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THE INTEGRABILITY OF THE GENERALIZED HAMILTONIAN SYSTEM

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This paper reviews the generalized Hamiltonian system and its connection to contact transformations. The generalized Hamiltonian system is related to Herglotz variational principle in the same way in which the Hamiltonian system is related to the classical variational principle. We prove a criterion for the integrability of the generalized Hamiltonian system in terms of a complete set of first integrals, and a method of generating such first integrals. These results are due to Gustav Herglotz.

Keywords: Herglotz variational principle, integrable systems, contact transformations, integrability, complete integrability, generalized Hamiltonian system

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1. Introduction

In 1932 Gustav Herglotz gave a series of lectures on contact transformations, the generalized Hamiltonian system

$$\begin{aligned}
\frac{d}{dt}x_j &= \frac{\partial \mathcal{H}}{\partial p_j}, \\
\frac{d}{dt}z &= p_j \frac{\partial \mathcal{H}}{\partial p_j} - \mathcal{H}, \\
\frac{d}{dt}p_j &= -\frac{\partial \mathcal{H}}{\partial x_j} - p_j \frac{\partial \mathcal{H}}{\partial z}, \quad j = 1, \ldots, n,
\end{aligned}$$

where $\mathcal{H}$ is a function of $x_1, \ldots, x_n, z, p_1, \ldots, p_n$, and the relationship between them. The generalized Hamiltonian system is closely related to the variational principle, proposed by Herglotz [15, 16]. It is very powerful for giving a variational description

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of nonconservative processes involving one independent variable. It is more general than the classical variational principle with one independent variable and contains it as a special case.

In the variational principle of Herglotz the functional z, whose extrema are sought, is defined by an ordinary differential equation rather than by an integral:

$$\frac{dz}{dt} = L(t, x, \dot{x}, z), \quad 0 \leq t \leq s,$$

where t is the only independent variable, $x \equiv (x^1, \ldots, x^n)$ are the argument functions of t, $\dot{x} = dx/dt$. We denote $z = z[x; s]$. Herglotz showed that the value of this functional is an extremum when its argument-functions $x^k(t)$ are solutions of the generalized Euler-Lagrange equations

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}^k} = 0, \quad k = 1, \ldots, n.$$

His lectures revealed the remarkable geometry which underlines the generalized Hamiltonian system and its integrability in terms of a complete set of first integrals. They provide a method for generating first integrals for such systems. In the present paper we review these results. The summation convention on repeated indices is used throughout the paper.

Furta et al. show in [4] a close link between the Herglotz variational principle and control and optimal control theories. It is also related to contact transformations, see Guenther et al. [14]. Herglotz's work was motivated by ideas from S. Lie [17,18] and others. For historical remarks through 1935 see Caratheodory [2]. The contact transformations, which can be derived from the generalized variational principle, have found applications in thermodynamics. Mrugala shows in [20] that the processes in equilibrium thermodynamics can be described by successions of contact transformations acting in a suitably defined thermodynamic phase space. The latter is endowed with a contact structure, closely related to the symplectic structure. In [5] and [7] Georgieva et al. formulated and proved first and second Noether-type theorems which yields a first integral corresponding to a known symmetry of the functional defined by the Herglotz variational principle; and an identity corresponding to an infinite-dimensional symmetry of the Herglotz functional. For a summary of the resent results related to the variational principle of Herglotz see [9].

In [6] Georgieva, Guenther and Bodurov introduce a new variational principle, which extends the Herglotz principle to one with several independent variables. In honor of Gustav Herglotz they named it in his name. This new varational principle contains as special cases both the classical variational principle with several independent variables and the Herglotz variational principle. It can describe not only all physical processes which the classical variational principle can, but also many others for which the classical variational principle is not applicable. It can give a variational description of nonconservative processes involving physical fields.

The *generalized variational principle with several independent variables* is as follows:

Let the functional $z = z[u; s]$ of $u = u(t, x)$ be defined by an integro-differential equation of the form

$$\frac{dz}{dt} = \int_{\Omega} \mathcal{L}(t, x, u, u_t, u_x, z)\, d^n x, \quad 0 \leq t \leq s, \tag{1.1}$$

where t and $x \equiv (x^1, \dots, x^n)$ are the independent variables, $u \equiv (u^1, \dots, u^m)$ are the argument functions, $u_x \equiv (u^1_x, \dots, u^m_x)$, $u_t \equiv (u^1_t, \dots, u^m_t)$ and $u^i_x \equiv (u^i_{x^1}, \dots, u^i_{x^n})$, $i = 1, \dots, m$, $d^n x \equiv dx^1 \dots dx^n$, and where the function $\mathcal{L}$ is at least twice differentiable with respect to u_x, u_t and once differentiable with respect to t, x, z. Let $\eta \equiv (\eta^1(t, x), \dots, \eta^m(t, x))$ have continuous first derivatives and otherwise be arbitrary except for the boundary conditions:

$$\begin{aligned} &\eta(0, x) = \eta(s, x) = 0, \\ &\eta(t, x) = 0 \quad \textit{for} \quad x \in \partial\Omega,\ 0 \leq t \leq s, \end{aligned}$$

where $\partial\Omega$ is the boundary of Ω. Then, the value of the functional $z[u; s]$ is an extremum for functions u which satisfy the condition

$$\frac{d}{d\varepsilon} z[u + \varepsilon\eta; s]\bigg|_{\varepsilon=0} = 0.$$

The function $\mathcal{L}$, just as in the classical case, is called the *Lagrangian density*. It should be observed that when a variation $\varepsilon\eta$ is applied to u, the integro-differential equation defining the functional z must be solved with the same fixed initial condition $z(0)$ at $t = 0$ and the solution evaluated at the same fixed final time $t = s$ for all varied argument functions $u + \varepsilon\eta$.

Every function $u \equiv (u^1, \dots, u^m)$, for which the functional z defined by the integro-differential equation (1.1) has an extremum, is a solution of

$$\frac{\partial \mathcal{L}}{\partial u^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u^i_t} - \frac{d}{dx^k}\frac{\partial \mathcal{L}}{\partial u^i_{x^k}} + \frac{\partial \mathcal{L}}{\partial u^i_t}\int_{\Omega} \frac{\partial \mathcal{L}}{\partial z}\, dx = 0, \quad i = 1, \dots, m. \tag{1.2}$$

These equations are called (in correspondence with the classical case) the generalized Euler-Lagrange equations.

It is important to observe that the definition of the functional z by the integro-differential equation reduces to the classical definition of a functional by an integral when $\mathcal{L}$ does not depend on z. Similarly, the generalized Euler-Lagrange equations reduce to the classical Euler-Lagrange equations when $\mathcal{L}$ does not depend on z.

Many examples of physical processes described with the generalized variational principle of Herglotz are available in the papers [3, 5–10, 12]. Here we give two applications for the convenience of the reader:

The first is the set of equations which describe the propagation of electromagnetic waves in a conductive medium

$$c^2 \nabla^2 \mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial t}^2 - \frac{\sigma}{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} = 0, \tag{1.3}$$

where $\mathbf{E} = (E^1, E^2, E^3)$ is the electric field vector, c is the velocity of the electromagnetic waves, σ is the electrical conductivity and ε is the dielectric constant of the medium. Exactly the same equation holds for the magnetic field vector $\mathbf{B} = (B^1, B^2, B^3)$. These equations are direct consequence of the Maxwell's equations in conjunction with the medium's property equations $\mathbf{J} = \sigma\mathbf{E}$ and $\rho = 0$, where $\mathbf{J} = (J^1, J^2, J^3)$ is the current density and ρ is the charge density. Equation (1.3) and the equation for the magnetic vector field $\mathbf{B} = (B^1, B^2, B^3)$ *can not* be described variationally via the classical variational principle, because their Frechet derivative is not selfadjoint. Remarkably, they can be described variationally via the variational principle proposed in [6]. In more detail:

