degrees we shall denote this structure by $\mathcal{G}_\omega$. When considering a local structure, one is usually concerned with questions about the definability of some classes of degrees, which have a natural definition either in the context of the global structure (for example the classes of the high and the low degrees) or in the context of the basic objects from which the degrees are built (for example the class of the Turing degrees containing a c.e. set).

Recall that a degree in the local structure is said to be *high$_n$* for some n *if and only if* its n-th jump is as high as possible. Similarly, a degree in the local structure is said to be *low$_n$* for some n *if and only if* its n-th jump is as low as possible. More formally, in the case of $\mathcal{G}_\omega$, a degree $\mathbf{a} \in \mathcal{G}_\omega$ is high$_n$ *if and only if* $\mathbf{a}^{(n)} = (\mathbf{0}'_\omega)^{(n)} = \mathbf{0}_\omega^{(n+1)}$, and is low$_n$ *if and only if* $\mathbf{a}^{(n)} = (\mathbf{0}'_\omega)^{(n)}$.

As usual, we denote by $\mathbf{H}_n$ the collection of all high$_n$ degrees, and by $\mathbf{L}_n$ the collection of all low$_n$ degrees. Also $\mathbf{H}$ stands for the union of all the classes $\mathbf{H}_n$ and analogously, $\mathbf{L}$ is the union of all of the classes $\mathbf{L}_n$. Finally, $\mathbf{I}$ will stay for the collection of the degrees that are neither high$_n$ nor low$_n$ for any n. The degrees in $\mathbf{I}$ shall be referred to as intermediate degrees.

Using the corresponding results for the structure of the enumeration degrees, it is easy to see that there exist intermediate degrees and for every natural number n, there are degrees in the local structure of the ω-enumeration degrees, that are high$_{(n+1)}$ (respectively low$_{(n+1)}$) but are not high$_n$ (respectively low$_n$).

Soskov and Ganchev [15] gave a characterisation of the classes $\mathbf{H}_n$ and $\mathbf{L}_n$ that does not involve directly the jump operation. Let us set $\mathbf{o}_n$ to be the least n-th jump invert of $\mathbf{0}_\omega^{(n+1)}$, i.e., $\mathbf{o}_n = \mathbf{I}^n(\mathbf{0}^{(n+1)})$. Note that $\mathbf{o}_n$ is the least element of the class $\mathbf{H}_n$. Thus for arbitrary $\mathbf{x} \in \mathcal{G}_\omega$,

$$\mathbf{x} \in \mathbf{H}_n \iff \mathbf{o}_n \leq_\omega \mathbf{x}. \tag{5.1}$$

In particular, since every high$_n$ degree is also high$_{(n+1)}$, $\mathbf{o}_{n+1} \leq_\omega \mathbf{o}_n$. On the other hand, since $\mathbf{H}_{n+1} \setminus \mathbf{H}_n \neq \emptyset$, the equality $\mathbf{o}_{n+1} = \mathbf{o}_n$ is impossible, so that

$$\mathbf{0}'_\omega = \mathbf{o}_0 >_\omega \mathbf{o}_1 >_\omega \mathbf{o}_2 >_\omega \cdots >_\omega \mathbf{o}_n >_\omega \cdots.$$

Recall that if a degree is beneath a least n-th jump invert above $\mathbf{a}$, then it itself is a least n-th jump invert above $\mathbf{a}$. In particular, if $\mathbf{y} \leq_\omega \mathbf{o}_n$, then $\mathbf{y} = \mathbf{I}^n(\mathbf{z})$ for some degree $\mathbf{0}_\omega^{(n)} \leq_\omega \mathbf{z} \leq_\omega \mathbf{0}_\omega^{(n+1)}$ or more concretely $\mathbf{y} = \mathbf{I}^n(\mathbf{y}^{(n)})$. On the other hand if $\mathbf{y} \in \mathcal{G}_\omega$ is a least n-th jump invert, then from the monotonicity of $\mathbf{I}^n$ we have $\mathbf{y} \leq_\omega \mathbf{o}_n$. Thus

$$\{\mathbf{y} \in \mathcal{G}_\omega \mid \mathbf{y} \leq_\omega \mathbf{o}_n\} = \{\mathbf{I}^n(\mathbf{z}) \mid \mathbf{0}_\omega^{(n)} \leq_\omega \mathbf{z} \leq_\omega \mathbf{0}_\omega^{(n+1)}\}.$$

In particular, since $\mathbf{I}^n$ is injective,

$$[\mathbf{0}_\omega, \mathbf{o}_n] \simeq [\mathbf{0}_\omega^{(n)}, \mathbf{0}_\omega^{(n+1)}].$$

Ganchev and M. Soskova [3] showed that for arbitrary $\mathbf{x} \in \mathcal{G}_\omega$,

$$\mathbf{I}^n(\mathbf{x}^{(n)}) = \mathbf{x} \wedge \mathbf{o}_n. \tag{5.2}$$

Indeed, let us take an arbitrary $\mathbf{x} \in \mathcal{G}_\omega$. Clearly $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{x}$ and $\mathbf{I}^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{o}_n$. On the other hand if $\mathbf{y}$ is such that $\mathbf{y} \leq_\omega \mathbf{x}$ and $\mathbf{y} \leq_\omega \mathbf{o}_n$, then from the second inequality we have $\mathbf{y} = \mathbf{I}^n(\mathbf{z})$ for some $\mathbf{z}$. This together with the first inequality gives us $\mathbf{z} = (\mathbf{I}^n(\mathbf{z}))^{(n)} = \mathbf{y}^{(n)} \leq_\omega \mathbf{x}^{(n)}$. Thus $\mathbf{y} = \mathbf{I}^n(\mathbf{z}) \leq_\omega \mathbf{I}^n(\mathbf{x}^{(n)})$.

