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НА

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ON THE STRUCTURE OF ($t \mod q$)-ARCS IN FINITE PROJECTIVE GEOMETRIES

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In this paper, we introduce constructions and structure results for ($t \mod q$)-arcs. We prove that all ($2 \mod q$)-arcs in $\mathrm{PG}(r, q)$ with $r \geq 3$ are lifted. We find all ($3 \mod 5$) plane arcs of small cardinality not exceeding 33 and prove that every ($3 \mod 5$)-arc in $\mathrm{PG}(3, 5)$ of size at most 158 is lifted. This result is applied further to rule out the existence of $(104, 22)$-arcs in $\mathrm{PG}(3, 5)$ which solves an open problem on the optimal size of fourdimensional linear codes over $\mathbb{F}_5$.

Keywords: finite projective geometries, arcs, blocking sets, divisible arcs, quasi-divisible arcs, the Griesmer bound, extendable arcs, minihypers

2000 Math. Subject Classification: Main 51A20, 51A21, 51A22, Secondary 94B65

1. INTRODUCTION

Consider the geometry $\Sigma = \mathrm{PG}(r, q)$, $r \geq 2$. Denote by $\mathcal{P}$ be the set of points and by $\mathcal{H}$ the set of hyperplanes of Σ. Every mapping $\mathcal{K} : \mathcal{P} \to \mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in Σ. This mapping is extended additively to every subset $\mathcal{Q}$ of $\mathcal{P}$ by $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$. The integer $n := \mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. For every set of points $\mathcal{Q} \subset \mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in Σ is called an (n, w)-multiarc (or simply (n, w)-arc) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \leq w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(r, q)$ is called an (n, w)-blocking set with respect to the hyperplanes (or (n, w)-minihyper) if (1) $\mathcal{K}(\mathcal{P}) = n$, (2) $\mathcal{K}(H) \geq w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0) = w$.

An (n, w)-arc $\mathcal{K}$ in Σ is called t-extendable, if there exists an $(n+t, w)$-arc $\mathcal{K}'$ in Σ with $\mathcal{K}'(P) \geq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. An arc is called simply extendable if it is 1-extendable. Similarly, an (n, w)-blocking set $\mathcal{K}$ in Σ is called t-reducible, if there exists an $(n-t, w)$-blocking set $\mathcal{K}'$ in Σ with $\mathcal{K}'(P) \leq \mathcal{K}(P)$ for every point $P \in \mathcal{P}$. A blocking set is called irreducible if it is not reducible.

Given a multiset $\mathcal{K}$ in Σ, we denote by a_i the number of hyperplanes H with $\mathcal{K}(H) = i$. The sequence $(a_i)_{i \geq 0}$ is called the spectrum of $\mathcal{K}$. An (n, w)-arc $\mathcal{K}$ with spectrum (a_i) is said to be divisible with divisor $\Delta > 1$ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$. Given an integer t with $1 \leq t \leq q-1$, we call the (n, w)-arc $\mathcal{K}$ with $w \equiv n + t \pmod{q}$ *t-quasidivisible with divisor* $\Delta > 1$ (or *t-quasidivisible modulo* Δ) if $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$.

Let t be a fixed non-negative integer. An arc $\mathcal{K}$ in Σ is called a $(t \mod q)$-arc if

(1) for every point $P \in \mathcal{P}$, $\mathcal{K}(P) \leq t$;

(2) for every subspace S of dimension at least 1, $\mathcal{K}(S) \equiv t \pmod{q}$.

These arcs arise naturally as certain duals of t-quasidivisible arcs. Let $\mathcal{K}$ be a t-quasidivisible (n, w)-arc with divisor q in Σ, $t < q$. Denote by $\widetilde{\mathcal{K}}$ the arc

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \to \quad \{0, 1, \ldots, t\} \\ H & \to \quad \widetilde{\mathcal{K}}(H) \equiv n + t - \mathcal{K}(H) \pmod{q} \end{cases}, \tag{1.1}$$

where $\mathcal{H}$ is the set of all hyperplanes in Σ. This means that hyperplanes of multiplicity congruent to $n + a \pmod{q}$ become $(t - a)$-points in the dual geometry. In particular, maximal hyperplanes are 0-points with respect to $\widetilde{\mathcal{K}}$. Then $\widetilde{\mathcal{K}}$ is a $(t \mod q)$-arc [7,8]. In the general case the cardinality of $\widetilde{\mathcal{K}}$ cannot be obtained from the parameters of $\mathcal{K}$. Extendability properties of $\mathcal{K}$ can be derived from the structure of $\widetilde{\mathcal{K}}$. In particular, $\mathcal{K}$ is extendable if it contains a hyperplane in its support. For a more detailed introduction to arcs and blocking sets and their relation to linear codes, we refer to[5,8].

The aim of this paper is to present various constructions and structure results for $(t \mod q)$-arcs. Section 2 contains general constructions for $(t \mod q)$-arcs. The most important is the so-called lifting construction, which is partly due to the fact that in dimension higher than 3 the only known $(t \mod q)$-arcs are sums of lifted arcs. In section 3, we prove that every $(2 \mod q)$-arc is lifted. This result implies Maruta's extendability result for linear codes with weights $-2, -1, 0 \mod q$ for q odd. In section 4, we characterize the $(3 \mod 5)$-arcs of small cardinality and prove that every $(3 \mod 5)$-arc in $\mathrm{PG}(3, 5)$ of size not exceeding 153 is lifted. In section 4, we apply the results from section 3 to rule out the existence of $(104, 22)$-arcs in $\mathrm{PG}(3, 5)$, or equivalently, of $[104, 4, 82]_5$-codes.

2. GENERAL CONSTRUCTIONS

In this section, we describe several constructions for $(t \mod q)$-arcs. We start with a straightforward observation.

Theorem 1. *Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a $(t_1 \mod q)$-arc (resp. $(t_2 \mod q)$-arc) in $\mathrm{PG}(r, q)$. If $t = t_1 + t_2 < q$, then $\mathcal{F}_1 + \mathcal{F}_2$ is a $(t \mod q)$ arc. In particular, the sum of t (not necessarily different) hyperplanes is a $(t \mod q)$-arc.*

The next construction is less obvious.

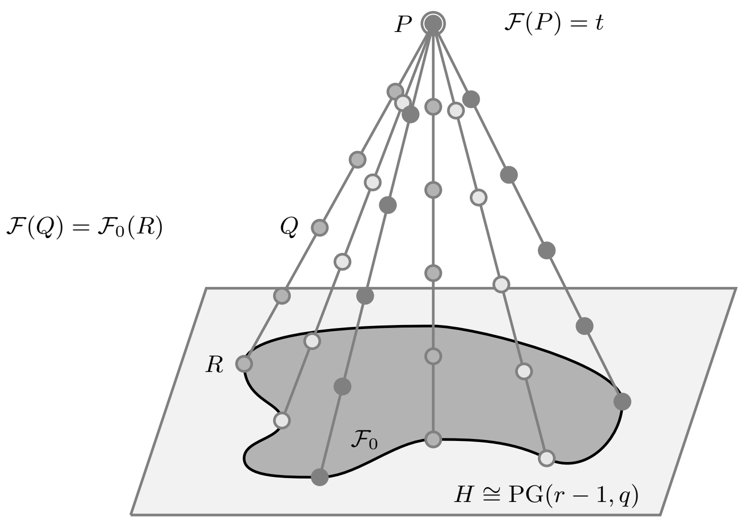
Theorem 2. *Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1, q)$ of $\Sigma = PG(r, q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{F}$ in Σ as follows:*

– $\mathcal{F}(P) = t$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle P, Q\rangle \cap H$.

Then the arc $\mathcal{F}$ is a $(t \mod q)$-arc in $\mathrm{PG}(r, q)$ of size $q|\mathcal{F}_0| + t$.

Proof. As already noted it is enough to prove that the multiplicity of every line is t modulo q. This is obvious for the lines through the point P. Now consider a line L in Σ which is not incident with P. Let π be the plane defined by P and L: $\pi = \langle L, P\rangle$. Set $L' = \pi \cap H$. Obviously, L contains points of the same multiplicities as L'. The multiplicity of L' is $\mathcal{F}(L') = \mathcal{F}_0(L) \equiv t \pmod q$ which proves the result. The construction is illustrated in the picture below. □



We call the $(t \mod q)$-arcs obtained by Theorem 2 *lifted arcs* and the point P – *lifting point.* We can have a more general notion of lifted arcs replacing the point P by a subspace U. Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in the subspace V of $\Sigma = \mathrm{PG}(r,q)$ and let U be a subspace with $\dim U + \dim V = r-1$, $U \cap V = \varnothing$. The arc $\mathcal{F}$ in Σ defined by

– $\mathcal{F}(P) = t$ for every point $P \in U$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle U, Q\rangle \cap V$

is called an arc lifted from the subspace U. Obviously $\mathcal{F}$ is also a $(t \mod q)$-arc. Let us note that if an arc is lifted from a subspace then it can be considered as lifted from any point of that subspace. We have also a partial converse of this observation.

Lemma 1. *Let the arc $\mathcal{F}$ be lifted from the points P and Q, $P \neq Q$. Then $\mathcal{F}$ is also lifted from the line PQ. In particular, the lifting points of a $(t \mod q)$-arc $\mathcal{F}$ form a subspace S and $\mathcal{F}$ is lifted from any point of S.*

Proof. All points on the line PQ are t-points. Let R be an arbitrary point in Σ. Then all points on PR (resp. QR) different from P (resp. Q) have the same multiplicity, a say. Then all points in the plane $\langle P, Q, R\rangle$ outside PQ have also multiplicity a, which proves the lemma. □

The sum of t hyperplanes can be viewed as the sum of lifted arcs. Remarkably, we do not know an example of a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$, with $r \geq 3$, that is not the sum of lifted arcs. It turns out that if in the geometry $\mathrm{PG}(r,q)$ there exist only lifted $(t \mod q)$-arcs then every $(t \mod q)$-arc in $\mathrm{PG}(r',q)$, $r' \geq r$, is also lifted.

Theorem 3. *Let $\mathcal{K}$ be a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ such that the restriction $\mathcal{K}|_H$ to every hyperplane H of $\mathrm{PG}(r,q)$ is also lifted. Then $\mathcal{K}$ is a lifted arc.*

Proof. Consider a $(t \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(r,q)$. Let S be an arbitrary subspace of $\mathrm{PG}(r,q)$ of codimension 2. Denote by H_i, $i = 0,\ldots,q$, the hyperplanes through S. The arcs $\mathcal{K}|_{H_i}$ are all lifted $(t \mod q)$-arcs. Let us denote by P_i, $i = 0,\ldots,q$, the corresponding lifting points.

Assume that for some indices, i and j say, $P_i \in S$ and $P_j \in H_j \setminus S$. Clearly, the line P_iP_j consists entirely of t-points. Let L be an arbitrary line in H_j incident with P_j and set $L \cap S = Q_j$. All points on the line P_jQ_j, different from P_j have the same multiplicity a, where $0 \leq a \leq t$. Thus all points in the plane $\langle P_i, P_j, Q_j\rangle$ outside P_iP_j are a points. Now it is clear that $\mathcal{K}|_{H_j}$ can be viewed as lifted from the line P_iP_j and hence from any point of P_iP_j.

Assume that $P_i \in H_i \setminus S$ for all $i = 0,\ldots,q$. If the points $P_0,\ldots,P_q$ are collinear then $\mathcal{K}$ is lifted from the line $\langle P_i \mid i = 0,\ldots,q\rangle$.

Now assume that the points P_i are not collinear. Then there exists a hyperplane H in $\mathrm{PG}(r,q)$ that does not contain any of the points P_i. Set $T = H \cap S$. If we denote $G_i = H \cap H_i$ then all the arcs $\mathcal{K}|_{G_i}$ are projectively equivalent to $\mathcal{K}_S$.

Let us first assume that the lifting point Q of $\mathcal{K}|_H$ is contained in $G_i \setminus T$. Set $Q_i = S \cap QP_i$. Obviously, P_iQ_i is a line of t-points. Consider an arbitrary line L

in H_i through P_i. If the points on L different from P_i are a-points then all points on the line through Q and $L \cap G_i$ different from Q are also a-points. Hence all points in the plane $\langle L, Q_i \rangle$ outside P_iQ_i are a-points and $\mathcal{K}|_{H_i}$ is lifted from P_iQ_i. Therefore it can be viewed as lifted from any point on P_iQ_i, in particular Q_i.

We have proved so far that without loss of generality we can assume that all points P_i are contained in S.

Consider the subspace T of S generated by the points P_i, $T = \langle P_i \mid i = 0, \ldots, q\}$. All points in T are of maximal multiplicity. Let $Q \in S \setminus T$ be a point of multiplicity a. All points from $\langle T, Q \rangle \setminus T$ also have multiplicity a. Hence the restriction $\mathcal{K}|_S$ is lifted from the subspace T. Since S was fixed arbitrarily, the restriction of $\mathcal{K}$ to any subspace of codimension 2 is a lifted arc.

We repeat this argument for the subspaces of smaller dimension. For subspaces of dimension 2 this means that all planes contain a line of t-points with all the remaining points of multiplicity a. It is easily checked that in such case we have a hyperplane of t-points and all the remaining points outside this hyperplane are a-points. But such an arc is obviously a lifted arc. □

In the plane case, non-trivial $(t \mod q)$-arcs can be constructed as σ-duals of certain blocking sets. Let $\mathcal{K}$ be a multiset in Σ. Consider a function σ such that $\sigma(\mathcal{K}(H))$ is a non-negative integer for all hyperplanes H. The multiset

$$\widetilde{\mathcal{K}}^\sigma : \begin{cases} \mathcal{H} & \rightarrow \quad \mathbb{N}_0 \\ H & \mapsto \quad \sigma(\mathcal{K}(H)) \end{cases} \tag{2.1}$$

in the dual space $\widetilde{\Sigma}$ is called the σ-*dual* of $\mathcal{K}$. If σ is a linear function, the parameters of $\widetilde{\mathcal{K}}^\sigma$, as well as its spectrum, are easily computed from the parameters and the spectrum of $\mathcal{K}$ (cf. [1,10]).

Theorem 4. *[7,8] Let $\mathcal{F}$ be a $(t \mod q)$-arc in $\mathrm{PG}(2, q)$ of size $mq + t$. Then the arc $\mathcal{F}^\sigma$ with $\sigma(x) = (x - t)/q$ is a $((m - t)q + m, m - t)$-blocking set in the dual plane. Moreover the multiplicities of the lines with respect to this blocking set belong to $\{m - t, m - t + 1, \ldots, m\}$.*

3. $(2 \mod Q)$-ARCS

Let us start by noting that an $(1 \mod q)$ arc is projective and hence either a hyperplane or the complete space [3,4]. For $t = 2$ and odd $q \geq 5$, the $(t \mod q)$-arcs were characterized by Maruta [13]. These are the following:

(I) a lifted arc from a 2-line; such an arc has $2q + 2$ points and there exist two possibilities

 (I-1) a double line, or

 (I-2) a sum of two different lines;

(II) a lifted arc from a $(q+2)$-line; such a line has i double points, $q-2i+2$ single points and $i-1$ 0-points, where $i=1,\dots,\frac{q+1}{2}$; we say that such an arc is of type (II-i) if it is lifted from a line with i double points;

(III) a lifted arc from a $(2q+2)$-line, i.e. the sum of two copies of the same plane;

(IV) an exceptional $(2 \mod q)$-arc for q odd; it consists of the points of an oval, a fixed tangent to this oval, and two copies of each internal point of the oval.

Now we are going to prove that in higher dimensions every $(2 \mod q)$-arc is a lifted arc. Consider a projection φ from a 2-point P onto some plane not incident with that point. Let L be a line incident with P. We have the following possibilities for the image of L:

type of L	multiplicity of L	type of $\varphi(L)$
$(2,0,\dots,0)$	2	ω
$(2,1,\dots,1)$	$q+2$	α
$(2,2,\dots,2)$	$2q+2$	β
$(2,\underbrace{2,\dots,2}_{i},\underbrace{1,\dots,1}_{q-2i},\underbrace{0,\dots,0}_{i})$	$q+2$	γ_i

Note that in type γ_i we have $i=1,\dots,\frac{q-1}{2}$. Now the images of the plane $(2 \mod q)$-arcs under φ are the following:

Type	the image of the plane arc	Remark
(I-1)	$(\beta,0,\dots,0)$	
(I-2)	$(\alpha,\alpha,0,\dots,0)$	
(II-i)	$(\underbrace{\beta,\dots,\beta}_{i},\underbrace{\alpha,\dots,\alpha}_{q-2i+1},\underbrace{0,\dots,0}_{i-1})$	projection from the exceptional 2-point
	$(\beta,\gamma_i,\gamma_i,\dots,\gamma_i)$ $i=1,\dots,\frac{q-1}{2}$	projection from all other 2-points
(III)	$(\beta,\beta,\dots,\beta)$	
(IV)	$(\alpha,\gamma_{\frac{q-1}{2}},\dots,\gamma_{\frac{q-1}{2}})$	from the 2-point on the oval
	$(\underbrace{\gamma_{\frac{q-1}{2}},\dots,\gamma_{\frac{q-1}{2}}}_{\frac{q+3}{2}},\underbrace{\gamma_{\frac{q-3}{2}},\dots,\gamma_{\frac{q-3}{2}}}_{\frac{q-1}{2}})$	from an internal point to the oval

Assume a $(2 \mod q)$-arc $\mathcal{K}$ in $\mathrm{PG}(3,q)$, q odd, is given and consider a projection from a 2-point P. The table above implies that

(i) no line in the projection plane is incident with points of type ω and points of type γ_i;

(ii) if on a line in the projection plane there exist points of type γ_i and points of type γ_j, $i \neq j$, then $i=\frac{q-3}{2}$, $j=\frac{q-1}{2}$.

Let us first assume that there exists a plane π such that $\mathcal{K}|_\pi$ is the exceptional arc (IV). Denote by φ a projection from the 2-point on the oval. Then the image of the plane π is of type $(\alpha, \gamma_{\frac{q-1}{2}}, \ldots, \gamma_{\frac{q-1}{2}})$. Denote by L the line of type α and fix a 1-point Q on this line. Assume there is a point of type β in the projection plane. Then the projection plane contains a line of type $(\beta, \alpha, \ldots, \alpha)$ and all the remaining lines through the type β point are of type $(\beta, \gamma_{\frac{q-1}{2}}, \ldots, \gamma_{\frac{q-1}{2}})$. Hence the lines in the projection plane through a point of type α are of the following types:

- $(\alpha, \alpha, \ldots, \alpha, \beta)$ – there is one such line;
- $(\alpha, \gamma_{\frac{q-2}{2}}, \ldots, \gamma_{\frac{q-2}{2}}, \gamma_{\frac{q-2}{2}})$ – there are q such lines.

Denote the points on the $2(q+1)$-line (the preimage of the point of type β) by $P_0, P_1, \ldots, P_q$. Assume that there is a point of type P_i such that all planes through QP_i (different from π_0) are not of type (IV). Then $\mathcal{K}$ is obviously lifted. If for all P_i there is a plane through QP_i such that the restriction of $\mathcal{K}$ to this plane is of type (IV) then projecting from each P_i we must get the same types of the lines in the projection plane (described above). Therefore no three of the q^2 1-points contained in the ovals, are collinear. Hence we can construct a q^2+2-cap taking these q^2 1-points and P_0, P_1, say. This is a contradiction since the maximal size of a cap in $\mathrm{PG}(3,q)$ is q^2+1.

We have proved that if there is a point of type β in the projection plane, then $\mathcal{K}$ is lifted. But there always must be a point of type β since the types $\gamma_{\frac{q-1}{2}}$ and ω are not compatible. Thus we have proved that if there exists a plane π such that $\mathcal{K}|_\pi$ is of type (IV) then $\mathcal{K}$ is a lifted arc.

Now assume that there is no plane such that $\mathcal{K}|_\pi$ is of type (IV). Now the restriction of $\mathcal{K}$ to any plane is a lifted arc and by Theorem 3 $\mathcal{K}$ is again lifted. We have proved the following lemma.

Lemma 2. *Let $\mathcal{K}$ be a $(2 \bmod q)$-arc in $\mathrm{PG}(3,q)$, q odd. Then $\mathcal{K}$ is a lifted arc.*

Now we proceed by induction on the dimension. Again by Theorem 3, we get that every $(2 \mod q)$-arc in a geometry of dimension at least 3 is lifted.

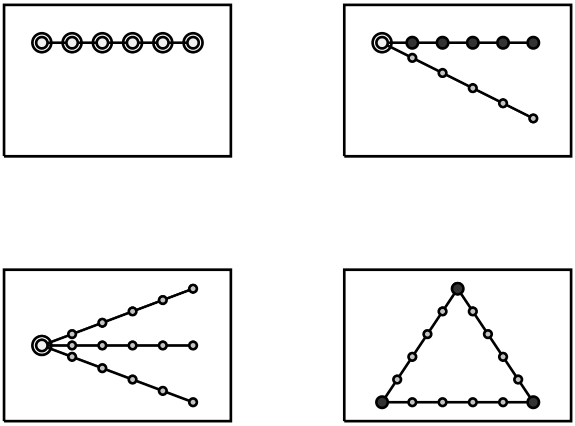
Theorem 5. *Let $\mathcal{K}$ be a $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, q odd, $r \geq 3$. Then $\mathcal{K}$ is a lifted arc. In particular, every $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 2$, has a hyperplane in its support.*

Remark. Theorem 5 provides alternative proof of Maruta's theorem on the extendability of codes with weights $-2, -1, 0 \pmod q$ [13]. The existence of such a code is equivalent to that of an arc $\mathcal{K}$ which is 2-quasidivisible modulo q. It was pointed out in [7,8] that for every t-quasidivisible arc $\mathcal{K}$ in Σ it is possible to define uniquely a $(t \mod q)$- arc $\widetilde{\mathcal{K}}$ in the dual geometry. If $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable. This is the fact established in Theorem 5.

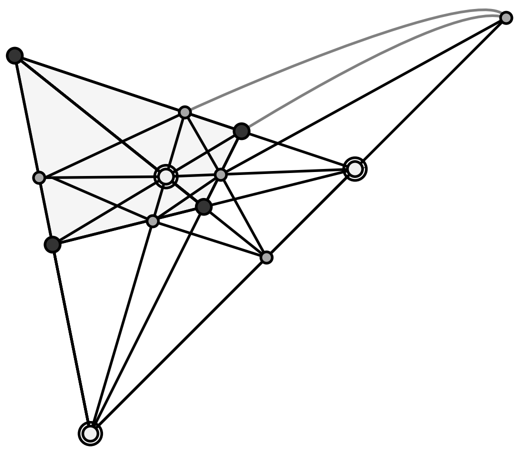
4. (3 mod Q)-ARCS

For values of t larger than 2 complete classification seems out of reach. However, it is still possible to obtain partial results on the structure of such arcs. In this section we classify some small (3 mod 5)-arcs in PG$(2,5)$. Due to Theorem 4, the classification of such arcs is equivalent to the classification of certain blocking sets with an additional restriction on the line multiplicities.

Arcs of cardinality 18. These arcs are $(18,3)$-blocking sets and hence the sum of three not necessarily different lines [9,11]. It is an easy check that there exist four (3 mod 5)-arcs of cardinality 18. They are given in the pictures below.



Arcs of cardinality 23. These arcs correspond to $(9,1)$-blocking sets with lines of multiplicity 1, 2, 3, 4. Hence blocking sets containing a full line do not give (3 mod 5)-arcs. Thus the only possibility is the projective triangle. Dualizing we get a (3 mod 5)-arc in which the 2-points form a complete quadrangle, the intersections of the diagonals are 3-points and the intersections of the diagonals with the sides of the quadrangle are 1-points. This arc is presented in the picture below. The doubly circled points are 3-points; the big black points are 2-points and the small gray points are 1-points.

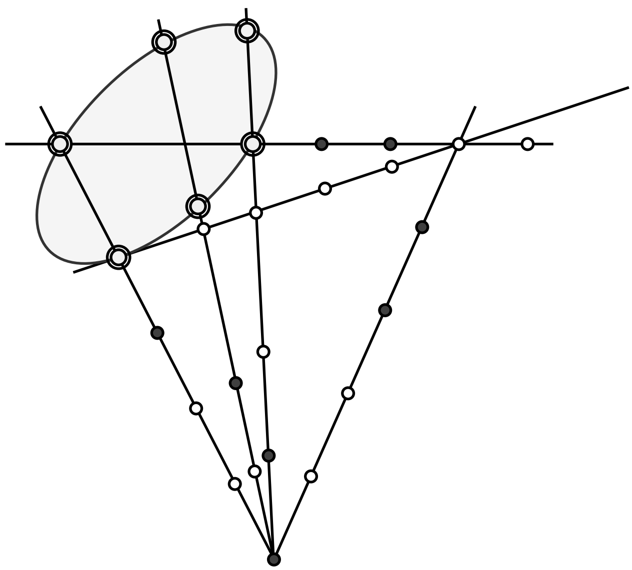


Arcs of cardinality 28. These arcs are obtained from (15,2)-blocking sets with lines of multiplicity 2, 3, 4, or 5. If such a blocking set does not have multiple points it is obtained as the complement of a (16,4)-arc. Such an arc should not have external lines since the maximal multiplicity of a line with respect to the blocking set is 5. The classification of the (16,4)-arcs is well-known. There exists exactly one such arc without external lines obtained by deleting the common points of six lines in general position from the plane. Now we are going to prove that a (15,2)-blocking set having points of multiplicity greater than 1 always has a line of multiplicity 6 and hence does not give a $(3 \mod 5)$-arc.

Let us note that such a blocking set cannot have a point of multiplicity 3. In this case the remaining 12 points would form a (12,2)-blocking set which is the sum of two lines and therefore has a line of multiplicity greater than 6.

Denote by Λ_i, $i = 0, 1, 2$, the number of i-points of a $(15, 2)$-blocking set. Clearly $\Lambda_2 \leq 6$ since the colinearity of three 2-points implies the existence of a 6-line. In the case of $\Lambda_2 = 4, 5, 6$, it is easily checked that the remaining 1-points cannot block twice each of the external lines. The remaining possibilities $1 \leq \Lambda_2 \leq 3$ are ruled out using additional arguments.

Thus the only $(3 \mod 5)$-arc of cardinality 28 has six 3-points forming an oval and ten 1-points that are the internal points to this oval.



Arcs of cardinality 33. If $\mathcal{F}$ is such an arc then $\mathcal{F}^{\sigma}$ is a $(21, 3)$-blocking set with line multiplicities $3, 4, 5, 6$. Again such a blocking set cannot have points of multiplicity 3 or larger since this would impose lines of multiplicity larger than 6 in $\mathcal{F}$.

Denote by Λ_i the number of points of multiplicity i. Since there cannot be five collinear 0-points we have $\Lambda_0 \leq 16$ and therefore $\Lambda_2 \leq 6$. We are going to rule out the case $\Lambda_2 = 6$. Assume there exist three collinear 2-points. There exist two lines containing three 2-points. They must necessarily meet in a 2-point. Now since a 0-point on a 6-line is incident with 3-lines only, a simple counting gives that the sixth point of multiplicity 2 is incident with three 2-lines. Counting the multiplicities through the exceptional 2-point, we get $21 \geq 3 \cdot 3 + 3 \cdot 6 - 5 \cdot 2 = 17$. Hence the 2-points form an oval. Now the ten external lines to the oval have to be blocked at least three times each by the 1-points. Since each 1-point blocks at most three external lines we need at least $3 \cdot 10/3$ such points, a contradiction since $\Lambda_1 = 9$.

The cases $3 \leq \Lambda_2 \leq 5$ are ruled out in a similar fashion.

For $\Lambda_2 = 0, 1, 2$ constructions are possible. In such case, $\mathcal{F}^\sigma$ is one of the following:

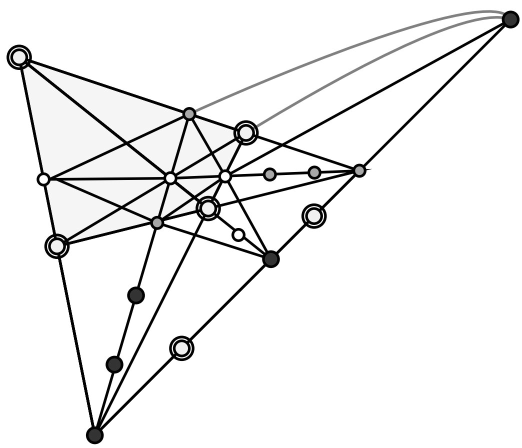
(1) the complements of the seven non-isomorphic $(10, 3)$-arcs; $\Lambda_2 = 0$;

(2) the complement of the $(11, 3)$-arc with four external lines and a double point – a point not on an external line, $\Lambda_2 = 1$;

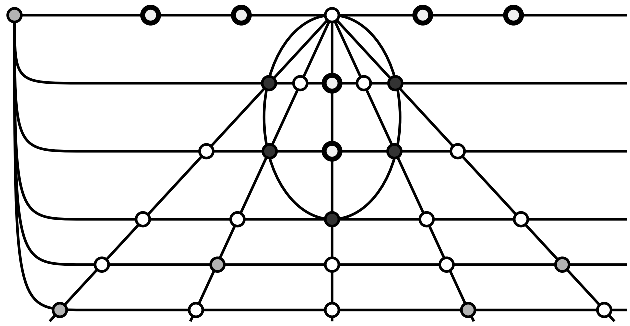
(3) one double point which forms an oval with five of the 0-points; the tangent in the 2-point is a 3-line, $\Lambda_2 = 1$;

(4) PG(2, 5) minus a triangle with vertices of multiplicity 2, 2, 1; $\Lambda_2 = 2$.

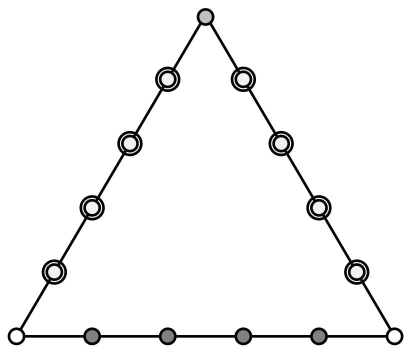
(2) The first (3 mod 5)-arc of cardinality 33 with one 13-line



(3) the second (3 mod 5)-arc of cardinality 33 with one 13-line



(4) $(33, \{3,8,13\})$-arc with two 13-lines



Arcs of cardinality 38. The (3 mod 5)-arcs of cardinality 38 can be derived from the (27,4)-blocking sets with line multiplicities $4, 5, 6, 7$ in $\mathrm{PG}(2,5)$. Such a blocking set does not have 3-points. Otherwise, removing a 3-point would give a $(24,4)$-blocking set which is a sum of line. This forces a line of multiplicity greater than 7. If there exist three collinear 2-points then $\Lambda_2 = 3$ and the corresponding line is a 7-line.