One can easily verify that this system is the system of generalized Euler-Lagrange equations for the functional z defined by the integro-differential equation with

$$\mathcal{L} = c^2 \frac{\partial E^i}{\partial x^j}\frac{\partial E^i}{\partial x^j} - \frac{\partial E^i}{\partial t}\frac{\partial E^i}{\partial t} + \alpha(x) z, \quad i, j = 1, 2, 3$$

and

$$\frac{\sigma}{\varepsilon} = \int_\Omega \alpha(x)\, d^3x = \text{const.}$$

As a second example of a physical process which *can not* be described variationally via the classical variational principle, but can be given a variational description with the variational principle of Bodurov [6] consider the nonlinear Schrödinger equation with electromagnetic interaction and losses or gains

$$i\frac{\partial \Psi}{\partial t} - \Phi\Psi + \mu\left(\frac{\partial}{\partial x^k} - iA_k\right)^2 \Psi - G(\Psi\Psi^*, x)\Psi - \beta\frac{i}{2}\Psi = 0, \quad \beta = \text{const}$$

for the wave function $\Psi(t, x^1, x^2, x^3)$ with electromagnetic interaction and losses or gains, where the summation index $k = 1, 2, 3$. Here $(\Phi(t, x^1, x^2, x^3), \mathbf{A}(t, x^1, x^2, x^3))$ is the electromagnetic potential, G is a real-valued function, and $\mathbf{A}$ is the vector potential $\mathbf{A} = (A_1, A_2, A_3)$. The losses $(\beta > 0)$ or gains $(\beta < 0)$ are represented with the term $-\beta\frac{i}{2}\Psi$. This equation does not have a variational description with the classical variational principle, because its Frechet derivative operator is not self-adjoint. In [3] such a description is presented for this process via the generalized variational principle of Herglotz with several independent variables, due to Bodurov. In [3] it is shown that the functional z is invariant under the gauge transformation

$$\Phi' = \Phi - \frac{\partial g}{\partial t}, \quad \mathbf{A}' = \mathbf{A} + \nabla_x g, \quad \Psi' = e^{ig}\Psi,$$

where $g = g(t, x^1, x^2, x^3)$ is an arbitrary function, and an identity is found using the main theorem in [12], due to Georgieva and Bodurov, which is satisfied by the four-potential

$$(\Phi(t, x^1, x^2, x^3), \mathbf{A}(t, x^1, x^2, x^3))$$

of the electromagnetic field. When the wave function Ψ is a solution to the nonlinear Schrödinger equation with electromagnetic interaction and losses or gains, this identity becomes

$$\frac{\partial Q_0}{\partial t} - \nabla_x \cdot \mathbf{Q} - \beta Q_0 = 0,$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3)$. In the classical case when $\beta = 0$ this is the common conservation law with Q_0-conserved density and $\mathbf{Q}$ conserved current. When $\beta \neq 0$ this identity becomes a continuity law – Q_0 is not conserved but generated or dissipated (depending on the sign of β) at a rate proportional to Q_0 itself.

In [10] Georgieva introduces a method for finding the variational symmetries of the functional in the generalized variational principle with several independent variables proposed by Bodurov et al. in [6]. In [12] Georgieva and Bodurov formulate and prove a theorem which gives an identity corresponding to an infinite-dimensional symmetry of that functional.

After this brief overview of the variational principle of Herglotz and its generalization to one with several independent variables, let us return to contact transformations, the generalized Hamiltonian system and their connection to the variational principle of Herglotz.

Let S be a continuously differentiable one-to-one transformation defined on a domain of $\mathbf{R}^n \times \mathbf{R}^1 \times \mathbf{R}^n$ with range in $\mathbf{R}^n \times \mathbf{R}^1 \times \mathbf{R}^n$ which we write in the form

$$S(x, z, p) = (X(x, z, p), Z(x, z, p), P(x, z, p)).$$

We assume that both it and its inverse are sufficiently differentiable so that the computations below make sense, and that the Jacobian is distinct than zero. Such a transformation is called an *element transformation*.

Definition 1.1. A *contact transformation* is an element transformation which is one-to-one, on to, and for which $p \cdot dx - dz = 0$ implies $P \cdot dX - dZ = 0$.

Theorem 1.1. *Equation* (1.1) *represents a contact transformation if and only if there is a function* $\rho = \rho(x, z, p) \neq 0$ *such that* $P \cdot dX - dZ = \rho(p \cdot dx - dz)$.

The proof can be found in [14].

Example 1.1. The Legendre transformation in 3-dimensional space

$$X = p, \quad Y = q, \quad Z = px + qy - z, \quad P = x, \quad Q = y$$

is a contact transformation, with $\rho = -1$.

2. Special contact transformations

Definition 2.1. A contact transformation of the form

$$X = \tilde{X}(x, p), \quad Z = \tilde{Z}(x, p) + z, \quad P = \tilde{P}(x, p) \tag{2.1}$$

is called a *special contact transformation*.

Some of the most important applications of special contact transformations are to Hamiltonian systems.

Theorem 2.1. *A (general) contact transformation U in the $(n+1)$-dimensional xz-space, $\mathbf{R}^{n+1}$, can be extended to a special contact transformation $\overline{U}$ in the $(n+2)$-dimensional $\bar{x}\bar{z}$-space, $\mathbf{R}^{n+2}$, which when restricted to the subspace $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+2}$ has the same effect as U.*

Proof. Let

$$U\colon X = X(x,z,p),\ Z = Z(x,z,p),\ P = P(x,z,p) \tag{2.2}$$

be a general contact transformation in $\mathbf{R}^{n+1}$. By Theorem 1.1, there is a function $\rho = \rho(x,z,p) \neq 0$ such that

$$P \cdot dX - dZ = \rho(p \cdot dx - dz). \tag{2.3}$$

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n, \bar{x}_{n+1})$ be a point in $\mathbf{R}^{n+1}$, where $\bar{x}_i = x_i$ for $i = 1, \dots, n$ and $\bar{x}_{n+1} = -z$. Then $(\bar{x}, \bar{z}) \in \mathbf{R}^{n+2}$. In the image space, we adjoin an additional coordinate, $\overline{Z}$, so that $(\overline{X}, \overline{Z})$ is in a domain in $\mathbf{R}^{n+2}$, where $\overline{X} = (\overline{X}_1, \dots, \overline{X}_n, \overline{X}_{n+1})$, $\overline{X}_i = X_i$ for $i = 1, \dots, n$ and $\overline{X}_{n+1} = -Z$. Next, let $\bar{p}_{n+1}$, $\overline{P}_{n+1}$ be direction coefficients. We shall choose $\overline{P}_{n+1}$ appropriately below. Equation (2.3) becomes

$$P_i d\overline{X}_i + d\overline{X}_{n+1} = \rho\left(p_i d\bar{x}_i + d\bar{x}_{n+1}\right). \tag{2.4}$$

Let $\bar{p}_1, \dots, \bar{p}_{n+1}$ be direction coefficients, where $\bar{p}_1, \dots, \bar{p}_n$ are related to $p_1, \dots, p_n$ by $\bar{p}_i = p_i \bar{p}_{n+1}$, $i = 1, \dots, n$. Also, define $\overline{P}_1, \dots, \overline{P}_n$ by $\overline{P}_i = P_i \overline{P}_{n+1}$, $i = 1, \dots, n$, and $\overline{P}_{n+1}$ is chosen as follows. From (2.4) we have

$$\frac{\overline{P}_i}{\overline{P}_{n+1}} d\overline{X}_i + d\overline{X}_{n+1} = \rho\left(\frac{\bar{p}_i}{\bar{p}_{n+1}} d\bar{x}_i + d\bar{x}_{n+1}\right)$$

or

$$\overline{P}_i\, d\overline{X}_i = \frac{\rho \overline{P}_{n+1}}{\bar{p}_{n+1}} \bar{p}_i\, d\bar{x}_i, \quad i = 1, \dots, n+1.$$