This gives us a characterisation of the low$_n$ degrees in terms of the partial order $\leq_\omega$ and the degrees $\mathbf{o}_n$, namely

$$\mathbf{x} \in \mathbf{L}_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_\omega. \tag{5.3}$$

They also show that for arbitrary $\mathbf{a} \in \mathcal{G}_\omega$, $\mathbf{a}$ is a degree in $\mathbf{D}_1$ *iff*

$$\forall \mathbf{x} \in \mathcal{G}_\omega(\mathbf{x} \vee \mathbf{o}_1 = \mathbf{a} \vee \mathbf{o}_1 \to \mathbf{x} \geq_\omega \mathbf{a}). \tag{5.4}$$

The formula (5.4) characterises the degrees in $\mathbf{D}_1 \cap \mathcal{G}_\omega$ in terms of the ordering $\leq_\omega$ and the degree $\mathbf{o}_1$.

Soskov and Ganchev [15] introduced the *almost zero* (*a.z.*) degrees. Following their lines, the degree $\mathbf{x}$ is *a.z. if and only if* there is a representative $\mathcal{X} \in \mathbf{x}$ such that

$$(\forall k)[P_k(\mathcal{X}) \equiv_e \emptyset^{(k)}]. \tag{5.5}$$

It is clear that the class of the *a.z.* degrees is downward closed. Further, one can easily show that the only *a.z.* degree $\mathbf{a}$ for which there is a natural number n such that $\mathbf{a}^{(n)} = \mathbf{0}_\omega^{(n)}$ is the least element $\mathbf{0}_\omega$. Note also that there are continuum many *a.z.* degrees and hence not all *a.z.* degrees are in $\mathcal{G}_\omega$.

The *a.z.* degrees in $\mathcal{G}_\omega$ are exactly the degrees bounded by every degree $\mathbf{o}_n$, i.e.

$$\mathbf{x} \in \mathcal{G}_e \text{ is } a.z. \iff (\forall n < \omega)[\mathbf{x} \leq_\omega \mathbf{o}_n]. \tag{5.6}$$

Further, the classes $\mathbf{H}$ and $\mathbf{L}$ can be characterised in terms of the ordering $\leq_\omega$ and the *a.z.* degrees [15], namely

$$\mathbf{a} \in \mathbf{H} \iff (\forall \mathbf{x} - a.z.)[\mathbf{x} \leq_\omega \mathbf{a}], \tag{5.7}$$

and

$$\mathbf{a} \in \mathbf{L} \iff (\forall \mathbf{x} - a.z.)[\mathbf{x} \leq_\omega \mathbf{a} \to \mathbf{x} = \mathbf{0}_\omega], \tag{5.8}$$

where all quantifiers are restricted to degrees in $\mathcal{G}_\omega$.

From the second equivalence it follows that the only low$_n$ *a.z.* degree is $\mathbf{0}_\omega$. Further, according to (5.1) no *a.z.* degree is high$_n$ for any n. Thus all *a.z.* degrees are intermediate degrees.

6. DEFINABILITY IN $\mathcal{G}_\omega$

We prove in this section that the set $\mathfrak{O} = \{\mathbf{o}_n | n < \omega\}$ is first order definable in $\mathcal{G}_\omega$. Thus, by (5.1) and (5.3), we may conclude the proof of the Theorem 1. For this purpose we shall need the notion of a *Kalimullin pair* (or $\mathcal{K}$-pair).

Definition 3. *Let $\mathcal{D} = (\mathbf{D}, \leq)$ be a partial order. The pair $\{\mathbf{a}, \mathbf{b}\}$ is said to be $\mathcal{K}$-pair (strictly) over $\mathbf{u}$ for $\mathcal{D}$, if $\mathbf{a}, \mathbf{b}, \mathbf{u} \in \mathbf{D}, \mathbf{u} \leq \mathbf{a}, \mathbf{b}$ ($\mathbf{u} \lneq \mathbf{a}, \mathbf{b}$) and for all $\mathbf{x} \in \mathbf{D}$ such that $\mathbf{u} \leq \mathbf{x}$, the least upper bounds $\mathbf{x} \vee \mathbf{a}, \mathbf{x} \vee \mathbf{b}$ and greatest lower bound $(\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})$ exist, and the following holds:*

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}). \tag{6.1}$$

Further, if $\mathcal{D} = (\mathbf{D}, \leq)$ is a partially ordered set and $\mathbf{u}, \mathbf{v} \in \mathbf{D}$, we shall use the notation $\mathbf{D}[\mathbf{u}, \mathbf{v}]$ for the set $\{\mathbf{x} \in \mathbf{D} | \mathbf{u} \leq \mathbf{x} \leq \mathbf{v}\}$ together with the partial order inherited from $\mathcal{D}$.