There exist a lot of such blocking sets and, consequently, (3 mod 5)-arcs of cardinality 38. In all cases, such arcs have a 13-line with a 0-point or an 8-line of type $(2,2,2,2,0,0)$, $(2,2,2,1,1,0)$ or $(3,3,2,0,0,0)$.

For instance, in the case of $\Lambda_2 = 0$ the blocking set consists of all points in the plane minus four points in general position. The corresponding (3 mod 5)-arc has a line of type $(2,2,2,1,1,0)$. In the case $\Lambda_2 = 6$ the 2-points form an oval. Th external points to this oval have to be blocked at least four times by the fifteen 1-points. An easy counting gives that we should take necessarily the ten internal points plus five external points. But now the six tangents cannot be blocked twice by six points not on the oval. The remaining cases are treated using similar arguments.

Now we can prove our main result for this section. The following observation turns out to be very useful. Let $\mathcal{F}$ be a $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ and consider a projection φ from a 0-point P onto some plane π not incident with P. Set

$$\mathcal{G} = \frac{1}{5}(\mathcal{F}^{\varphi} - 3). \tag{4.1}$$

Now lines through P of multiplicity $3+5i$, $i = 0, 1, 2$, become i-points. The following lemma restricts the possible structure of $\mathcal{G}$.

Lemma 3. *Let XY and XZ be 2-lines in π with respect to $\mathcal{G}$ and let there exist an 1-point $U \neq X, Y, Z$ which is incident with a 2-line. Then U does not lie on a tangent of $\mathcal{G}$.*

Proof. Let t be the tangent through U and let $t \cap XY = V$, $t \cap XZ = W$. Obviously $V \neq X, Y$, $W \neq X, Z$. Since U, V, W are on 2-lines that are the image of 28-planes, they are the image of 2-lines without 2-points. Then the preimage of t is a 23-plane with at most three 2-points, a contradiction since a 23-plane contains four 2-points. □

Theorem 6. *Every $(3 \mod 5)$-arc $\mathcal{F}$ in $\mathrm{PG}(3,5)$ with $|\mathcal{F}| \leq 158$ is a lifted arc. In particular, $|\mathcal{F}| = 93, 118,$ or 143.*

Proof. Assume there exists a 13-line L with 0-point. By the classification of the plane $(3 \mod 5)$-arcs we have that all planes through such a line have multiplicity at least 33. If there exists a 33-plane, π say, through L then it must be of type (2), (3), or (4). In the first two cases π is incident with an 8-line of type (3,3,2,0,0,0), while in the third case it is incident with a line of type (2,2,2,2,0,0). Planes of multiplicity less than 33 do not contain such lines. Hence $|\mathcal{F}| \geq 8 + 6 \cdot 25 = 158$. If all planes through L are of cardinality ≥ 38, then again $|\mathcal{F}| \geq 13 + 6 \cdot 25 = 163$, a contradiction.

If $|\mathcal{F}| = 158$ then there exists a 33-plane of the type (2), (3), or (4). Assume there exists a 33-plane of type (2). It contains a line of type (2,2,2,1,1,0). Consider a projection from the 0-point on this line. The induced arc has thirteen 8-points and eighteen 3-points. We cannot have a line incident with one, two or six 8-points. Now by an easy counting we get that there are no lines with four or five 8-points, a contradiction since the number of 8-points is 13 and the largest $(n, 3)$-arc has 11 points.

The case of planes with 33 points of type (3) and (4) are ruled out in a similar way. Thus 0-points are incident with 3- or 8-lines only.

Further, a line containing a 0-point has multiplicity at most 48. It is easily checked that 48-planes are impossible. In such a plane each 8-line is incident with exactly two 0-points and the 0-points must form an oval. But an oval in $\mathrm{PG}(2,5)$ has 6 points while a 48-plane has seven 0-points.

The restriction of $\mathcal{F}$ to a 43-plane in which every line through a 0-point is necessarily lifted from an 8-line. The planes through a line of type (3,1,1,1,1,1) in a 43-plane are either lifted 43-planes or 18-planes that are again lifted. Hence in such case $\mathcal{F}$ is necessarily a lifted arc of size 118 or 143 since the 3-point on the line of type (3,1,1,1,1,1) is the lifting point of the 43- as well as of the 18-planes.

Finally, the fact that 38-planes have either a 13-line with a 0-point or an 8-line of type $(2,2,2,2,0,0)$, $(2,2,2,1,1,0)$ or $(3,3,2,0,0,0)$ implies that such planes are impossible if $|\mathcal{F}| \leq 158$. Thus we can assume with no loss of generality that every plane incident with a 0-point has multiplicity 18, 23, 28, or 33. Moreover, a 33-plane should necessarily be of type (1).

Now consider the arc $\mathcal{G}$ defined in (4.1). Since it does not have 4-lines $|\mathcal{G}| \leq 11$ and $\mathcal{F} \leq 148$. These cases are ruled out easily by Lemma 3. □

5. AN EXAMPLE FROM CODING THEORY

One of the forms of the main problem of coding theory is to determine the minimal length of an $[n,k,d]_q$-code for fixed q,k and d [10]. For codes over $\mathbb{F}_5$ of dimension 4 there exist four values of d for which $n_5(4,d)$ is not decided [12]. The results from the previous section enable us to solve one of the four open cases. We can rule out the existence of codes with parameters $[104,4,82]_5$ which implies that $n_5(4,82) = 105$.

The approach to this problem is geometric. The existence of a $[104,4,82]_5$-code is equivalent to the existence of a $(104,22)$-arc in $\mathrm{PG}(3,5)$ (cf. [2,6,10]). Such a hypothetical arc will turn out to be non-extendable.

Assume that $\mathcal{K}$ is a $(104,22)$-arc in $\mathrm{PG}(3,5)$. Let us denote by δ_i, $i = 0,1,2$, the maximal multiplicity of an i-dimensional subspace in $\mathrm{PG}(3,5)$. In the following lemma, we summarize the straightforward properties of $(104,22)$-arcs.

Lemma 4. *Let $\mathcal{K}$ be a $(104,22)$-arc with spectrum (a_i). Then*

(a) $\delta_0 = 1$, $\delta_1 = 5$, $\delta_2 = 22$;

(b) The maximal multiplicity of a line in an m-plane is $\lfloor (6+m)/5 \rfloor$;

(c) There do not exist planes with $2,3,7,8,12,13,17,18$ *points.*

(d) $a_0 = 0$.

(e) $a_1 = 0$.

(f) $a_4 = a_5 = 0$

(g) The spectrum of $\mathcal{K}$ satisfies the following identity

$$\sum_{i=0}^{20} \binom{22-i}{2} a_i = 468. \tag{5.1}$$

By Lemma 4, a (104, 22)-arc $\mathcal{K}$ is 3-quasidivisible. Moreover, 0-points with respect to the dual arc $\widetilde{\mathcal{K}}$ must come necessarily from maximal planes. This forces certain restrictions on the structure of $\widetilde{\mathcal{K}}$ described in the lemma below.

Lemma 5. *Let $\mathcal{K}$ be a* $(104, 22)$*-arc in* $\mathrm{PG}(3,5)$*. Then there exists no plane $\widetilde{P}$ in the dual space such that $\widetilde{\mathcal{K}}|\widetilde{P}$ is $3\chi_{\widetilde{L}}$ for some line $\widetilde{L}$ in the dual space.*

Proof. Let X be a point in $\mathrm{PG}(3,5)$. Summing up the multiplicities of all planes through X, we have:

$$\sum_{H: H \ni X} \mathcal{K}(H) = 6|\mathcal{K}| + 25\mathcal{K}(X).$$

On the other hand, a point $\widetilde{H}$ in the dual space with $\widetilde{K}(\widetilde{H}) = 0$ comes necessarily from a maximal plane. For the points on the line L with $\widetilde{K}(\widetilde{L}) = 18$ we have

$$\sum_{\widetilde{P}: \widetilde{P} \in \widetilde{L}} \mathcal{K}(\widetilde{P}) = |\mathcal{K}| + 5\mathcal{K}(L).$$

This implies that

$$6|\mathcal{K}| + 25\mathcal{K}(X) = 25 \cdot 22 + |\mathcal{K}| + 5\mathcal{K}(L),$$

which gives

$$649 \geq 6|\mathcal{K}| + \mathcal{K}(X) = 654 + 5\mathcal{K}(L),$$

a contradiction. □

Lemma 6. *Let $\mathcal{K}$ be a* $(104, 22)$*-arc in* $\mathrm{PG}(3,5)$*. Then $|\widetilde{\mathcal{K}}| \geq 163$.*

Proof. This follows by Lemma 5, Theorem 6 and the fact that a (104, 22)-arc is not extendable. □

We can use Lemma 6 together with the necessary condition (5.1) to restrict further the possible multiplicities of planes. Our key observation is that if a 5-tuple of planes through a a line L in H_0 gives a high contribution to the left-hand side of (5.1) then $\widetilde{K}(\widetilde{L})$ is small.

Lemma 7. *Let $\mathcal{K}$ be a* $(104, 22)$*-arc in* $\mathrm{PG}(3,5)$*. Then $a_6 = 0$.*

Proof. Let H_0 be a 6-plane. Then $\mathcal{K}|_{H_0}$ is a $(6,2)$-arc and has spectrum $a_2 = 15, a_1 = 6, a_0 = 10$. Consider an arbitrary line L in H_0. By Theorem 4, if L is a 2-line with respect to $\mathcal{K}$, then it is a 3-line with respect to $\widetilde{\mathcal{K}}$; similarly, if L is a 1-line it is a 3-line with respect to $\widetilde{\mathcal{K}}$ (since 22-planes do not have 1-lines) and, finally, if it is a 0-line with respect to $\mathcal{K}$, it is a 3-, 8- or 13-line with respect to $\widetilde{\mathcal{K}}$.

In the case of $\mathcal{K}(L) = 0$ and $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$ the maximal contribution of the planes through L is 66 obtained for

$$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5)) = (6, 22, 22, 22, 22, 10);$$

if $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$ the maximal contribution of the planes through L is 31 obtained for

$$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5)) = (6, 22, 22, 21, 19, 14);$$

and if $\widetilde{\mathcal{K}}(\widetilde{L}) = 13$ the maximal contribution of the planes through L is 13 obtained for

$$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5)) = (6, 22, 19, 19, 19, 19).$$

Let us denote by x the number of 0-lines L of H_0 with $\widetilde{\mathcal{K}}(H_0) = 3$ and by y the number of such lines with $\widetilde{\mathcal{K}}(H_0) = 8$. Counting the contribution of the different planes through the lines of L we get

$$\binom{16}{2} + 15 \cdot 1 + 6 \cdot 3 + 66x + 31y + 13(10 - x - y) \geq 468,$$

whence $53x + 18y \geq 185$. On the other hand, we have

$$|\widetilde{\mathcal{K}}| = 121 \cdot 2 + 2x + 7y + 12(10 - x - y) = 163 - 5x - 10y.$$

Since $\mathcal{K}$ is not extendable we have $|\widetilde{\mathcal{K}}| \geq 163$, and hence $x + 2y \leq 0$, i.e. $x = y = 0$, a contradiction to $53x + 18y \geq 185$. □

Lemma 8. *Let $\mathcal{K}$ be a $(104, 22)$-arc in* PG$(3, 5)$*. Then* $a_9 = a_{10} = a_{11} = 0$.

Proof. We use the classification of the (9,3), (10,3)- and (11,3)-arcs made in [5]. We will demonstrate only the non-existence of 9-planes of type C4 (we use the notation from [5]). The non-existence of 9-planes of the other three types, as well as the non-existence of 10- and 11-planes, is done analogously.

Let H_0 be a 9-plane and let $\mathcal{K}|_{H_0}$ be a $(9,3)$-arc of type C4. For a arbitrarily fixed line L in H_0 we denote by $H_1, \ldots, H_5$ the other 5 planes through L. We have the following possibilities:

$\mathcal{K}(L)$	$\widetilde{\mathcal{K}}(\widetilde{L})$	η_i	$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5))$
3	3	0	(22,22,22,22,22,9)
2	8	4	(22,22,22,2,19,9)
1	8	15	(21,21,21,21,16,9)
1	13	7	(21,21,20,19,19,9)
0	8	79	(22,22,22,19,10,9)
0	13	34	(22,21,19,19,14,9)
0	18	15	(19,19,19,19,19,9)

Counting the contribution of the planes through the different lines in H_0 to the left-hand side of (5.1), we get

$$\binom{12}{3} + 7 \cdot 0 + 15 \cdot 4 + 15x + 7 \cdot (3 - x) + 79u + 34v + 15(6 - u - v) \geq 468,$$

whence $8x + 64u + 19v \geq 219$.

On the other hand, computing the cardinality of $\widetilde{\mathcal{K}}$ and taking into account that $\widetilde{\mathcal{K}}(\widetilde{H}) = 3$, we get

$$3 + 7 \cdot 0 + 15 \cdot 5 + 5x + 10(3 - x) + 5u + 10v + 15(6 - u - v) \geq 163,$$

whence $x + 2u + v \leq 7$. Now we have the chain of inequalities

$$224 \geq 32x + 64u + 32v \geq 8x + 64u + 19v \geq 219.$$

This implies that $x = v = 0$, which in turn gives $224 \geq 64u \geq 219$, a contradiction since u is an integer. □

Now using once more the same idea we can prove the nonexistence of $(104, 22)$-arcs.

Theorem 7. *There is no* $(104, 22)$*-arc in* $\mathrm{PG}(3, 5)$.

Proof. We apply the above technique to the three non-isomorphic $(22, 5)$-arcs. Their spectra are given below.

Type	a_0	a_1	a_2	a_3	a_4	a_5
D1	1	0	1	0	15	14
D2	1	0	0	3	12	15
D3	0	0	3	4	6	18

Let H_0 be a fixed 22-plane. For a line L in H_0 we have the following possibilities:

$\mathcal{K}(L)$	$\widetilde{\mathcal{K}}(\widetilde{L})$	η_i	$(\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5))$
5	3	3	(22,22,22,22,22,19)
4	3	28	(22,22,22,22,22,14)
4	8	7	(22,22,22,22,22,14)
3	3	36	(22,22,22,22,16,15)
3	8	32	(22,22,22,20,19,14)
3	13	13	(22,20,20,19,19,19)
2	3	45	(22,22,22,16,16,16)
2	8	57	(22,22,22,20,14,14)
2	13	37	(22,21,19,19,19,14)
0	8	86	(22,22,16,16,14,14)
0	13	87	(22,21,19,14,14,14)

(D1) Denote by x the number of lines L in H_0 of multiplicity 4 for which $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$. Counting the contribution of the planes through the different lines in H_0 to the left-hand side of (5.1), we get

$$14 \cdot 3 + 28x + 7(15 - x) + 1 \cdot 57 + 1 \cdot 87 \geq 468,$$

whence $21x \geq 177$, i.e. $x \geq 9$. On the other hand,

$$|\widetilde{\mathcal{K}}| \leq 14 \cdot 3 + x \cdot 3 + (15 - x) \cdot 8 + 13 = 13,$$

whence $|\widetilde{\mathcal{K}}| \leq 188 - 5x$. This implies $188 - 5x \geq 163$, i.e. $x \leq 5$, a contradiction.

(D2) Denote by x the number of 4-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$; by u – the number of 3-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$, and by v – the number of 3-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$. Again counting the contribution to the left-hand side of (5.1), we have

$$15 \cdot 3 + x \cdot 28 + (12 - x) \cdot 7 + u \cdot 36 + v \cdot 32 + (3 - u - v)12 + 1 \cdot 87 \geq 468,$$

whence $21x + 24u + 20v \geq 216$. On the other hand,

$$|\widetilde{\mathcal{K}}| = 15 \cdot 3 + 3x + 8(12 - x) + 3u + 8v + (3 - u - v) \cdot 13 + 13 \geq 163,$$

$x + 2u + v \leq 6$. Now we get

$$126 \geq 21x + 42u + 21v \geq 21x + 24u + 20v \geq 216,$$

a contradiction.

(D3) Let x, u and v be as above. Denote also by s the number of 2-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 3$, and by t – the number of 2-lines L with $\widetilde{\mathcal{K}}(\widetilde{L}) = 8$. Once again:

$$18 \cdot 3 + x28 + (6 - x) \cdot 7 + u36 + v32 + (4 - u - v)12 + s \cdot 45 + t \cdot 57 + (3 - s - t)37 \geq 468,$$

whence $21x + 24u + 20v + 8s + 20t \geq 213$. On the other hand

$$|\widetilde{\mathcal{K}}| = 18 \cdot 3 + 3x + 8(6 - x) + 3u + 8v + 13(4 - u - v) + 3s + 8t + (13(3 - s - t) \geq 163,$$

hence $x + 2u + v + 2s + t \leq 4$. This implies

$$84 \geq 21x + 42u + 21v + 42s + 21t \geq 21x + 24u + 21v + 42s + 21t \geq 213,$$

a contradiction. □

Corollary 1. *There exists no* $[104, 4, 82]_5$*-code and, consequently,* $n_5(4, 82) = 105$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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METRIC CONNECTIONS ON ALMOST COMPLEX NORDEN METRIC MANIFOLDS

MARTA TEOFILOVA

Two families of metric connections on almost complex Norden metric manifolds are introduced and studied. These connections are constructed by means of the two Lie 1-forms naturally existing on the manifolds. Invariant tensors under the transformations of the Levi-Civita connection into the introduced metric connections are obtained.

Keywords: Metric connection, complex manifold, Norden metric.

2000 Math. Subject Classification: 53C15, 53C50.

1. INTRODUCTION

Linear connections with non-vanishing torsion tensor are widely studied. In particular, two types of such connections are well known – semi-symmetric and quarter-symmetric, introduced by Friedmann and Schouten in [1] and Golab in [5], respectively. The torsion tensors of such connections are constructed by means of a 1-form and a tensor of type (1,1).

On the other hand, an object of extensive research on pseudo Riemannian manifolds are linear connections which preserve the metric tensor by covariant differentiation called metric connections. It is well known that the Levi-Civita connection of the pseudo Riemannian metric is the unique linear connection which is simultaneously metric and symmetric (i.e. torsion-free). Metric connections with non-zero torsion tensor are introduced by Hayden in [7].

By combining these both ideas, numerous authors studied semi-symmetric metric and quarter-symmetric metric connections, e.g. [8], [10], [12], [13], [17], [18]. Tripathi [16] generalized the concept of various metric and non-metric connections.

In the present work, we aim to study metric connections on almost complex Norden metric (B-metric) manifolds. These manifolds are introduced by Norden in [11] and studied for the first time in [6] under the name generalized B-manifolds. Since on such manifolds, there exist two Lie 1-forms, they can be used to generate metric connections with torsion tensors of special types, e.g. semi-symmetric, quarter-symmetric or others.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we construct a 4-parametric family of metric connection on almost complex Norden metric manifolds with non-vanishing Lie 1-forms. These connections are composed by two semi-symmetric and two quarter-symmetric metric connections. We obtain necessary and sufficient conditions for the introduced connections to be invariant under the transformation of the Levi-Civita connections of the Norden metrics on a class complex Norden metric manifolds. Also, we consider tensors which under certain conditions are invariant under the transformation of the Levi-Civita connection into the constructed metric connections. In Section 4 we introduce a 2-parametric family of metric connections which are neither semi-symmetric nor quarter-symmetric and study some of their curvature properties.

2. PRELIMINARIES

A triple (M, J, g) is called an almost complex Norden metric (B-metric) manifold [6, 11] if M is a differentiable even-dimensional manifold ($\dim M = 2n$), J is an endomorphism of the tangent bundle TM, and g is a pseudo Riemannian metric on M, compatible with J, such that the following relations are satisfied:

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y). \tag{2.1}$$

Here and further on, by x, y, z, u we denote differentiable vector fields on M, i.e. elements in the Lie algebra $\mathfrak{X}(M)$, or vectors in the tangent space T_pM at an arbitrary point $p \in M$.

Equalities (2.1) imply $g(Jx, y) = g(x, Jy)$. Hence the tensor $\widetilde{g}$ defined by

$$\widetilde{g}(x, y) = g(x, Jy) \tag{2.2}$$

is symmetric and is known as the associated (twin) metric of g. This tensor also satisfies the Norden metric property, i.e. $\widetilde{g}(Jx, Jy) = -\widetilde{g}(x, y)$. Both metrics, g and $\widetilde{g}$, are necessarily of neutral signature (n, n).

The fundamental tensor F of type (0,3) is defined by

$$F(x, y, z) = (\nabla_x\widetilde{g})(y, z) = g((\nabla_xJ)y, z), \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. This tensor has the following properties $F(x, y, z) = F(x, z, y) = F(x, Jy, Jz)$.

Let $\{e_i\}$ $(i = 1, 2, ..., 2n)$ be an arbitrary basis of T_pM. The components of the inverse matrix of g with respect to this basis are denoted by g^{ij}. The Lie 1-forms θ and θ^* associated with F are defined by:

$$\theta(x) = g^{ij}F(e_i, e_j, x), \qquad \theta^*(x) = \theta(Jx). \tag{2.4}$$

We denote by Ω the Lie vector corresponding to θ, i.e. $\theta(x) = g(x, \Omega)$. Then, the vector $\Omega^* = J\Omega$ corresponds to θ^*.

A classification of the almost complex Norden metric manifolds with respect to the properties of F is introduced by Ganchev and Borisov in [2]. This classification consists of 8 classes: 3 basic classes $\mathcal{W}_i$ $(i = 1, 2, 3)$, their pairwise direct sums $\mathcal{W}_i \oplus \mathcal{W}_j$, the widest class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ and the class of the Kähler Norden metric manifolds $\mathcal{W}_0$ which is contained in all of the other classes and is characterized by $F = 0$ (i.e. $\nabla J = 0$). Two of the basic classes ($\mathcal{W}_1$ and $\mathcal{W}_2$) are integrable, i.e. with a vanishing Nijenhuis tensor (complex Norden metric manifolds). One of the integrable classes is said to be a main class because its characteristic condition is an explicit expression of F by means of the other structural tensors. This class is denoted by $\mathcal{W}_1$ and is defined by the condition

$$\begin{aligned} F(x, y, z) = \tfrac{1}{2n}[&g(x, y)\theta(z) + g(x, Jy)\theta(Jz) \\ &+ g(x, z)\theta(y) + g(x, Jz)\theta(Jy)]. \end{aligned} \tag{2.5}$$

A $\mathcal{W}_1$-manifold with closed Lie 1-forms θ and θ^* is called a conformal Kähler Norden metric manifold. The manifolds in this class are conformally equivalent to Kähler Norden metric manifolds by the usual conformal transformation of the metric g [3].

Since ∇ is symmetric, the 1-forms θ and θ^* are closed if and only if $(\nabla_x\theta)y = (\nabla_y\theta)x$ and $(\nabla_x\theta^*)y = (\nabla_y\theta^*)x$. By (2.3) and (2.4) it it easy to compute that $(\nabla_x\theta^*)y = (\nabla_x\theta)Jy + F(x, y, \Omega)$. Then, because of (2.5), a necessary and sufficient condition for the Lie 1-forms θ and θ^* to be closed on a $\mathcal{W}_1$-manifold is given by

$$(\nabla_x\theta)y = (\nabla_y\theta)x, \qquad (\nabla_x\theta)Jy = (\nabla_y\theta)Jx. \tag{2.6}$$

The Lie 1-forms θ and θ^* vanish on the manifolds in the classes $\mathcal{W}_0$, $\mathcal{W}_2$, $\mathcal{W}_3$ [2]. Hence, the widest class with zero Lie 1-forms is $\mathcal{W}_2 \oplus \mathcal{W}_3$.

3. SEMI-SYMMETRIC AND QUARTER-SYMMETRIC METRIC CONNECTIONS ON ALMOST COMPLEX NORDEN METRIC MANIFOLDS

Let us recall some basic definitions.

Definition 1. A linear connection ∇' on an almost complex Norden metric manifold (M, J, g) is called: (i) *almost complex* if $\nabla'J = 0$ (if the manifold is complex, a connection with this property is called *complex*); (ii) *metric* if $\nabla'g = 0$; (iii) *natural* if $\nabla'J = \nabla'g = 0$.

Because of (2.2), it is easy to prove that the defining condition in (iii) is equivalent to $\nabla' J = \nabla' \widetilde{g} = 0$ and also to $\nabla' g = \nabla' \widetilde{g} = 0$.

Definition 2. A non-symmetric linear connection ∇' with torsion tensor T is called: (i) *semi-symmetric* if $T(x,y) = \pi(y)x - \pi(x)y$; (ii) *quarter-symmetric* if $T(x,y) = \pi(y)\varphi x - \pi(x)\varphi y$, where π is a 1-form and φ is a tensor of type (1,1). In particular, if $\varphi = \mathrm{id}$, i.e. $\varphi x = x$ for all $x \in \mathfrak{X}(M)$, then a quarter-symmetric connection reduces to a semi-symmetric connection.

Further in this section, we study metric connections with torsion tensors of special types, namely semi-symmetric and quarter-symmetric ones.

Let us consider a linear connection ∇' with deformation tensor Q defined by

$$\nabla'_x y = \nabla_x y + Q(x,y). \tag{3.1}$$

Since $\nabla g = 0$, we compute $(\nabla'_x g)(y,z) = -Q(x,y,z) - Q(x,z,y)$, where $Q(x,y,z) = g(Q(x,y),z)$ is the deformation (0,3)-type tensor. Then, a necessary and sufficient condition for the connection ∇' to be metric is given by

$$Q(x,y,z) = -Q(x,z,y). \tag{3.2}$$

Because ∇ is symmetric, i.e. $\nabla_x y - \nabla_y x = [x,y]$, the torsion tensor T of ∇' is given by $T(x,y) = \nabla'_x y - \nabla'_y x - [x,y] = Q(x,y) - Q(y,x)$. Then, its corresponding (0,3)-type has the form $T(x,y,z) = g(T(x,y),z) = Q(x,y,z) - Q(y,x,z)$.

It is known that ∇' is a metric connection if and only if the following relation between its deformation tensor Q and its torsion tensor T exists

$$Q(x,y,z) = \tfrac{1}{2}\left[T(x,y,z) + T(z,y,x) - T(x,z,y)\right]. \tag{3.3}$$

Let ∇' be a semi-symmetric metric connection generated by the 1-form π, i.e. its torsion tensor is given by $T(x,y,z) = \pi(y)g(x,z) - \pi(x)g(y,z)$. Then, by applying the last equality to (3.3) we obtain the form of the deformation tensor Q of such a connection as follows [17]

$$Q(x,y,z) = \pi(y)g(x,z) - \pi(z)g(x,y). \tag{3.4}$$

We remark that the last formula is valid on an arbitrary pseudo Riemannian manifold with metric tensor g and 1-form π.

Next, we consider the case of a quarter-symmetric metric connection generated by the 1-form π and the almost complex structure J on an almost complex Norden metric manifold. Because of (2.1) and Definition 2, the torsion tensor T of such a connection is given by $T(x,y,z) = \pi(y)g(x,Jz) - \pi(x)g(y,Jz)$. Then, by (3.3) we obtain the form of the deformation tensor of a quarter-symmetric metric connection as follows

$$Q(x,y,z) = \pi(y)g(x,Jz) - \pi(z)g(x,Jy). \tag{3.5}$$

Let us remark that the last formula differs in the case of an almost Hermitian manifold because on such manifolds the tensor $\Phi(x,y)=g(Jx,y)$ is a 2-form.

Since on almost complex Norden metric manifolds which are not in the class $\mathcal{W}_2\oplus\mathcal{W}_3$, there exist two non-vanishing Lie 1-forms θ and θ^*, they can be used to generate semi-symmetric and quarter-symmetric metric connections. Four metric connections of these types can be constructed – two generated by θ (one semi-symmetric and one quarter-symmetric) and two of the same types generated by θ^*. Then, if we compose an arbitrary linear combination of the expressions in the right-hand sides of (3.4) and (3.5), and by doing so replace π with θ and θ^*, we obtain a 4-parametric family of metric connections on such manifolds. Thus, we proved the following statement.

Theorem 1. *On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, there exists a 4-parametric family ($\lambda_i\in\mathbb{R}$, $i=1,2,3,4$) of metric connections ∇' defined by (3.1) with deformation tensor Q given by*

$$\begin{aligned} Q(x,y)=&\lambda_1[\theta(y)x-g(x,y)\Omega]+\lambda_2[\theta(Jy)x-g(x,y)J\Omega]\\ &+\lambda_3[\theta(y)Jx-g(x,Jy)\Omega]+\lambda_4[\theta(Jy)Jx-g(x,Jy)J\Omega]. \end{aligned} \tag{3.6}$$

Let us have a more detailed look at the four metric connections which give rise to the family ∇' and consider if they can be natural.

If $\lambda_1\neq 0$, $\lambda_i=0$ $(i=2,3,4)$, we obtain a 1-parametric family of semi-symmetric metric connections ∇^1 generated by θ. These connections cannot be complex for any values of λ_1.

If $\lambda_2\neq 0$, $\lambda_i=0$ $(i=1,3,4)$, we obtain a 1-parametric family of semi-symmetric metric connections ∇^2 generated by θ^*. Because of formula (2.5), the connections ∇^2 are complex and hence natural on $\mathcal{W}_1$-manifolds if and only if $\lambda_2=\frac{1}{2n}$.

If $\lambda_3\neq 0$, $\lambda_i=0$ $(i=1,2,4)$, we obtain a 1-parametric family of quarter-symmetric metric connections ∇^3 generated by θ. Because of (2.5), these connections are complex and hence natural on $\mathcal{W}_1$-manifolds if and only if $\lambda_3=-\frac{1}{2n}$.

If $\lambda_4\neq 0$, $\lambda_i=0$ $(i=1,2,3)$, we obtain a 1-parametric family of quarter-symmetric metric connections ∇^4 generated by θ^*. The connections ∇^4 cannot be complex for any values of λ_4.