The transformation $U\colon (x,z,p) \to (X,Z,P)$ is extended to the transformation $\overline{U}\colon (\bar{x}, \bar{z}, \bar{p}) \to (\overline{X}, \overline{Z}, \overline{P})$ by adjoining to the $2n+1$ equations defining U, the two additional equations $\overline{Z} = \bar{z}$, $\overline{P}_{n+1} = (1/\rho)\bar{p}_{n+1}$. The system of equations

$$\begin{aligned}
\overline{X}_j &= X_j(x, -x_{n+1}, p) \equiv \overline{X}_j(\bar{x}, \bar{p}), \quad j = 1, \dots, n,\\
\overline{X}_{n+1} &= -Z(x, -x_{n+1}, p) \equiv \overline{X}_{n+1}(\bar{x}, \bar{p}),\\
\overline{Z} &= 0 + \bar{z} \quad (\text{i.e. } \tilde{Z}(\bar{x}, \bar{p}) = 0),\\
\overline{P}_j &= (\bar{p}_{n+1}/\rho) P_j(x, -x_{n+1}, p) \equiv \overline{P}_j(\bar{x}, \bar{p}), \qquad j = 1, \dots, n,\\
\overline{P}_{n+1} &= (1/\rho)\bar{p}_{n+1} \equiv \overline{P}_{n+1}(\bar{x}, \bar{p})
\end{aligned} \tag{2.5}$$

is a special contact transformation in $\mathbf{R}^{n+2}$ which satisfies $\overline{P} \cdot d\overline{X} = \bar{p} \cdot d\bar{x}$. Conversely, when restricted to $\mathbf{R}^{n+1}$, (2.5) defines a contact transformation which coincides with (2.2). □

Example 2.1. The extension $\overline{U}$ of the Legendre transformation in Example 1.1 is

$$\begin{aligned}
\overline{X}_i &= X_i = p_i = \frac{\bar{p}_i}{\bar{p}_{n+1}}, \quad i = 1, \dots, n, \\
\overline{X}_{n+1} &= -Z = -\left(\frac{\bar{p}_i}{\bar{p}_{n+1}} \bar{x}_i + \bar{x}_{n+1}\right), \\
\overline{Z} &= \bar{z}, \\
\overline{P}_i &= P_i \overline{P}_{n+1} = x_i \overline{P}_{n+1} = \bar{x}_i \overline{P}_{n+1} = -\bar{x}_i \bar{p}_{n+1}, \quad i = 1, \dots, n, \\
\overline{P}_{n+1} &= -\bar{p}_{n+1}.
\end{aligned}$$

Returning now to transformations in $\mathbf{R}^{n+1}$, we drop the bar notation.

Theorem 2.2. *An element transformation of the form* (2.1)

$$X = X(x, p), \quad Z = Z(x, p) + z, \quad P = P(x, p)$$

is a special contact transformation if and only if the equation

$$P \cdot dX - p \cdot dx = d(Z - z) = dZ \tag{2.6}$$

holds, where dZ is the total differential of a function Z of (x, p).

Condition (2.6) yields

$$\left(P_i \frac{\partial X_i}{\partial x_j} - p_j\right) dx_j + P_i \frac{\partial X_i}{\partial p_j} dp_j = \frac{\partial Z}{\partial x_j} dx_j + \frac{\partial Z}{\partial p_j} dp_j$$

or, comparing coefficients,

$$\begin{aligned}
\frac{\partial Z}{\partial x_j} &= P_i \frac{\partial X_i}{\partial x_j} - p_j, \quad j = 1, \dots, n, \\
\frac{\partial Z}{\partial p_j} &= P_i \frac{\partial X_i}{\partial p_j}, \qquad\quad j = 1, \dots, n.
\end{aligned} \tag{2.7}$$

These conditions characterize contact transformations of the form $X = X(x, p)$, $P = P(x, p)$ in the $2n$-dimensional xp-space. Such transformations are also referred to as *canonical transformations.*

Using the equivalence of the mixed second partial derivatives for Z and (2.7) one obtains conditions on $(X(x, p), P(x, p))$ that are independent of Z:

$$\begin{aligned}
\frac{\partial P_i}{\partial x_k} \frac{\partial X_i}{\partial x_j} - \frac{\partial P_i}{\partial x_j} \frac{\partial X_i}{\partial x_k} &= 0, \quad j, k = 1, \dots, n, \\
\frac{\partial P_i}{\partial p_k} \frac{\partial X_i}{\partial x_j} - \frac{\partial P_i}{\partial x_j} \frac{\partial X_i}{\partial p_k} &= \delta_{jk}, \quad j, k = 1, \dots, n, \\
\frac{\partial P_i}{\partial p_k} \frac{\partial X_i}{\partial p_j} - \frac{\partial P_i}{\partial p_j} \frac{\partial X_i}{\partial p_k} &= 0, \quad j, k = 1, \dots, n,
\end{aligned}$$

where δ_{jk} is the Kronecker delta.

3. Characterization of the general contact transformation

Consider the transformation $(x, z, p) \to (\bar{x}, \bar{z}, \bar{p})$, where we use the bar notation:

$$\bar{x}_i = x_i, \quad \bar{x}_{n+1} = -z, \quad \bar{p}_i = \bar{p}_{n+1} p_i, \quad i = 1, \dots, n. \tag{3.1}$$

Let $f = f(x, z, p)$, $g = g(x, z, p)$ be two differentiable functions, and

$$\begin{aligned} f(x, z, p) &= f(x_1, \dots, x_n, z, p_1, \dots, p_n) \\ &= f(\bar{x}_1, \dots, \bar{x}_n, -\bar{x}_{n+1}, \bar{p}_1/\bar{p}_{n+1}, \dots, \bar{p}_n/\bar{p}_{n+1}) \\ &\equiv \bar{f}(\bar{x}, \bar{p}) \end{aligned}$$

and similarly $g(x, z, p) \equiv \bar{g}(\bar{x}, \bar{p})$. The Poisson bracket for the pair of functions $\bar{f}$ and $\bar{g}$ is given by

$$[\bar{f}, \bar{g}]_{\bar{x}\bar{p}} = \frac{\partial \bar{f}}{\partial \bar{x}_j} \frac{\partial \bar{g}}{\partial \bar{p}_j} - \frac{\partial \bar{f}}{\partial \bar{p}_j} \frac{\partial \bar{g}}{\partial \bar{x}_j}. \tag{3.2}$$

We may now rewrite this expression in terms of the original variables

$$\frac{\partial \bar{f}}{\partial \bar{x}_i} = \frac{\partial f}{\partial x_i}, \quad \frac{\partial \bar{f}}{\partial \bar{p}_i} = \frac{1}{\bar{p}_{n+1}} \frac{\partial f}{\partial p_i}, \quad i = 1, \dots, n,$$
$$\frac{\partial \bar{f}}{\partial \bar{x}_{n+1}} = -\frac{\partial f}{\partial z}, \quad \frac{\partial \bar{f}}{\partial \bar{p}_{n+1}} = -\frac{\partial f}{\partial p_i} \frac{\bar{p}_i}{\bar{p}_{n+1}^2} = -\frac{1}{\bar{p}_{n+1}} p_i \frac{\partial f}{\partial p_i}, \quad p_i = \frac{\bar{p}_i}{\bar{p}_{n+1}},$$

and similar formulas for $\bar{g}$ hold. Then formula (3.2) takes the form

$$\begin{aligned} [\bar{f}, \bar{g}]_{\bar{x}\bar{p}} &= \frac{1}{\bar{p}_{n+1}} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right) + \frac{1}{\bar{p}_{n+1}} \left(\frac{\partial f}{\partial z} p_j \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial z} p_j \frac{\partial f}{\partial p_j} \right) \\ &= \frac{1}{\bar{p}_{n+1}} \left(\left(\frac{\partial f}{\partial x_j} + p_j \frac{\partial f}{\partial z} \right) \frac{\partial g}{\partial p_j} - \left(\frac{\partial g}{\partial x_j} + p_j \frac{\partial g}{\partial z} \right) \frac{\partial f}{\partial p_j} \right). \end{aligned} \tag{3.3}$$

The symbol

$$\{f, g\}_{xzp} = \left(\frac{\partial f}{\partial x_j} + p_j \frac{\partial f}{\partial z} \right) \frac{\partial g}{\partial p_j} - \left(\frac{\partial g}{\partial x_j} + p_j \frac{\partial g}{\partial z} \right) \frac{\partial f}{\partial p_j} \tag{3.4}$$

is called the *Mayer bracket* of f and g. Equation (3.3) in terms of the Mayer bracket takes the form

$$[\bar{f}, \bar{g}]_{\bar{x}\bar{p}} = \frac{1}{\bar{p}_{n+1}} \{f, g\}_{xzp}. \tag{3.5}$$

The Mayer bracket satisfies properties similar to those of the Poisson bracket.