Clearly, there exists a first order formula $\mathcal{K}$ of two free variables such that if $\mathcal{D}$ has a least element $\mathbf{0}_{\mathcal{D}}$, then

$$\mathcal{D} \models \mathcal{K}(\mathbf{a}, \mathbf{b}) \iff \{\mathbf{a}, \mathbf{b}\} \text{ is a } \mathcal{K}\text{-pair strictly over } \mathbf{0}_{\mathcal{D}} \text{ for } \mathcal{D}.$$

Also, we shall use the fact that for each $\mathbf{a} \in \mathbf{D}$, the set

$$\mathcal{I} = \{\mathbf{b} \mid \{\mathbf{a}, \mathbf{b}\} \text{ is a } \mathcal{K}\text{-pair strictly over } \mathbf{0}_{\mathcal{D}} \text{ for } \mathcal{D}\}$$

is either empty or ideal, see for example [5].

The starting step of the first order definition in $\mathcal{G}_\omega$ of the set $\mathfrak{O}$ is the characterisation of the $\mathcal{K}$-pairs in $\mathcal{G}_\omega$, due to Ganchev and M. Soskova [3]. According to it, whenever $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair in $\mathcal{G}_\omega$ strictly over $\mathbf{0}_\omega$, then either $\mathbf{a}$ and $\mathbf{b}$ are both *a.z.* or the $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ is inherited from the structure $\mathcal{D}_e$, i.e. there exist sets A, B and a natural number n such that:

1. $\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ and $A' = B' = \emptyset^{(n+1)}$;
2. $\{\mathbf{d}_e(A), \mathbf{d}_e(B)\}$ is a $\mathcal{K}$-pair in $\mathbf{D}_e[\mathbf{0}_e^{(n)}, \mathbf{0}_e^{(n+1)}]$ strictly over $\mathbf{0}_e^{(n)}$;
3. $\mathbf{a} = \mathbf{I}^n(\kappa(\mathbf{d}_e(A)))$ and $\mathbf{b} = \mathbf{I}^n(\kappa(\mathbf{d}_e(B)))$.

It is known [3] that every two degrees $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$, which are inherited from $\mathcal{D}_e$ in the above sense, form a $\mathcal{K}$-pair in $\mathcal{G}_\omega$ strictly over $\mathbf{0}_\omega$.

Note that by definitions of the embedding κ and the least jump inversion operation (4.3) the last condition of the above characterisation of the $\mathcal{K}$-pairs in the local theory is equivalent to the fact that the degrees $\mathbf{a}$ and $\mathbf{b}$ contain respectively the sequences $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots, \emptyset, \ldots)$ and $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, B, \emptyset, \ldots, \emptyset, \ldots)$.

Using the above characterisation, one can prove that for each $n \geq 0$, $\mathbf{o}_{n+1}$ is the greatest degree (in $\mathcal{G}_\omega$) which is the least upper bound of a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly above $\mathbf{0}_\omega$ such that $(\forall \mathbf{x} \lneq_\omega \mathbf{o}_n)[\mathbf{a} \vee \mathbf{x} \lneq_\omega \mathbf{o}_n]$. Since $\mathbf{o}_0$ is the greatest degree in $\mathcal{G}_\omega$, it follows that for each natural number n, $\mathbf{o}_n$ is first order definable in $\mathcal{G}_\omega$.

Note that the *a.z.* degrees are closed under the least upper bound operation and no $\mathbf{o}_n$ is *a.z.*, thus if $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{o}_n$, then $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair.

Now we shall show how to separate in $\mathcal{G}_\omega$ the inherited $\mathcal{K}$-pairs from those formed by *a.z.* degrees. Suppose that $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair and let $A, B \subseteq \omega$ and $n < \omega$ be the corresponding witnesses for this. It is known by the a result of Kent and Sorbi [6], that every nonzero enumeration degree $\mathbf{x} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}_e']$ bounds a nonzero nonsplittable[2] degree $\mathbf{y} \in \mathbf{D}_e[\mathbf{0}_e, \mathbf{0}'_e]$. Relativising this result over $\mathbf{0}_e^{(n)}$ we conclude that there are sets $A_0, B_0 \subseteq \omega$ such that $\emptyset^{(n)} <_e A_0 \leq_e A$, $\emptyset^{(n)} <_e B_0 \leq_e B$, such that both $\mathbf{d}_e(A_0)$ and $\mathbf{d}_e(B_0)$ are nonsplittable over $\mathbf{0}_e^{(n)}$. But then the degrees $\mathbf{a}_0 = \mathbf{I}^n(\kappa(\mathbf{d}_e(A_0)))$ and $\mathbf{b}_0 = \mathbf{I}^n(\kappa(\mathbf{d}_e(B_0)))$ are nonsplittable. Indeed, assume without loss of generality that $\mathbf{a}_0$ is splittable. Then $\mathbf{a}_0 = \mathbf{c} \vee \mathbf{d}$ for some $\mathbf{0}_\omega <_\omega \mathbf{c}, \mathbf{d} <_\omega \mathbf{a}_0$ and let $\mathcal{C} = \{C_m\}_{m<\omega} \in \mathbf{c}, \mathcal{D} = \{D_m\}_{m<\omega} \in \mathbf{d}$. According to (4.3) and (3.2), $(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A_0, \emptyset^{(n+1)}, \ldots) \in \mathbf{a}_0$, so that

$$\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D}) \leq_e (\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A_0, \emptyset^{(n+1)}, \ldots),$$