Let us remark that the connections ∇^2 and ∇^3 for $\lambda_2=\frac{1}{2n}$ and $\lambda_3=-\frac{1}{2n}$, respectively, i.e. $\nabla^{2n}_x y=\nabla_x y+\frac{1}{2n}[\theta(Jy)x-g(x,y)J\Omega]$ and $\nabla^{3n}_x y=\nabla_x y-\frac{1}{2n}[\theta(y)Jx-g(x,Jy)\Omega]$, are part of a 2-parametric family of natural connections ∇^n introduced and studied on $\mathcal{W}_1$-manifolds in [15]. These connections are defined by

$$\begin{aligned} \nabla^n_x y=\nabla_x y+\tfrac{1-2p}{2n}\,[\theta(Jy)x-g(x,y)J\Omega]+\tfrac{1}{n}\,\{p\,[g(x,Jy)\Omega-\theta(y)Jx]\\ +q\,[g(x,y)\Omega-g(x,Jy)J\Omega-\theta(y)x+\theta(Jy)Jx]\}\,,\quad p,q\in\mathbb{R}. \end{aligned} \tag{3.7}$$

The connections in question are obtained from (3.7) for $p=q=0$ and $p=\frac{1}{2}$, $q=0$, respectively. In the same work, we discussed that the B-connection ∇^c

[3, 4], known as Lichnerowicz first canonical connection in the Hermitian geometry, is also a member of the family (3.7) and is obtained for $p = \frac{1}{4}$, $q = 0$. Then, the B-connection is the average connection of ∇^{2n} and ∇^{3n}, i.e. $\nabla^c = \frac{1}{2}(\nabla^{2n} + \nabla^{3n})$.

3.1. INVARIANT METRIC CONNECTIONS UNDER THE TRANSFORMATION OF THE LEVI-CIVITA CONNECTIONS GENERATED BY THE NORDEN METRICS

Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of the metric tensors g and $\widetilde{g}$, respectively. The transformation of these connections $\nabla \to \widetilde{\nabla}$ is studied on $\mathcal{W}_1$-manifolds in [14]. In the same work, the following relations are obtained:

$$\widetilde{\nabla}_x y = \nabla_x y + \frac{1}{2n}[g(x, Jy)\Omega - g(x, y)J\Omega], \qquad \widetilde{\theta} = \theta, \quad \widetilde{\Omega} = -J\Omega, \tag{3.8}$$

where $\widetilde{\theta}$ and $\widetilde{\Omega}$ are the Lie 1-form and its corresponding vector defined analogously to (2.4). The general case of an almost complex Norden metric manifold is studied in [9], where it is proved that all classes of these manifolds are invariant under the considered transformation.

Then, on a $\mathcal{W}_1$-manifold $(M, J, \widetilde{g})$ corresponding to a $\mathcal{W}_1$-manifold (M, J, g) one can construct a 4-parametric family of metric connections $\widetilde{\nabla}'$ (i.e. with the property $\widetilde{\nabla}'\widetilde{g} = 0$) defined analogously to ∇' by replacing ∇, g and θ with $\widetilde{\nabla}$, $\widetilde{g}$ and $\widetilde{\theta}$, respectively, in (3.1) and (3.6).

The following theorem is valid.

Theorem 2. *The metric connections ∇' defined by (3.1) and (3.6) are invariant under the transformation $\nabla \to \widetilde{\nabla}$ on $\mathcal{W}_1$-manifolds if and only if they coincide with the natural connections ∇^n defined by (3.7).*

Proof. Let ∇' be invariant under the transformation $\nabla \to \widetilde{\nabla}$, i.e. $\widetilde{\nabla}' = \nabla'$. Then, both metric tensors g and $\widetilde{g}$ are parallel with respect to ∇' and $\widetilde{\nabla}'$. Because of (2.2), this means that the connections ∇' and $\widetilde{\nabla}'$ are metric and complex and hence natural. By (3.1), (3.6) and (2.3) we compute

$$\begin{aligned}(\nabla'_x J)y =&(\nabla_x J)y + Q(x, Jy) - JQ(x, y)\\ =&(\lambda_1 + \lambda_4)[g(x, y)J\Omega - g(x, Jy)\Omega + \theta(Jy)x - \theta(y)Jx]\\ &+ (\lambda_3 - \lambda_2 + \frac{1}{2n})[g(x, y)\Omega + g(x, Jy)J\Omega + \theta(y)x + \theta(Jy)Jx].\end{aligned}$$

The last equality implies that $\nabla' J = 0$ if and only if $\lambda_2 = \lambda_3 + \frac{1}{2n}$ and $\lambda_1 = -\lambda_4$. In this case, by substituting $\lambda_3 = -\frac{p}{n}$ and $\lambda_4 = \frac{q}{n}$ in (3.6) we establish that ∇' coincide with ∇^n.

To prove the reverse statement, by (3.7) and (3.8) we obtain $\widetilde{\nabla}^n$ and verify that $\widetilde{\nabla}^n = \nabla^n$, i.e. ∇^n and hence ∇' are invariant under the transformation $\nabla \to \widetilde{\nabla}$. □

Let us consider the covariant derivatives of θ and θ^* with respect to the metric connections ∇'. Because of (3.1), we compute: $(\nabla'_x\theta)y = (\nabla_x\theta)y - Q(x, y, \Omega)$ and $(\nabla'_x\theta^*)y = (\nabla_x\theta^*)y - Q(x, y, J\Omega)$. Thus, we have:

$$\begin{aligned}(\nabla'_x\theta)y - (\nabla'_y\theta)x &= (\nabla_x\theta)y - (\nabla_y\theta)x + Q(y, x, \Omega) - Q(x, y, \Omega),\\ (\nabla'_x\theta^*)y - (\nabla'_y\theta^*)x &= (\nabla_x\theta^*)y - (\nabla_y\theta^*)x + Q(y, x, J\Omega) - Q(x, y, J\Omega).\end{aligned} \tag{3.9}$$

We introduce the following (0,2)-type tensors:

$$S(x, y) = (\nabla_x\theta)y - (\nabla_y\theta)x, \qquad S_*(x, y) = (\nabla_x\theta^*)y - (\nabla_y\theta^*)x. \tag{3.10}$$

By replacing ∇ with ∇' in (3.10), we obtain analogous tensors to S and S_* with respect to the connections ∇' which we denote by S' and S'_*. Then, equalities (3.9) yield that $S = S'$ and $S_* = S'_*$ if and only if $Q(x, y, \Omega) = Q(y, x, \Omega)$ and $Q(x, y, J\Omega) = Q(y, x, J\Omega)$, respectively. Having in mind (3.6), the last two conditions hold if and only if $\lambda_3 = \lambda_2$ and $\lambda_4 = -\lambda_1$, respectively. Hence, we proved the following result.

Theorem 3. *The tensors S and S_* given by (3.10) are invariant under the transformation of the Levi-Civita connection ∇ into the 4-parametric family of metric connections ∇' defined by (3.1) and (3.6) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^* if and only if the parameters satisfy the following conditions: $\lambda_3 = \lambda_2$ and $\lambda_4 = -\lambda_1$.*

Under the assumptions of the last theorem we obtain

Corollary 1. *The invariant tensors S and S_* vanish on a $\mathcal{W}_1$-manifold if and only if both Lie 1-forms θ and θ^* are closed, i.e. the manifold is conformal Kählerian.*

Further, let us consider another tensor of type (0,2) constructed by the covariant derivatives of θ and θ^* which is defined by

$$P(x, y) = (\nabla_x\theta)Jy + (\nabla_x\theta^*)y. \tag{3.11}$$

The analogous tensor P' with respect to the metric connections ∇' is given by $P'(x, y) = (\nabla'_x\theta)Jy + (\nabla'_x\theta^*)y = (\nabla_x\theta)Jy + (\nabla_x\theta^*)y - Q(x, y, J\Omega) - Q(x, Jy, \Omega)$. Hence, $P'(x, y) = P(x, y) - Q(x, Jy, \Omega) - Q(x, y, J\Omega)$, and therefore $P' = P$ if and only if

$$Q(x, Jy, \Omega) = -Q(x, y, J\Omega). \tag{3.12}$$

By applying (3.6) to the last equality we prove the following

Theorem 4. *The tensor P is invariant under the transformation of the Levi-Civita connection ∇ into the 4-parametric family of metric connections ∇' defined by (3.1) and (3.6) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^* if and only if the parameters satisfy the following conditions: $\lambda_3 = -\lambda_2$ and $\lambda_4 = \lambda_1$.*

If the assumptions of the last theorem are satisfied, by (2.5) we get

Corollary 2. *The invariant tensor P defined by (3.11) vanishes on a $\mathcal{W}_1$-manifold if and only if*

$$(\nabla_x\theta)y = \frac{1}{4n}\{g(x,Jy)\theta(\Omega) - g(x,y)\theta(J\Omega) + \theta(x)\theta(Jy) - \theta(Jx)\theta(y)\}. \quad (3.13)$$

Equalities (2.6) and (3.13) yield that if P vanishes on a $\mathcal{W}_1$-manifold, the Lie 1-form θ^* is closed but θ is not closed. Also, if $P = 0$ the following relations are valid: $\operatorname{div}\Omega = -\frac{\theta(J\Omega)}{2}$ and $\operatorname{div}(J\Omega) = \frac{\theta(\Omega)}{2}$, where $\operatorname{div}\Omega = g^{ij}(\nabla_{e_i}\theta)e_j$ and $\operatorname{div}(J\Omega) = g^{ij}(\nabla_{e_i}\theta^*)e_j$.

4. OTHER TYPES OF METRIC CONNECTIONS CONSTRUCTED BY THE LIE 1-FORMS

In this section, by means of the Lie 1-forms θ and θ^* and their corresponding Lie vectors Ω and $J\Omega$, we construct a 2-parametric family of metric connection which are neither semi-symmetric, nor quarter-symmetric.

On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, let us consider a family of metric connections $\widehat{\nabla}$ defined by (3.1) with torsion tensors T given by

$$T(x,y) = [\theta(x)\theta(Jy) - \theta(Jx)\theta(y)][s\Omega + tJ\Omega], \quad s,t \in \mathbb{R}. \quad (4.1)$$

Then, by (3.3) and (4.1) we obtain the form of the deformation tensor Q of the metric connections $\widehat{\nabla}$. Thus, we establish the truthfulness of the following theorem.

Theorem 5. *On an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*, there exists a 2-parametric family ($s,t \in \mathbb{R}$) of metric connections $\widehat{\nabla}$ defined by*

$$\widehat{\nabla}_x y = \nabla_x y + Q(x,y), \quad Q(x,y) = [s\theta(x) + t\theta(Jx)][\theta(Jy)\Omega - \theta(y)J\Omega]. \quad (4.2)$$

Let R be the curvature tensor of the Levi-Civita connection ∇, i.e. $R(x,y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x,y]}z$. The corresponding tensor of type (0,4) is denoted by the same letter and is defined by $R(x,y,z,u) = g(R(x,y)z,u)$. Analogously, let us denote by $\widehat{R}$ the curvature tensor of $\widehat{\nabla}$ and define its corresponding (0,4)-tensor with respect to g by $\widehat{R}(x,y,z,u) = g(\widehat{R}(x,y)z,u)$. Then, the following relation between R and $\widehat{R}$ is known to be valid: $\widehat{R}(x,y,z,u) = R(x,y,z,u) + (\nabla_x Q)(y,z,u) - (\nabla_y Q)(x,z,u) + Q(x,Q(y,z),u) - Q(y,Q(x,z),u)$. If we apply (4.2) to the last formula and take into account (2.6), we obtain the following

Theorem 6. *On a conformal Kähler Norden metric manifold, the tensors R $\widehat{R}$ of ∇ and $\widehat{\nabla}$, respectively, are related as follows*

$$\widehat{R}(x,y,z,u) = R(x,y,z,u) + [s\theta(y) + t\theta(Jy)][\theta(u)(\nabla_x\theta^*)z - \theta(z)(\nabla_x\theta^*)u$$
$$+\theta^*(z)(\nabla_x\theta)u - \theta^*(u)(\nabla_x\theta)z] - [s\theta(x) + t\theta(Jx)][\theta(u)(\nabla_y\theta^*)z - \theta(z)(\nabla_y\theta^*)u$$
$$+\theta^*(z)(\nabla_y\theta)u - \theta^*(u)(\nabla_y\theta)z].$$

Let us remark that $\widehat{R}(x,y,z,u)$ is anti-symmetric by its first and last pair of arguments, but since it does not satisfy Bianchi's first identity it is not a curvature-like tensor of type (0,4).

Having in mind the definitions of the scalar curvatures τ and $\widehat{\tau}$ of R and $\widehat{R}$, respectively, i.e. $\tau = g^{is}g^{jk}R(e_i,e_j,e_k,e_s)$ and $\widehat{\tau} = g^{is}g^{jk}\widehat{R}(e_i,e_j,e_k,e_s)$, from Theorem 6 we obtain $\widehat{\tau} = \tau + \frac{2s}{n}[\theta(\Omega)^2 + \theta(J\Omega)^2] + 2\mathrm{div}\,\Omega\,[s\theta(J\Omega) - t\theta(\Omega)] - 2\mathrm{div}\,(J\Omega)\,[s\theta(\Omega) + t\theta(J\Omega)]$.

Let us consider the transformation of the Levi-Civita connection ∇ into the metric connections $\widehat{\nabla}$ and the tensor P defined in the previous section by (3.11). Having in mind (4.2), we establish that the deformation tensor Q of $\widehat{\nabla}$ satisfies property (3.12) for all $s,t \in \mathbb{R}$. Thus, we proved

Theorem 7. *The tensor P given by (3.11) is invariant under the transformation of the Levi-Civita connection ∇ into the 2-parametric family of metric connections $\widehat{\nabla}$ defined by (4.2) on an almost complex Norden metric manifold with non-vanishing Lie 1-forms θ and θ^*.*

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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LOOPS RELATED TO REFLECTION GEOMETRIES

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1. INTRODUCTION

In the book "Theory of K-Loops" by Hubert Kiechle we find the following construction theorem for left-loops and loops (cf. [5] (2.7)):

Theorem 1. *Let* $(G, \cdot)$ *be a group,* 1 *the neutral element,* U *a subgroup of* G *and let* $L \subseteq G$ *be a transversal of the pair* (G, U)*, i.e.*

(T1) $\forall x \in G \ \exists_1 \bar{x} \in L$ *with* $\bar{x} \cdot U = x \cdot U$.

(T2) $1 \in L$.

For $t, s \in L$ *let* $t \oplus s := \overline{t \cdot s}$*. Then* $(L, \oplus)$ *is a left loop (called derived left-loop), i.e.*

1) $\forall l \in L : 1 \oplus l = l \oplus 1 = l$.

2) $\forall s, t \in L, \exists_1 x \in L$ *with* $s \oplus x = t$.

$(L, \oplus)$ *is a loop (i.e. also the equation* $x \oplus s = t$ *has an unique solution) if and only if the set* L *satisfies the condition (T3) - stronger as (T1) - (then* L *is called L-transversal).*

(T3) $\forall x, y \in G : |L \cap x \cdot U \cdot y| = 1$.

The next theorem tells us that any left-loop can be obtained by the method of Theorem 1 (cf. [5] (2.6), (2.7)):

Theorem 2. *Let* $(L,+)$ *be a left-loop, for* $a \in L$ *let* $a^+ : L \to L\ ;\ x \mapsto a+x$, $L^+ := \{a^+ \mid a \in L\}$, *let* $G :=< L^+ >$ *be the group generated by the left-translations* a^+ *and let* $U := \{\xi \in G \mid \xi(o) = o\}$. *Then* L^+ *is a transversal of* (G,U) *and the derived left-loop is isomorphic to* $(L,+)$.

Under the notion "reflection geometry" or "Sperner plane" one finds all absolute planes. To any reflection geometry $(G,\mathfrak{D},\mathfrak{B})$ we associate firstly a so called *kinematic fibration* $\mathfrak{F}$ of the group $\mathfrak{D}^2$ and then a *kinematic space* $(\mathfrak{D}^2,\mathfrak{G},\cdot)$. To certain subgroups $F \in \mathfrak{F}$ we find in form of a plane $< \varepsilon >$ of the kinematic space $(\mathfrak{D}^2,\mathfrak{G},\cdot)$ a transversal of (F). With this method we can associate to each reflection geometry in a natural way loops $(L,\oplus)$, which shall be studied in this paper. Also in the paper [6] by S. Pasotti, S. Pianta and E. Zizioli we find constructions of loops related to hyperbolic planes using transversals.

2. REFLECTION GROUPS , REFLECTION GEOMETRIES AND SPERNER PLANES

We recall some notions and facts taken from [1], [3] and [4]. Let $(G,\cdot)$ be a group, $J := \{\gamma \in G \mid \gamma^2 = id \neq \gamma\}$ and $\mathfrak{D} \subseteq J$ such that $< \mathfrak{D} >= G$, i.e. $\mathfrak{D}$ is a system of generators of G.

A subset $\mathfrak{b} \subseteq \mathfrak{D}$ is called *pencil* if there are $A,B \in \mathfrak{D}$, $A \neq B$ with $\mathfrak{b} = \overbrace{A,B} := \{X \in \mathfrak{D} \mid A\cdot B\cdot X \in J\}$. Let $\mathfrak{B}$ be the set of all pencils. A pencil $\mathfrak{b}$ is called *proper* or also *projective* if for all $\mathfrak{x} \in \mathfrak{B}$, $\mathfrak{b} \cap \mathfrak{x} \neq \emptyset$. Let $\mathfrak{B}_o$ be the set of all proper pencils.

We claim (*Three reflection Axiom*):

(S) If $\mathfrak{b}$ is a pencil and $A,B,C \in \mathfrak{b}$, then $A\cdot B\cdot C \in \mathfrak{D}$ and then if $\mathfrak{B}_o \neq \emptyset$, we call the pair $(G,\mathfrak{D})$ *reflection group.*

Proposition 1. If $(G,\mathfrak{D})$ is a reflection group then

1. $\mathfrak{D}^4 = \mathfrak{D}^2$, i. e. $\mathfrak{D}^2$ is a subgroup of G.
2. For $\mathfrak{b} \in \mathfrak{B}$ the set $\mathfrak{b}^2 := \{X\cdot Y \mid X,Y \in \mathfrak{b}\}$ is a commutative subgroup of $\mathfrak{D}^2$.
3. For $\mathfrak{b}_1,\mathfrak{b}_2 \in \mathfrak{B}$ we have: $\mathfrak{b}_1^2 \cap \mathfrak{b}_2^2 = \{1\} \iff \mathfrak{b}_1 \neq \mathfrak{b}_2$.
4. For $\xi \in G$ and $\mathfrak{b} \in \mathfrak{B}$ we have $\xi\cdot\mathfrak{b}\cdot\xi^{-1} \in \mathfrak{B}$.

To a reflection group $(G,\mathfrak{D})$ we associate the following geometric structure $(G,\mathfrak{D},\mathfrak{B})$ called *reflection geometry*: The elements of $\mathfrak{D}$ are called *lines*, of $\mathfrak{B}$ *points* and of $\mathfrak{B}_o$ *projective points.* The incidence between a line $L \in \mathfrak{D}$ and a point

$\mathfrak{b} \in \mathfrak{B}$ is given by $L \in \mathfrak{b}$. A reflection geometry $(G, \mathfrak{D}, \mathfrak{B})$ is called *Sperner plane* if every line $G \in \mathfrak{D}$ is incident with in at least three distinct projective points.

If in a reflection geometry there exist more than one pencil then there exist $A, B, C \in \mathfrak{D}$ with $A \cdot B \cdot C \neq C \cdot B \cdot A$.

Let $A, B \in \mathfrak{D}$ with $A \neq B$ and $\mathfrak{b} := \overbrace{A, B}$. We call A and B *orthogonal* and denote that by $A \perp B$ if $A \cdot B \in J$ and then $\mathfrak{b}$ is called an *orthogonal pencil.*

A Sperner plane is called *regular* if for every $A \in \mathfrak{D}$, the set $A^{\perp} := \{X \in \mathfrak{D} \mid X \perp A\}$ is a pencil and then $A^{\perp}$ is called the *pole* of the line A.

Let $\mathfrak{P} := \mathfrak{D}^2 \cap J$. If $p \in \mathfrak{P}$ then $\mathfrak{p} := \{X \in \mathfrak{D} \mid p \cdot X \in J\}$ is an orthogonal pencil. Hence the set of all orthogonal pencils can be identified with the set $\mathfrak{P}$ of all involutions contained in the group $\mathfrak{D}^2$.

Proposition 2. Let $(G, \mathfrak{D}, \mathfrak{B})$ be a Sperner plane and let $\alpha \in G$ then

1. If $(G, \mathfrak{D}, \mathfrak{B})$ is regular then $\mathfrak{B}_o \subseteq \mathfrak{P}$.
2. The map
$$\widetilde{\alpha} \;:\; G \;\rightarrow\; G \;:\; \xi \;\mapsto\; \alpha \cdot \xi \cdot \alpha^{-1}$$
maps lines onto lines, points onto points, preserves incidence and orthogonality hence $\widetilde{\alpha}$ is a motion of the Sperner plane.

If $\alpha \in \mathfrak{D}$ resp. $\alpha \in \mathfrak{P}$ then $\widetilde{\alpha}$ is called *line-reflection* resp. *point-reflection.* If to $a, b \in \mathfrak{P}$ there is a $m \in \mathfrak{P}$ with $\widetilde{m}(a) = b$ then m is called *midpoint* of a and b. A regular Sperner plane with $\mathfrak{B}_o = \mathfrak{P}$ is called *midpoint plane* if for any two distinct points there exists exactly one midpoint.

Proposition 3. Let $(G, \mathfrak{D}, \mathfrak{B})$ be a midpoint plane, let $o \in \mathfrak{P}$ be fixed and for $x \in \mathfrak{P}$ let x' be the midpoint of o and x and let $x^+ := \widetilde{x'} \circ \widetilde{o}$. If for $a, b \in \mathfrak{P}$

$$a \oplus b := a^+(b)$$

then $(\mathfrak{P}, \oplus)$ is a loop, even a K-loop.

3. THE KINEMATIC SPACE OF A SPERNER PLANE

Now we associate to a reflection group $(G, \mathfrak{D})$ a spatial structure. By 3. and 4. of Proposition 1 the set $\mathfrak{F} := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}\}$ forms a *kinematic fibration of the group* $\mathfrak{D}^2$, i.e.

1) $\bigcup \mathfrak{F} = \mathfrak{D}^2$.

2) $\forall U, V \in \mathfrak{F}, \quad U \cap V = \{1\}$ or $U = V$.

3) If $\xi \in \mathfrak{D}^2$ and $U \in \mathfrak{F}$ then $\xi \cdot U \cdot \xi^{-1} \in \mathfrak{F}$.

Therefore if we call the elements of $\mathfrak{D}^2$ *points* and of $\mathfrak{G} := \{\gamma \cdot F \mid \gamma \in \mathfrak{D}^2, F \in \mathfrak{F}\}$ *lines* then $(\mathfrak{D}^2, \mathfrak{G})$ is an incidence space and the triple $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ is called *kinematic space* (cf. [2]).

A subset $\Delta \subseteq \mathfrak{D}^2$ is called *subspace* if for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$ the line $\overline{\alpha, \beta}$ joining the points α and β is contained in Δ.

If $\alpha \in \mathfrak{D}^2 \setminus \{1\}$ let $[\alpha]$ denote the unique fiber of $\mathfrak{F}$ with $\alpha \in [\alpha]$.

By Proposition 8. of [4] we have

Proposition 4. $\alpha, \beta, \gamma \in \mathfrak{D}^2$ are collinear if and only if

$$\beta \cdot \alpha^{-1} \cdot \gamma = \gamma \cdot \alpha^{-1} \cdot \beta.$$

Let $\mathfrak{F}_o := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}_o\}$ the subset of fibers coming from proper pencils. The elements of the subset $\mathfrak{G}_o := \{\xi \cdot U \mid \xi \in \mathfrak{D}^2,\ U \in \mathfrak{F}_o\}$ are called *projective lines.*

For $\varepsilon \in \mathfrak{D}^3$ let $<\varepsilon> := \{\xi \in \mathfrak{D}^2 \mid \varepsilon \cdot \xi \in \mathfrak{D}\} = \varepsilon^{-1} \cdot \mathfrak{D}$.

By Proposition 10 of [4] we have

Proposition 5. Let $\epsilon \in \mathfrak{D}^3$, $F \in \mathfrak{F}_o$ a projective fiber, $\xi \in \mathfrak{D}^2,\ G := \xi \cdot F$ hence $G \in \mathfrak{G}_o$ then

1. $<\varepsilon> \cap F \neq \emptyset$.
2. $<\varepsilon> \ \cap G \neq \emptyset$.
3. If $F \subseteq <\varepsilon>$ and $G \neq F$ then $|<\varepsilon> \ \cap G| = 1$.
4. $<\varepsilon>$ is a plane.
5. If G is contained in the plane $<\varepsilon>$ then G meets any line $A \in \mathfrak{G}$ which is contained in the plane $<\varepsilon>$.

4. PARALLELISMS IN A KINEMATIC SPACE

Let $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ be the kinematic space belonging to the kinematic fibration $\mathfrak{F} := \{\mathfrak{b}^2 \mid \mathfrak{b} \in \mathfrak{B}\}$ of the group $\mathfrak{D}^2$. Then for $G \in \mathfrak{G}$, $G^{-1} \cdot G \in \mathfrak{F}$ and therefore we can define a left and a right parallelism. For $A, B \in \mathfrak{G}$ let

$$A \parallel_l B \Longleftrightarrow A^{-1} \cdot A = B^{-1} \cdot B, \qquad A \parallel_r B \Longleftrightarrow A \cdot A^{-1} = B \cdot B^{-1}.$$

If $A \in \mathfrak{G}$ and $\beta \in \mathfrak{D}^2$ then there is exactly one line $B := (\beta \parallel_l A)$ with $B \parallel_l A$ and $\beta \in B$, namely $B := (\beta \parallel_l A) = \beta \cdot A^{-1} \cdot A$.

By the last remarks of [4] we have:

Theorem 3. *Let $\varepsilon \in \mathfrak{D}^3$ and let $G \in \mathfrak{G}_o$ a projective line. Then:*

1. *If* $H \in \mathfrak{G}$ *and* $H \parallel_l G$ *or* $H \parallel_r G$ *then* $H \in \mathfrak{G}_o$.

2. *There is exactly one projective line denoted by* $(\varepsilon \parallel_l G)$ *resp.* $(\varepsilon \parallel_r G)$ *contained in* $< \varepsilon >$ *such that* $(\varepsilon \parallel_l G) \parallel_l G$ *resp.* $(\varepsilon \parallel_r G) \parallel_r G$.

3. *If* $L \in \mathfrak{G}$ *with* $L \parallel_l G$ *and* $L \neq (\varepsilon \parallel_l G)$ *or with* $L \parallel_r G$ *and* $L \neq (\varepsilon \parallel_r G)$ *then* $|L\cap < \varepsilon > | = 1$.

Theorem 4. *Let* $(G, \mathfrak{D}, \mathfrak{B})$ *be a Sperner plane, let* $\mathfrak{b} \in \mathfrak{B}_o$ *be a proper pencil, let* $F := \mathfrak{b}^2$ *, let* $A \in \mathfrak{b}$ *and let* $T := (< A > \backslash F) \cup \{1\}$. *Then*

1. F *is a commutative subgroup of* $\mathfrak{D}^2$ *and a projective line of the corresponding kinematic space* $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$,

2. $< A >$ *is a plane of* $(\mathfrak{D}^2, \mathfrak{G}, \cdot)$ *with* $F \subseteq < A >$.

3. $\forall \alpha, \beta \in \mathfrak{D}^2,\ \alpha \cdot F \cdot \beta$ *is a projective line and* $\alpha \cdot F \cdot \beta \cap < A > \neq \emptyset$.

4. *For each* $\alpha \in \mathfrak{D}^2 \setminus F$ *the line* $(\alpha \parallel_l F) = \alpha \cdot F$ *meets the plane* $< A >$ *in exactly one point* $\bar{\alpha}$ *and we have* $\bar{\alpha} = \alpha \Leftrightarrow \alpha \in A$. *For* $\alpha \in F$ *let* $\bar{\alpha} := 1$.

5. T *is a transversal of* $(\mathfrak{D}^2, F)$ *hence* $(T, \oplus)$ *with* $\alpha \oplus \beta := \overline{\alpha \cdot \beta}$ *for* $\alpha, \beta \in T$ *is a left loop.*

6. T *is a L-transversal of* $(\mathfrak{D}^2, F)$ *hence* $(T, \oplus)$ *is a loop and* 1 *is the neutral element of* $(T, \oplus)$.

5. PROPERTIES OF THE LOOP OF A SPERNER PLANE

Let $(T, \oplus)$ be the loop corresponding to a Sperner plane according to Theorem 4. Then if $\alpha, \beta \in T$ and if $\alpha \cdot \beta \in T$ we obtain $\alpha \oplus \beta = \alpha \cdot \beta$. Now let $\alpha \in T \setminus \{1\}$ and $[\alpha] = \overline{1, \alpha}$. Then $[\alpha]$ is a subgroup of $\mathfrak{D}^2$ and $[\alpha] \subseteq T$. Hence on $[\alpha]$ coincide the operations $\cdot$ and $\oplus$ and we have:

Theorem 5. $(T, \oplus)$ *is a fibered loop with the fibration* $\mathfrak{F}' := \mathfrak{F} \setminus \{F\}$ *and each fiber* $X \in \mathfrak{F}'$ *is a commutative subgroup of the loop* $(T, \oplus)$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ACUTE TRIANGLES IN THE CONTEXT OF THE ILLUMINATION PROBLEM

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We consider strong *at*-subsets of the Euclidean space $\mathbf{R}^n$ and estimate from below the growth of the maximal cardinality of such subsets (our method essentially differs from that of [6]). We then apply some properties of strong *at*-sets to the illumination problem.

Keywords: *at*-set, strong *at*-set, illumination problem

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1. INTRODUCTION AND RESULTS

Let X be a subset of of the n-dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 2$.

We shall say that X is an *at*-set (in $\mathbf{R}^n$) if any three-element subset of X forms either an acute-angled triangle or a right-angled triangle.

We shall say that X is a strong *at*-set (in $\mathbf{R}^n$) if any three-element subset of X forms an acute-angled triangle.

It directly follows from the above definitions that each subset of an *at*-set (respectively, of a strong *at*-set) is also an *at*-set (respectively, a strong *at*-set).