Theorem 3.1. *Let f, g, h be differentiable functions of the variables (x, y, z) and let α be a constant. Then:*

i) $\{f, g\} = -\{g, f\}$, $\{f, f\} = 0$;

ii) $\{\alpha, f\} = 0$, $\{\alpha f, g\} = \alpha\{f, g\}$;

iii) $\{f+g,h\} = \{f,h\} + \{g,h\}$;

iv) $\{fg,h\} = g\{f,h\} + f\{g,h\}$;

v) *The Jacobi identity holds in the form*

$$\{f\{g,h\}\} + \{g\{h,f\}\} + \{h\{f,g\}\} + f_z\{g,h\} + g_z\{h,f\} + h_z\{f,g\} = 0.$$

Note that the subscripts xzp have been dropped.

Equation (3.5) leads to a formula describing how the Mayer bracket changes under a contact transformation.

Theorem 3.2. *The element transformation*

$$X = X(x,z,p), \quad Z = Z(x,z,p), \quad Y = Y(x,z,p)$$

is a contact transformation with multiplier ρ if and only if up to a factor $1/\rho$ it leaves the Mayer bracket of two arbitrary differentiable functions invariant

$$\{F,G\}_{XZP} = \frac{1}{\rho}\{f,g\}_{xzp}.$$

Proof. Let

$$X = X(x,z,p), \quad Z = Z(x,z,p), \quad P = P(x,z,p)$$

be contact transformation, and let

$$x = x(X,Z,P), \quad z = z(X,Z,P), \quad p = p(X,Z,P)$$

be its inverse. Set

$$\begin{aligned} F(X,Z,P) &= F(X(x,z,p),Z(x,z,p),P(x,z,p)) \equiv f(x,z,p),\\ G(X,Z,P) &= G(X(x,z,p),Z(x,z,p),P(x,z,p)) \equiv g(x,z,p). \end{aligned}$$

Now lift the variables one dimension and set

$$\overline{X}_i = X_i, \quad \overline{X}_{n+1} = -Z, \quad \overline{P}_i = \overline{P}_{n+1}P_i, \quad \overline{P}_{n+1} = \frac{1}{\rho}\bar{p}_{n+1}.$$

We use the fact that canonical transformations preserve the form of the Poisson bracket, and formula (3.5) to get

$$\frac{1}{\bar{p}_{n+1}}\{f,g\}_{xzp} = [\bar{f},\bar{g}]_{\bar{x}\bar{p}} = [\bar{F},\bar{G}]_{\overline{XP}} = \frac{1}{\overline{P}_{n+1}}\{F,G\}_{XZP},$$

or since $\overline{P}_{n+1}/\bar{p}_{n+1} = 1/\rho$, $\{F,G\}_{XZP} = (1/\rho)\{f,g\}_{xzp}$. □

Theorem 3.2 suggests that the Mayer bracket plays the same role for general contact transformations as the Poisson bracket plays for the special (canonical) transformations.

Theorem 3.3. *In order for the one-to-one element transformation $X = X(x,z,p)$, $Z = Z(x,z,p)$, $Y = Y(x,z,p)$, which satisfies the relationship*

$$P_j\, dX_j - dZ = \rho(p_k\, dx_k - dz) \tag{3.6}$$

with $\rho(x,z,p) \neq 0$, to be a contact transformation, it is necessary and sufficient that the following relations are satisfied:

$$\begin{aligned} \{X_i, X_j\}_{xzp} &= 0, && i,j = 1,\dots,n,\\ \{X_i, P_j\}_{xzp} &= \rho\delta_{ij}, && i,j = 1,\dots,n,\\ \{X_i, Z\}_{xzp} &= 0, && i = 1,\dots,n,\\ \{P_i, P_j\}_{xzp} &= 0, && i,j = 1,\dots,n,\\ \{P_i, Z\}_{xzp} &= -\rho P_i, && i = 1,\dots,n. \end{aligned} \tag{3.7}$$

Moreover, the following conditions hold:

$$\begin{aligned} \{\rho, X_j\}_{xzp} &= \rho\frac{\partial X_j}{\partial z},\\ \{\rho, Z\}_{xzp} &= \rho\frac{\partial Z}{\partial z} - \rho^2,\\ \{\rho, P_j\}_{xzp} &= \rho\frac{\partial P_j}{\partial z}. \end{aligned} \tag{3.8}$$

Proof. Notice that $\{X_i, X_j\}_{xzp} = \rho\{X_i, X_j\}_{XZP} = 0$, $i,j = 1,\dots,n$. The rest of the equations (3.7) are obtained similarly. The derivations of equations (3.8) are lengthy and can be found in [14]. □

Corollary 3.1. *The functions (X,P) of a contact transformation are independent of z if and only if ρ is a constant.*

Proof. We observe that

$$\begin{aligned} \frac{\partial\rho}{\partial X_j} &= \frac{\partial P_j}{\partial z} + \frac{1}{\rho}\frac{\partial\rho}{\partial z}P_j, && j = 1,\dots,n,\\ \frac{\partial\rho}{\partial P_j} &= -\frac{\partial X_j}{\partial z}, && j = 1,\dots,n,\\ \frac{\partial\rho}{\partial Z} &= -\frac{1}{\rho}\frac{\partial\rho}{\partial z}. \end{aligned}$$

We will show how to obtain the second of these equations

$$\{\rho, X_j\}_{XZP} = \left(\frac{\partial\rho}{\partial X_i} + P_i\frac{\partial\rho}{\partial Z}\right)\frac{\partial X_j}{\partial P_i} - \left(\frac{\partial X_j}{\partial X_i} + P_i\frac{\partial X_j}{\partial Z}\right)\frac{\partial\rho}{\partial P_i} = -\frac{\partial\rho}{\partial P_j},$$

since $\partial X_j/\partial P_i = \partial X_j/\partial Z = 0$ and $\partial X_i/\partial X_j = \delta_{ij}$. Also, by one of the identities (3.8), $\{\rho, X_j\}_{XZP} = (1/\rho)\{\rho, X_j\}_{xzp} = \partial X_j/\partial z$. Similar calculations produce the other two equations. □

Theorem 3.4. *Let $X_1, \dots, X_n, Z$ be $n+1$ independent functions which are pairwise in involution with respect to the Mayer bracket. Then there is precisely one contact transformation for which these are the first $n+1$ functions and the remaining $n+1$ functions $P_1, \dots, P_n, \rho$ may be obtained by solving a linear system of equations.*

Proof. If the $n+1$ independent functions $Z, X_1, \dots, X_n$ of (x, z, p) are pairwise in involution, that is if they satisfy $\{Z, X_i\}_{xzp} = 0$, $\{X_i, X_j\}_{xzp} = 0$, then the functions $P_1, \dots, P_n, \rho$ can be calculated as follows. By equating coefficients in the defining identity (3.6) for a contact transformation, we obtain the system

$$\begin{aligned} \frac{P_i}{\rho}\frac{\partial X_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial Z}{\partial x_j} &= p_j, \qquad j = 1, \dots, n, \\ \frac{P_i}{\rho}\frac{\partial X_i}{\partial z} - \frac{1}{\rho}\frac{\partial Z}{\partial z} &= -1, \\ \frac{P_i}{\rho}\frac{\partial X_i}{\partial p_j} - \frac{1}{\rho}\frac{\partial Z}{\partial p_j} &= 0, \qquad j = 1, \dots, n. \end{aligned} \tag{3.9}$$