From here $P_n(\mathcal{C}) \oplus P_n(\mathcal{D}) \leq_e A_0$,

$$\mathcal{P}(\mathcal{C}) \equiv_e (\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, P_n(\mathcal{C}), \emptyset^{(n+1)}, \ldots)$$

and

$$\mathcal{P}(\mathcal{D}) \equiv_e (\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, P_n(\mathcal{D}), \emptyset^{(n+1)}, \ldots).$$

Since $\mathbf{a}_0 \leq_\omega \mathbf{c} \vee \mathbf{d}$, then $(\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}, A_0, \emptyset^{(n+1)}, \ldots) \leq_e \mathcal{P}(\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D}))$. Then we have that $A_0 \leq_e P_n(\mathcal{P}(\mathcal{C}) \oplus \mathcal{P}(\mathcal{D})) \equiv_e P_n(\mathcal{C}) \oplus P_n(\mathcal{D})$, [3], so finally $A_0 \equiv_e P_n(\mathcal{C}) \oplus P_n(\mathcal{D})$. Since $\mathbf{d}_e(A_0)$ is a nonsplittable degree over $\mathbf{0}_e^{(n)}$, either $P_n(\mathcal{C}) \equiv_e A_0$ or $P_n(\mathcal{D}) \equiv_e A_0$. In the first case we have that $\mathbf{a}_0 = \mathbf{c}$, and in the second – $\mathbf{a}_0 = \mathbf{d}$, i.e., we reach a contradiction.

Thus, if $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$, then both $\mathbf{a}$ and $\mathbf{b}$ bound nonzero nonsplitting degrees. Next we shall see that if $\{\mathbf{a}, \mathbf{b}\}$ is a *a.z.* $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ then neither $\mathbf{a}$ nor $\mathbf{b}$ bounds a nonzero nonsplitting degree. Moreover, the following property holds for every *a.z.* degree in $\mathcal{D}_\omega$.

Lemma 4. *Every nonzero a.z. degree in $\mathcal{D}_\omega$ is splittable.*

Proof. Let $\mathbf{a}$ be a nonzero a.z. degree and let $\mathcal{A} \in \mathbf{a}$ satisfy (5.5). We shall construct sequences $\mathcal{B}$ and $\mathcal{C}$ such that $\emptyset_\omega \lneq_\omega \mathcal{B}, \mathcal{C} \lneq_\omega \mathcal{A}$ and $\mathcal{B} \oplus \mathcal{C} \equiv_\omega \mathcal{A}$. We shall construct $\mathcal{B} = \{B_k\}_{k<\omega}$ and $\mathcal{C} = \{C_k\}_{k<\omega}$ using induction on k. For every k we shall set either $B_k = \emptyset$ and $C_k = A_k$ or $B_k = A_k$ and $C_k = \emptyset$. This condition will ensure that $\mathcal{B} \oplus \mathcal{C} = \mathcal{A}$. So, in order to build $\mathcal{B}$ and $\mathcal{C}$ as desired, it suffices that $\mathcal{B}, \mathcal{C} \leq_\omega \mathcal{A}$ and that the following requirements are satisfied:

$$R_{2e} : \exists k \left(\varphi_e(k) \uparrow \ \vee \ A_k \neq W_{\varphi_e(k)}(P_k(\mathcal{B})) \right),$$

[2]Let $\mathcal{D} = (\mathbf{D}, \mathbf{0}, \leq, \vee)$ be an upper semilattice with a least element. Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}$ be such that $\mathbf{b} \leq \mathbf{a}$. We shall say that $\mathbf{a}$ is *splittable over* $\mathbf{b}$ if and only if there are $\mathbf{x}, \mathbf{y} \in \mathbf{D}$ such that

$$\mathbf{b} \leq \mathbf{x}, \mathbf{y} < \mathbf{a} = \mathbf{x} \vee \mathbf{y}.$$

When there are not such $\mathbf{x}$ and $\mathbf{y}$ we shall say that $\mathbf{a}$ is *nonsplittable over* $\mathbf{b}$. In the case when $\mathbf{b}$ is the least element we shall say only that $\mathbf{a}$ is splittable or nonsplittable.

[3]the last equivalence can be easily verified using induction on $n < \omega$.

$$R_{2e+1} : \exists k \left(\varphi_e(k)\uparrow \ \vee A_k \neq W_{\varphi_e(k)}(P_k(\mathcal{C}))\right).$$

Note that $\mathcal{B}, \mathcal{C} \leq_\omega \mathcal{A}$ gives us automatically, that $\mathcal{B}$ and $\mathcal{C}$ satisfy (5.5). The requirement R_{2e} ensures that $\mathcal{A}$ can not be uniformly reduced to $\mathcal{P}(\mathcal{B})$ using the e-th computable function. Similarly, R_{2e+1} expresses that $\mathcal{A}$ can not be uniformly reduced to $\mathcal{P}(\mathcal{C})$ using the e-th computable function.