It is natural to envisage the question concerning the maximal value among the cardinalities of *at*-subsets of $\mathbf{R}^n$.

Denote by $q(n)$ the maximum of the cardinalities of all *at*-sets in $\mathbf{R}^n$. Answering two questions posed by P. Erdös and V. L. Klee, it was demonstrated in the paper by Danzer and Grünbaum [5] that the inequality $q(n) \leq 2^n$ holds true. Moreover, the equality $\operatorname{card}(X) = 2^n$ for an *at*-set $X \subset \mathbf{R}^n$ is valid if and only if X coincides with the set of all vertices of some right rectangular n-dimensional parallelepiped in $\mathbf{R}^n$. Thus, one can directly see that $q(n)$ has an exponential growth with respect to n. For more details, see the above-mentioned paper [5] or [2] or Chapter 15 of the remarkable book [1].

Denote by $k(n)$ the maximum of the cardinalities of all strong *at*-sets in $\mathbf{R}^n$. It is easy to show that $k(2) = 3$ and it is also known that $k(3) = 5$. It immediately follows from the result of Danzer and Grünbaum [5] that one of the upper bounds for $k(n)$ is $2^n - 1$, i.e., one has the trivial inequality

$$k(n) \leq 2^n - 1.$$

In the general case the precise value of $k(n)$ is still unknown. However, it was proved that $k(n)$ also has an exponential growth with respect to n; in this connection, see [6] or Chapter 15 of the same book [1].

It should be noticed that in [1] and [6] an exponential growth of $k(n)$ is proved with the aid of a probabilistic argument which seems to be somewhat artificial in this case. Indeed, a deterministic proof of the same fact can be presented by using another approach. We would like to give below a sketch of a different proof of the same fact. The suggested proof is simple, purely combinatorial, and so does not rely on any facts from probability theory.

In what follows, the symbol V_n will stand for the set of all vertices of the unit cube $C_n = [0, 1]^n$ of the space $\mathbf{R}^n$, so we have $\operatorname{card}(V_n) = 2^n$. First of all, we are going to present a precise formula for the total number r_n of right-angled triangles whose vertices belong to V_n. Clearly, this number coincides with the total number of all right-angled triangles whose vertices belong to the set of vertices of any n-dimensional right rectangular parallelepiped P in $\mathbf{R}^n$.

Let t_n stand for the number of all right-angled triangles in C_n, the right angle of which is a fixed vertex v from V_n and the other two vertices also belong to V_n. Consider some facet C_{n-1} of C_n incident to v. Obviously, we have t_{n-1} right angles with the same vertex v, all of which lie in C_{n-1}. Further, each of the above-mentioned angles is a projection of exactly two right angles which do not lie in C_{n-1}. Besides, there are precisely $2^{n-1} - 1$ right angles, all of which have a fixed common side, namely, the edge of C_n passing through v and orthogonal to C_{n-1}.

Thus, we come to the following recurrence formula:

$$t_n = 3t_{n-1} + 2^{n-1} - 1.$$

This formula allows us to readily deduce (e.g., by induction) that

$$t_n = (3^n + 1)/2 - 2^n.$$

Therefore, ranging v over the whole of V_n, we finally get

$$r_n = 2^n((3^n + 1)/2 - 2^n).$$

As an immediate consequence of the above formula, we obtain that the total number of all those acute-angled triangles whose vertices belong to V_n is equal to

$$\frac{2^n!}{3!(2^n-3)!} - r_n = \frac{2^n!}{3!(2^n-3)!} - 2^n((3^n+1)/2 - 2^n).$$

Now, let us try to apply the formula for r_n in evaluating from below the function $k(n) = k$.

Let $X_1, X_2, \dots, X_p$ be an injective enumeration of all $(k+1)$-element subsets of V_n, so

$$p = \frac{2^n!}{(k+1)!(2^n-(k+1))!},$$

and let, for each natural index $i \in [1,p]$, the symbol a_i denote the number of the right-angled triangles in X_i. Since no X_i is a strong *at*-set, we obviously may write

$$1 \leq a_i \qquad (1 \leq i \leq p).$$

At the same time, it is clear that

$$a_1 + a_2 + \cdots + a_p = \frac{(2^n-3)!}{(k-2)!(2^n-3-(k-2))!} \cdot r_n.$$

The above equality is easily deduced if we consider the set of all pairs (Z, X_i), where X_i ranges over the family of all $(k+1)$-element subsets of V_n and Z is a three-element subset of X_i which forms a right-angled triangle. Calculating in two possible ways the cardinality of the set of all these pairs, we come to the required equality.

Now, since we have the trivial inequality

$$\frac{(2^n-3)!}{(k-2)!(2^n-3-(k-2))!} \leq \frac{(2^n)!}{(k-2)!(2^n-(k-2))!},$$

we infer that

$$a_1 + a_2 + \ldots + a_p \leq \frac{(2^n)!}{(k-2)!(2^n-(k-2))!} \cdot r_n.$$

Consequently,

$$\frac{2^n!}{(k+1)!(2^n-(k+1))!} \leq \frac{2^n!}{(k-2)!(2^n-(k-2))!} \cdot 2^n((3^n+1)/2 - 2^n).$$

The last inequality directly implies

$$(2^n-(k+1))^3 \leq (k+1)^3 \cdot 2^n((3^n+1)/2 - 2^n)$$

or, equivalently,

$$\frac{2^n}{1 + (2^n((3^n+1)/2 - 2^n))^{1/3}} \leq k+1.$$

Further, taking into account the two trivial relations

$$1 + (2^n((3^n + 1)/2 - 2^n))^{1/3} \leq 2 \cdot (2^n((3^n + 1)/2 - 2^n))^{1/3},$$

$$(3^n + 1)/2 - 2^n < 3^n,$$

we can conclude that

$$\frac{1}{2} \cdot (\frac{2}{6^{1/3}})^n \leq k + 1.$$

Since $2 > 6^{1/3}$, we see that $k + 1$ (and, consequently, $k = k(n)$) has an exponential growth with respect to n.

Remark 1. The argument presented above and the argument given in [6] are not effective in the sense that they do not allow one to indicate or geometrically describe any strong *at*-subset X of V_n whose cardinality is of an exponential growth with respect to n. In this connection, it would be interesting to have some concrete examples of such subsets X of V_n and to give their geometric characterization.

Remark 2. The notions of *at*-sets and of strong *at*-sets can be introduced for any Hilbert space H over the field $\mathbf{R}$ of all real numbers. In this more general situation the question concerning maximal cardinality of such sets also makes sense and deserves to be investigated. In particular, for an infinite-dimensional H the question is interesting from the purely set-theoretical view-point.

Strong *at*-sets in $\mathbf{R}^n$ are also of interest in connection with the well-known problem of illumination of the boundary of a compact convex body in $\mathbf{R}^n$. There is a rich literature devoted to this important problem of combinatorial geometry. See, for example, [2], [3], and [4].

Actually, the famous hypothesis of Hadwiger says that the minimum number of rays in $\mathbf{R}^n$ which suffice to illuminate the boundary of every compact convex body in $\mathbf{R}^n$ is equal to 2^n and, moreover, any n-dimensional parallelepiped P in $\mathbf{R}^n$ needs at least 2^n rays. Notice that the set of all singular boundary points of P is infinite (moreover, it is of cardinality continuum).

In this context, we would like to recall the following old result of Hadwiger.

Theorem 1. *If the boundary of a convex body $T \subset \mathbf{R}^n$ is smooth, then $n + 1$ rays in $\mathbf{R}^n$ suffice to illuminate the boundary of T.*

Actually, Theorem 1 states that if $n + 1$ rays $l_1, l_2, \ldots, l_n, l_{n+1}$ are given in $\mathbf{R}^n$, which have common end-point 0 and do not lie in a closed half-space of $\mathbf{R}^n$, then $l_1, l_2, \ldots, l_n, l_{n+1}$ are enough to illuminate the boundary of any convex smooth body in $\mathbf{R}^n$ (the compactness of the body is not required here).

Recall also that Hadwiger's above-mentioned result was strengthened by Boltyanskii (see, e.g., [2]). Namely, Boltyanskii established the following statement.

Theorem 2. *If the boundary of a convex body $T \subset \mathbf{R}^n$ has at most n singular points, then $n + 1$ rays in $\mathbf{R}^n$ suffice to illuminate the boundary of T.*

Boltyanskii's theorem does not admit further generalizations to the case where the boundary of a compact convex body $T \subset \mathbf{R}^n$ may have more than n singular points (see [4] and [7]). In addition to this, the $n+1$ rays of Theorem 2 substantially depend on the convex body T.

It is natural to ask whether there is a compact convex body in $\mathbf{R}^n$ with a finite number of singular boundary points, which needs a large number of rays for illuminating its boundary (i.e., the number of illuminating rays must be of an exponential growth with respect to the dimension n of $\mathbf{R}^n$).

Let X be a strong *at*-subset of $\mathbf{R}^n$ with cardinality equal to $k(n)$. Recall that $k(n)$ is of an exponential growth with respect to n. By starting with this X, one can obtain the following statement.

Theorem 3. *There exists a compact convex body $B \subset \mathbf{R}^n$ such that:*

(1) X coincides with the set of all singular boundary points of B;

(2) at least $k(n)$ rays are necessary to illuminate the boundary of B.

Let us present a sketch of the proof of Theorem 3.

Denote by M the convex hull of the set X. Clearly, M is an n-dimensional convex polyhedron in $\mathbf{R}^n$ and the set of all vertices of M coincides with X. For every point $x \in X$, denote by $M(x)$ the polyhedral angle of M with vertex x, and let $C(x)$ be a convex cone with the same vertex x, such that $M(x) \subset C(x)$. We may assume that the conical hypersurface of $C(x)$ is smooth (of course, except for its vertex x). If each $C(x)$ slightly differs from $M(x)$, then the boundary of the compact convex body

$$B' = \cap\{C(x) : x \in X\}$$

has isolated singular points x, where $x \in X$, and continuum many other singular points y, where $y \in Y$. We may suppose that the distance between the sets X and Y is strictly positive. Now, all singular boundary points of B' belonging to Y can be deleted by using a standard trick, without touching the points of X. So, proceeding in this way, we will be able to replace B' by the compact convex body B satisfying both conditions (1) and (2) of Theorem 3.

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ON VOLUME TYPE FUNCTIONALS IN EUCLIDEAN GEOMETRY

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The paper is concerned with some aspects of the theory of volumes in Euclidean space. In this context, it is shown that there exists a solution of Cauchy's functional equation, which is absolutely nonmeasurable with respect to the class of all translation invariant measures on the real line $\mathbf{R}$, extending the Lebesgue measure on $\mathbf{R}$.

Keywords: volume, measure, additive function

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The concept of volume for sufficiently simple geometric figures is one of the most important in classical geometry of Euclidean spaces. Discussions of this concept occupy a substantial place in all standard university lecture courses in Euclidean geometry. There are many text-books, manuals and monographs devoted to the subject (see, for example, [2], [3], [5], [6], [7], [8], [10], [11], [12]).

The deep notion of volume type functionals is closely tied with several interesting and important geometric topics, such as equidecomposability theory (including well-known paradoxes about partitions of certain geometric bodies), dissections of figures into finitely many other figures of a prescribed type, the behavior of the volume function under Minkowskis sum of point sets, etc.

One of the principal problems which arises here is to extend the function of elementary volume for simple geometric figures to a volume defined for a maximally large class of figures. This problem is successfully solved within framework of modern theory of invariant measures and its solution heavily depends on purely algebraic properties of a basic group of transformations of the Euclidean space.

These aspects are partially touched upon in H. Hadwiger's widely known monograph [3] in which the role of nontrivial solutions of Cauchy's functional equation is shown and stressed. As was proved by Frechet, all of those solutions are nonmeasurable in the Lebesgue sense. However, a much deeper result about such solutions can be established (see Theorem 2 below).

The present paper is devoted to some aspects highlighting profound connections between elementary theory of volume with general methods of the theory of additive functions having bad descriptive properties.

Throughout this article, we use the following standard notation:

$\mathbf{N}$ is the set of all natural numbers;

$\mathbf{Q}$ is the set of all rational numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{R}^n$ is the n-dimensional Euclidean space ($n \geq 1$);

dom(μ) is the domain of a given measure μ on $\mathbf{R}^n$.;

ran(f) is the range of a given function f;

λ_n is the classical Lebesgue measure on $\mathbf{R}^n$.

Let D_n be the group of all isometric transformations of $\mathbf{R^n}$ and let S_n be the ring of sets generated by the collection of all coordinate parallelepipeds of $\mathbf{R^n}$.

Let G be a subgroup of D_n. A functional V_n is called an elementary G-volume on $\mathbf{R^n}$ if the following four conditions hold:

(1) V_n is non-negative:

$$(\forall X)(X \in S_n \Rightarrow V_n(X) \geq 0);$$

(2) V_n is additive:

$$(\forall X)(\forall Y)(X \in S_n \wedge Y \in S_n \wedge X \cap Y = \emptyset \Rightarrow V_n(X \cup Y) = V_n(X) + V_n(Y));$$

(3) V_n is G-invariant:

$$(\forall g)(\forall X)(g \in G \wedge X \in S_n \Rightarrow V_n(g(X)) = V_n(X));$$

(4) $V_n(\triangle_n) = 1$, where $\triangle_n = [0,1]^n$ denotes the unit coordinate cube in $\mathbf{R^n}$.

The above-mentioned conditions are usually treated as Axioms of Invariant Finitely Additive Measure (see, for instance, [3], [5]).

If condition (2) is replaced by the countable additivity condition, then we obtain the definition of a G-measure (cf. [8]).

It is well known that the classical Jordan measure on $\mathbf{R^n}$ is a natural example of G-volume in $\mathbf{R^n}$. Respectively, a certain extension of Jordan measure to a sufficiently large class of subsets of $\mathbf{R^n}$ is the standard Lebesgue measure (see [3], [4]). In some sense, the latter class of sets is maximal, because within the framework of constructive methods it is impossible to further enlarge this class. This result is due to R. Solovay who was able to construct a model of set theory with a restricted (countable) version of the Axiom of Choice, in which all subsets of the space $\mathbf{R^n}$ turn out to be measurable in the Lebesgue sense (see [13]).

Notice that, by using the Zorn lemma, any D_n-volume on $\mathbf{R^n}$ $(n \geq 3)$ can be extended to a maximal (by the inclusion relation) D_n-volume on $\mathbf{R^n}$, but the geometrical structure of the domain of such a maximal D_n-volume is not known and this problem seems to be of some interest.

It is well known that if an additive function

$$f : \mathbf{R} \to \mathbf{R}$$

satisfied one of the following conditions, then there exists a real constant k such that $f(x) = k \cdot x$ for all $x \in \mathbf{R}$:

(a) f is continuous at a point of $\mathbf{R}$;

(b) f is monotone on an interval of positive length;

(c) f is bounded from above (or below) on an interval of positive length;

(d) f is locally integrable in the Lebesgue sense;

(e) f is Lebesgue measurable;

(f) f is a Borel function;

(g) f has the Baire property.

It is clear that if a functional V_1 satisfied the condition (1) (that is, $V_n \geq 0$), then

$$V_1 = kx, \qquad (k \in \mathbf{R}).$$

From the measure-theoretical point of view, there are many interesting and important facts concerning G-volumes. The most famous among them is due to Banach.

Theorem 1 (Banach). *In the cases $n = 1$ and $n = 2$ there exists a non-negative additive functional defined on the family of all bounded subsets of the Euclidean space $\mathbf{R}^n$, invariant under the group of all isometries of $\mathbf{R}^n$ and extending the Lebesgue measure λ_n.*

The proof of Theorem 1 can be found e.g. in [1], [8].

It directly follows from this theorem that if X and Y are two Lebesgue measurable subsets of $\mathbf{R}^n$ (n = 1,2) such that $\lambda_n(X) \neq \lambda_n(Y)$, then X and Y are not finitely equidecomposable subsets of $\mathbf{R}^n$.

Example 1. In the case $n \geq 3$, we have no analogous result because of the famous Banach-Tarski paradox. As a remark, notice that if $n \geq 3$, then the group D_n possesses paradoxical properties which are implied by the fact that this group contains a free subgroup generated by two independent rotations. Actually, just from the latter circumstance follows the Banach-Tarski paradox stating that any two bounded subsets of the Euclidean space $\mathbf{R^n}$, $n \geq 3$ with nonempty interiors are equivalent by finite decompositions. Namely, if A and B are two bounded subsets of $\mathbf{R}^n$ ($n \geq 3$), both of which have a nonempty interior, then there are partitions of A and B into a finite number of disjoint subsets

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k,$$

$$B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k,$$

such that for each $i \in [1, k]$ the sets A_i and B_i are D_n-congruent.

The proof of this paradox is essentially based on the Hausdorff theorem (see [1]) which states that if one removes a certain countable subset of the sphere S^2 in $\mathbf{R^3}$, then the remainder can be divided into three disjoint subsets A, B and C such that A, B, C and $B \cup C$ are mutually congruent under the group of all rotations of $\mathbf{R}^3$ about its origin. In particular, it follows from the above theorem that on S^2 there is no finitely additive non-negative normalized functional defined on all of its subsets such that the values of this functional on congruent sets are equal to each other.

From the Hausdorff theorem also follows that on the Euclidean space $\mathbf{R}^n$ ($n \geq 3$) there exists no D_n-volume defined on the family of all subsets of $\mathbf{R}^n$.

Example 2. There is also a somewhat paradoxical result in the case of the plane $\mathbf{R}^2$. Namely, let G denote the group of all those affine transformations of $\mathbf{R}^2$ which preserve the area, i.e., g belongs to G if and only if $|det(g)| = 1$ (this group is much wider than D_2). According to the theorem of von Neumann, if A and B are two bounded subsets of the plane $\mathbf{R}^2$, both of which have nonempty interiors, then there are partitions of A and B into a finite number of disjoint subsets

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k,$$

$$B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k,$$

such that for each $i \in [1, k]$ the sets A_i and B_i are G-congruent.

A detailed discussion of Example 2 see in [3].

Let us restrict our further considerations to the ring of all polyhedrons in the space $\mathbf{R}^n$ (see, e. g., [3]).

Hilbert's third problem is formulated as follows:

Given any two polyhedrons of equal volume, is it always possible to cut the first into finitely many polyhedrons which can be reassembled to yield the second?

Two polyhedrons are equidecomposable if the first of them admits a cutting into finitely many polyhedrons which can be reassembled to yield the second one. Obviously, any two equidecomposable polyhedrons have the same volume. The converse assertion is not true. For example, the unit cube in $\mathbf{R}^3$ and a regular tetrahedron of volume 1 are not equidecomposable, which gives a negative solution of Hilbert's third problem.

In connection with this problem, for every polyhedron P, Dehn introduced some kinds of functionals, now widely known as the Dehn invariants, which are defined as follows:

$$D_f(P) = \sum l(e) f(\alpha), \qquad (*)$$

where $l(e)$ is the length of an edge e of P, α is the value of the dihedral angle of P between the two faces meeting at e, and f is an additive function such that $f(\pi) = 0$.

In other words, a function f is any solution of the Cauchy functional equation

$$f(x+y) = f(x) + f(y),$$

such that $f(\pi) = 0$.

Notice also that, the sum in $(*)$ is taken over all edges of the polyhedron P.

It is well known that any nonzero additive function f which participates in Dehn invariants is nonmeasurable in the Lebesgue sense (see [3], [9]).

Let M be the class of all those measures on $\mathbf{R}$ which are translation invariant and extend the Lebesgue measure λ_1.

We shall say that a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable with respect to M if, for every measure $\mu \in M$, this f is not μ-measurable.

Theorem 2. *There exists an additive function $f : \mathbf{R} \to \mathbf{Q}$ which is absolutely nonmeasurable with respect to the class of all translation invariant measures on the real line $\mathbf{R}$, extending the Lebesgue measure λ_1.*

Proof. For establishing this fact, consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$. Take an arbitrary element $e \in \mathbf{Q} \setminus \{0\}$. It is well known that, one-element set $\{e\}$ can be extended to a basis of $\mathbf{R}$, that is there exists a Hamel basis $\{e_i : i \in I\}$ for $\mathbf{R}$, containing e. The latter means that $\{e_i : i \in I\}$ is a maximal (with respect to inclusion) linearly independent (over $\mathbf{Q}$) family of elements of $\mathbf{R}$ and $e \in \{e_i : i \in I\}$. Now, find the index $i_0 \in I$ for which $e_{i_0} = e$ and consider the vector subspace V of $\mathbf{R}$ generated by the family $\{e_i : i \in I \setminus \{i_0\}\}$. It is obvious that V turns out to be a vector space in $\mathbf{R}$, complementary to the vector subspace $\mathbf{Q}$. In other words, we have the representation

$$\mathbf{R} = V + \mathbf{Q}, \qquad (V \cap \mathbf{Q} = \{0\})$$

of the space $\mathbf{R}$ in the form of a direct sum of its two vector subspaces. In particular, for each $x \in \mathbf{R}$, the relation

$$\mathrm{card}(\mathrm{V} \cap (\mathrm{x} + \mathbf{Q})) = 1$$

is true, from which it follows that V is a certain Vitali subset of $\mathbf{R}$ (see, e. g., [1], [5]).

For any $x \in \mathbf{R}$, we have the unique representation

$$x = v + q \qquad (v \in V, q \in \mathbf{Q}).$$

Consider a function

$$f : \mathbf{R} \to \mathbf{Q}$$

defined by the formula:

$$f(x) = q \qquad (x \in \mathbf{R}).$$

Obviously,

$$f(x + y) = f(x) + f(y) \qquad (x \in \mathbf{R}, y \in \mathbf{R}).$$

We thus conclude that f turns out to be an additive functional on $\mathbf{R}$ and $ran(f) = \mathbf{Q}$. Also, a straightforward verification shows that f is not measurable with respect to every translation invariant measure on the real line $\mathbf{R}$, extending the Lebesgue measure. This follows from the fact, that

$$f^{-1}(0) = V$$

and V is nonmeasurable with respect to every translation invariant measure μ on the real line $\mathbf{R}$, extending the Lebesgue measure λ_1. In other words, we always have $f^{-1} \notin dom(\mu)$ (compare with Theorem 1).

This finishes the proof of the Theorem 2. □

Example 3. There exist many nontrivial solutions of Cauchy functional equation which are not absolutely nonmeasurable with respect to the class M. Moreover, most of solutions of Cauchy functional equation are not absolutely nonmeasurable with respect to M.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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MULTITIME ACTION RECURRENCES ON A MONOID

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The discrete multitime multiple recurrences are common in analysis of algorithms, computational biology, information theory, queueing theory, filters theory, statistical physics etc. We discuss in detail the cases of recurrences on a monoid, highlighting in particular algebraical aspects and original theorems on existence and uniqueness of solutions.

Keywords: multitime recurrences, recurrence on a monoid, matrix 3-sequence
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1. DISCRETE MULTITIME MULTIPLE RECURRENCE

Generically, we refer to *discrete multitime multiple recurrences* of the form

$$x(t+1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \in \mathbb{Z}^m,\ t \geq t_0,\ \forall \alpha \in \{1, 2, \dots, m\}, \tag{1.1}$$

where $F_\alpha \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_1\big\} \times M \to M$, $\alpha \in \{1, 2, \dots, m\}$, $m \in \mathbb{N}^*$, $t_0, t_1 \in \mathbb{Z}^m$, $t_0 \geq t_1$; $1_\alpha = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^m$, i.e., 1_α has 1 on the position α and 0 otherwise; M is a nonvoid set. The unknown function is an m-sequence $x \colon \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\} \to M$.

Let us start by presenting two results on the existence and uniqueness of the recurrence (1.1) solutions (see [4]).

Proposition 1. If for any $(t_0, x_0) \in \{t \in \mathbb{Z}^m \mid t \geq t_1\} \times M$, there exists at least one solution $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$ which verifies the recurrence (1.1) and the condition $x(t_0) = x_0$, then

$$F_\alpha(t + 1_\beta, F_\beta(t, x)) = F_\beta(t + 1_\alpha, F_\alpha(t, x)), \tag{1.2}$$
$$\forall t \geq t_1,\ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

Theorem 1. *We consider the functions* $F_\alpha\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \times M \to M$, $\alpha \in \{1, 2, \ldots, m\}$, *such that,* $\forall t \geq t_0$, $\forall x \in M$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$, *the relations* (1.2) *are fulfilled.*

Then, for any $x_0 \in M$, *there exists a unique function*

$$x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M,$$

which verifies

$$x(t + 1_\alpha) = F_\alpha(t, x(t)), \quad \forall t \geq t_0, \quad \forall \alpha \in \{1, 2, \ldots, m\},$$

and the condition $x(t_0) = x_0$.

2. MULTITIME RECURRENCES ON A MONOID

A monoid is an algebraic structure with a single associative binary operation and an identity element. Monoids are used in computer science, both in its foundational aspects and in practical programming.

Our aim is to analyse a multitime recurrence on a monoid $(N, \cdot, e)$. We consider $\eta\colon N \times M \to M$, an action of the monoid N on the set M, i.e.

$$\eta(ab, x) = \eta\big(a, (b, x)\big),\ \ \eta(e, x) = x, \quad \forall a, b \in N, \forall x \in M. \tag{2.1}$$

We will use the more convenient notation

$$\eta(a, x) = ax, \quad \forall a \in N,\ \forall x \in M$$

(not to be confused with the operation of monoid N). The relations (2.1) become

$$(ab)x = a(bx),\ \ ex = x, \quad \forall a, b \in N, \forall x \in M.$$

The action functions $a_\alpha\colon \{t \in \mathbb{Z}^m \mid t \geq t_1\} \to N$, $\alpha \in \{1, 2, \ldots, m\}$ (with $t_1 \in \mathbb{Z}^m$) define the action recurrence

$$x(t + 1_\alpha) = a_\alpha(t)x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{2.2}$$

with the unknown function $x\colon \{t \in \mathbb{Z}^m \mid t \geq t_0\} \to M$, $t_0 \in \mathbb{Z}^m$, $t_0 \geq t_1$.

Introducing the set

$$\mathcal{Z} := \{t \in \mathbb{Z}^m \mid t \geq t_1\}$$

and using Proposition 1 and Theorem 1, one can prove easily the following result (see [5]):

Theorem 2. *a) If, for any* $(t_0, x_0) \in \mathcal{Z} \times M$*, there exists at least one function* $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to M$*, which, for any* $t \geq t_0$*, verifies the recurrence* (2.2) *and the condition* $x(t_0) = x_0$*, then*

$$a_\alpha(t + 1_\beta)a_\beta(t)x = a_\beta(t + 1_\alpha)a_\alpha(t)x, \tag{2.3}$$

$$\forall t \in \mathcal{Z},\ \forall x \in M, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

b) If the relations (2.3) *are satisfied, then, for any* $(t_0, x_0) \in \mathcal{Z} \times M$*, there exists a unique function* $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to M$*, which, for any* $t \geq t_0$*, verifies the recurrence* (2.2) *and the condition* $x(t_0) = x_0$*.*

For any point $t = (t^1, \ldots, t^m) \in \mathbb{N}^m$, it is useful to denote

$$|t| := t^1 + \ldots + t^m.$$

Theorem 3. *Suppose that the monoid* N *is commutative. Let us consider the function (sequence)* $r\colon \mathbb{N} \to N$ *and the elements* $q_\alpha \in N$*,* $\alpha \in \{1, 2, \ldots, m\}$*,* $p_{\alpha\beta} \in N$*,* $\alpha, \beta \in \{1, 2, \ldots, m\}$*, with* $p_{\alpha\beta} = p_{\beta\alpha}$*,* $\forall \alpha, \beta$*.*

For each index $\alpha \in \{1, 2, \ldots, m\}$*, we define the function*

$$a_\alpha\colon \mathbb{N}^m \to N,\ a_\alpha(t) = q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|),\ \forall t = (t^1, \ldots, t^m) \in \mathbb{N}^m.$$

We shall consider the recurrence (2.2) *defined by these functions.*

In the previous conditions, for any $x_0 \in M$*, there exists a unique m-sequence* $x\colon \mathbb{N}^m \to M$*, which, for any* $t \in \mathbb{N}^m$ *verifies the recurrence* (2.2)*, as well as the condition* $x(0) = x_0$*. This m-sequence is defined by*

$$x(t) = \prod_{\alpha=1}^{m} q_\alpha^{t^\alpha} \cdot \prod_{\alpha=1}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha - 1)}{2}} \cdot \prod_{1 \leq \alpha < \beta \leq m} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0,\ \forall t \in \mathbb{N}^m \setminus \{0\} \tag{2.4}$$

(if $m = 1$*, then the factor* $\prod_{1 \leq \alpha < \beta \leq m} p_{\alpha\beta}^{t^\alpha t^\beta}$ *does not appear).*

Proof. For any α, β, we have

$$a_\alpha(t + 1_\beta) = p_{\alpha\beta} \cdot q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t| + 1), \quad a_\beta(t) = q_\beta \cdot p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} \cdot r(|t|),$$

$$a_\alpha(t + 1_\beta)a_\beta(t) = p_{\alpha\beta} \cdot q_\alpha q_\beta \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} \cdot r(|t|) r(|t| + 1).$$

It follows that

$$a_\beta(t + 1_\alpha)a_\alpha(t) = p_{\beta\alpha} \cdot q_\beta q_\alpha \cdot p_{\beta 1}^{t^1} p_{\beta 2}^{t^2} \cdot \ldots \cdot p_{\beta m}^{t^m} p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|) r(|t| + 1).$$

Since $p_{\alpha\beta} = p_{\beta\alpha}$, we can write

$$a_\alpha(t + 1_\beta)a_\beta(t) = a_\beta(t + 1_\alpha)a_\alpha(t), \quad \forall t \in \mathbb{N}^m,\ \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

We deduce that the relations (2.3) are satisfied. According to Theorem 2 (with $t_0 = t_1 = 0$), there exists a unique function $x\colon \mathbb{N}^m \to M$, which, for any $t \in \mathbb{N}^m$ verifies the recurrence (2.2), and the condition $x(0) = x_0$.

It is sufficient to show that the function defined by the formula (2.4), for $t \in \mathbb{N}^m \setminus \{0\}$, and $x(0) = x_0$, verifies the recurrence relation (2.2).

We shall verify the case $m \geq 2$; the case $m = 1$ is treated similarly.