Since $X_1, \dots, X_n, Z$ are functionally independent, the rank of the matrix

$$\begin{pmatrix} X_x & Z_x \\ X_z & Z_z \end{pmatrix} = \begin{pmatrix} \frac{\partial X_i}{\partial x_j} & \frac{\partial Z}{\partial x_j} \\ \frac{\partial X_i}{\partial z} & \frac{\partial Z}{\partial z} \end{pmatrix} \tag{3.10}$$

is $n+1$, so the first $n+1$ equations in the above system can be solved for P_i/ρ, $i = 1, \dots, n$, and $1/\rho$. We now must show that the last n equations in system (3.9) are satisfied identically. For that consider the expression

$$\begin{aligned} \frac{P_j}{\rho}\{X_i, X_j\} - \frac{1}{\rho}\{X_i, Z\} &= \left(\frac{\partial X_i}{\partial x_k} + p_k\frac{\partial X_i}{\partial z}\right)\left(\frac{P_j}{\rho}\frac{\partial X_j}{\partial p_k} - \frac{1}{\rho}\frac{\partial Z}{\partial p_k}\right) \\ &\quad - \left(\frac{\partial X_j}{\partial x_k}\frac{P_j}{\rho} - \frac{1}{\rho}\frac{\partial Z}{\partial x_k}\right)\frac{\partial X_i}{\partial p_k} - p_k\left(\frac{\partial X_j}{\partial z}\frac{P_j}{\rho} - \frac{\partial Z}{\partial z}\frac{1}{\rho}\right)\frac{\partial X_i}{\partial p_k}. \end{aligned}$$

Taking in consideration the validity of the first two equations in system (3.9) and that $\{X_i, X_j\} = 0$ and $\{X_i, Z\} = 0$, we obtain

$$\left(\frac{\partial X_i}{\partial x_k} + p_k\frac{\partial X_i}{\partial z}\right)\left(\frac{P_j}{\rho}\frac{\partial X_j}{\partial p_k} - \frac{1}{\rho}\frac{\partial Z}{\partial p_k}\right) = 0. \tag{3.11}$$

Since the columns of the matrix (3.10) are linearly independent, identity (3.11) implies that the last n equations in system (3.9) are identically satisfied. □

4. One-parameter families of contact transformations

We now consider the special system of $2n+1$ differential equations for $2n+1$ unknowns $X = (X_1, \dots, X_n)$, Z, $X = (P_1, \dots, P_n)$

$$\dot{X} = \xi(X, Z, P, t), \quad \dot{Z} = \zeta(X, Z, P, t), \quad \dot{P} = \pi(X, Z, P, t), \tag{4.1}$$

which satisfy the initial conditions

$$X = x, \quad Z = z, \quad P = p, \quad \text{when} \quad t = 0. \tag{4.2}$$

The functions $\xi = (\xi_1, \dots, \xi_n)$, ζ, $\pi = (\pi_1, \dots, \pi_n)$ are all assumed to be continuously differentiable. The solutions to (4.1), (4.2)

$$X = X(x, z, p, t), \quad Z = Z(x, z, p, t), \quad P = P(x, z, p, t) \tag{4.3}$$

determine a family of transformations

$$S_t\colon (x, z, p) \to (X, Z, P). \tag{4.4}$$

In this section we give the necessary and sufficient conditions for the transformations (4.4) to be contact transformations uniformly in t.

Theorem 4.1. *In order for solution (4.3) of system (4.1) to represent a one-parameter family of contact transformations containing the identity, it is necessary that (4.1) be a canonical system, that is, that there exists a function, $\mathcal{H} = \mathcal{H}(X, Z, P, t)$ called the characteristic function, such that the system (4.1) has the form*

$$\begin{aligned} \frac{d}{dt}X_j &= \frac{\partial \mathcal{H}}{\partial P_j}, \\ \frac{d}{dt}Z &= P_j \frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}, \\ \frac{d}{dt}P_j &= -\frac{\partial \mathcal{H}}{\partial X_j} - P_j \frac{\partial \mathcal{H}}{\partial Z}, \quad j = 1, \dots, n. \end{aligned} \tag{4.5}$$

Proof. In Section 3 we had found that the transformations must satisfy

$$P \cdot dX - dZ = \rho(p \cdot dx - dz), \quad \rho \neq 0. \tag{4.6}$$

(4.6) is supposed to hold when the differentials are calculated only with respect to the spatial variables. When X, Z, P also depend on t, then dZ is given by

$$dZ = \frac{\partial Z}{\partial x_j} dx_j + \frac{\partial Z}{\partial z} dz + \frac{\partial Z}{\partial p_j} dp_j + \frac{\partial Z}{\partial t} dt.$$

A similar assertion holds for the dX_i. Thus, condition (4.6) must be replaced by

$$P_i\, dX_i - dZ - \left(P_i \frac{\partial X_i}{\partial t} - \frac{\partial Z}{\partial t} \right) dt = \rho(p_i\, dx_i - dz). \tag{4.7}$$

By (4.1), $\partial X_i/\partial t = \xi_i(X, Z, P, t)$, $\partial Z/\partial t = \zeta(X, Z, P, t)$. Let us introduce the function

$$\mathcal{H} \equiv \mathcal{H}(X, Z, P, t) \equiv P_i \xi_i(X, Z, P, t) - \zeta(X, Z, P, t). \tag{4.8}$$

Then relation (4.7) takes the form

$$P \cdot dX - dZ = \rho(p \cdot dx - dz) + \mathcal{H}\, dt. \tag{4.9}$$

If $dt = 0$, equation (4.9) reduces to (4.6). (4.9) represents a system of $2n+2$ equations relating the variables (X, Z, P, t) with those of (x, z, p, t), which is obtained by expanding the differentials and comparing coefficients. To obtain the conditions we seek, we shall rewrite these conditions in the (X, Z, P, t) variables. This is most simply done by working directly with (4.9). First differentiate (4.9) with respect to t and note that the differential operator, d, commutes with the differentiation d/dt. This leads to

$$\pi_j\, dX_j + P_j\, d\xi_j - d\zeta = \dot{\rho}(p_j\, dx_j - dz) + \dot{\mathcal{H}}\, dt, \tag{4.10}$$

where $\partial P_j/\partial t = \pi_j(X, Z, P, t)$, the dot, as usual, represents d/dt. From (4.9) and (4.10) we obtain

$$\pi_j\, dX_j + P_j\, d\xi_j - d\zeta - \dot{\mathcal{H}}\, dt = \frac{\dot{\rho}}{\rho}\left(P_j\, dX_j - dZ - \mathcal{H}\, dt\right). \tag{4.11}$$

From (4.8) we find $d\mathcal{H} = \xi_j\, dP_j + P_j\, d\xi_j - d\zeta$ so that (4.11) takes the form

$$d\mathcal{H} + \pi_j\, dX_j - \xi_j\, dP_j = \frac{\dot{\rho}}{\rho}\left(P_j\, dX_j - dZ\right) + \left(\dot{\mathcal{H}} - \frac{\dot{\rho}}{\rho}\mathcal{H}\right) dt. \tag{4.12}$$

Expand $d\mathcal{H}$ in the form

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial X_j}\, dX_j + \frac{\partial \mathcal{H}}{\partial Z}\, dZ + \frac{\partial \mathcal{H}}{\partial P_j}\, dP_j + \frac{\partial \mathcal{H}}{\partial t}\, dt,$$

insert the result into (4.12) and compare coefficients to obtain the following system

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial X_j} &= -\pi_j + \frac{\dot{\rho}}{\rho}P_j, & \frac{\partial \mathcal{H}}{\partial P_j} &= \xi_j, \\ \frac{\partial \mathcal{H}}{\partial Z} &= -\frac{\dot{\rho}}{\rho}, & \frac{\partial \mathcal{H}}{\partial t} &= \dot{\mathcal{H}} - \frac{\dot{\rho}}{\rho}\mathcal{H}. \end{aligned} \tag{4.13}$$