The construction: During the construction we shall use a global variable $\mathfrak{r}$ which shall show us the least requirement that is (possibly) not yet satisfied. We start by setting $\mathfrak{r} = 0$. Also we set $B_0 = B_1 = \emptyset$, $C_0 = A_0$ and $C_1 = A_1$. Let us suppose that $k \geq 2$ and that B_s and C_s are defined for $s \leq k$. Note that our assumption yields that for $s \leq k$, $P_s(\mathcal{B})$ and $P_s(\mathcal{C})$ are defined as well.

Case 1: $\mathfrak{r} = 2e$. If $\varphi_e(k-2)\uparrow$ or $A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$, set $B_k = A_k$, $C_k = \emptyset$ and augment $\mathfrak{r}$ by 1. Otherwise set $B_k = \emptyset$, $C_k = A_k$ and keep $\mathfrak{r}$ the same.

Case 2: $\mathfrak{r} = 2e+1$. If $\varphi_e(k-2)\uparrow$ or $A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{C}))$, set $B_k = \emptyset$, $C_k = A_k$ and augment $\mathfrak{r}$ by 1. Otherwise set $B_k = A_k$, $C_k = \emptyset$ and keep $\mathfrak{r}$ the same.

End of construction.

First of all let us note that, according to the definition of the jump sequence $\mathcal{P}(\mathcal{A})$, $\emptyset'' \leq_e P_k(\mathcal{A})$ for $k \geq 2$ uniformly in k. Hence for $k \geq 2$, given any enumeration of $P_k(\mathcal{A})$ we can uniformly decide if $\varphi_e(k-2)\uparrow$. Further, for $k \geq 2$, $P_{k-2}(\mathcal{A})'' \leq_e P_k(\mathcal{A})$ uniformly in k. These properties of $P_k(\mathcal{A})$ and a simple induction on $k \geq 2$ yield that given any enumeration of $P_k(\mathcal{A})$, we can uniformly answer to the questions

$$\varphi_e(k-2)\uparrow \ \vee A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$$

and

$$\varphi_e(k-2)\uparrow \ \vee A_{k-2} \neq W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{C})).$$

In particular, any enumeration of $P_k(\mathcal{A})$ can compute uniformly the value of $\mathfrak{r}$ at stage k and hence it can compute uniformly B_k and C_k. Therefore $\mathcal{B}, \mathcal{C} \leq_\omega \mathcal{A}$.

It remains to prove that all the requirements are satisfied. Towards a contradiction assume that some requirement is not fulfilled and let n be the least index of such a requirement. Note that the construction yields that at some stage m, the global variable $\mathfrak{r}$ has been set to be equal to n, and from then on $\mathfrak{r}$ has never changed its value. First let us suppose that $n = 2e$ for some natural number e. Then for every $k > m$, $A_{k-2} = W_{\varphi_e(k-2)}(P_{k-2}(\mathcal{B}))$, so that $B_k = \emptyset$ for $k > m$ and $A_k \leq_e P_k(\mathcal{B})$ uniformly in $k > m$. On the other hand for $0 \leq k \leq m$,

$$B_k \leq_e P_k(\mathcal{A}) \leq_e \emptyset^{(k)},$$

which together with our previous observation yields $\mathcal{B} \leq_\omega \emptyset_\omega$ and $\mathcal{A} \leq_\omega \mathcal{B}$. Thus $\mathcal{A} \leq_\omega \emptyset_\omega$, contradicting the choice of $\mathcal{A}$.

If $n = 2e + 1$, we obtain in a quite similar way $\mathcal{A} \leq_\omega \emptyset_\omega$, contradicting once again the choice of $\mathcal{A}$. Therefore our assumption that some of the requirements is not satisfied is incorrect, and hence $\emptyset_\omega \lneq_\omega \mathcal{B}, \mathcal{C} \lneq_\omega \mathcal{A}$. □

Thus, we have obtained that every inherited $\mathcal{K}$-pair bounds a nonsplitting degree, whereas every *a.z.* is splittable. Therefore we may define a first order formula $\mathcal{K}_{inh}$ separating the inherited $\mathcal{K}$-pairs from the ones formed by *a.z.* degree by setting

$$\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) = \mathcal{K}(\mathbf{a}, \mathbf{b}) \ \& \ (\exists \mathbf{x})[x \leq_\omega \mathbf{a} \ \& \ (\forall \mathbf{u}, \mathbf{v})[\mathbf{u}, \mathbf{v} <_\omega \mathbf{x} \to \mathbf{u} \vee \mathbf{v} <_\omega \mathbf{x}]].$$

Now we have the instrument needed for the definition of the set $\mathfrak{O}$. Recall that every degree $\mathbf{o}_n$ is the least upper bound of an inherited $\mathcal{K}$-pair, so that we need just to focus on the properties of the least upper bounds of such $\mathcal{K}$-pairs.