We fix $\gamma \in \{1, 2, \ldots, m\}$. We show that for any $t \in \mathbb{N}^m$, we have

$$x(t + 1_\gamma) = a_\gamma(t)x(t).$$

Let $t \in \mathbb{N}^m \setminus \{0\}$. We need the set $\mathcal{P}_{m,2}$ of the subsets with two elements from the set $\{1, 2, \ldots, m\}$, i.e., $\mathcal{P}_{m,2} = \Big\{\{\alpha, \beta\} \subseteq \{1, 2, \ldots, m\} \,\Big|\, \alpha \neq \beta\Big\}$.

Since $p_{\alpha\beta}^{t^\alpha t^\beta} = p_{\beta\alpha}^{t^\beta t^\alpha}$, we observe that in the product $\prod\limits_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta}$ the factors $p_{\alpha\beta}^{t^\alpha t^\beta}$ occur, taken over all distinct elements $\{\alpha, \beta\}$ of the set $\mathcal{P}_{m,2}$.

If $m \geq 3$, we can write

$$\mathcal{P}_{m,2} = \Big\{\{\gamma, \alpha\} \,\Big|\, \alpha \in \{1, 2, \ldots, m\},\, \alpha \neq \gamma\Big\} \cup$$
$$\cup \Big\{\{\alpha, \beta\} \subseteq \{1, 2, \ldots, m\} \,\Big|\, \alpha \neq \gamma,\, \beta \neq \gamma\Big\},$$

hence we have

$$\prod_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta} = \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta}. \tag{2.5}$$

For $m = 2$, one obtains the relation (2.5), but without the factor $\prod\limits_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta}$.

For $m = 2$, we denote $\prod\limits_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} := e$. With this convention, it follows that the relation (2.5) is satisfied for any $m \geq 2$. The relation (2.4) becomes

$$\begin{aligned} x(t) =& q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{t^\gamma(t^\gamma-1)}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \\ &\cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0; \end{aligned}$$

$$\begin{aligned} x(t + 1_\gamma) =& q_\gamma \cdot q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{(t^\gamma+1)t^\gamma}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \\ &\cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot p_{\gamma\alpha}^{t^\alpha}) \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\, \beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|} r(j) \cdot x_0. \end{aligned}$$

Since $\dfrac{(t^\gamma+1)t^\gamma}{2} = t^\gamma + \dfrac{t^\gamma(t^\gamma-1)}{2}$, it follows that

$$x(t+1_\gamma) = q_\gamma \cdot p_{\gamma\gamma}^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\alpha} \cdot r(|t|) \cdot q_\gamma^{t^\gamma} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} q_\alpha^{t^\alpha} \cdot (p_{\gamma\gamma})^{\frac{t^\gamma(t^\gamma-1)}{2}} \cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}}$$

$$\cdot \prod_{\substack{\alpha=1\\ \alpha\neq\gamma}}^{m} p_{\gamma\alpha}^{t^\gamma t^\alpha} \cdot \prod_{\substack{1\leq\alpha<\beta\leq m\\ \alpha\neq\gamma,\,\beta\neq\gamma}} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j) \cdot x_0 = a_\gamma(t)x(t)$$

For $t = 1_\gamma$, the relation (2.4) reads as

$$x(1_\gamma) = q_\gamma \cdot r(0) \cdot x_0 = a_\gamma(0)x(0),$$

hence the equality $x(t+1_\gamma) = a_\gamma(t)x(t)$ is true also for $t = 0$. □

Remark 1. If we use additive notation, i.e., the operation on N is denoted by " + " and $\eta(a,x) = a + x$ $(a \in N,\ x \in M)$, then the recurrence relation (2.2) reads as

$$x(t+1_\alpha) = a_\alpha(t) + x(t), \quad \forall\alpha \in \{1,2,\dots,m\}.$$

In Theorem 3, we have $a_\alpha(t) = q_\alpha + t^1 p_{\alpha 1} + t^2 p_{\alpha 2} + \ldots + t^m p_{\alpha m} + r(|t|)$. Formula (2.4) can be written

$$x(t) = \sum_{\alpha=1}^{m}\left(t^\alpha q_\alpha + \frac{t^\alpha(t^\alpha-1)}{2}p_{\alpha\alpha}\right) + \sum_{1\leq\alpha<\beta\leq m} t^\alpha t^\beta p_{\alpha\beta} + \sum_{j=0}^{|t|-1} r(j) + x_0.$$

Corollary 1. *Let $(M,\cdot)$ be a semigroup. We consider the function (sequence) $r\colon \mathbb{N} \to \mathbb{N}^*$ and the elements $q_\alpha \in \mathbb{N}^*$, $\alpha \in \{1,2,\dots,m\}$, $p_{\alpha\beta} \in \mathbb{N}^*$, $\alpha,\beta \in \{1,2,\dots,m\}$, with $p_{\alpha\beta} = p_{\beta\alpha}$, $\forall\alpha,\beta$.*

For each index $\alpha \in \{1,2,\dots,m\}$, we define the function

$$a_\alpha\colon \mathbb{N}^m \to \mathbb{N}^*,\ a_\alpha(t) = q_\alpha \cdot p_{\alpha 1}^{t^1} p_{\alpha 2}^{t^2} \cdot \ldots \cdot p_{\alpha m}^{t^m} \cdot r(|t|),\ \forall t = (t^1,\dots,t^m) \in \mathbb{N}^m.$$

Then, for $x_0 \in M$, there exists a unique m-sequence $x\colon \mathbb{N}^m \to M$, which, for any $t \in \mathbb{N}^m$, verifies

$$x(t+1_\alpha) = x(t)^{a_\alpha(t)}, \quad \forall\alpha \in \{1,2,\dots,m\}, \tag{2.6}$$

and the condition $x(0) = x_0$. For any $t \in \mathbb{N}^m \setminus \{0\}$, we have

$$x(t) = x_0^{\left(\prod_{\alpha=1}^{m} q_\alpha^{t^\alpha} \cdot \prod_{\alpha=1}^{m} (p_{\alpha\alpha})^{\frac{t^\alpha(t^\alpha-1)}{2}} \cdot \prod_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta} \cdot \prod_{j=0}^{|t|-1} r(j)\right)} \tag{2.7}$$

(if $m = 1$, the factor $\prod_{1\leq\alpha<\beta\leq m} p_{\alpha\beta}^{t^\alpha t^\beta}$ does not appear).

Proof. We apply Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{N}^*, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{N}^* \times M \to M, \quad \eta(a, x) = x^a, \quad \forall a \in \mathbb{N}^*, \forall x \in M.$$

□

Remark 2. *a*) If in Corollary 1 the semigroup $(M, \cdot)$ is a monoid, then we can consider: $q_\alpha \in \mathbb{N}$, $p_{\alpha\beta} \in \mathbb{N}$ (with $p_{\alpha\beta} = p_{\beta\alpha}$) and $r\colon \mathbb{N} \to \mathbb{N}$ (i.e., q_α, $p_{\alpha\beta}$, $r(j)$ can be eventually zero). The conclusion in Corollary 1 reads similarly and the solution of the recurrence (2.6) is defined by the formula (2.7).

The proof follows by applying Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{N}, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{N} \times M \to M, \quad \eta(a, x) = x^a, \quad \forall a \in \mathbb{N}, \forall x \in M.$$

b) If in Corollary 1 the semigroup $(M, \cdot)$ is a monoid, and the element x_0 of M is chosen invertible, then we can consider $q_\alpha \in \mathbb{Z}$, $p_{\alpha\beta} \in \mathbb{Z}$ (with $p_{\alpha\beta} = p_{\beta\alpha}$) and $r\colon \mathbb{N} \to \mathbb{Z}$ (i.e., q_α, $p_{\alpha\beta}$, $r(j)$ are integers). The conclusion in Corollary 1 writes similarly, and the solution of the recurrence (2.6) is defined also by the formula (2.7).

The proof can be obtained by applying Theorem 3 to the commutative monoid $\big(N, \cdot, e\big) = \big(\mathbb{Z}, \cdot, 1\big)$ and the action

$$\eta\colon \mathbb{Z} \times U(M) \to U(M), \quad \eta(a, x) = x^a, \quad \forall a \in \mathbb{Z}, \forall x \in U(M),$$

where $U(M)$ is the set of invertible elements in M; the set $U(M)$, with operation induced by that of M, is a group.

Many other original results, regarding the multitime recurrences, can be found in [3]-[8]. Some related sources are [1], [2], [9]-[12].

3. EXAMPLE OF MATRIX 3-SEQUENCE

Let us determine the matrix 3-sequence $X\colon \mathbb{N}^3 \to \mathcal{M}_2(\mathbb{R})$, which, for any $t = (t^1, t^2, t^3) \in \mathbb{N}^3$, verifies the recurrence relations

$$\begin{cases} X(t^1+1, t^2, t^3) = X(t^1, t^2, t^3)^{2^{t^1} \cdot 3^{t^2+1} \cdot 7^{t^3} \cdot (t^1+t^2+t^3+1)}, \\ X(t^1, t^2+1, t^3) = X(t^1, t^2, t^3)^{5 \cdot 3^{t^1} \cdot 2^{t^3} \cdot (t^1+t^2+t^3+1)}, \\ X(t^1, t^2, t^3+1) = X(t^1, t^2, t^3)^{11 \cdot 7^{t^1} \cdot 2^{t^2} \cdot (t^1+t^2+t^3+1)}, \end{cases}$$

and the condition $X(0,0,0) = A := \begin{pmatrix} 1 & -2 \\ 4 & 7 \end{pmatrix}$.

We are in the assumptions of Corollary 1, with $(M,\cdot) = \big(\mathcal{M}_2(\mathbb{R}),\cdot\big)$ and

$$q_1 = 3, \quad q_2 = 5, \quad q_3 = 11,$$

$$p_{11} = 2, \quad p_{22} = p_{33} = 1, \quad p_{12} = p_{21} = 3, \quad p_{13} = p_{31} = 7, \quad p_{23} = p_{32} = 2,$$
$$r\colon \mathbb{N} \to \mathbb{N}^*, \quad r(j) = j+1, \quad \forall j \in \mathbb{N}.$$

According to Corollary 1, for any $(t^1,t^2,t^3) \in \mathbb{N}^3 \setminus \{(0,0,0)\}$, we have

$$X(t^1,t^2,t^3) = A^{3^{t^1}\cdot 5^{t^2}\cdot 11^{t^3}\cdot 2^{\frac{t^1(t^1-1)}{2}}\cdot 3^{t^1t^2}\cdot 7^{t^1t^3}\cdot 2^{t^2t^3}\cdot (t^1+t^2+t^3)!},$$

$$X(t^1,t^2,t^3) = A^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!},$$

relation which is also true for $(t^1,t^2,t^3) = (0,0,0)$.

By induction one shows that

$$A^n = \begin{pmatrix} 2\cdot 3^n - 5^n & 3^n - 5^n \\ 2(5^n - 3^n) & 2\cdot 5^n - 3^n \end{pmatrix}, \quad \forall n \in \mathbb{N},$$

$$A^n = 3^n \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + 5^n \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}, \quad \forall n \in \mathbb{N}.$$

It follows that, for any $(t^1,t^2,t^3) \in \mathbb{N}^3$, the general term is

$$X(t^1,t^2,t^3) = 3^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!}\cdot \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$
$$+ 5^{2^{\frac{t^1(t^1-1)}{2}+t^2t^3}\cdot 3^{t^1(t^2+1)}\cdot 5^{t^2}\cdot 7^{t^1t^3}\cdot 11^{t^3}\cdot (t^1+t^2+t^3)!}\cdot \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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TOPOLOGY AND DIFFERENTIAL STRUCTURE ON DESCARTES FOLIUM

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The Descartes Folium carries families of group laws, defined entirely in terms of algebraic operations over the base field. To highlight these structures and their properties, we introduced and studied new topologies and differential structures on Descartes Folium.

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1. DESCARTES FOLIUM

Let $\mathbb{K}$ be a field with $\operatorname{char}\mathbb{K} \neq 3$. Our theory (see also [2], [3], [8]) refers to the *Descartes Folium* which is a non-smooth curve

$$DF : x^3 + y^3 - 3axy = 0 \subset \mathbb{A}^2_{\mathbb{K}},\ a \in \mathbb{K} \setminus \{0\}$$

and to its *projective closure* defined by homogenization, i.e.,

$$\overline{DF} : x^3 + y^3 - 3axyz = 0 \subset \mathbb{P}^2_{\mathbb{K}}$$

and called the *projective Descartes Folium*, too. DF, resp $\overline{DF}$, has only a non-smooth point, namely $O = (0, 0) \in DF \subset \overline{DF}$.

Suppose now $\mathbb{K} = \mathbb{R}$. We have a bijective map p:

$$\begin{array}{cccc} DF & \left(x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}\right) & (x,y) \in DF \setminus \{(0,0)\} & (0,0) \\ \wr\uparrow p & \uparrow & \downarrow & \downarrow \\ \mathbb{R} \setminus \{-1\} & t & t = \frac{y}{x} & t = 0 \end{array}$$

On the other hand, the function $t \to \tau = t + 1$, with the inverse $t = \tau - 1 \overset{\alpha}{\longleftarrow} \tau$, establishes a bijection

$$\mathbb{R} \setminus \{-1\} \underset{\sim}{\overset{\alpha}{\longleftarrow}} \mathbb{R} \setminus \{0\}.$$

It appears the bijective map $p\alpha$:

$$\begin{array}{cccc} DF & \left(x = \frac{3a(\tau-1)}{1+(\tau-1)^3}, y = \frac{3a(\tau-1)^2}{1+(\tau-1)^3}\right) & (x,y) \in DF \setminus \{(0,0)\} & (0,0) \\ \wr\uparrow p\,\alpha & \uparrow & \downarrow & \downarrow \\ R \setminus \{0\} & \tau & \tau = \frac{y}{x} + 1 & 1 \end{array}$$

It follows that the natural group law $\cdot$ on $R \setminus \{0\}$ transfers to a group composition law $\circ$ on DF defined as follows:

$$(p\alpha)(\tau) \circ (p\alpha)(\tau') \overset{def}{=} (p\alpha)(\tau\tau').$$

Consequently, we have an isomorphism of groups:

$$(DF, \circ) \simeq (\mathbb{R} \setminus \{0\}, \cdot)$$

$$(p\,\alpha)(\tau) \leftarrow \tau.$$

Remarks. (i) For each $t \in \mathbb{R} \setminus \{-1, 0\}$, we can observe that $p(t) \in DF \setminus \{0\}$ is the intersection point in $\mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2$, of DF with the affine straight line $y - tx = 0$, different of $O = (0,0)$. In this way, p admits a geometric definition.

(ii) p (and $p\alpha$) can be related to the normalization morphism of an algebraic curve ([4]). Namely, p can be uniquely extended as algebraic map to a (non-bijective) algebraic map $\bar{p}$:

$$\begin{array}{ccc} \overline{DF} & (x = 3at, y = 3at^2, z = 1 + t^3) & (x,y,z) \in \overline{DF} \setminus \{O\} \\ \bar{p}\uparrow & \uparrow & \downarrow \\ P^1_{\mathbb{R}} = A^1_{\mathbb{R}} \cup \{\infty\} & t \in A^1_{\mathbb{R}} & t = \frac{y}{x}\ , \end{array}$$

where we indicated the definition of $\bar{p}$ and of a partial inverse of $\bar{p}$.

If we pass from the base field $\mathbb{R}$ to its algebraic closure $\overline{\mathbb{R}} = \mathbb{C} \supset \mathbb{R}$, then DF, $\overline{DF}$, $\bar{p}$, α are all defined over $\mathbb{C}$, by the same equations and formulae. According to [4], one verifies that the pair $(\mathbb{P}^1_{\mathbb{C}}, \bar{p})$ is a normalization of the algebraic $\mathbb{C}$-variety $\overline{DF}$. Recall that a normalization pair is uniquely determined up to an isomorphism of algebraic $\mathbb{C}$-varieties.

We have a similar situation with the pair $(\mathbb{P}^1_{\mathbb{C}}, \bar{p}\alpha)$, which is also a normalization of $\overline{DF}$ over $\mathbb{C}$. Based on this natural relation of p and $p\alpha$, with the normalizations of $\overline{DF}$ over $\mathbb{C}$, we will say shortly in the following that p and $p\alpha$ are also "normalization maps".

So the algebraic maps p and $p\alpha$ are natural and then the group composition law $\circ$ on DF is still natural. But the group $(DF, \circ)$ *is not* a Lie group over $\mathbb{R}$ because DF is not a topological manifold w.r.t. the induced real topology of $\mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2 \supset DF$ (If $U \subset DF$ is a small connected open subset w.r.t. this topology, containing $O = (0,0)$, then $U \setminus \{O\}$ has at least 4 connected components).

2. BRANCHES OF DESCARTES FOLIUM

Suppose $\mathbb{K} = \mathbb{R}$. Let us consider the branches of the non-smooth point $O = (0,0)$ of DF as follows: (i) the "South branch" $S = p(-1,1)$; (ii) the "West branch" $W = p(1,\infty) \cup \{O\} \cup p(-\infty,-1)$.

We have $S \cap W = \{O\}$ and $S \cup W = DF \setminus \{V\}$, where $V = p(1) = \left(\frac{3}{2}, \frac{3}{2}\right)$ is the "vertex" of DF. The branches S and W are symmetric w.r.t. the first bisector $x - y = 0$ of $\mathbb{A}^2_{\mathbb{R}}$. This means that applying the symmetry σ w.r.t. the bisector $x - y = 0$, given by

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{R}} & \overset{\sigma}{\xrightarrow{\sim}} & \mathbb{A}^2_{\mathbb{R}} \\ (x,y) & \longrightarrow & (y,x), \end{array}$$

we have $\sigma(DF) = DF$ and the branches S and W interchange by σ (i.e., $\sigma(S) = W$ and $\sigma(W) = S$).

By the parametrization p, the point $O = p(0)$ is reached on the branch S and it is not reached on the branch W.

Let us consider the parametrization $p' = \sigma p$ of DF. Then, by using the interchange of S and W by σ, it follows that $W = p'(-1,1)$ and $S = p'(1,\infty) \cup \{O\} \cup p'(-\infty,-1)$. By the parametrization $p' = \sigma p$, the point $\{O\}$ is reached only on the branch W (not on the branch S).

It is easy to see that the pair $((\mathbb{A}^1_{\mathbb{R}} \setminus \{-1\}) \cup \{\infty\}, p' = \sigma p)$ is also related to a normalization of $\overline{DF}$ over $\mathbb{C}$, as p and $p\alpha$. We will say shortly, in the following, that p' is also a "normalization map".

3. SOME TOPOLOGIES ON AFFINE DESCARTES FOLIUM

Concerning the topological properties of the map p, we have the following

Proposition 1. *Suppose $\mathbb{K} = \mathbb{R}$ and $DF \subset \mathbb{A}^2_{\mathbb{R}}$ endowed with the topology τ induced by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Then*

(i) the bijective map $p : \mathbb{R} \setminus \{-1\} \xrightarrow{\sim} DF$ is continuous but not a homeomorphism;

(ii) $p|_{\mathbb{R}\setminus\{0,-1\}} : \mathbb{R} \setminus \{0, -1\} \xrightarrow{\sim} DF \setminus \{O\}$ *is a homeomorphism.*

Similar properties hold for the map $p' = \sigma p : \mathbb{R} \setminus \{-1\} \xrightarrow{\sim} DF$.

Proof. (i) Suppose, by contrary, that $p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$ is a homeomorphism. Then $p\alpha : \mathbb{R} \setminus \{0\} \xrightarrow{\sim} DF$ is also a homeomorphism. Since $(\mathbb{R} \setminus \{0\}, +)$ is a topological group and $p\alpha$ is a group isomorphism onto $(DF, \circ)$, it follows easy that $(DF, \circ)$ is a topological group, which is not possible.

An alternative proof based on the different connectivity properties of $\mathbb{R} \setminus \{0\}$ and DF can be done.

(ii) The inverse map

$$\begin{array}{ccc} \mathbb{R} \setminus \{0, -1\} & \overset{p^{-1}}{\underset{\sim}{\longleftarrow}} & DF \setminus \{O\} \\ t = \frac{y}{x} & \longleftarrow & (x, y) \end{array}$$

is also continuous. □

In the previous Proposition we have worked with the topology τ on DF which is induced on $DF \subset \mathbb{A}^2_{\mathbb{R}}$ by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Now let us change the topology τ on DF with the topology τ_S (resp. τ_W) defined as follows:

Definition 1. *τ_S (resp. τ_W) is the image on DF of the real topology of $\mathbb{R}\setminus\{-1\}$ by the bijective map $p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$ (resp. by $p' = \sigma p : \mathbb{R}\setminus\{-1\} \xrightarrow{\sim} DF$).*

Hence the new topology τ_S (resp. τ_W) on DF is obtained by carrying the real topology of an open subset of DF by the normalization map p (resp. p'). It follows that the topology τ_S (resp. τ_W) is separated, paracompact and locally compact, and with countable basis, as well as the fact that $S = p(-1, 1)$ (resp. $W = p'(-1, 1)$) is open in DF w.r.t. τ_S (resp. τ_W). Moreover, the topological space (DF, τ_S) (resp. (DF, τ_W)) has two connected components.

3.1. SOME PROPERTIES OF TOPOLOGY τ_S (RESP. τ_W)

Theorem 1. *(i) τ_S (resp. τ_W) is a finer topology than τ (i.e., $\tau_S, \tau_W \succ \tau$).*

(ii) The induced topology $\tau_S|_{DF\setminus\{O\}}$ $(\tau_W|_{DF\setminus\{O\}})$ on $DF \setminus \{O\} \subset \mathbb{A}^2_{\mathbb{R}}$ is that induced on $DF \setminus \{O\}$ by the real topology of $\mathbb{A}^2_{\mathbb{R}}$. Equivalently,

$$\tau_S|_{DF\setminus\{O\}} = \tau_{DF\setminus\{O\}} \ \left(\textit{resp.}\, \tau_W|_{DF\setminus\{O\}} = \tau_{DF\setminus\{O\}}\right).$$

(iii) If $\{U_n\}_{n\in\mathbb{N}}$ is a fundamental system of open neighborhoods of O in $\mathbb{A}^2_{\mathbb{R}}$, with respect to the real topology, then $\{U_n \cap S\}_{n\in\mathbb{N}}$ (resp. $\{U_n \cap W\}_{n\in\mathbb{N}}$) is a fundamental system of open neighborhoods of $O \in DF$, in DF, with respect to the topology τ_S (resp. τ_W).

(iv)

$$\tau \cup \{U \cap S \,|\, U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}$$
$$\big(\text{resp. } \tau \cup \{U \cap W \,|\, U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}\big)$$

is a basis for the topology τ_S (resp. τ_W). Moreover, for each $V \in \tau_S$ (resp. τ_W),

$$V = (U' \cap DF) \cup (U \cap S) \ \ (\text{resp. } V = (U' \cap DF) \cup (U \cap W)),$$

with $U', U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open subsets.

(v) Let

$$\pi : DF \to \mathbb{R}, \pi(x,y) = \begin{cases} \frac{y}{x} & if \quad (x,y) \neq O \\ 0 & if \quad (x,y) = O \end{cases}$$
$$\left(\text{resp. } \pi' : DF \to \mathbb{R}, \pi'(x,y) = \begin{cases} \frac{x}{y} & if \quad (x,y) \neq O \\ 0 & if \quad (x,y) = O \end{cases}\right).$$

Then τ_S (resp. τ_W) is the weakest topology on DF such that π (resp. π') is continuous ($\mathbb{R}$ endowed with the real topology).

(vi) $\{O\} \subset W$ (resp. $\{O\} \subset S$) is a connected component of the subspace W (resp. S) w.r.t. the topology τ_S (resp. τ_W). Moreover

$$W = p(1,\infty) \cup \{O\} \cup p(-\infty,-1)$$
$$(\text{resp. } S = p'(1,\infty) \cup \{O\} \cup p'(-\infty,-1))$$

is the representation of W (resp. S) as the union of its connected components w.r.t. τ_S (resp. τ_W). On the other hand S (resp. W) is connected w.r.t. τ_S (resp. τ_W).

Proof. Properties (i), (ii) and (v) are direct consequences of the definition of τ_S (resp. τ_W) and of the fact that the maps

$$p, p' : \mathbb{R} \setminus \{-1\} \xrightarrow{\sim} DF$$

(having $\pi, \pi' : DF \to \mathbb{R} \setminus \{-1\}$ as inverse maps) are continuous and

$$p, p' : \mathbb{R} \setminus \{0, -1\} \xrightarrow{\sim} DF \setminus \{O\}$$

are homeomorphisms, where DF (resp. $DF \setminus \{O\}$) above is endowed with the topology τ (resp. $\tau|_{DF\setminus\{O\}}$).

For property (iii), let us point out firstly that $U \cap S$ (resp. $U \cap W$) is an open subset of DF w.r.t. τ_S (resp. τ_W), in particular an open neighborhood of the point $O \in DF$ w.r.t. τ_S (resp. τ_W), if $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ is an open subset w.r.t. τ_S (resp. τ_W), resp. an open neighborhood of the point O in $\mathbb{A}^2_{\mathbb{R}}$. In fact, S (resp. W) is open in DF w.r.t. τ_S (resp. τ_W) and $U \cap DF \in \tau \subseteq \tau_S$ (resp. $U \cap DF \in \tau_W$) and so $U \cap S = (U \cap DF) \cap S \in \tau_S$ (resp. τ_W).

To end the proof of (iii) it suffices to resume to the topology τ_S and to show that for an open neighborhood V of O in DF w.r.t τ_S, there exists an open neighborhood

U of O in $\mathbb{A}^2_{\mathbb{R}}$ such that $V \supseteq U \cap S$. Indeed, we can reduce the situation to the case $V = p((-\delta, \delta))$, with $0 < \delta < 1$, because always for such V we have $V \supseteq p((-\delta, \delta))$, with $0 < \delta < 1$, and $p((-\delta, \delta))$ is an open neighborhood of O in DF w.r.t. τ_S.

For $t \in p((-\delta, \delta))$, with $0 < \delta < 1$, we have $|t| < \delta < 1$ and from the relation $x = \frac{3at}{1+t^3}$, where $t = \pi(x, y)$, with $(x, y) \in S$, it follows $3at = (1 + t^3)x$ and

$$3|a|\,|t| \leq |1 + t^3|\,|x| \leq (1 + |t|^3)|x| < (1 + \delta^3)|x| < 2|x|.$$

Hence $|t| < \frac{2}{3|a|}\,|x|$. If we consider ϵ, with $0 < \epsilon < \delta$, and

$$U = \{(x, y) \in \mathbb{A}^2_{\mathbb{R}} = \mathbb{R}^2\,|x| < \frac{3|a|}{2}\,\epsilon\},$$

then $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ is open w.r.t. the standard real topology and we have $|t| < \epsilon$, for each $(x, y) \in U \cap S$, i.e., $|\pi(x, y)| < \epsilon$, for $(x, y) \in U \cap S$. Therefore $\pi(U \cap S) \subset (-\epsilon, \epsilon) \subset (-\delta, \delta)$. Since $\pi = p^{-1}$, we have then

$$U \cap S = p\pi(U \cap S) \subset p(-\epsilon, \epsilon) \subset p(-\delta, \delta) = V.$$

For property (iv), recall firstly that

$$\tau = \{U \cap DF \mid U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}.$$

Also, we resume to the topology τ_S. Then the family

$$\tau \cup \{U \cap S \mid U \subseteq \mathbb{A}^2_{\mathbb{R}} \text{ open subset}\}$$

is closed w.r.t. the finite intersections.

Let $V \subseteq DF$ be an open subset w.r.t. τ_S. If $O \in V$, then $V \supseteq U \cap S$ with $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open and $O \in U$, according to (iii) and its proof. If $P \in V$, $P \neq O$, then

$$V \supseteq V \cap (DF \setminus \{O\}) \in \tau$$

according to (ii) and $P \in V \cap (DF \setminus \{O\})$. It follows that

$$V = (V \cap (DF \setminus \{O\}) \cup (U \cap S),$$

where $V \cap (DF \setminus \{O\}) \in \tau$ (hence $V \cap (DF \setminus \{O\}) = U' \cap DF$ with $U' \subseteq \mathbb{A}^2_{\mathbb{R}}$ open) and $U \subseteq \mathbb{A}^2_{\mathbb{R}}$ open such that $O \in U$. The proof of (iv) is achieved.

For property (vi), we use the fact that $S \cap W = \{O\}$ and then for an open neighborhood $U \subset \mathbb{A}^2_{\mathbb{R}}$ of O w.r.t. the real topology of $\mathbb{A}^n_{\mathbb{R}}$, $U \cap S$ (resp. $U \cap W$) is an open neighborhood of $O \in DF$ w.r.t. τ_S (resp. τ_W) and $(U \cap S) \cap W = \{O\}$ (resp. $(U \cap W) \cap S = \{O\}$). Hence $\{O\}$ is open in W (resp. in S) w.r.t. τ_S (resp. τ_W) and so it is a connected component of W (resp. S), because $\{O\}$ is also closed in W (resp. S) w.r.t the separated topology τ_S (resp. τ_W). The connectivity of $p(1, \infty)$, $p(-\infty, -1)$, $S = p(-1, 1)$ (resp. $p'(1, \infty)$, $p'(-\infty, -1)$, $W = p'(-1, 1)$) w.r.t. τ_S (resp. τ_W) is clear because p (resp. p') is a homeomorphism. $\square$

Comment (ii) in conjunction to (iii), as well as (iv), determine completely the topology τ_S (resp. τ_W) by means of the real topology of the ambient space $\mathbb{A}^2_{\mathbb{R}}$, DF and its branch S (resp. W).

3.2. DIFFERENTIAL STRUCTURES ON AFFINE DESCARTES FOLIUM

On the topological space (DF, τ_S) (resp. (DF, τ_W)) we can introduce a structure $\mathcal{A}_S$ (resp $\mathcal{A}_W$) of smooth *differential manifold* by means of the simple atlas $\{(DF, \pi)\}$, (resp. $\{(DF, \pi')\}$ having only one chart, where

$$\pi : DF \xrightarrow{\sim} \mathbb{R} \setminus \{-1\} \subset \mathbb{R} \;\; (\text{resp. } \pi' : DF \xrightarrow{\sim} \mathbb{R} \setminus \{-1\} \subset \mathbb{R})$$

is the bijective map defined above, i.e.,

$$\pi : DF \to \mathbb{R},\; \pi(x, y) = \begin{cases} \frac{y}{x} & if \quad (x, y) \neq O \\ 0 & if \quad (x, y) = O \end{cases}$$

$$\left(\text{resp. } \pi' : DF \to \mathbb{R},\; \pi'(x, y) = \begin{cases} \frac{x}{y} & if \quad (x, y) \neq O \\ 0 & if \quad (x, y) = O \end{cases}\right).$$

Recall that the inverse of the map π (resp. π') is the map p (resp. $p' = \sigma p$) and all are continuous, hence homeomorphisms. In this way,

$$DF \;\; \underset{p}{\overset{\pi}{\rightleftarrows}} \;\; \mathbb{R} \setminus \{-1\} \;\; \left(\text{resp.} DF \;\; \underset{p'}{\overset{\pi'}{\rightleftarrows}} \;\; \mathbb{R} \setminus \{-1\}\right)$$

become diffeomorphisms of differentiable manifolds.