The ξ_j and π_j are obtained directly from (4.13) by eliminating the quotient $\dot{\rho}/\rho$ and solving. To obtain ζ combine (4.8) with (4.13). We find

$$\begin{aligned} \xi_j &= \frac{\partial \mathcal{H}}{\partial P_j}, \\ \zeta &= P_j \xi_j - \mathcal{H} = P_j \frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}, \\ \pi_j &= -\frac{\partial \mathcal{H}}{\partial X_j} - P_j \frac{\partial \mathcal{H}}{\partial Z}, \qquad j = 1, \ldots, n, \end{aligned} \tag{4.14}$$

which is system (4.5). □

The converse of this theorem is also valid. We state and prove

Theorem 4.2. *The solution to the canonical equations (4.5), which satisfy the initial conditions (4.2), generates a one-parameter family of contact transformations, which for $t = 0$ contains the identity.*

Proof. We must show that every solution of (4.5) and (4.2) satisfies the strip condition (4.9). For notational purposes let us set $\Omega = \Omega(t) \equiv P_j\, dX_j - dZ - \mathcal{H}\, dt$ and $\Omega(0) \equiv \omega = p_j\, dx_j - dz$. Then the strip condition (4.9) takes the form $\Omega(t) = \rho\omega$. Set up a differential equation for Ω making use of (4.5). The proof is simply a calculation. We find $\dot{\Omega} = \dot{P}_j\, dX_j + P_j\, d\dot{X}_j - d\dot{Z} - \dot{\mathcal{H}}\, dt$. Since $\mathcal{H} = P_j\, d\dot{X}_j - d\dot{Z}$,

$$\begin{aligned}
\dot{\Omega} &= \dot{P}_j\, dX_j - \dot{X}_j P_j \\
&= -\left(\frac{\partial \mathcal{H}}{\partial X_j} + P_j \frac{\partial \mathcal{H}}{\partial Z}\right) dX_j - \frac{\partial \mathcal{H}}{\partial P_j}\, dP_j \\
&= -\left(\frac{\partial \mathcal{H}}{\partial X_j}\, dX_j + \frac{\partial \mathcal{H}}{\partial P_j}\, dP_j\right) - \frac{\partial \mathcal{H}}{\partial Z}\,(P_j\, dX_j) \\
&= -\left(\frac{\partial \mathcal{H}}{\partial X_j}\, dX_j + \frac{\partial \mathcal{H}}{\partial P_j}\, dP_j\right) + \frac{\partial \mathcal{H}}{\partial Z}\, dZ + \frac{\partial \mathcal{H}}{\partial t}\, dt \\
&\quad - \frac{\partial \mathcal{H}}{\partial Z}\,(P_j\, dX_j) + \frac{\partial \mathcal{H}}{\partial Z}\, dZ + \frac{\partial \mathcal{H}}{\partial t}\, dt \\
&= -\, d\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\,(P_j\, dX_j - dZ - \mathcal{H}\, dt) - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\, dt + \frac{\partial \mathcal{H}}{\partial t}\, dt \\
&= -\, d\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\Omega - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\, dt + \frac{\partial \mathcal{H}}{\partial t}\, dt.
\end{aligned}$$

Thus we obtain the ODE for Ω

$$\dot{\Omega} = -d\mathcal{H} - \frac{\partial \mathcal{H}}{\partial Z}\Omega - \frac{\partial \mathcal{H}}{\partial Z}\mathcal{H}\, dt + \frac{\partial \mathcal{H}}{\partial t}\, dt.$$

Next we calculate, using (4.5)

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \frac{\partial \mathcal{H}}{\partial X_j}\frac{dX_j}{dt} + \frac{\partial \mathcal{H}}{\partial P_j}\frac{dP_j}{dt} + \frac{\partial \mathcal{H}}{\partial Z}\frac{dZ}{dt} + \frac{\partial \mathcal{H}}{\partial t} \\
&= \frac{\partial \mathcal{H}}{\partial X_j}\frac{\partial \mathcal{H}}{\partial P_j} - \frac{\partial \mathcal{H}}{\partial P_j}\left(\frac{\partial \mathcal{H}}{\partial X_j} + P_j\frac{\partial \mathcal{H}}{\partial Z}\right) + \frac{\partial \mathcal{H}}{\partial Z}\left(P_j\frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}\right) + \frac{\partial \mathcal{H}}{\partial t} \\
&= -\mathcal{H}\frac{\partial \mathcal{H}}{\partial Z} + \frac{\partial \mathcal{H}}{\partial t}.
\end{aligned}$$

Thus,

$$d\mathcal{H} = -\mathcal{H}\frac{\partial \mathcal{H}}{\partial Z}\, dt + \frac{\partial \mathcal{H}}{\partial t}\, dt$$

and so from the previous calculation

$$\dot{\Omega} = -\frac{\partial \mathcal{H}}{\partial Z}\Omega.$$

We integrate to obtain $\Omega = \rho\omega$, where

$$\rho = \exp\left(-\int_0^t \frac{\partial \mathcal{H}}{\partial Z}\, dt\right), \tag{4.15}$$

which proves the assertion. □

We close this section with a few remarks on the characteristic function $\mathcal{H} = \mathcal{H}(X, Z, P, t)$. From the forth equation in (4.13), we have

$$\rho \frac{\partial \mathcal{H}}{\partial t} = \rho \dot{\mathcal{H}} - \dot{\rho}\mathcal{H}.$$

Divide by ρ^2 to find

$$\frac{1}{\rho}\frac{\partial \mathcal{H}}{\partial t} = \frac{\rho \dot{\mathcal{H}} - \dot{\rho}\mathcal{H}}{\rho^2} = \frac{d}{dt}\left(\frac{\mathcal{H}}{\rho}\right).$$

Integrate with respect to t to find

$$\frac{\mathcal{H}}{\rho} - \frac{\mathcal{H}^0}{\rho^0} = \int_0^t \frac{1}{\rho}\frac{\partial \mathcal{H}}{\partial t}\, dt, \tag{4.16}$$

where the superscript indicates that the arguments of $\mathcal{H}$ and ρ are to be taken at $t = 0$: $\rho^0 = \rho(x, z, p, 0)$, $\mathcal{H}^0 = \mathcal{H}(x, z, p, 0)$. The fact that $\rho^0 = 1$ is a consequence of (4.15).

We consider two special cases.

Case 1. $\partial\mathcal{H}/\partial t = 0$ so that $\mathcal{H}$ does not depend explicitly on t.

Then the family $\{S_t\}$ represents a one-parameter group of contact transformations. (The proof can be found in [14].) Relation (4.16) implies that

$$\mathcal{H}(X, Z, P) = \mathcal{H}^0(x, z, p)\rho(x, z, p). \tag{4.17}$$

(4.17) has a geometric interpretation. Let us think of the parameter t as the time and the curve along which $(X, Z, P) = S_t(x, z, p)$ moves in $\mathbf{R}^{2n+1}$ as its orbit under the group of contact transformations. Along this orbit the function $\mathcal{H}(X, Z, P)$, up to the factor $\mathcal{H}^0$, coincides with $\rho(X, Z, P)$.

If in particular $\mathcal{H}^0 = 0$ at a point (x, z, p), then $\mathcal{H}(X, Z, P) = 0$ along the whole orbit through it. The strip condition is along the orbit. If we think of (X, Z, P) as an element in $\mathbf{R}^{n+1}$, then we refer to the orbit as an orbital strip of the group of contact transformations in $\mathbf{R}^{n+1}$. For points on the orbital strip, the second equation in (4.5) simplifies to

$$\frac{dZ}{dt} = P_j \frac{\partial \mathcal{H}}{\partial P_j}, \quad j = 1, \ldots, n.$$

Case 2. $\partial\mathcal{H}/\partial Z = 0$ so that $\mathcal{H}$ does not depend explicitly on Z and by (4.15) $\rho = \rho(X, Z, P, t) \equiv 1$.