Suppose that $\{\mathbf{a}, \mathbf{b}\}$ is an inherited $\mathcal{K}$-pair and let $A, B \subseteq \omega$ and $n < \omega$ be witnesses for this. Since

$$(\underbrace{\emptyset, \emptyset' \ldots, \emptyset^{(m-1)}}_{m}, \emptyset^{(m+1)}, \emptyset^{(m+2)}, \ldots) \in \mathbf{o}_m,$$

$$(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, A, \emptyset^{(n+1)}, \ldots) \in \mathbf{a},$$

$$(\underbrace{\emptyset, \emptyset', \ldots, \emptyset^{(n-1)}}_{n}, B, \emptyset^{(n+1)}, \ldots) \in \mathbf{b},$$

$\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ and $A' = B' = \emptyset^{(n+1)}$, we have $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_m$ for $m \leq n$. On the other hand, $m > n$ implies that $\mathbf{a}, \mathbf{b} \not\leq_\omega \mathbf{o}_m$ and $\mathbf{o}_m \not\leq_\omega \mathbf{a}, \mathbf{b}$, for otherwise we would have $A \leq_e \emptyset^{(n)}$ and $\emptyset^{(m+1)} \leq_e \emptyset^{(m)}$, respectively.

Hence, for every $m < \omega$ and every inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$, either $\mathbf{a}, \mathbf{b} <_\omega \mathbf{o}_m$ or $\mathbf{a}, \mathbf{b} |_\omega \mathbf{o}_m$.

Now we claim that whenever $\mathbf{x}$ is the least upper bound of an inherited $\mathcal{K}$-pair and $\mathbf{x}$ is not $\mathbf{o}_m$ for any natural number m, there exists an inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{a} |_\omega \mathbf{x}$ and $\mathbf{b} \leq_\omega \mathbf{x}$. Indeed, suppose that $\mathbf{x} = \mathbf{c} \vee \mathbf{d}$ for some inherited $\mathcal{K}$-pair $\{\mathbf{c}, \mathbf{d}\}$ and for all $m < \omega$, $\mathbf{x} \neq \mathbf{o}_m$. Let the sets C, D and the natural number n be witnessing that the $\mathcal{K}$-pair is inherited. Then the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{n}, C \oplus D, \emptyset, \ldots)$ is an element of the degree $\mathbf{x}$. Note that $C, D \leq_e \emptyset^{(n+1)}$ and $\mathbf{x} = \mathbf{c} \vee \mathbf{d} \neq \mathbf{o}_n$, so $C \oplus D \lneq_e \emptyset^{(n+1)}$. Since $\mathbf{c}$ and $\mathbf{d}$ are not *a.z.*, we have that C and D are low over $\emptyset^{(n)}$ and hence $C, D \in \Delta^0_2(\emptyset^{(n)})$. But then we have also $C \oplus D \in \Delta^0_2(\emptyset^{(n)})$. In what follows we shall need the following result due to Ganchev and M. Soskova [3].

Theorem 5. *For every total*[4] *enumeration degree* $\mathbf{g}$ *and every degree* $\mathbf{e}$*, such that* $\mathbf{g} \leq_e \mathbf{e}$ *and* $\mathbf{e}$ *contains a set* Δ_2^0 *relative to* $\mathbf{g}$*, there is a* $\mathcal{K}$*-pair* $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ *in* $\mathbf{D}_e[\mathbf{g}, \mathbf{g}']$ *strictly over* $\mathbf{g}$*, such that* $\widetilde{\mathbf{a}} \vee \mathbf{e} = \mathbf{g}'$*. In the case when* $\mathbf{e} \not\leq_e \mathbf{g}'$ *we additionally have that* $\widetilde{\mathbf{a}}|_e\mathbf{e}$ *and* $\widetilde{\mathbf{b}} \leq_e \mathbf{e}$ *(since* $\mathbf{e} = (\widetilde{\mathbf{a}} \vee \mathbf{e}) \wedge (\widetilde{\mathbf{b}} \vee \mathbf{e})$ *and* $\widetilde{\mathbf{a}} \vee \mathbf{e} = \mathbf{g}'$*).*

Now let $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ be the corresponding $\mathcal{K}$-pair for $\mathbf{g} = \mathbf{0}_e^{(n)}$ and $\mathbf{e} = \mathbf{d}_e(C \oplus D)$. Let A and B be sets having enumeration degrees $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ respectively. Then the ω-enumeration degrees $\mathbf{a} = \mathbf{d}_\omega(\underbrace{\emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots)$ and $\mathbf{b} = \mathbf{d}_\omega(\underbrace{\emptyset, \ldots, \emptyset}_{n}, B, \emptyset, \ldots)$ form an inherited $\mathcal{K}$-pair for $\mathcal{G}_\omega$ such that $\mathbf{a}|_\omega\mathbf{x}$ and $\mathbf{b} \leq_\omega \mathbf{x}$.

Thus we have proven that a degree $\mathbf{x} \leq_\omega \mathbf{0}_\omega'$ is $\mathbf{o}_n$ for some natural number n if and only if $\mathbf{x}$ is the least upper bound of an inherited $\mathcal{K}$-pair and for each inherited $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ either $\mathbf{a}, \mathbf{b} <_\omega \mathbf{x}$ or $\mathbf{a}, \mathbf{b}|_\omega\mathbf{x}$. Namely,

$$\mathbf{x} \in \mathfrak{O} \iff (\exists \mathbf{a}, \mathbf{b})[\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) \ \& \ \mathbf{x} = \mathbf{a} \vee \mathbf{b}] \&$$

$$(\forall \mathbf{a}, \mathbf{b})[\mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}) \rightarrow \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{x} \ \vee \ \mathbf{a}, \mathbf{b}|_\omega\mathbf{x}].$$

This gives us a first order definability in $\mathcal{G}_\omega$ of the set $\mathfrak{O}$ as well as of the classes $\mathbf{H}$ and $\mathbf{L}$. A direct consequence of the latter and (5.6) is the following corollary.