In particular, $p\alpha : \mathbb{R} \setminus \{0\} \xrightarrow{\sim} DF$ is then also a diffeomorphism, where DF is endowed with the topology τ_S and the atlas $\{(DF, \pi)\}$. Since

$$p\alpha : (\mathbb{R} \setminus \{0\}, \cdot) \xrightarrow{\sim} (DF, \circ)$$

is a group isomorphism, it follows directly

Theorem 2. *(i)* $(DF, \circ)$ *is a Lie group over* $\mathbb{R}$ *(in particular a topological group), where* DF *is endowed with the topology* τ_S *and the differential manifold structure given by the atlas* $\{(DF, \pi)\}$.

(ii)

$$p\alpha : (\mathbb{R} \setminus \{0\}, \cdot) \xrightarrow{\sim} (DF, \circ)$$

is then an isomorphism of Lie groups over $\mathbb{R}$ *(in particular an isomorphism of topological groups).*

4. COMMENTS

The original ideas regarding the group laws on elliptic curves and algebraic (Lie) groups are found in [1], [4]-[7], [10],[11].

In this paper, and also in our papers (see also [2], [3], [8]), the Descartes Folium is just one example of (irreducible) projective plane algebraic cubic curve with singularity (node), which support group laws. Some of these structures require to introduce and to exploit new topologies and some differential structures. One of our aims is to extend the study to the family of cubic curves

$$C_{a,b} : x^3 + y^3 - 3axy = b,$$

which bifurcates in smooth curves and non-smooth ones (as Descartes Folium). This could help us to understand whether the Cryptography on Descartes Folium is simpler than that on Elliptic curves.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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EQUIVALENT RELIABILITY POLYNOMIALS

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Looking for geometric modeling of reliability polynomials, we discuss three important ideas:
(i) find equivalent reliability polynomials via diffeomorphisms;
(ii) cover a reliability hypersurface by probability straight lines;
(iii) cover a reliability hypersurface by exponential decay curves.
In this paper we shall prove that two reliability polynomials, attached to some electric systems used inside aircrafts, are equivalent via an algebraic diffeomorphism. Also, we introduce the X-loxodromic curves on an equi-reliable hypersurface, which are constrained paths (evolutions) that are equi-reliable.

Keywords: Aircraft designs, reliability polynomials, algebraic diffeomorphism.

2000 Math. Subject Classification: 60K10, 62N05, 90B25.

1. BRIDGE STRUCTURE AND RELIABILITY POLYNOMIAL

In some engineering systems [1, 2, 4, 6], units may be connected in a bridge configuration as shown in Figure 1 which represents a three-phase electrical generator, part of the airplane power system, powered by a three-phase electric motor [3].

Theorem 1. *If $R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_7, R_8$ are the reliabilities of the arcs (paths) in the bridge system in Figure1, then the reliability polynomial P of*

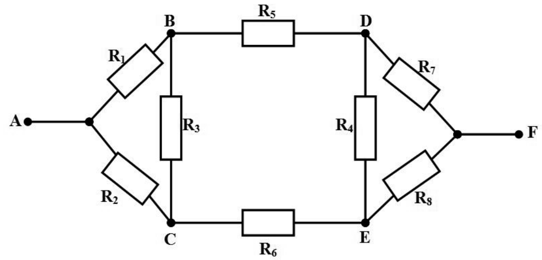

Figure 1: A bridge network

the system is

$$\begin{aligned} P = & R_1R_5R_7 + R_2R_6R_8 + R_2R_3R_5R_7 + R_1R_3R_6R_8 + R_1R_4R_5R_8 \\ & + R_2R_4R_6R_7 - R_1R_2R_3R_5R_7 - R_1R_2R_3R_6R_8 + R_1R_3R_4R_6R_7 \\ & + R_2R_3R_4R_5R_8 - R_1R_4R_5R_7R_8 - R_2R_4R_6R_7R_8 - R_1R_2R_3R_4R_5R_8 \\ & - R_1R_2R_3R_4R_6R_7 - R_1R_2R_4R_5R_6R_7 - R_1R_2R_4R_5R_6R_8 \\ & - R_1R_3R_4R_5R_6R_7 - R_1R_3R_4R_5R_6R_8 - R_2R_3R_4R_5R_6R_7 \\ & - R_2R_3R_4R_5R_6R_8 - R_1R_2R_5R_6R_7R_8 - R_1R_3R_4R_6R_7R_8 \\ & - R_2R_3R_4R_5R_7R_8 - R_1R_3R_5R_6R_7R_8 - R_2R_3R_5R_6R_7R_8 \\ & + 2R_1R_2R_3R_4R_5R_6R_7 + 2R_1R_2R_3R_4R_5R_6R_8 \\ & + R_1R_2R_3R_4R_5R_7R_8 + R_1R_2R_3R_4R_6R_7R_8 + 2R_1R_2R_3R_5R_6R_7R_8 \\ & + 2R_1R_2R_4R_5R_6R_7R_8 + 2R_1R_3R_4R_5R_6R_7R_8 \\ & + 2R_2R_3R_4R_5R_6R_7R_8 - 4R_1R_2R_3R_4R_5R_6R_7R_8. \end{aligned} \tag{1}$$

This polynomial is very long, and this lead to difficulties in its computation and geometrical interpretation. For these reasons, we shall introduce an equivalent reliability polynomial which is simpler. Perhaps, in engineering judgment, the best way to do this is to use Delta-Star Technique.

2. DELTA-STAR TECHNIQUE FOR SIMPLIFIED EQUIVALENT RELIABILITY POLYNOMIAL

The system in Figure 1 can be transformed into its equivalent series and parallel form by using Delta-star technique [5], see Figure 2. The reliability polynomial of the system in Figure 2 is

$$Q = R_5R_AR_BR_DR_F + R_6R_AR_CR_ER_F - R_5R_6R_AR_BR_CR_DR_ER_F. \tag{2}$$

Computationally, this method has some advantages: once a bridge network is transformed to its equivalent parallel and series form, the network reduction approach can be applied to obtain network reliability [7, 8]. Nonetheless, the

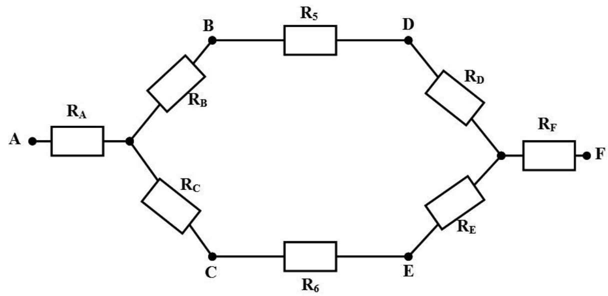

Figure 2: A simplified network

Delta-star method can easily handle networks containing more than one bridge configurations. Furthermore, it can be applied to bridge networks composed of devices having two mutually exclusive failure modes [8, 9].

To obtain the reliability polynomial Q, we observe that the Delta configurations $A, B; B, C; A, C$, respectively $D, F; D, E; E, F$ are replaced by star configurations A, B, C respectively D, E, F. The connection between them (see Figures 1 and 2) is given by the equations

$$R_A R_B = 1 - (1 - R_{AB})(1 - R_{AC} R_{BC}) , \tag{3}$$

$$R_B R_C = 1 - (1 - R_{BC})(1 - R_{AC} R_{AB}) , \tag{4}$$

$$R_A R_C = 1 - (1 - R_{AC})(1 - R_{AB} R_{BC}) \tag{5}$$

for the first triangle, and similar equations for the second one.

Solving equations (3) – (5), we obtain the following star equivalent reliabilities

$$R_A = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{BC})(1 - R_{AC} R_{AB})}} ,$$

$$R_B = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{BC})(1 - R_{AC} R_{AB})]}{1 - (1 - R_{AC})(1 - R_{AB} R_{BC})}} ,$$

$$R_C = \sqrt{\frac{[1 - (1 - R_{BC})(1 - R_{AC} R_{AB})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{AB})(1 - R_{AC} R_{BC})}} .$$

The transformation Delta-star equations applied to R_A, R_B, R_C and R_D, R_E, R_F, gives a simple configuration, so by using the above results, the equivalent to the network complex system in Figure 1 is shown in Figure 2.

In mathematical terms, we use a diffeomorphism to replace the initial reliability polynomial by a simpler ones. This diffeomorphism maps the unit hypercube into itself.

Lemma 1. *The mapping*

$$(R_{BC}, R_{AC}, R_{AB}, R_{EF}, R_{DF}, R_{DE}) \mapsto (R_A, R_B, R_C, R_D, R_E, R_F),$$

defined by formulas (3) - (5) *and the analogous formulae relating* (R_{EF}, R_{DF}, R_{DE}) *and* (R_D, R_E, R_F)*, transforms the unit hypercube into itself.*

Proof. For simplicity, let us denote $R_{AB} = c$, $R_{BC} = a$ and $R_{AC} = b$, where $a, b, c \in [0, 1]$ and $a \neq 0$ or $bc \neq 0$. We introduce the function

$$f(a, b, c) = \frac{(1 - (1 - c)(1 - ba))(1 - (1 - b)(1 - ca))}{1 - (1 - a)(1 - bc)}.$$

In view of the assumptions for a, b and c, both the numerator and denominator in the right-hand side are non-negative, therefore $f \geq 0$. We shall show that $f \leq 1$. After simplification, we obtain

$$f(a, b, c) = \frac{(c + ab - abc)(b + ac - abc)}{a + bc - abc}.$$

We consider separately two cases.

Case 1: $bc = 0$**.** In this case

$$f(a, b, c) = \frac{(c + ab)(b + ac)}{a} = \frac{ac^2 + ab^2}{a} = b^2 + c^2 \leq 1,$$

since either b or c is zero, and the other summand does not exceed 1.

Case 2: $bc > 0$**.** In this case the denominator of f is $a + (1 - a)bc > 0$, and the inequality $f(a, b, c) \leq 1$ is equivalent to $(c+ab-abc)(b+ac-abc)-(a+bc-abc) \leq 0$, or, after simplification, to

$$a\Big[abc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1\Big] \leq 0.$$

Since $a \geq 0$, we need to show that the expression in the brackets in non-positive. The latter is seen as follows:

$$\begin{aligned} &abc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1 \\ &\quad \leq bc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1 \\ &\quad = (1 - b)(1 - c)(bc - b - c - 1) \leq 0. \end{aligned}$$

The lemma is proved. □

Theorem 2. *The reliability polynomials P and Q are equivalent via the algebraic diffeomorphism defined by formulas* (3) - (5) *and their analogues relating* $(R_D,\ R_E,\ R_F)$ *and* $(R_{EF},\ R_{DF},\ R_{DE})$*.*

Generally, a convenient algebraic diffeomorphism is the one which possesses the following three properties: (i) it transforms the unit hypercube into a subset of the unit hypercube; (ii) the number of terms in Q is smaller than the number of terms in P; (iii) the degree of Q is smaller than or equal to the degree of P.

2.1. EQUI-RELIABLE LOXODROMIC CURVES

For some geometrical concepts, we consider the reliability polynomial (2).

First, we rewrite this reliability polynomial (2) by replacing the indices A, B, C, D, E, F by numbers: $R_1 = R_A$, $R_2 = R_B$, $R_3 = R_C$, $R_4 = R_D$, $R_7 = R_E$, $R_8 = R_F$. In this way we obtain the polynomial

$$Q = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8. \qquad (6)$$

In $\mathbb{R}^8$, let us consider the constant level algebraic hypersurfaces

$$c = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8,$$

which will be called *equi-reliable hypersurfaces* [4]. The normal vector field is

$$N = \left(\frac{\partial Q}{\partial R_1}, \frac{\partial Q}{\partial R_2}, \frac{\partial Q}{\partial R_3}, \frac{\partial Q}{\partial R_4}, \frac{\partial Q}{\partial R_5}, \frac{\partial Q}{\partial R_6}, \frac{\partial Q}{\partial R_7}, \frac{\partial Q}{\partial R_8}\right).$$

Consequently, the vector field

$$X = \left(-\frac{\partial Q}{\partial R_2}, \frac{\partial Q}{\partial R_1}, 0, 0, 0, 0, 0, 0\right)$$

is tangent to equi-reliable hypersurfaces.

Let Y be a significant vector field tangent to equi-reliable hypersurfaces, i.e., $< N, Y >= 0$. A curve $\gamma(t) = (R_1(t), R_2(t), R_3(t), R_4(t), R_5(t), R_6(t), R_7(t), R_8(t))$ in an equi-reliable hypersurface is called *Y-loxodroma* if

$$< \dot\gamma(t), Y(\gamma(t)) >= const.$$

For example, the X-loxodromic curves satisfy the first order ODE

$$-(R_1R_4R_5 - R_1R_3R_4R_5R_6R_7)(t)\dot R_1(t)$$

$$+(R_2R_4R_5 + R_3R_6R_7 - R_2R_3R_4R_5R_6R_7)(t)\dot R_2(t) = 0.$$

Along each Y-loxodroma the reliability is constant. Consequently, the previous Y-loxodromic curves are locally constrained paths (evolutions) that are equi-reliable.

Let $\gamma(t)$ be an *X-loxodroma.* The curve $\gamma(t)\exp(-\lambda t)$, $\lambda > 0$ is a decay curve that is necessary when we built the pullback reliability (to compute mean time to failure (MTTF)).

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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EXISTENCE OF CONTINUOUS SOLUTIONS OF A PERTURBED LINEAR VOLTERRA INTEGRAL EQUATIONS

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In this paper we study the existence of continuous solutions on a compact interval of perturbed linear Volterra integral equations. The existence of such a solution is based on the well-known Leray–Schauder principle for a fixed point in Banach space. A special interest is devoted to the study of the uniqueness of continuous solution. Our numerical approach is based on a fixed point method and we apply quadrature rules to approximate the solution for the perturbed linear Volterra integral equations. The convergence of the numerical scheme is proved. Some numerical examples are given to show the applicability and accuracy of the numerical method and to validate the theoretical results.

Keywords: perturbed linear Volterra integral equation, Leray–Schauder principle, compact operator, fixed point method

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1. INTRODUCTION

Integral equations play a very important role in nonlinear analysis and have found numerous applications in engineering, mathematical physics, economics, etc. (see [2], [4], [10]). Many other applications in science are described by integral equations or integro-differential equations such as the Volterra‘s population growth model, biological species living together, propagation of stocked fish in a new lake, the heat radiation and so on [5], [6].

The existence of solutions of nonlinear integral equations has been considered in many papers and books [3], [4], [8]. In this paper, we show that under some assumptions the perturbed linear Volterra integral equation has an unique continuous solution in a bounded and closed interval.

We propose a numerical scheme to approximate the solution of this integral equation [11] and present some numerical examples to show the accuracy of our numerical method.

2. PRELIMINARIES

Let X be an arbitrary Banach space with a norm $\|\cdot\|$. By $C(X, X)$ we denote the space of all continuous operators acting in X. Set $\mathbb{R}_+ = [0, +\infty)$.

By $C([a, b]) = \{x : [a, b] \to \mathbb{R} \text{ is continuous}\}$ we denote the Banach space with the norm $\|x\|_\infty = \max_{t \in [a,b]} |x(t)|$.

As usual, $L_p([a, b]) = \{x : [a, b] \to \mathbb{R}\,; \int_a^b |x(t)|^p \, ds < \infty\}$ stands for the Banach space with norm $\|x\|_p = \left(\int_a^b |x(t)|^p \, ds \right)^{1/p}$, $p \geq 1$.

For $r > 0$, we set $B_r = \{x \in C([a, b]); \|x\|_\infty \leq r\}$, i.e., B_r is a closed ball.

We consider the perturbed linear Volterra integral equation

$$x(t) = f(t) + \int_a^t K(t, s)x(s)ds + \int_a^t V(t, s)g(s, x(s))ds\,, \tag{2.1}$$

with given functions $f \in C([a, b])$, $g(\cdot, \cdot) : [a, b] \times \mathbb{R} \to \mathbb{R}$ and kernels $K(\cdot, \cdot)$, $V(\cdot, \cdot)$: $[a, b] \times [a, b] \to \mathbb{R}$.

We should mention that an extensive amount of work has been done on the existence and uniqueness of solutions of some special cases of Volterra integral equations, see for example [1], [3], [7], [8].

By using the following Leray–Schauder principle, we prove the existence of a solution of perturbed linear Volterra integral equation (2.1).

Theorem 1 ([7] (Leray-Schauder principle))**.** *Let X be a Banach space and the operator $T \in C(X, X)$ be compact. Suppose that any solution x of $x = \lambda T x$, $0 \leq \lambda \leq 1$ satisfies the a priori bound $\|x\| \leq M$ for some constant $M > 0$. Then T has a fixed point.*

Define the operator T on $C([a, b])$ by

$$Tx(t) = f(t) + \int_a^t K(t, s)x(s)ds + \int_a^t V(t, s)g(s, x(s))ds\,. \tag{2.2}$$

3. THE EXISTENCE OF A SOLUTION

Theorem 2. *Let the following conditions be fulfilled:*

1). The function $g(s,x)$ satisfies

$$\sup_{s\in[a,b],x\in\mathbb{R}}\left(|g(s,x)|\,,\left|\frac{\partial g}{\partial x}(s,x)\right|\right)\le G(s)\phi(|x|), \tag{3.1}$$

where $G(\cdot)$ is a positive measurable function and $\phi(\cdot)$ is positive and continuous function satisfying

$$\lim_{y\to+\infty}\frac{\phi(y)}{y}=L<\infty. \tag{3.2}$$

2). The kernels $K(t,s)$ and $V(t,s)$ are continuous with respect to t and satisfy

$$|K(t,s)|\le K_1(t)K_2(s), \qquad |V(t,s)|\le V_1(t)V_2(s), \tag{3.3}$$

where $K_1(\cdot),V_1(\cdot)\in C([a,b])$ and $K_2(\cdot),G(\cdot)V_2(\cdot)\in L_1([a,b])$.

Then the equation (2.1) has a solution in $C([a,b])$.

Proof. We observe that condition (3.2) implies the existence of a positive real number $A>0$ such that $\left|\frac{\phi(u)}{u}\right|\le\frac{3}{2}L=L'$, for all $u\ge A$.

First, we shall prove that the operator $T:C([a,b])\to C([a,b])$ is continuous.

Let $x\in C([a,b])$. Hence for all $s\in[a,b]$, one conclude that $|x(s)|$ is contained in a compact set of $\mathbb{R}_+$. Moreover, $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, then one concludes that there exists a positive constant N_ϕ, such that $\phi(|x(s)|\le N_\phi$. Let $h>0$. On using assumptions 1) and 2) and applying the dominated convergence theorem, and using that $f\in C([a,b])$, we have

$$\begin{aligned}&\lim_{h\to 0}|Tx(t+h)-Tx(t)|\le\lim_{h\to 0}|f(t+h)-f(t)|\\&+\|x\|_\infty\int_a^t\lim_{h\to 0}|K(t+h,s)-K(t,s)|\,ds+\|K_1\|_\infty\|x\|_\infty\lim_{h\to 0}\int_t^{t+h}K_2(s)ds\\&+N_\phi\int_a^t\lim_{h\to 0}|V(t+h,s)-V(t,s)|\,G(s)ds+\|V_1\|_\infty N_\phi\lim_{h\to 0}\int_t^{t+h}V_2(s)G(s)ds=0\,.\end{aligned}$$

Next, we shall prove that the operator T is continuous over $C([a,b])$.

Let $\{x_n\}_{n=1}^\infty\in C([a,b])$ be a sequence converging uniformly to x. Since $C([a,b])$ is complete, then $x\in C([a,b])$. Hence, for all $n\in N$, for all $s\in[a,b]$ and $\Theta_s\in[0,1]$, one concludes that $|\Theta_s x_n(s)+(1-\Theta_s)x(s)|$ is contained in a compact set of $\mathbb{R}_+$. Moreover, $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, therefore there exists a positive constant M_ϕ such that $\phi(|\Theta_s x_n(s)+(1-\Theta_s)x(s)|)\le M_\phi$.

From assumptions 1) and 2) for each $t \in [a,b]$ we have

$$
\begin{aligned}
&|Tx_n(t) - Tx(t)| \\
&\le \|x_n - x\|_\infty \left[\int_a^t |K(t,s)|\, ds + \int_a^t |V(t,s)| \left| \frac{\partial g}{\partial x}(s, \Theta_s x_n(s) + (1-\Theta_s)x(s)) \right| ds \right] \\
&\le \|x_n - x\|_\infty \left[\|K_1\|_\infty \|K_2\|_1 + \|V_1\|_\infty \|V_2 \cdot G\|_1 M_\phi \right].
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} \|Tx_n - Tx\|_\infty = 0$ or, equivalently, T is continuous over $C([a,b])$.

Next, we shall prove that the operator T is compact on $C([a,b])$. Let us set $E := \{Tx\,; x \in B_r\}$.

On using Arzella - Ascoli theorem, the compactness of the set E will be ensured if we show that E is equicontinuous and uniformly bounded.

Let $x \in B_r$. Since $\phi(\cdot)$ is continuous over $\mathbb{R}_+$, there exists a positive constant P_ϕ such that $\phi(|x(s)|) \le P_\phi$ for each $s \in [a,b]$. From assumptions 1) and 2), for every $t \in [a,b]$ we have

$$
\begin{aligned}
|Tx(t)| &\le \|f\|_\infty + \int_a^t K_1(t)K_2(s)\,|x(s)|\,ds + \int_a^t V_1(t)V_2(s)G(s)\phi(|x(s)|)ds \\
&\le \|f\|_\infty + \|K_1\|_\infty \|K_2\|_1 \|x\|_\infty + \|V_1\|_\infty \|V_2 G\|_1 P_\phi.
\end{aligned}
$$

Consequently, E is uniformly bounded.

Let $x \in B_r$, $t', t'' \in [a,b]$ and $t' < t''$. From condition 1) and 2) we obtain

$$
\begin{aligned}
&|Tx(t'') - Tx(t')| \le |f(t'') - f(t')| \\
&+ \|x\|_\infty \int_a^{t'} |K(t'',s) - K(t',s)|\,ds + \|x\|_\infty \|K_1\|_\infty \int_{t'}^{t''} K_2(s)ds \\
&+ P_\phi \int_a^{t'} |V(t'',s) - V(t',s)|\, G(s)ds + P_\phi \|V_1\|_\infty \int_{t'}^{t''} V_2(s)G(s)ds.
\end{aligned}
$$

By applying the dominated convergence theorem to the right-hand side of the above inequality, one concludes that $\lim_{t'\to t''} |Tx(t'') - Tx(t')| = 0$.

Next, we shall prove that any solution of the equation $x = \lambda Tx$, $0 \le \lambda \le 1$ is bounded by the same constant $M > 0$. Let

$$
M_1 = |\lambda|\, \|f\|_\infty + |\lambda|\, \|V_1\|_\infty \|V_2 G\|_1 \sup_{u\in[0,A]} \phi(u)\,, \tag{3.4}
$$

$$
M_2 = \max\{\lambda \|K_1\|_\infty , \lambda \|V_1\|_\infty\}\,, \tag{3.5}
$$

$$
M = M_1 \exp(M_2 \left[\|K_2\|_1 + \|V_2 G\|_1 L'\right])\,, \tag{3.6}
$$

$$Q(s) = K_2(s) + V_2(s)G(s)L'. \tag{3.7}$$

Let $x \in C([a,b])$ be a solution of $x = \lambda Tx$ for some $0 \le \lambda \le 1$, then we have

$$|x(t)| \le |\lambda| \, \|f\|_\infty + |\lambda| \, \|K_1\|_\infty \int_a^b K_2(s) \, |x(s)| \, ds$$

$$+ |\lambda| \, \|V_1\|_\infty \int_a^b V_2(s)G(s) \sup_{u\in[0,A]} \phi(u) ds + |\lambda \| | \, V_1\|_\infty \int_a^b V_2(s)G(s)L' \, |x(s)| \, ds$$

$$\le |\lambda| \left[\|f|_\infty + \|V_1\|_\infty \, \|V_2 G\|_1 \sup_{u\in[0,A]} \phi(u) \right] + M_2 \int_a^b [K_2(s) + V_2(s)G(s)L'] \, |x(s)| \, ds.$$

Hence, from (3.4), (3.5), (3.7) we get that $|x(t)| \le M_1 + M_2 \int_a^b Q(s) \, |x(s)| \, ds$.

By using the general version of Gronwall‘s inequality together with the previous inequality, one concludes that

$$|x(t)| \le M_1 \exp(M_2 \int_a^b Q(s) ds) = M_1 \exp(M_2 \left[\|K_2\|_1 + \|V_2 G\|_1 \, L' \right]) = M.$$

Since M_1 and M_2 do not depend on x, we conclude that the solutions of $x = \lambda Tx$, $0 \le \lambda \le 1$ are uniformly bounded by the same constant M. Now the Leray–Schauder principle implies that T has a fixed point in $C([a,b])$. □

4. NUMERICAL APPROACH AND ITS CONVERGENCE

In the proof of Theorem 2 we have shown that the continuous solutions of $x = Tx$ are uniformly bounded by the same constant M, and consequently they are contained in a closed ball B_M. We choose an initial function $x_0 \in B_M$ and construct the sequence $\{x_n(t)\}_{n=0}^\infty$ as follows

$$x_{n+1}(t) = Tx_n(t), \qquad n \ge 0, \; t \in [a,b]. \tag{4.1}$$

In the next theorem we show that under certain assumptions the sequence $\{x_n(t)\}_{n=0}^\infty$ constructed by (4.1) converges to the unique fixed point $\tilde{x}$ of T.

Theorem 3. *Let the following conditions be fulfilled.*

1. *The conditions of Theorem 2 hold.*
2. *The functions $K_2(\cdot), V_2(\cdot)G(\cdot) \in L_p([a,b])$ for some $p \ge 1$;*
3. *With the constant M defined by* (3.6), *the following inequality holds:*

$$\left[\|K_1\|_\infty \, \|K_2\|_p + \|V_1\|_\infty \, \|V_2 \cdot G\|_p \max_{u\in[-M,M]} \phi(|u|) \right] (b-a)^{\frac{1}{q}} < 1,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \geq 1$.

Then the operator T *is a contractive mapping in* B_M *and has exactly one fixed point, say,* $\tilde{x}(t)$. *Moreover, the generated by* (4.1) *sequence* $\{x_n(t)\}_{n=0}^{\infty}$ *convergence to this fixed point, i.e.*

$$\lim_{n\to\infty} x_n(t) = \tilde{x}(t) \quad \textit{for every} \quad t \in [a,b], \tag{4.2}$$

and

$$\|x_n - \tilde{x}\|_\infty \leq \frac{L^n}{1-L} \|x_1 - x_0\|_\infty , \tag{4.3}$$

where $0 < L < 1$ *is the contraction constant of* T.

Proof: Suppose that $x, y \in B_M$. For all $s \in [a,b]$ and $\Theta_s \in [0,1]$, there holds $|\Theta_s x(s) + (1-\Theta_s)y(s)) \leq M$. For $t \in [a,b]$, using assumptions 1) and 2) of Theorem 2 and Holder's inequality, we obtain

$$|Tx(t) - Ty(t)| \leq \int_a^t K_1(t)K_2(s)\,|x(s) - y(s)|\,ds$$

$$+ \int_a^t V_1(t)V_2(s)\left|\frac{\partial g}{\partial x}(s, \Theta_s x(s) + (1-\Theta_s)y(s))\right| |x(s) - y(s)|\,ds$$

$$\leq \|x - y\|_\infty \left[\|K_1\|_\infty \|K_2\|_p (b-a)^{\frac{1}{q}} + \|V_1\|_\infty \max_{u\in[-M,M]} \phi(|u|)\,\|V_2 G\|_p (b-a)^{\frac{1}{q}}\right],$$

where $1/p + 1/q = 1$, $p \geq 1$.

Let $L = (b-a)^{\frac{1}{q}} \left[\|K_1\|_\infty \|K_2\|_p + \|V_1\|_\infty \|V_2 G\|_p \max_{u\in[-M,M]} \phi(|u|)\right]$. By assumption 3), we have $L < 1$, hence the operator T satisfies the Lipschitz condition

$$\|Tx - Ty\|_\infty \leq L\,\|x - y\|_\infty . \tag{4.4}$$

If we assume that T has two fixed point $\tilde{x}, \tilde{y} \in B_M$, we would have

$$\|\tilde{x} - \tilde{y}\|_\infty = \|T\tilde{x} - T\tilde{y}\|_\infty \leq L\,\|\tilde{x} - \tilde{y}\|_\infty , \tag{4.5}$$

and since $0 < L < 1$, it follows that $\tilde{x} \equiv \tilde{y}$. Hence, the operator T has a unique fixed point in B_M.

Finally, relations (4.2), (4.3) are proved in a standard way by using equation (4.4) and [11, p.267, Theorem 5.2.3.], with $X = C([a,b])$. $\square$

From (3.4) and (3.6) it follows that $f \in B_M$, hence we can choose f as an initial function, $x_0 \equiv f$. We apply quadrature formulae such as trapezoidal, Simpson and "3/8"-rule to evaluate numerically the integrals in the operator T.