The canonical equations (4.5) reduce to

$$\frac{dX_j}{dt} = \frac{\partial \mathcal{H}}{\partial P_j}, \quad \frac{dP_j}{dt} = -\frac{\partial \mathcal{H}}{\partial X_j} \tag{4.18}$$

together with the additional equation

$$\frac{dZ}{dt} = P_j \frac{\partial \mathcal{H}}{\partial P_j} - \mathcal{H}, \quad j = 1, \ldots, n \tag{4.19}$$

for the construction of Z.

The transformations determined by (4.18) are the special, or xp-transformations which commute with translations along the z-axis. Equation (4.9) in this case reads

$$P_j\,dX_j - p_j\,dx_j = d(Z-z) + \mathcal{H}\,dt.$$

If in addition, $\partial\mathcal{H}/\partial t = 0$, then $\mathcal{H} = \mathcal{H}^0$. The family determined by solutions to (4.18) is a group of contact transformations which on the orbit passing through (x,z,p) satisfies $\mathcal{H}(X,Z,P) = \mathcal{H}^0(x,z,p)$.

5. Transformations of canonical differential equations

In this section we show that the form of the generalized Hamiltonian system is preserved by contact transformations.

Consider the generalized Hamiltonian system

$$\begin{aligned} \dot{x}_j &= \frac{\partial\mathcal{H}}{\partial p_j}, \\ \dot{z} &= p_j\frac{\partial\mathcal{H}}{\partial p_j} - \mathcal{H}, \\ \dot{p}_j &= -\frac{\partial\mathcal{H}}{\partial x_j} - p_j\frac{\partial\mathcal{H}}{\partial z}, \quad j=1,\dots,n, \end{aligned} \tag{5.1}$$

where $H = H(x,z,p,t)$, and with initial values

$$x_j(0) = x_j^0, \quad z(0) = z^0, \quad p_j(0) = p_j^0, \quad j=1,\dots,n. \tag{5.2}$$

Theorem 5.1. *If system (5.1) with initial values (5.2) is transformed with the contact transformation*

$$T_t\colon \begin{aligned} X_j &= X_j(x,z,p,t), \\ Z &= Z(x,z,p,t), \\ P_j &= P_j(x,z,p,t), \quad j=1,\dots,n, \end{aligned} \tag{5.3}$$

then the transformed system is a generalized Hamiltonian system with characteristic function $\sigma H + K$, where $K(X,Z,P,t)$ and $\sigma(X,Z,P,t)$ are the characteristic function and the multiplier of the contact transformation T_t, i.e., $P_j\,dX_j - dZ - K\,dt = \sigma(p_j\,dx_j - dz)$.

Proof. Let S_t denote the contact transformation defined by the solution of the generalized Hamiltonian system (5.1)–(5.2), i.e., let $(x,z,p) = S_t(x^0,z^0,p^0)$. Now carry out the substitution indicated by (5.3). The initial values transform as follows

$$(X^0,Z^0,P^0) = T_0(x^0,z^0,p^0) = \big(X(x^0,z^0,p^0,0), Z(x^0,z^0,p^0,0), P(x^0,z^0,p^0,0)\big)$$

and the solutions to (5.1)– (5.2) transform to functions of (X^0,Z^0,P^0,t) according to

$$(X,Z,P) = T_tS_tT_0^{-1}(X^0,Z^0,P^0). \tag{5.4}$$

Let $S_t^* \equiv T_t S_t T_0^{-1}$. $\{S_t^*\}$ is a one parameter family of contact transformations, so there exists a canonical system for it which is determined by a characteristic function $\mathcal{H}^* = \mathcal{H}^*(X, Z, P, t)$. We must determine $\mathcal{H}^*$ in terms of known quantities.

Since T_0 is a contact transformation, we find from (5.3)

$$P_j^0\, dX_j^0 - dZ^0 = \sigma^0\big(p_j^0\, dx_j^0 - dz^0\big), \tag{5.5}$$

where $\sigma^0 = \sigma(X^0, Z^0, P^0, 0)$. Further,

$$p_j\, dx_j - dz = \rho\big(p_j^0\, dx_j^0 - dz^0\big) + \mathcal{H}\, dt. \tag{5.6}$$

Now, using (5.3), (5.6), and (5.5) we find

$$\begin{aligned} P_j\, dX_j - dZ &= \sigma\big(p_j\, dx_j - dz\big) + K\, dt \\ &= \sigma\left(\rho(p_j^0\, dx_j^0 - dz^0) + \mathcal{H}\, dt\right) + K\, dt \\ &= \sigma\left(\frac{\rho}{\sigma^0}(P_j^0\, dX_j^0 - dZ^0) + \mathcal{H}\, dt\right) + K\, dt \\ &= \frac{\sigma\rho}{\sigma^0}(P_j^0\, dX_j^0 - dZ^0) + (\sigma\mathcal{H} + K)\, dt. \end{aligned}$$

The coefficient of dt represents the desired characteristic function $\mathcal{H}^*{=}\mathcal{H}^*(X, Z, P, t)$ $= (\sigma\mathcal{H} + K)$. Observe that σ and K are already evaluated at (X, Z, P, t). The function H, initially evaluated at (x, z, p, t) must simply be rewritten in terms of the new variables $(X, Z, P) = T_t^{-1}(x, z, p)$. Having determined the characteristic function $\mathcal{H}^*$ we can rewrite the system (5.1) immediately in terms of the new variables. □

We close this section with a final remark. Suppose $\mathcal{H}$ is independent of z so that the canonical equations are

$$\dot{x}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial x_j}. \tag{5.7}$$

Now make the substitution

$$X = X(x, p), \quad P = P(x, p) \quad \text{with} \quad P \cdot dX = p \cdot dx. \tag{5.8}$$

This is a special contact transformation which is independent of the parameter t. Then $\sigma = 1$, $K = 0$ and $\mathcal{H}^*$ is obtained by evaluating $\mathcal{H}$ at $x = x(X, P)$, $p = p(X, P)$ and the canonical equations in the (X, P) variables are

$$\dot{X}_j = \frac{\partial \mathcal{H}}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial \mathcal{H}}{\partial X_j}. \tag{5.9}$$

Since (5.7) transforms in (5.9) with $\mathcal{H}^*$ arising from $\mathcal{H}$ by means of (5.8), the special contact transformation is also called a canonical transformation.

6. Liouville-type integrability theorem

This section reviews two remarkable theorems. The first gives a necessary and sufficient condition for the integrability of the generalized Hamiltonian system in terms of a complete set of first integrals. The second provides a method of generating such first integrals.

Consider the generalized Hamiltonian system (5.1). The following theorem extends the classical theorem of Liouville which gives a necessary and sufficient condition for the integrability of the classical Hamiltonian system in terms of a complete set of first integrals which are in involution with respect to the Poisson bracket.

Theorem 6.1. *Suppose $X_1(x,z,p,t),\dots,X_n(x,z,p,t)$, $Z(x,z,p,t)$ are $n+1$ independent first integrals for (5.1) which are pairwise in involution with respect to the Mayer bracket:*

$$\{X_i,X_j\}_{xzp}=0,\qquad i,j=1,\dots,n,$$
$$\{X_i,Z\}_{xzp}=0,\qquad i=1,\dots,n.$$

Then the general solution to the system (5.1) can be constructed by means of a quadrature.