Corollary 6. *The set of all a.z. degrees is first order definable in* $\mathcal{G}_\omega$*.*

7. DEFINABILITY OF $\mathbf{0}_\omega'$

In this section we characterise the class of the $\mathcal{K}$-pairs strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Namely, we shall show that either such a $\mathcal{K}$-pair consists of *a.z.* degrees, or it is inherited. As a consequence of this characterisation and the fact that $\mathbf{0}_\omega'$ bounds the elements of all inherited $\mathcal{K}$-pairs we shall find a first order definition of the first jump of the least element in the structure $\mathcal{D}_\omega$.

First, let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ respectively. Using the connections between the $\mathcal{K}$-pairs in $\mathcal{D}_e$ and $\mathcal{G}_\omega$ derived in [3], we are able to conclude that for each $n < \omega$, $\{\mathbf{d}_e(P_n(\mathcal{A})), \mathbf{d}_e(P_n(\mathcal{B}))\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_e^{(n)}$ for $\mathbf{D}_e[\geq \mathbf{0}_e^{(n)}]$. Hence by [5] each of $\mathbf{d}_e(P_n(\mathcal{A}))$ and $\mathbf{d}_e(P_n(\mathcal{B}))$ is quasiminimal over $\mathbf{0}_e^{(n)}$ (the enumeration degree $\mathbf{a}$ is quasiminimal over the enumeration degree $\mathbf{b} <_e \mathbf{a}$ if and only if there is no total $\mathbf{b} \not\leq_e \mathbf{c} \leq_e \mathbf{a}$). Since for each n, $\mathbf{0}_e^{(n+1)} \leq_e \mathbf{d}_e(P_n(\mathcal{A}))' \leq_e \mathbf{d}_e(P_{n+1}(\mathcal{A}))$ and $\mathbf{d}_e(P_n(\mathcal{A}))'$ is total (since every jump is total), then for each n, $P_n(\mathcal{A})' \equiv_e \emptyset^{(n+1)}$. The same equivalence obviously holds also for $P_n(\mathcal{B})'$.

[4]An enumeration degree is said to be total if and only if there exists a set A such that the degree contains the set A^+. With other words a degree is total if and only if it is an image of a Turing degree under the Rogers' embedding $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$. For example, for each n, the degree $\mathbf{0}_e^{(n)}$ is total.

Having in mind the last observation, consider a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ and suppose that at least one of the degrees $\mathbf{a}$ or $\mathbf{b}$ is not *a.z.*. Without loss of generality, suppose that $\mathbf{a}$ is not *a.z.* degree and let $\mathcal{A} \in \mathbf{a}$. Therefore there is $n < \omega$ such that $P_n(\mathcal{A}) \not\equiv_e \emptyset^{(n)}$. Let $\mathbf{a}^-$ be the ω-enumeration degree which contains the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{n}, P_n(\mathcal{A}), \emptyset, \ldots)$. Note that $\mathbf{a}^-$ is bellow $\mathbf{0}'_\omega$ and that $\{\mathbf{a}^-, \mathbf{b}\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Since $\mathbf{0}_e^{(n)} \leq_e \mathbf{d}_e(P_n(\mathcal{A})) \leq_e \mathbf{0}_e^{(n+1)}$, then the equality $\mathbf{d}_e(P_n(\mathcal{A}))' = \mathbf{0}_e^{(n+1)}$, together with Theorem 5, yields

$$\mathbf{d}_e(P_n(\mathcal{A})) \vee \mathbf{x} = \mathbf{x}' = \mathbf{0}_e^{(n+1)}$$

for some $\mathbf{0}_e^{(n)} \lneq_e \mathbf{x} \leq_e \mathbf{0}_e^{(n+1)}$. So for the ω-enumeration degree $\kappa(\mathbf{x})$, we have that $\mathbf{0}_\omega^{(n)} \leq_\omega \kappa(\mathbf{x}) \leq_\omega \mathbf{0}_\omega^{(n+1)}$ and since the jump inversion operation is monotone, $\mathbf{I}^n(\kappa(\mathbf{x})) \leq_\omega \mathbf{0}'_\omega$. Therefore,

$$\mathbf{I}^n(\kappa(\mathbf{x})) = (\mathbf{I}^n(\kappa(\mathbf{x})) \vee \mathbf{a}^-) \wedge (\mathbf{I}^n(\kappa(\mathbf{x})) \vee \mathbf{b}).$$

Hence, using (4.4) and (4.2), we obtain

$$\kappa(\mathbf{x}) = (\kappa(\mathbf{x}) \vee (\mathbf{a}^-)^{(n)}) \wedge (\kappa(\mathbf{x}) \vee \mathbf{b}^{(n)}).$$