We construct an uniform mesh on $[a,b]$ with stepsize h: $s_k = a + (k-1)h$, $k = \overline{1,n}$, where $a + nh \le b < a + (n+1)h$. We put $t = s_k$ in (2.1) and obtain the following nonlinear integral system for the unknowns $x_k = x(s_k)$, $k = \overline{1,n}$:

$$\begin{aligned} x_1 &= f(s_1) = f(a)\,, \\ x_k &= f(s_k) + \int\limits_a^{s_k} K(s_k,s)\,x(s)\,ds + \int\limits_a^{s_k} V(s_k,s)\,g(s,x(s))\,ds\,, \quad k = \overline{2,n}\,. \end{aligned} \tag{4.6}$$

We apply quadrature rules for each k with nodes $s_1, s_2, \ldots, s_k$ and coefficients $h.A_{kj}$, $k = \overline{2,n}, j = \overline{1,k}$ to approximate the integrals in (4.6) for $k = \overline{2,n}$:

$$x_k = f(s_k) + h\sum_{j=1}^{k} A_{kj}\,K(s_k,s_j)\,x_j + h\sum_{j=1}^{k} A_{kj}\,V(s_k,s_j)\,g(s_j,x_j) + R_k(x)\,,$$

where $R_k(x) = O(h^r)$ is the error term due to the quadrature rule. We denote

$$F_k(x_1,\ldots,x_n) = f(s_k) + h\sum_{j=1}^{k} A_{kj}\,K(s_k,s_j)\,x_j + h\sum_{j=1}^{k} A_{kj}\,V(s_k,s_j)\,g(s_j,x_j)\,.$$

In our calculations we choose two different schemes for coefficients A_{kj}. The first scheme is constructed on the base of the trapezium quadrature formulas. The other is based on the Simpson rule and the 3/8-rule (also called Simpson 3/8) [11, Section 3.1].

The fixed point method with initial condition $x_k^0 = f(s_k)$, $k = \overline{1,n}$ is as follows:

$$x_k^{i+1} = F_k(x_1^i,\ldots,x_n^i)\,, \quad k = \overline{2,n}\,, \quad i = 0,1,2,\ldots\,.$$

The convergence of the numerical iterations is proved by Theorem 3.

4.2. EXPERIMENTAL RESULTS

We have tested the efficiency of the proposed numerical scheme on two Volterra integral equations given in the examples below. In our numerical scheme the iterations stop when $E^{i+1} = \|x^{i+1} - x^i\| = \max\limits_j |x_j^{i+1} - x_j^i| \le \varepsilon$, where $\varepsilon = 10^{-7}$ is the chosen precision. All routines have been written in the software system Wolfram Mathematica 9.0.

Example 1. Consider the perturbed linear Volterra integral equation

$$x(t) = \frac{1}{t+1} + \int\limits_0^t \frac{t+1}{s^2+s+1}x(s)\,ds + \int\limits_0^t \frac{1}{t+1}\frac{1}{\sqrt{s^2+s+2}}\sqrt{1+x(s)}\,ds$$

with the exact solution $x^{ex}(t) = t^2 + t + 1$ for $t \in [0, 1]$. Obviously, the assumptions of Theorem 3 are fulfilled.

In Table 1 are shown the errors E^i for some iterations with different quadrature methods and different choice of the step h. It is seen that the iterations in the fixed point method are more effective for reaching the desired precision than the scheme with choice of a grid and use of quadrature rule. The left panel of Figure 1 shows the first approximations x^i, $i = \overline{0, 4}$. The exact and the approximate solution based on Simpson quadrature rule with $h = 0.2$ are shown in the right panel of Figure 1. Good agreement is demonstrated. The last approximation, the errors between the last approximate solutions E_k^{12}, $k = \overline{1, 6}$ and with the exact solution $E_k^{ex} = x_k^{12} - x_k^{ex}$, $k = \overline{1, 6}$ are listed in Table 2.

Table 1: Numerical results for Example 1.

	$h = 0.2$		$h = 0.02$		$h = 0.001$	
	Trapezoid	Simpson	Trapezoid	Simpson	Trapezoid	Simpson
$i = 1$	1.2981	1.2863	1.2862	1.2861	1.2861	1.2861
$i = 3$	0.2942	0.2997	0.3002	0.3003	0.3003	0.3003
$i = 9$	1.42E-5	8.36E-6	5.56E-6	5.49E-6	5.49E-6	5.48E-6
$i = 12$	4.02E-8	1.60E-8				

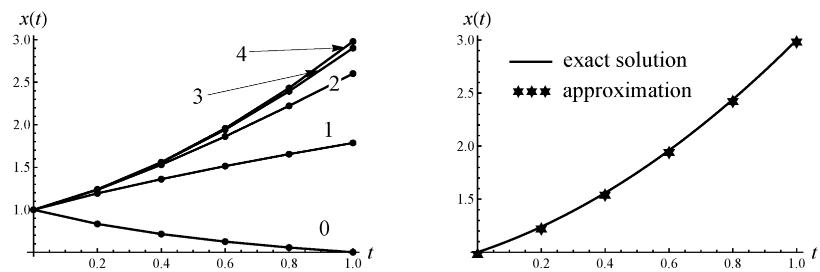

Figure. 1. Left: First 5 approximations; Right: 12-th approximation and the exact solution. Both graphics demonstrate the results using Simpson's rules with $h = 0.2$ for Example 1.

Table 2: Comparison with exact solution in Example 1.

k	1	2	3	4	5	6
x_k^{12}	1.	1.24	1.56	1.96	2.44	3.
E_k^{12}	0.	1.750E-11	1.297E-10	8.309E-10	4.031E-9	1.605E-8
E_k^{ex}	0.	-2.283E-12	-1.693E-11	-1.092E-10	-5.359E-10	-2.173E-9

Example 2. Consider the perturbed linear Volterra integral equation

$$x(t) = \frac{t^2}{4}e^t + \int_0^t \frac{(t+s)^2}{4}e^{t-s}x(s)\,ds + \int_0^t \frac{s\ln(1+t)}{(1+t^2)(1+s^2)}\frac{se^{-2s}}{x(s)}\,ds\,, \quad t \in [0,1]$$

It is easily seen that the functions occurring in this integral equation satisfy the assumptions of Theorem 3.

Table 3: Numerical results for Example 2.

	$h=0.2$		$h=0.02$		$h=0.001$	
	Trapezoid	Simpson	Trapezoid	Simpson	Trapezoid	Simpson
$i=1$	8.2666	8.2443	8.2444	8.2442	8.2442	8.2442
$i=10$	0.0122	0.0008	6.50E-5	5.76E-5	5.76E-5	5.76E-5
$i=14$	0.0003	4.19E-6	1.16E-8			
$i=17$	2.05E-5	7.99E-8				
$i=23$	8.40E-8					

Table 3 shows the results analogous to those in Table 1 but for Example 2. Here the role of the quadrature rule and the stepsize of the grid is significant. In Figure 2 (left panel), the approximations obtained in the first five iterations are shown. The decrease of the error with increasing the number of iterations is shown in the right panel of Figure 2, where $E^{10} = 6.06E-5$ and $E^{14} = 1.07E-8$.

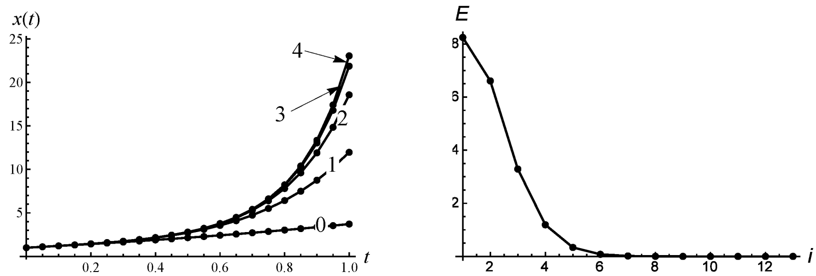

Figure. 2. Left: First 5 approximations; Right: The maximum error related to each iteration. Both graphics demonstrate the results using Simpson's rules with $h = 0.05$ for Example 2.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
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MANIFOLDS ADMITTING A STRUCTURE OF FOUR DIMENTIONAL ALGEBRA OF AFFINORS

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The purpose of this note is to describe some properties of manifolds endowed with an almost tangent structure T, $T^2 = 0$ and an almost complex structure J, $J^2 = -E$, $E = id$.
We consider a linear connection ∇ on N, which is compatible with the algebraic structure, i.e. $\nabla J = 0$, $\nabla T = 0$. The existence of ideals in corresponding algebra implies the existence of autoparallel submanifolds of N.

Keywords: Four dimentional associative algebra, affinely connected manifold, algebra of fiber-preserving operators

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1. ALGEBRAIC PRELIMINARIES

Let us consider a real associative algebra $\mathfrak{A}$ with the unit element e and two generators i, ε satisfying

$$i^2 = -e, \quad \varepsilon^2 = 0,$$

under the requirement $dim\, \mathfrak{A} = 4$ [1].

We distinguish three cases described by the relations

$$i\varepsilon = \varepsilon i\,, \tag{1.1}$$

$$i\varepsilon = -\varepsilon i\,, \tag{1.2}$$

$$i\varepsilon + \varepsilon i = e. \tag{1.3}$$

The corresponding algebras are denoted by $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, respectively.

Proposition 1. The algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ possess nontrivial ideals while $\mathfrak{A}_3$ is a simple algebra.

Proof. Let us denote $i\varepsilon = \varepsilon i = k$. Then we have the following table of multiplications of $\mathfrak{A}_1$

	e	i	ε	k
e	e	i	ε	k
i	i	$-e$	k	ε
ε	ε	k	0	0
k	k	$-\varepsilon$	0	0

Obviously, $\{e, i, \varepsilon, k\}$ is a basis of $\mathfrak{A}_1$ and $\{\varepsilon, k\}$ is an ideal with zero-multiplication.

Similarly to the previous case, $\mathfrak{A}_2$ admits an ideal, too.

Now we consider the algebra $\mathfrak{A}_3$. The mapping

$$\varphi : \mathfrak{A}_3 \to M(2)\,,$$

where $M(2)$ is the algebra of (2×2) real matrices defined by

$$\varphi(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varphi(\varepsilon) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

is an isomorphism. It is well-known that the algebra $M(2)$ is simple. That completes the proof. ☐

2. MANIFOLDS OVER ALGEBRAS

Let N be a manifold of class C^∞, TN - the tangent bundle of N, $\mathfrak{X}(N)$ - the $F(N)$ - module of global sections of TN, F(N)-the ring the smooth functions on N. Let $A(TN)$ be the algebra of $F(N)$ - linear operators of $\mathfrak{X}(N)$. It can be identified with the algebra of fiber–preserving automorphisms of TN.

Let us consider a real associative algebra $\mathfrak{A}$ with unit element e. A morphism of algebras $\Phi : \mathfrak{A} \to A(TN)$ such that $\Phi(e) = I$, the identity operator of $A(TN)$ will be called an $\mathfrak{A}$ - *structure* on N. A linear connection ∇ on N is said to be *compatible* with the $\mathfrak{A}$ - *structure* if $\nabla\Phi(a) = 0$, for all $a \in \mathfrak{A}$, i.e. each operator $\Phi(a)$ is parallel with respect to ∇. An $\mathfrak{A}$ - *structure* is said to be integrable if for each point p exists a neighborhood U, such that the operator $\Phi(a)$ for all $a \in \mathfrak{A}$ have constant components in corresponding coordinate chart.

If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, we define a distribution D in TN as follows:

$$D_p = \{\Phi(b)v \in T_pN;\ for\ all\ b \in B\ and\ v \in T_pN\}\,.$$

In other words, at each point $p \in N$, D_p is the image of T_pN by the operators corresponding to the elements of $\mathfrak{B}$. This distribution is invariant with respect to all operators $\Phi(a)$, $a \in \mathfrak{A}$.

Proposition 2. Let $\mathfrak{A}$ is associative unitary R-algebra, N be a manifold with $\mathfrak{A}$ - *structure* and ∇ be a linear connection on N. If $\nabla\Phi(a_i) = 0$ for all basis elements a_i of $\mathfrak{A}$ then ∇ is compatible with $\mathfrak{A}$.□

Proof. The operator $\nabla : \mathfrak{D}(N) \to \mathfrak{D}(N)$ is a differentiation of the tensor algebra on N. If $\Phi(a_i) = A_i \in \mathfrak{D}^1_1(N)$, $i = 1, 2, 3, 4$, it follows that

$$\nabla_X(A_iA_j) = \nabla_X(A_i)A_j + A_i\nabla_X(A_j) = 0$$

□

The following theorem is proved in [2], p. 118.

Theorem 1. *Let (M, ∇) be an affinely connected analytical manifold equipped with an $\mathfrak{A}$ - structure compatible with ∇. Then the following properties are satisfied:*

1. The distribution D is involutive;

2. If N' is a maximal integral submanifold of D through any point of N, then it is autoparallel submanifold of N;

3. On each N' acts the quotient - algebra $\mathfrak{A}/O(\mathfrak{B})$, where $O(\mathfrak{B})$ is the annihilator of the ideal $\mathfrak{B}$ in algebra $\mathfrak{A}$.

3. ALGEBRAIC STRUCTURES $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$

The integrability conditions of these structures are given in [1].

According to the previous notations, we set $\Phi(e) = I$, $\Phi(i) = J$ and $\Phi(\varepsilon) = T$, by I we denote the unit matrix and we set $JT = K$. Moreover, we suppose that

$$Im\,T = Ker\,T = \frac{1}{2}\,dim\,N.$$

Theorem 2. *Let N be a manifold with an integrable algebraic structure of type $\mathfrak{A}_i$, $(i = 1, 2)$ and ∇ be torsion-free connection compatible with the algebraic structure, i.e. $\nabla J = 0, \nabla T = 0$. Then there exists an $\mathfrak{A}_i$ - invariant foliation N' in N, i.e. at any point $p \in N' \subset N$ the tangent space T_pN' is invariant with respect to J and T.*

Proof. Case (1): $J^2 = -I, T^2 = 0, JT = TJ$ and $Ker\, T = Im\, T$. We denote by D the distribution $Ker\, T = Im\, T$. It can be easily seen that the following holds: $JD \subseteq D, TD \subseteq D$. This implies that $n \equiv 0 \pmod 4$, so we can write $n = 4m$.

For any point of N there exist an open neighborhood with a chart $(x^1, \dots, x^{4m})$ on it such that with respect to the basis $\partial/\partial x^1, \dots, \partial/\partial x^{4m}$ the tensors J and T have matrix expression:

$$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}.$$

We denote $(x^1, \dots, x^{4m}) = (x^i, x^{i+m}, x^{i+2m}, x^{i+3m})$, $(i = 1, \dots, m)$. Every integral submanifold N' of $D = Im\, T$ has coordinates $(x_0^i, x_0^{i+m}, x^{i+2m}, x^{i+3m})$.

We have $\mathfrak{A} = \{I, J, T, K\}$, $\mathfrak{B} = \{T, K\}$ - an ideal, the annihilator $O(\mathfrak{B}) = \mathfrak{B}$, $\mathfrak{A}/O(\mathfrak{B}) \approx \{I, J\}$.

The restriction of J on D is the following

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v^{i+2m} \\ v^{i+3m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v^{i+3m} \\ v^{i+2m} \end{pmatrix}.$$

Here $v = (0, 0, v^{i+2m}, v^{i+3m}) \in D$ and by I we denote the unit $(n \times n)$- matrix.

Case (2): $JT = -TJ$.

Let M be a manifold provided with a $\mathfrak{A}_2$ - *structure*. Similarly to the previous case, one may choose an atlas, such that with respect to any chart $U_x \subset N$ the operators J and T have the form

$$\begin{pmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}.$$

Theorem 2 is proved. □

Now we start considering the last case.

Case (3): N is a manifold provided with a couple J, T of tensor fields of type (1,1), satisfying $J^2 = -I$, $T^2 = 0$ and $JT + TJ = I$. Here it is not necessary to require that $Ker\, T = Im\, T$, because it follows from the relation between J and T. Obviously, we can write $n = 2m$.

Proposition 3. An $\mathfrak{A}_3$ - structure on a smooth manifold N may be given equivalently:

1. By the operators P and Q, such that

$$P^2 = I, \quad Q^2 = I \quad \text{and} \quad PQ + QP = 0,$$

2. By the operators J and P, such that

$$J^2 = -I, \quad P^2 = I \quad \text{and} \quad JP = -PJ.$$

Proof: 1. If we set $P = JT - TJ$ and $Q = J + 2T$, by using the characteristic identity in *Case (3)* we have

$$\begin{aligned} P^2 &= (JT - TJ)(JT - TJ) = JTJT - JTTJ - TJJT + TJTJ \\ &= JT(I - TJ) + TJ(I - JT) = JT + TJ = I\,, \\ Q^2 &= (J + 2T)(J + 2T) = J^2 + 2JT + 2TJ = -I + 2I = I\,. \end{aligned}$$

2. In analogy with the previous case we have

$$\begin{aligned} JP &= J(JT - TJ) = J^2T - JTJ = -T - J(I - JT) \\ &= -T - J - T = -J - 2T\,, \\ PJ &= (JT - TJ)J = JTJ + T = J(I - JT) + T = J + 2T\,. \end{aligned}$$

Remark 1. In [1] the structure $\{J, P\}$ is called a *complex product structure.*

The next theorem is a modification of the result of A. Andrada [3].

Theorem 3. *Let N be a manifold with an $\mathfrak{A}_3$ -structure, given by the operators $\{J, T\}$. Then:*

1. There exists a unique torsion-free connection ∇ with respect to which J and T are parallel;

2. The leaves of the distribution $\mathfrak{D} = Im\,T$ are flat autoparallel submanifolds of N.

Proof. The connection ∇, which preserves the tensor fields J and P is given by

$$\begin{aligned} \nabla_x Y =& \frac{1}{4}\{[X, Y] - [PX, PY] + P[X, PY] - P[PX, Y] \\ & - J[X, JY] - J[PX, Q] + Q[X, QY] + Q[PX, JY]\,, \end{aligned}$$

where $Q = -JP$.

Since $T = \frac{1}{2}(PJ - J)$, it follows that $\nabla P = \nabla J = 0$. Our assertion follows from Proposition 2.

We may choose an atlas on N, whose Jacobian matrices are local constant. Then the operators J and T have the following form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix},$$

where I is the unit $(n \times n)$ - matrix. The theorem is proved. □

Remark 2. Another proof of the existence and uniqueness of ∇ is given in [1].

Remark 3. In this case the distribution $D = ImT$ is not invariant with respect to the operator J.

Remark 4. As it is shown in [1], [3] the connection ∇ does not need to be flat.

An essential property of the tangent bundle TM is the fact that it bears a *tangent structure*. More precisely, let $\pi : TM \to M$ and $K : TTM \to TM$ be natural projection and connection maps of ∇, respectively. If X is a vector field on M, we may define vertical lift X^v and horizontal lift X^h on TM by the relations

$$(d\pi)X^v = 0, \qquad KX^v = X\,,$$
$$(d\pi)X^h = X, \qquad KX^h = 0\,.$$

From a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{X}(M)$ we get the basis of $\mathfrak{X}(TM) : \{X_k^h, X_n^v\}$, $k = 1, \ldots, n$. With respect to this basis the tangent structure has the matrix expression mentioned above. We define

$$\tilde{J} : X^h \to X^v, \quad X^v \to -X^h, \quad \tilde{J}^2 = -I.$$

By setting $J = -\tilde{J}$, this leads us to the $\mathfrak{A}_3$ algebra.

Theorem 4. *The manifold TM can be endowed with integrable operators P, Q, subject to the relations*

$$P^2 = I, \;\; Q^2 = I, \;\; PQ = QP = 0.$$

Proof. Let us set $P = JT - TJ$ and $Q = J + 2T$. By using the identity in Case (3) of Theorem 2, we can easily verify our statement.

The integrability of J and T implies the integrability of P and Q. □

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LOXODROMES ON CANAL SURFACES IN EUCLIDEAN 3-SPACE

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We obtain the differential equations of loxodromes on canal surfaces which cut all meridians or parallels at a constant angle in the Euclidean 3-space. Also we compute the arc-lengths of loxodromes and give some examples by using Mathematica to illustrate our results.

Keywords: Loxodrome, canal surface, Euclidean space

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1. INTRODUCTION

Loxodromes are curves which intersects all meridians or parallels at a constant angle on the Earth's surface. As a result of this, loxodromes do not require a change of course [4]. Therefore they are usually used in navigation.

The equations of the loxodromes on rotational surfaces were found by Noble [6].

A natural generalization of rotational surfaces is helicoidal surfaces. Also there are a lot of helicoidal objects and structures which are related to navigation in nature, science and engineering, for example; creeper plants, helicoidal staircases, moving walkways, parking garage ramps, helicoidal railways and so on [1].

The differential equations of the loxodromes on helicoidal surfaces in Euclidean 3-space $\mathbb{E}^3$ were obtained by Babaarslan and Yayli [1].

A canal surface in $\mathbb{E}^3$ can be defined as envelope of a moving sphere whose trajectory of centers is a spine curve $m(u)$ with varying radius $r(u)$. When $r(u)$ is a constant function, the canal surfaces reduce to pipe surfaces [8].

A lot of objects and structures may be represented by using canal surfaces, for example; blending surfaces and transition surfaces between pipes [7], hoses, ropes [2], 3D fonts, brass instrument, internal organs of the body in solid modeling [3], helical channel [5] and tunnels. Some particular examples of canal surfaces are cylinder, cone, torus, sphere, pipe and Dupin cyclide. Hence canal surfaces are often used in Computer Aided Geometric Design and Computer Aided Manufacturing [3].

Internal organs of the body, helical channel and tunnels are especially interesting examples of canal surfaces on which navigation is possible.

In this paper, we investigate the differential equations of loxodromes on the canal surfaces. Also we compute the arc-lengths of loxodromes and give some examples by using Mathematica.

2. LOXODROMES ON CANAL SURFACES

The parametrization of a canal surface in $\mathbb{E}^3$ is

$$C(u,v) = m(u)+r(u)\big(\sqrt{(1-r'(u)^2)}n(u)\cos v+\sqrt{(1-r'(u)^2)}b(u)\sin v-t(u)r'(u)\big),$$

where u is arc-length parameter, $0 \le v < 2\pi$, t, n and b are the unit tangent, principal normal and binormal vectors of the spine curve $m(u)$, respectively.

The coefficients of first fundamental form of the canal surface C with respect to the base $\{C_u, C_v\}$ are

$$E = \langle C_u, C_u\rangle = (1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2,$$

$$F = \langle C_u, C_v\rangle = g^2\tau + gh\kappa\sin v,$$

$$G = \langle C_v, C_v\rangle = g^2.$$

Thus the first fundamental form of the canal surface C is given by the following equation

$$\begin{aligned} ds^2 &= Edu^2 + Fdudv + Gdv^2 \\ &= \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)du^2 \\ &\quad +2(g^2\tau + gh\kappa\sin v)dudv + g^2dv^2, \end{aligned}$$

where $g = g(u) = r(u)\sqrt{1-r'(u)^2}$ and $h = h(u) = r(u)r'(u)$; $\kappa = \kappa(u)$ and $\tau = \tau(u)$ are the curvature and the torsion of $m(u)$, respectively [8].

We recall that when $1-\kappa g\cos v - h' \neq 0$, the canal surface is regular. Also, a regular canal surface is developable if and only if it is a cylinder or a cone [8].

The arc-length of any curve on the canal surface $C(u,v)$ between u_1 and u_2 is given by

$$s = \int_{u_1}^{u_2} \sqrt{H(u,v)}du,$$

where

$$\begin{aligned}H(u,v) =&(1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\\ &+ 2(g^2\tau + gh\kappa\sin v)\frac{dv}{du} + g^2\Big(\frac{dv}{du}\Big)^2.\end{aligned}$$

A curve on the canal surface $C(u,v)$ in $\mathbb{E}^3$ which cuts all meridians (v=constant) or parallels (u=constant) at a constant angle is called *a loxodrome*.

Let us assume that a loxodrome $\alpha(t)$ is the image of a curve $(u(t), v(t))$ which lies on the (uv)-plane under H. The tangent vector $\alpha'(t)$ has coordinates (u', v') and the tangent vector C_u has coordinates $(1,0)$ with respect to the basis $\{C_u, C_v\}$. Thus, at the intersection point $C(u,v)$, we have

$$\begin{aligned}\cos\theta &= \frac{Edu + Fdv}{\sqrt{E^2du^2 + 2EFdudv + EGdv^2}}\\ &= \frac{K(u,v)}{\sqrt{L(u,v)}},\end{aligned}$$

where

$$\begin{aligned}K(u,v) &= \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)du\\ &\quad +\big(g^2\tau + gh\kappa\sin v\big)dv,\end{aligned}$$

$$\begin{aligned}L(u,v) &= \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)^2du^2\\ &\quad +2\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)\\ &\qquad \times(g^2\tau + gh\kappa\sin v)dudv\\ &\quad +\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2dv^2.\end{aligned}$$

From this equation, the differential equation of the loxodrome on the canal surface which cuts all meridians at a constant angle θ is

$$A\Big(\frac{dv}{du}\Big)^2 + B\frac{dv}{du} = C \tag{2.1}$$

with

$$\begin{aligned}A =&\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2\cos^2\theta\\ &- (g^2\tau + gh\kappa\sin v)^2,\end{aligned}$$

$$B = -2\big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big) \times (g^2\tau + gh\kappa\sin v)\sin^2\theta\,,$$
$$C = \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)^2\sin^2\theta.$$

Also, the angle γ between the loxodrome and any parallel (u=constant) is defined by the following equation

$$\cos\gamma = \frac{Fdu + Gdv}{\sqrt{EGdu^2 + 2FGdudv + G^2dv^2}} = \frac{M(u,v)}{\sqrt{N(u,v)}},$$

where

$$M(u,v) = \big(g^2\tau + gh\kappa\sin v\big)du + g^2dv,$$

$$N(u,v) = \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2du^2 + 2(g^2\tau + gh\kappa\sin v)g^2dudv + g^4dv^2.$$

From this equation, the differential equation of the loxodrome on the canal surface which cuts all parallels at a constant angle γ is given by

$$\widetilde{A}\,\Big(\frac{du}{dv}\Big)^2 + \widetilde{B}\,\frac{du}{dv} = \widetilde{C} \tag{2.2}$$

with

$$\widetilde{A} = \big((1-\kappa g\cos v - h')^2 + (g\tau + h\kappa\sin v)^2 + (g' - h\kappa\cos v)^2\big)g^2\cos^2\gamma - (g^2\tau + gh\kappa\sin v)^2,$$
$$\widetilde{B} = -2(g^2\tau + gh\kappa\sin v)g^2\sin^2\gamma,$$
$$\widetilde{C} = g^4\sin^2\gamma.$$

If we take $r = r(u)$=constant in equations (2.1) and (2.2), respectively, then we obtain the differential equation of the loxodrome on pipe surfaces which cut all meridians (resp., parallels) at a constant angle θ (resp., γ).

We have not succeeded in finding the general solutions of the differential equations (2.1) and (2.2) by using analytical methods, so it remains an open problem. To illustrate the obtained results, we give some examples produced with the help of MATHEMATICA.

Example 2.1. Let us consider the spine curve $m(u) = (0, 0, u + 1)$. Taking $r(u) = 1$, $\theta = \pi/4$, $u \in (0, 2)$, $v \in (0, 2\pi)$ and $v(0) = 0$, the arc-length of the loxodrome is computed as $2\sqrt{2}$. The loxodrome, the meridian ($v = 1$) and the canal surface are shown in Figure 1.

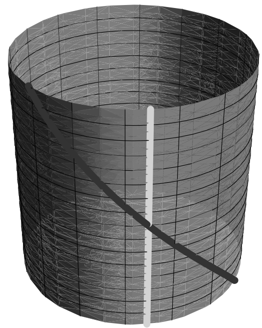

Figure. 1. The loxodrome (blue) and the meridian (green) on the canal surface (cylinder)

Example 2.2. Let us consider the spine curve $m(u) = (0, 0, u)$. Taking $r(u) = u/2$, $\theta = \pi/3$, $u \in (0, 7)$, $u_0 = 1$ and $v \in (0, 2\pi)$, the arc-length of the loxodrome is computed as $7\sqrt{3}$. Also the loxodrome, the meridian ($v = 3$) and the canal surface are shown in Figure 2.

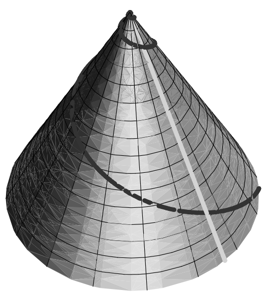

Figure. 2. The loxodrome (blue) and the meridian (green) on the canal surface (cone)

Example 2.3. Let us consider the spine curve $m(u) = (\cos u, \sin u, 0)$. Taking $r(u) = 1/2$, $\gamma = \pi/6$, $v \in (-\pi, \pi)$, $u \in (0, 2\pi)$ and $u(0) = 0$, the arc-length of the loxodrome is computed as $2\sqrt{3}\pi/3$. The loxodrome, the parallel $(u = 1/2)$ and the canal surface are depicted in Figure 3.

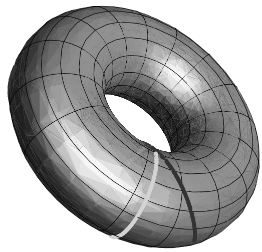

Figure. 3. The loxodrome (blue) and the parallel (green) on the canal surface (torus)

Example 2.4. Consider the spine curve $m(u) = \big(\cos(u/\sqrt{2}), \sin(u/\sqrt{2}), u/\sqrt{2}\big)$. Taking $r(u) = 1$, $\gamma = \pi/2$, $v \in (-2\pi, \pi)$, $u \in (0, 4\pi)$ and $u(0) = 0$, the arc-length of the loxodrome is computed as 14.2074. Also the loxodrome, the parallel $(u = 2\pi)$ and the canal surface are shown in Figure 4.

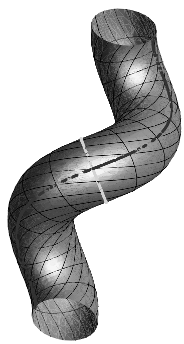

Figure. 4. The loxodrome (blue) and the parallel (green) on the canal surface (helical)

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DIRECT AND INDIRECT METHODS OF PROOF. THE LEHMUS-STEINER THEOREM

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We describe and discuss the different methods of proof of a given statement and illustrate by logical models the essence of specific types of proofs, especially of direct and indirect methods of proof.

Two direct proofs of Lehmus-Steiner's theorem are proposed.

Keywords: Direct and indirect methods of proof, logical models, direct proof of Lehmus-Steiner's theorem, Stewart's theorem

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1. INTRODUCTION

In most fields of study, knowledge is acquired by way of observations, by reasoning about the results of observations and by studying the observations, methods and theories of other fields and practices.

Ancient Egyptian, Babylonian and Chinese mathematics consisted of rules for measuring land, computing taxes, predicting eclipses, solving equations and so on.