Proof. Construct the functions $P_1,\dots,P_n$ so that (X,Z,P) is a contact transformation (follow the procedure in the proof of Theorem 3.4). Let $H^*(X,Z,P)$ be the characteristic function of this contact transformation. Along a solution of the system (5.1), $X_i=c_i$, $Z=\gamma$, where c_i and γ are constants, so that $\dot X_i=\dot Z=0$. From the proof of Theorem 5.1 we know that

$$\frac{\partial H^*}{\partial P_j}=\dot X_i=0,\quad j=1,\dots,n,$$

hence $H^*=H^*(c,\gamma,t)$. In the new variables

$$\dot P_j=-\frac{\partial H^*}{\partial c_j}-P_j\frac{\partial H^*}{\partial \gamma},\quad j=1,\dots,n,$$

which is immediately solvable. The complete solution is given by

$$X_i(x,z,p,t)=c_i,\quad i=1\dots,n,$$
$$Z(x,z,p,t)=\gamma,$$
$$P_i(x,z,p,t)=-\left(\int\exp\left(\int\tfrac{\partial H^*(c,\gamma,t)}{\partial\gamma}\,dt\right)\tfrac{\partial H^*}{\partial c_j}(c,\gamma,t)\,dt\right)\Big/\exp\left(\int\tfrac{\partial H^*}{\partial\gamma}\,dt\right),\ i=1,\dots,n.$$

We now solve this system for $x_1,\dots,x_n,z,p_1,\dots,p_n$. □

Theorem 6.2. $F(x,z,p,t)=\mathrm{const}$ *is a first integral for the generalized Hamiltonian system (5.1) if and only if it satisfies $F_t+\{F,H\}-F_zH=0$, where $\{F,H\}$ is the Mayer bracket of F and the characteristic function H of system (5.1). The subscripts denote partial differentiation.*

Proof. We observe that if $F(x,z,p,t)$ is a first integral for system (6.1), then

$$\begin{aligned} 0 &= F_t + \frac{\partial F}{\partial x_j}\dot{x}_j + \frac{\partial F}{\partial z}\dot{z} + \frac{\partial F}{\partial p_j}\dot{p}_j \\ &= F_t + F_{x_j}H_{p_j} + F_z(p_j H_{p_j} - H) + F_{p_j}(-H_{x_j} - p_j H_z) \\ &= F_t + (F_{x_j} + p_j F_z)H_{p_j} - (H_{x_j} + p_j H_z)F_{p_j} - F_z H. \end{aligned}$$

We therefore obtain the equality $F_t + \{F,H\} - F_z H = 0$, which is a first order partial differential equation for F having (5.1) as its system of characteristic equations. □

The following theorem gives a method for generating first integrals for the generalized Hamiltonian system.

Theorem 6.3. *If $F(x,z,p,t) = \alpha$, $G(x,z,p,t) = \beta$, with α and β constants, are first integrals for the system (5.1), then $\rho\{F,G\}$ is also a first integral for the same system.*

Proof. Let $F(x,z,p,t) = \alpha$, $G(x,z,p,t) = \beta$, α and β constants, be two first integrals for system (5.1). The Jacobi identity for the Mayer bracket is

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} + F_z\{G,H\} + G_z\{H,F\} + H_z\{F,G\} = 0.$$

Replace $\{F,H\}$ and $\{G,H\}$ using the identity provided by Theorem 6.2 and rearrange to obtain the identity

$$-\frac{\partial}{\partial t}\{F,G\} - \{\{F,G\},H\} + H\frac{\partial}{\partial z}\{F,G\} + \frac{\partial H}{\partial z}\{F,G\} = 0.$$

We can rewrite this identity as $d(\rho\{F,G\})/dt = 0$, where $\rho = \exp\big(-\int_0^t \frac{\partial H}{\partial z}\,d\tau\big)$ and conclude that along a solution, $\rho\{F,G\}$ is a constant. □

7. The connection with the variational principle of Herglotz

Let us denote by $\mathcal{L} \equiv \mathcal{L}(x,\dot{x},z,t) \equiv \mathcal{L}(x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n,z,t)$ the Lagrange function, or Lagrangian, of the variables $(x,\dot{x},z,t)$, $z = z(t)$ is a scalar valued function of t. The variable z is to be determined as the solution to the differential equation

$$\dot{z} = \mathcal{L}(x,\dot{x},z,t). \tag{7.1}$$

Observe that (7.1) represents a family of differential equations, since for each $x(t)$ a different differential equation arises, that is, given $x(t)$, $z(t)$ is determined by (7.1) so that $z(t)$ depends on $x(t)$. A fact which we make explicit by writing $z = z[x;t] = z(x,\dot{x},t)$. Problem (7.1) is a kind of control problem. The differential equation for z describes a process which depends on $(x,\dot{x})$ and which in turn can be chosen, that is they give us the opportunity to control or guide the process and are therefore referred to as controls.

Theorem 7.1. *The functions (x, z) for which the functional z has stationary values satisfies the following system of ordinary differential equations*

$$\begin{aligned} \dot{p}_j &= \mathcal{L}_j + \mathcal{L}_z p_j, \quad j = 1, \dots, n, \\ \dot{z} &= \mathcal{L} \quad \text{with} \quad \mathcal{L}_j \equiv \frac{\partial \mathcal{L}}{\partial x_j}, \quad p_j \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_j}. \end{aligned} \tag{7.2}$$

The proof of this theorem can be found in [14].

Herglotz named equations (7.2) *generalized Euler-Lagrange equations.*

Theorem 7.2. *Let $\mathcal{L} = \mathcal{L}(x, \dot{x}, z, t)$ and suppose* $\det(\partial^2\mathcal{L}/\partial\dot{x}_i\partial\dot{x}_j) \neq 0$. *Then the solutions to (7.2) determine a family of contact transformations. If $\mathcal{L}$ is independent of t, the family is a one-parameter group.*

The proof can be found in [9].

Observe that system (7.2) is a generalized Hamiltonian system with

$$H(x, p, z, t) \equiv p_j \dot{x}_j - L(x, \dot{x}, z, t), \quad p_j \equiv \frac{\partial L}{\partial \dot{x}_j}.$$

We can summarize these considerations in the following general statement.

The following four kinds of problems are equivalent:

- *Variational problems for the functional z defined by the differential equation (7.1).*
- *Euler-Lagrange equations for the stationary values of the functional z defined by (7.1).*
- *The generalized Hamiltonian system.*
- *One parameter families of contact transformations.*

Example 7.1. Consider the Lagrangian function $L = m\dot{x}^2/2 - lx^2/2 - \alpha z$, where m, l, α are positive constants. Then $L_{\dot{x}} = m\dot{x} = p$, $L_x = -lx$, $L_z = -\alpha$. The Hamiltonian or characteristic function H is

$$H = H(x, p, z) = \frac{p^2}{2m} + \frac{lx^2}{2} + \alpha z.$$

The canonical system is

$$\begin{aligned} \dot{x} &= \frac{p}{m}, \\ \dot{z} &= \frac{p^2}{2m} - \frac{lx^2}{2} - \alpha z, \\ \dot{p} &= -(lx + \alpha p) \end{aligned}$$

and the Lagrange equation is $m\ddot{x} = -lx - \alpha m\dot{x}$ or

$$\ddot{x} + \alpha\dot{x} + \omega^2 x = 0, \quad \text{where} \quad \omega^2 = \frac{l}{m},$$

which is the equation of the damped harmonic oscillator.

Conclusion. Remarks for further research

Every reader of this paper will find suitable directions for his/her research, nevertheless, I like to mention a few. It will be valuable to see how these results extend to the variational principle of Herglots with several independent variables, which is so useful for the variational description of physical fields.

Another direction is to find methods for solving the generalized Euler-Lagrange equations obtained from Herglotz variational principle, perhaps using the results in this paper. In addition, to find further properties of the generalized Euler-Lagrange equations.

Is it possible to extend, in an appropriate sense, the variational principle of Herglotz to evolution equations? If so, then to investigate their relationship to the Hamiltonian evolution equations.

A more in-depth treatment of the theory of contact transformations, the generalized Hamiltonian system and the variational principle of Herglotz can be found in [14].

In the last 12 years or so about 200 new applications of the variational principle of Herglotz and the variational principle which generalizes it to one with several independent variables were published. They are in theoretical and applied physics, quantum mechanics, field theory, chemistry, mathematics, cosmology, dynamical systems, and many more branches of the exact sciences. I like to mention [1] and [25].

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