By the choice of the degree $\mathbf{x}$, we have that $\kappa(\mathbf{x}) \vee (\mathbf{a}^-)^{(n)} = \mathbf{0}_\omega^{(n+1)}$. Therefore $\mathbf{b}^{(n)} \leq_\omega \kappa(\mathbf{x})$. But $\kappa(\mathbf{x})' = \mathbf{0}_\omega^{(n+1)}$, so we may conclude that $\mathbf{b}^{(n+1)} = \mathbf{0}_\omega^{(n+1)}$. From here, noting that $\mathbf{b} \neq \mathbf{0}_\omega$ and recalling that for each nonzero *a.z.* degree $\mathbf{p}$ and each $n < \omega$, $\mathbf{p}^{(n)} \not\leq_\omega \mathbf{0}_\omega^{(n)}$, we conclude that $\mathbf{b}$ is also not *a.z.* degree.

Therefore, there is $m < \omega$ such that $P_m(\mathcal{B}) \not\equiv_e \emptyset^{(m)}$. Let $\mathbf{b}^-$ be the degree containing the sequence $(\underbrace{\emptyset, \ldots, \emptyset}_{m}, P_m(\mathcal{B}), \emptyset, \ldots)$. Then $\mathbf{a}^-, \mathbf{b}^- \leq_\omega \mathbf{0}'_\omega$ and $\{\mathbf{a}^-, \mathbf{b}^-\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Note that $\{\mathbf{a}^-, \mathbf{b}^-\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ also for $\mathcal{G}_\omega$, whose elements are not *a.z.*. From the characterisation of the $\mathcal{K}$-pairs for $\mathcal{G}_\omega$ noted in the previous section, we conclude that $m = n$. Because of the choice of n and m, we have that for all $k \neq n$, $P_k(\mathcal{A}) \equiv_e P_k(\mathcal{B}) \equiv_e \emptyset^{(k)}$. Therefore $\mathbf{a} = \mathbf{a}^- \vee \mathbf{p}$ and $\mathbf{b} = \mathbf{b}^- \vee \mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are both *a.z.*. But $\mathbf{p} \leq_\omega \mathbf{a}$, so if $\mathbf{p} \neq \mathbf{0}_\omega$ then $\{\mathbf{p}, \mathbf{b}\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Now since $\mathbf{b}$ is not *a.z.* we conclude that $\mathbf{p}$ is not *a.z.*. A contradiction. So $\mathbf{p}$ must be equal to $\mathbf{0}_\omega$. Analogously, $\mathbf{q} = \mathbf{0}_\omega$ and hence $\mathbf{a} = \mathbf{a}^-, \mathbf{b} = \mathbf{b}^-$. So we have the following characterisation of the $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$.

Theorem 7. *Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$. Then exactly one of the following assertions holds:*

1. *Both $\mathbf{a}$ and $\mathbf{b}$ are a.z..*
2. *There is a natural number $n < \omega$ and sets $A, B \subseteq \omega$ such that*
   - *$\emptyset^{(n)} <_e A, B \leq_e \emptyset^{(n+1)}$ and $A' = B' = \emptyset^{(n+1)}$;*

- $\{\mathbf{d}_e(A), \mathbf{d}_e(B)\}$ *is $\mathcal{K}$-pair strictly over* $\mathbf{0}_e^{(n)}$ *for* $\mathbf{D}_e[\geq \mathbf{0}_e^{(n)}]$*;*
- $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, A, \emptyset, \ldots, \emptyset, \ldots) \in \mathbf{a}$ *and* $(\underbrace{\emptyset, \emptyset, \ldots, \emptyset}_{n}, B, \emptyset, \ldots, \emptyset, \ldots) \in \mathbf{b}$.

Note that each $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$, whose elements are both not *a.z.*, is an inherited $\mathcal{K}$-pair for $\mathcal{G}_\omega$ and hence its elements are bellow $\mathbf{0}'_\omega$. So, by the observations in the previous section, each of its elements bounds a nonzero nonsplitting degree. Now, recalling Lemma 4, we have that the $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ consists of non *a.z.* elements iff,

$$\mathcal{D}_\omega \models \mathcal{K}_{inh}(\mathbf{a}, \mathbf{b}),$$

where $\mathcal{K}_{inh}$ is the corresponding formula from the previous section. Since the elements of each inherited $\mathcal{K}$-pair are both below $\mathbf{0}'_\omega$ then their least upper bounds are also below $\mathbf{0}'_\omega$.

Now note that, by Kalimullin [5], $\mathbf{0}'_e$ can be split by a $\mathcal{K}$-pair $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ strictly over $\mathbf{0}_e$ for $\mathcal{D}_e$ such that $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ are low. Then $\kappa(\widetilde{\mathbf{a}})$ and $\kappa(\widetilde{\mathbf{b}})$ are not *a.z.* degrees and $\{\kappa(\widetilde{\mathbf{a}}), \kappa(\widetilde{\mathbf{b}})\}$ is a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$ with $\kappa(\widetilde{\mathbf{a}}) \vee \kappa(\widetilde{\mathbf{b}}) = \mathbf{0}'_\omega$. Thus we may define $\mathbf{0}'_\omega$ as the greatest degree, which is a least upper bound of the elements of a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{D}_\omega$, whose elements are both not *a.z.*. Thus Theorem 2 is proved.

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