The ancient Greeks found that in arithmetic and geometry it was possible to *prove* that the observation results are true. They found that some truths in mathematics were *obvious* and that many of the others could be shown to follow logically from the obvious ones.

On the other hand, Physics, Biology, Economics and other sciences *discover general truths* relying on observations. Besides, not any general truth can be proved to be true - it can only be tested for contradictions and inconsistencies. If a scientific theory is accepted because observations have agreed with it, there is in principle small doubt that a new observation will not agree with the theory, even if all previous observations have agreed with that theory. However, if a result is proved thoroughly and correctly, that cannot happen.

Under what conditions can we be sure that the steps in our investigations are correct? Are we really sure that what seems to be obvious to us is in fact true? Can we expect *all* mathematical truths to follow from the obvious ones? These questions are not easy to answer.

Disputes and mistakes about what is obvious could be avoided by laying down certain basic notions, relations and statements, called *axioms* (postulates assumed true, but unprovable) for each branch of mathematics, and agreeing that proofs of assertions must be derived from these. To axiomatize a system of knowledge means to show that its claims can be derived from a small, well-understood set of axioms (see also [1]).

Any axiomatic system is subordinated to some conditions.

- The system must be consistent, to lack contradiction, i. e. the ability to derive both a statement and its negation from the system's axioms.

  Consistency is a necessary requirement for the system.

- Each axiom has to be independent, i. e. not to be a theorem that can be derived from other axioms in the system.

  However, independence is not a necessary requirement for the system.

- The system can be complete, i. e. for every statement, either itself or its negation is derivable.

There is no longer an assumption that axioms are *true in any sense*; this allows parallel mathematical theories to be built on alternative sets of axioms (for instance *Axiomatic set theory, Number theory*). *Euclidean* and *Non-Euclidean geometry* have a common basic set of axioms; the differences between these important geometries are based on their alternative axioms of parallel lines.

Another way to avoid mistakes about what is obvious in mathematics could be the use of rules of inference with purely formal content.

In mathematical logic a propositional calculus (also called sentential calculus or sentential logic) is a formal system in which formulas of a formal language may be interpreted to represent propositions.

In [7, 8, 9, 10] we explain methods, based on logical laws, for composition and proof of equivalent and inverse problems.

In [8] we discuss a way of generating groups of equivalent problems. The method we propound is based on the logical equivalences

$$p \wedge \neg q \to r \;\Leftrightarrow\; p \wedge \neg r \to q \;\Leftrightarrow\; p \to q \vee r,$$

where p, q, r are statements.

Using the sentential logic, in [9] and [10] we propose a new problems composing technology as an interpretation of specific logical models. Clarifying and using the logical equivalence (see also [12])

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow t \wedge (p \vee q) \to r, \qquad (*)$$

we give an algorithm for composition of *inverse* problems with a given logical structure that is based on the steps below.

- Formulating and proving *generating* problems with logical structures of the statements as those at the left hand side of (*).
- Formulating a problem with logical structure $t \wedge (p \vee q) \to r$ of the statement.
- Formulating and proving the *inverse* problem with logical structure $t \wedge r \to p \vee q$.

In [7], besides the generalization of criteria A and D for congruence of triangles, we also illustrate the above algorithm by suitable groups of examples.

In Section 2 of the present paper we describe and discuss different methods of proof of implicative statements and illustrate by logical models the essence of specific types of proofs, especially of direct and indirect proofs.

In Section 3 we propose *direct proofs* of Lehmus-Steiner's theorem that differ from any proofs we have come across.

Our investigations in this field are appropriate for training of mathematics students and teachers.

2. TYPES OF PROOFS

Both discovery and proof are integral parts of problem solving. The *discovery* is thinking of possible solutions, and the *proving* ensures that the proposed solution actually solves the problem.

Proofs are logical descriptions of deductive reasoning and are distinguished from inductive or empirical arguments; a proof must demonstrate that a statement is always true (occasionally by listing all possible cases and showing that it holds in each).

An unproven statement that is believed true is known as a *conjecture*.

The *objects* of proofs are premises, conclusions, axioms, theorems (propositions derived earlier from axioms), definitions and evidence from the real world.

The abilities (techniques) to have a working knowledge of these objects include

- Rules of inference: simple valid argument forms. They may be divided into *basic rules*, which are fundamental to logic and cannot be eliminated without losing the ability to express some valid argument forms, and *derived rules*, which can be proven by the basic rules.

  To summarize, the rules of inference are logical rules which allow the deduction of conclusions from premises.

- Laws of logical equivalence.

Different methods of proof combine these objects and techniques in different ways to create valid arguments.

According to *Euclid* a precise proof of a given statement has the following structure:

- *Premises*: These include given axioms and theorems, true statements, strict restrictions for the validity of the given statement, chosen suitable denotations. (*It is given...*)

- *Statement*: Strict formulation of the submitted statement. (*It is to be proved that...*)

- *Proof*: Establishing the truth of the submitted statement using premises, conclusions, rules of inference and logical laws.

Let now P and Q be statements. In order to establish the truth of the implication $P \to Q$, we discuss different methods of proof. Occasionally, it may be helpful first to rephrase certain statements, to clarify that they are really formulated in an implicative form.

If "not" is put in front of a statement P, it negates the statement. $\neg P$ is sometimes called the *negation* (or *contradictory*) of P. For any statement P either P or $\neg P$ is true and the other is false.

Formal proofs. The concept of a proof is formalized in the field of mathematical logic. Purely formal proofs, written in symbolic language instead of natural language, are considered in proof theory. A formal proof is defined as a sequence of formulas in a formal language, in which each formula is a logical consequence of preceding formulas.

In a formal proof the statements P and Q aren't necessarily related comprehensively to each other. Only the structure of the statements and the logical rules that allow the deduction of conclusions from premises are important.

Hence, to prove formally that an argument Q is valid or the conclusion follows logically from the hypotheses P, we have to

- assume the hypotheses P are true,

- use the formal rules of inference and logical equivalences to determine that the conclusion Q is true.

The following logical equivalences illustrate a formal proof:

$$\neg(P \to Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow \neg(\neg P) \wedge \neg Q \Leftrightarrow P \wedge \neg Q.$$

Vacuous proof. A vacuous proof of an implication happens when the hypothesis of the implication is always false, i. e. if we know one of the hypotheses in P is false then $P \to Q$ is vacuously true.

For instance, in the implication $(P \wedge \neg P) \to Q$ the hypotheses form a contradiction. Hence, Q follows from the hypotheses vacuously.

Trivial proofs. An implication is trivially true when its conclusion is always true. Consider an implication $P \to Q$. If it can be shown (independently of P) that Q is true, then the implication is always true.

The form of the trivial proof $Q \to (P \to Q)$ is, in fact, a tautology.

Proofs of equivalences. For equivalence proofs or proofs of statements of the form P *if and only if* Q there are two methods.

- Truth table.

- Using direct or indirect methods and the equivalence

$$(P \leftrightarrow Q) \Leftrightarrow (P \to Q) \wedge (Q \to P).$$

  Thus, the proposition P *if and only if* Q can be proved if both the implication $P \to Q$ and the implication $Q \to P$ are proved. This is the definition of the *biconditional* statement.

Proof by cases. If the hypothesis P can be separated into cases $p_1 \vee p_2 \vee \ldots \vee p_k$, each of the propositions $p_1 \to Q$, $p_2 \to Q, \ldots, p_k \to Q$, is to be proved separately. A statement $P \to Q$ is true if all possible cases are true.

The logical equivalences in this case are (see also [12], p. 81)

$$p_1 \to Q \wedge p_2 \to Q \wedge \ldots \wedge p_k \to Q \Leftrightarrow p_1 \vee p_2 \vee \ldots \vee p_k \to Q \Leftrightarrow P \to Q.$$

Different methods may be used to prove the different cases.

Direct proof. In mathematics and logic, a direct proof is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually existing lemmas and theorems.

The methods of proof of these established facts, lemmas, propositions and theorems are of *no importance*. Their truth or falsehood are to be accepted *without any effort.*

However, it is exceptionally important that the actual proof of the given statement consists of straightforward combinations of these facts *without making any further assumptions.*

Thus, to prove an implication $P \to Q$ directly, we assume that statement P holds and try to deduce that statement Q *must* follow.

The structure of the direct proof is:

- *Given* - a statement of the form $P \to Q$.
- *Assumption* - the hypotheses in P are true.
- *Proof* - using the rules of inference, axioms, theorems and any logical equivalences to establish in a straightforward way the truth of the conclusion Q.

Indirect proof. It is often very difficult to give a direct proof to $P \to Q$. The connection between P and Q might not be suitable to this approach.

Indirect proof is a type of proof in which a statement to be proved is assumed false and if the assumption leads to an impossibility, then the statement assumed false has been proved to be true.

There are four possible implications we can derive from the implication $P \to Q$, namely

- *Conversion* (the *converse*): $Q \to P$,
- *Inversion* (the *inverse*): $\neg P \to \neg Q$,
- *Negation*: $\neg(P \to Q)$,
- *Contraposition* (the *opposite*, *contrapositive*): $\neg Q \to \neg P$.

The implications $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent.

The implications $Q \to P$ and $\neg P \to \neg Q$ are logically equivalent too, but they are not equivalent to the implication $P \to Q$.

The two most common indirect methods of proof are called *Proof by Contraposition* and *Proof by Contradiction*. These methods of indirect proof differ from each other in the assumptions we do as premisses.

Proof by contraposition. In logic, contraposition is a law that says that a conditional statement is logically equivalent to its contrapositive. This is often called the *law of contrapositive*, or the *modus tollens* (*denying the consequent*) rule of inference.

The structure of this indirect proof is:

- We consider an implication $P \to Q$.
- Its contrapositive (opposite) $\neg Q \to \neg P$ is logically equivalent to the original implication, i.e.

$$\neg Q \to \neg P \;\Leftrightarrow\; P \to Q.$$

- We prove that *if* $\neg Q$ *is true* (the assumption), then $\neg P$ is true.

Therefore, a proof by contraposition is a *direct* proof of the contrapositive.

The proof of Lehmus-Steiner's theorem in [11] is an illustration of a proof by contraposition.

Proof by contradiction. In logic, proof by contradiction is a form of proof, and more specifically a form of indirect proof, that establishes the truth or validity of a proposition by showing that the proposition's being false would imply a contradiction. Proof by contradiction is also known as *indirect* proof, *apagogical argument*, proof by assuming the opposite, and *reductio ad impossibility.* It is a particular kind of the more general form of argument known as *reductio ad absurdum.*

We assume the proposition $P \to Q$ is false by assuming the negation of the conclusion Q and the premise P are true, and then using $P \wedge \neg Q$ to derive a contradiction.

Hence, the structure of this indirect proof is:

- We use the equivalence $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.
- The negation of the last disjunction is $P \wedge \neg Q$, i. e. $\neg(P \to Q) \Leftrightarrow (P \wedge \neg Q)$.
- To prove the original implication $P \to Q$, we show that *if its negation* $P \wedge \neg Q$ *is true* (the assumption), then this leads to a contradiction.

In other words, to prove the implication $P \to Q$ by contradiction, we assume the hypothesis P and the negation of the conclusion $\neg Q$ both hold and show that this is a contradiction (see also [12], p. 188).

A logical base of this method are equivalences of the form

$$\begin{aligned} P \to Q \quad &\Leftrightarrow \neg Q \wedge P \to \neg P \quad &&\Leftrightarrow \neg(P \to Q) \to \neg P; \\ P \to Q \quad &\Leftrightarrow \neg Q \wedge P \to Q \quad &&\Leftrightarrow \neg(P \to Q) \to Q. \end{aligned}$$

Let now T be a valid theorem, statement, axiom or definition of a notion in the corresponding system of knowledge. The following equivalences can also be logical base of a Proof by Contradiction of the implication $P \to Q$.

$$P \to Q \;\Leftrightarrow \neg Q \wedge P \to \neg T \;\Leftrightarrow \neg(P \to Q) \to \neg T.$$

The theoretical base of this method of proof is the *law of excluded middle* (or the *principle of excluded middle*). It states that for any proposition, either that proposition is true, or its negation is true. The law is also known as the *law* (or *principle*) *of the excluded third.*

Examples of indirect proofs of Lehmus-Steiner's theorem are given in [4].

There exist also examples of indirect proofs of implications $P \to Q$ in which the statement $\neg Q$ can be separated into cases $q_1 \vee q_2 \vee \ldots \vee q_k$, $k \geq 2$, $k \in \mathbb{N}$. In such a case each of the propositions $P \to q_1$, $P \to q_2, \ldots, P \to q_k$ is to be proved separately to be false. If moreover the premise P is true it follows that all the statements q_i, $i = 1, \ldots, k$, are false and the conclusion Q is true, i. e.

$$\neg(\neg Q) \Leftrightarrow \neg(q_1 \vee q_2 \vee \ldots \vee q_k) \Leftrightarrow \neg q_1 \wedge \neg q_2 \wedge \ldots \wedge \neg q_k \Leftrightarrow Q.$$

The logical equivalences in this case are (see also [12], p. 81)

$$P \to \neg q_1 \wedge P \to \neg q_2 \wedge \ldots \wedge P \to \neg q_k \Leftrightarrow P \to \neg q_1 \wedge \neg q_2 \wedge \ldots \wedge \neg q_k \Leftrightarrow P \to Q.$$

The indirect proof of Lehmus-Steiner's theorem given in [3] has in fact logical structure as the described above although this is not mentioned by the authors.

Proof by construction. In mathematics, a *constructive* proof is a method of proof that demonstrates the existence of a mathematical object by creating or providing a method for creating the object.

In other words, *proof by construction* (proof by example) is the construction of a concrete example with a property to show that something having that property exists.

A simple constructive proof of Lehmus-Steiner's Theorem is given in [13].

Nonconstructive proof. A nonconstructive proof establishes that a mathematical object with a certain property exists without explaining how such an object can be found. This often takes the form of a proof by contradiction in which the nonexistence of the object is proven to be impossible.

Proof by counterexamples. We can *disprove* something by showing a single counter example, i. e. one finds an example to show that something is not true.

However, we cannot prove something by example.

Mathematical induction. In proof by mathematical induction, a single *base case* is proved, and an *induction rule* is proved, which establishes that a certain case implies the next case. Applying the induction rule repeatedly, starting from the independently proved base case, proves many, often infinitely many, other cases. Since the base case is true, the infinity of other cases must also be true, even if all of them cannot be proved directly because of their infinite number.

The mathematical induction is a method of mathematical proof *typically* used to establish a given statement for all natural numbers. It is a form of direct proof and it is done in three steps.

Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ be the set of natural numbers, and $P(n)$ be a mathematical statement involving the natural number $n \geq k$, $k, n \in \mathbb{N}$, k suitably fixed.

- The first step, known as the *base step*, is to prove the given statement for the first possible (admissible) natural number k, i.e. to show that $P(k)$ is true for $n = k$.

- The second step, known as the *inductive hypothesis*, is to assume that for a natural number $i \geq k$ the statement $P(i),\ i \in \mathbb{N}$ is true.

- The third step, known as the *inductive step*, is to prove that the given statement $P(i)$ (just assumed to be true) for any one natural number i implies that the given statement for the next natural number $P(i+1)$ is true, i. e. to prove that $P(i) \to P(i+1)$.

From these three steps, mathematical induction is the rule from which we infer that the given statement $P(n)$ is established for all natural numbers $n \geq k$.

3. THE LEHMUS-STEINER THEOREM

The Lehmus-Steiner theorem states:.

Theorem 3.1. *If the straight line segments bisecting the angles at the base of a triangle and terminating at the opposite sides are equal, then the triangle is isosceles.*

This so–called *equal internal bisectors theorem* was communicated by Professor Lehmus (1780–1863) of Berlin to Jacob Steiner (1796–1867) in the year 1840 with a request for a pure geometrical proof of it. The request was complied with at the time, but Steiner's proof was not published till some years later. After giving his proof, Steiner considered also the case when the angles below the base are bisected; he generalized the theorem somewhat; found an external case where the theorem is not true; finally he discussed the case of the spherical triangle. His solution by the method of *proof by contraposition* [11] is considered to be the most elementary one at that time.

Since then many mathematicians have published analytical and geometrical solutions of this "elementary" theorem.

Does there exist a proof of this theorem which is *direct*? This problem was set in a Cambridge Examination Paper in England around 1850. In 1853, the famous Bristish mathematician James Joseph Sylvester (1814–1897) intended to show that *no direct proof can exist*, but he was not very successful. Since then, there have been a number of direct proofs published, but generally speaking they require some other results which have not been proved directly.

A simple, constructive proof, based mainly on Euclid's Book III, is given in [13].

McBride's paper [5] contains a short history of the theorem, a selection from the numerous other solutions that have been published, some discussion of the logical points raised, and a list of references to the extensive literature on the subject. For the long history of this remarkable theorem see also [6].

Below we propose two strictly direct proofs of Lehmus-Steiner's theorem.

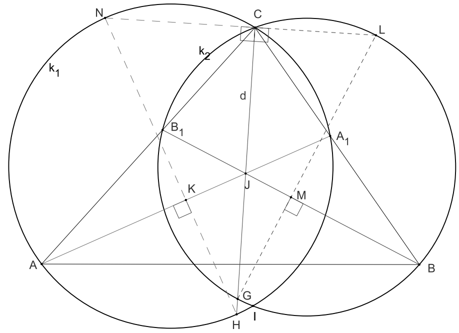

Figure 1.

3.1. FIRST PROOF OF THE THEOREM OF LEHMUS-STEINER

Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be the internal bisectors in $\triangle ABC$, $AA_1 = BB_1$ and $AA_1 \cap BB_1 = J$. Then CJ is the internal bisector of $\angle ACB$. We use the denotation $\gamma := \angle ACJ = \angle BCJ$.

Let also k_1 be the circumscribing circle of $\triangle ACA_1$, and k_2 the circumscribing circle of $\triangle BCB_1$ (Figure 1). First we need the following

Proposition 3.2. *The cut loci of points, from which two equal segments appear under the same angle, are equal arcs of congruent circles.*

Proof of Proposition 3.2. Consider $\triangle ACA_1$ and $\triangle BC_1B_1$, where $\angle ACA_1 = \angle BC_1B_1 = 2\gamma$ and $AA_1 = BB_1$. Let k_1 with center O_1 be the circumscribing circle of $\triangle ACA_1$, and k_2 with center O_2 the circumscribing circle of $\triangle BC_1B_1$ (Figure 2).

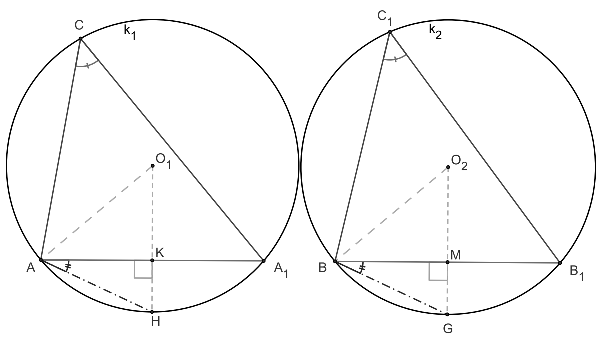

Figure 2.

The cut loci of points, from which the equal segments AA_1 and BB_1 appear under the same angle 2γ, are respectively the arcs $\widehat{ACA_1}$ in k_1 and $\widehat{BC_1B_1}$ in k_2.

The perpendicular line O_1K ($K \in AA_1$) from O_1 to the chord AA_1 cuts the arc $\widehat{AA_1}$ in k_1 at its midpoint H, the perpendicular line O_2M ($M \in BB_1$) from O_2 to the chord BB_1 cuts the arc $\widehat{BB_1}$ in k_2 at its midpoint G.

The right angled triangles $\triangle AKH$ and $\triangle BMG$ are congruent, because of $AK = BM$ (as a half of equal chords) and $\angle KAH = \angle MBG = \gamma$. Hence, $AH = BG$ and $\angle AHK = \angle BGM$.

Then, the isosceles triangles $\triangle AO_1H$ and $\triangle BO_2G$ are congruent and the circles k_1 and k_2 have equal radii.

This proves the assertion of the proposition. □

Since the equal segments AA_1 and BB_1 in $\triangle ABC$ (fig. 1) appear under the same angle 2γ from C, the circles k_1 and k_2 have equal radii (Proposition 3.2).

Let now $CJ \cap k_1 = H$ and $CJ \cap k_2 = G$.

The points H and G lie on the same ray $CJ^{\rightarrow}$. Since CJ bisects the angles $\angle ACA_1$ and $\angle BCB_1$, the point H is midpoint of the arc $\widehat{AA_1}$ in k_1, and the point G is midpoint of $\widehat{BB_1}$ in k_2.

Let K be the midpoint of the chord AA_1, M be the midpoint of the chord BB_1, $HK \cap k_1 = N$ and $GM \cap k_2 = L$. Hence, the segments HN and GL are diameters of the circles k_1 and k_2 respectively. The triangles $\triangle CHN$ and $\triangle CGL$ are right angled with right angles at the vertex C.

The quadrilateral $CJKN$ can be inscribed in a circle and it follows that

$$|HK||HN| = |HJ||HC|. \tag{1}$$

The quadrilateral $CJML$ can be inscribed in a circle and it follows that

$$|GM||GL| = |GJ||GC|. \tag{2}$$

Remark 3.3. The equalities (1) and (2) are also a consequence of the similarities $\triangle HKJ \sim \triangle HCN$ and $\triangle GMJ \sim \triangle GCL$.

Since the circles k_1 and k_2 have equal radii and the chords AA_1 and BB_1 are equal, then $HK = GM$ and $HN = GL$. If we put $d = |CJ| > 0$, $x = |HJ| > 0$, $y = |GJ| > 0$, then $|HC| = x + d$ and $|GC| = y + d$.

The left-hand sides of equalities (1) and (2) are equal, so are their right hand sides. Hence

$$x(x + d) = y(y + d) \;\Leftrightarrow\; (x - y)(x + y + d) = 0. \tag{3}$$

Since $x + y + d \neq 0$, equality (3) is equivalent to the equality

$$x - y = 0 \,.\, \frac{1}{x + y + d} = 0,$$

which directly implies $x = y$.

Remark 3.4. If we denote the equal positive left-hand sides of equalities (1) and (2) by a^2, we get respectively the quadratic equations

$$\begin{aligned} x^2+dx-a^2=0 \quad &\Leftrightarrow \left(x+\frac{d}{2}\right)^2-\left(\frac{\sqrt{4\,a^2+d^2}}{2}\right)^2=0 \\ &\Leftrightarrow \left(x+\frac{\sqrt{4\,a^2+d^2}+d}{2}\right)\left(x-\frac{\sqrt{4\,a^2+d^2}-d}{2}\right)=0 \\ &\Leftrightarrow x-\frac{\sqrt{4\,a^2+d^2}-d}{2}=0\,.\left(x+\frac{\sqrt{4\,a^2+d^2}+d}{2}\right)^{-1}=0, \end{aligned}$$

and, analogously,

$$y^2+dy-a^2=0,$$

with the same solution

$$x=y=\frac{1}{2}\left(\sqrt{4a^2+d^2}-d\right).$$

Hence, the points H and G, which lie on the same ray, coincide and CG is the common chord of the circles k_1 and k_2.

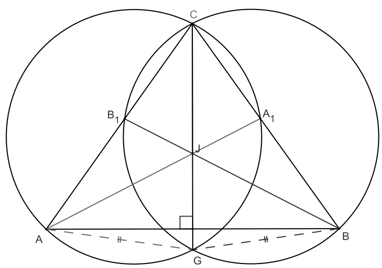

Figure 3.

As a consequence of the conditions

- CG is a common side,
- $\angle ACG=\angle BCG$ (CG is the bisector of $\angle ACB$),
- $\angle CAG=\angle CBG$ (CG is the common chord of two circles with equal radii, hence $\widehat{CA_1G}=\widehat{CB_1G}$),

the triangles $\triangle AGC$ and $\triangle BGC$ are congruent (Figure 3).

Thus, $CA=CB$ and $\triangle ABC$ is isosceles. The direct proof of Theorem 3.1 is complete. □

Remark 3.5. In this proof, the condition that the segments AA_1 and BB_1 are internal bisectors of the angles based at AB in $\triangle ABC$ is not necessary.

It is only of importance that they are equal by length cevians and their intersection point lies on the bisector of $\angle ACB$.

We recall that a *cevian* is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension).

In fact we proved directly the following

Theorem 3.6. *If in a $\triangle ABC$ the segments AA_1 $(A_1 \in BC)$ and BB_1 $(B_1 \in AC)$ interesct at a point on the bisector of $\angle ACB$ and are equal by length, then $\triangle ABC$ is isosceles.*

3.2. SECOND PROOF OF THE THEOREM OF LEHMUS-STEINER

The idea for this proof comes from Problem 2.1–2.16 in [2]: *Find a direct proof of Lehmus-Steiner's theorem as a consequence of Stewart's theorem.*

We need the notion *algebraic measure* (*relative measure*) of a line segment.

On any straight line there are two (opposite to each other) directions. The *axis* is a couple of a straight line and a fixed (positive) direction on it.

Let g^+ denotes any axis. For any non zero line segment MN on g^+ we can define its *relative* (*algebraic*) *measure* by $\overline{MN} = \varepsilon|MN|$, where $\varepsilon = +1$ in case $\overrightarrow{MN}$ has the same direction as g^+, and $\varepsilon = -1$ in case $\overrightarrow{MN}$ has the opposite direction with respect to g^+.

Stewart's theorem yields a relation between the lengths of the sides of a triangle and the length of a cevian.

Let in $\triangle ABC$ the line segment CP, $P \in AB$, be a cevian (more general, let $\{C; A, B, P\}$ be a quadruple of points such that A, B, P are collinear).

Theorem 3.7 (Theorem of Stewart)**.** *If A, B, P are three collinear points and C is any point then*

$$|CA|^2 \cdot \overline{BP} + |CB|^2 \cdot \overline{PA} + |CP|^2 \cdot \overline{AB} + \overline{BP} \cdot \overline{PA} \cdot \overline{AB} = 0.$$

Remark 3.8. Using the Pythagoras theorem, the proof of Steward's theorem is a simply verification.

In what follows we prove the *equal internal bisectors theorem* in the following formulation.

Theorem 3.9. *The straight line segments bisecting the angles at the base of a triangle and terminating at the opposite sides are equal if and only if the triangle is isosceles.*

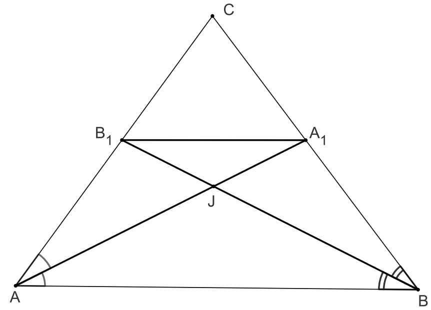

Figure 4.

Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in a triangle ABC (Figure 4).

Since the triples $\{B, A_1, C\}$ and $\{A, B_1, C,\}$ consist of collinear points there exist integers α and β such that

$$\begin{aligned} \overline{BA_1} &= \alpha\,\overline{BC}, \qquad \overline{A_1C} = (1-\alpha)\,\overline{BC}, \quad 0<\alpha<1; \\ \overline{AB_1} &= \beta\,\overline{AC}, \qquad \overline{B_1C} = (1-\beta)\,\overline{AC}, \quad 0<\beta<1. \end{aligned} \tag{4}$$

Using the fact that $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ are the internal bisectors of $\angle CAB$ and $\angle CBA$ in a triangle ABC, i. e. that

$$\frac{\overline{CA_1}}{\overline{A_1B}} = \frac{|CA|}{|AB|}, \quad \frac{\overline{CB_1}}{\overline{B_1A}} = \frac{|CB|}{|BA|},$$

from relations (4) we obtain

$$\begin{aligned} \alpha &= \frac{|AB|}{|AB|+|AC|}, \qquad 1-\alpha = \frac{|AC|}{|AB|+|AC|}, \\ \beta &= \frac{|AB|}{|AB|+|BC|}, \qquad 1-\beta = \frac{|BC|}{|AB|+|BC|}. \end{aligned} \tag{5}$$

Applying Stewart's theorem for the quadruple $\{A; B, A_1, C\}$

$$|AB|^2.\overline{A_1C} + |AA_1|^2.\overline{CB} + |AC|^2.\overline{BA_1} + \overline{A_1C}.\overline{CB}.\overline{BA_1} = 0,$$

and for the quadruple $\{B; A, B_1, C\}$

$$|BA|^2.\overline{B_1C} + |BB_1|^2.\overline{CA} + |BC|^2.\overline{AB_1} + \overline{B_1C}.\overline{CA}.\overline{AB_1} = 0,$$

from (4) and (5) we get

$$|AA_1|^2 = \frac{|AB||AC|}{(|AB|+|AC|)^2}\,\{(|AB|+|AC|)^2 - |BC|^2\},$$

$$|BB_1|^2 = \frac{|AB||BC|}{(|AB|+|BC|)^2}\,\{(|AB|+|BC|)^2 - |AC|^2\},$$

and finally

$$(|AA_1| - |BB_1|)\,\frac{(|AA_1| + |BB_1|)}{|AB|} = (|AC| - |BC|)$$
$$\times \left\{1+\frac{|AC||BC|(|AB|^2+|AC|^2+|BC|^2+2|AB|(|AC|+|BC|)+|AC||BC|)}{(|AB|+|BC|)^2(|AB|+|AC|)^2}\right\}.$$

Using the denotations

$$X := \frac{(|AA_1| + |BB_1|)}{|AB|}$$

and

$$Y := \left\{1+\frac{|AC||BC|(|AB|^2+|AC|^2+|BC|^2+2|AB|(|AC|+|BC|)+|AC||BC|)}{(|AB|+|BC|)^2(|AB|+|AC|)^2}\right\},$$

we rewrite the last equation in the form

$$(|AA_1| - |BB_1|)\,X = (|AC| - |BC|)\,Y.$$

Since $X \neq 0$ and $Y \neq 0$, the latter equation is equivalent to the equation

$$(|AA_1| - |BB_1|)\,\frac{X}{Y} = |AC| - |BC|. \tag{6}$$

Now, from (6) we see that $|\,AA_1| = |BB_1| \;\Leftrightarrow\; |AC| = |BC|$, which completes this direct proof of Lehmus-Steiner's theorem. $\square$

Remark 3.10. • In this proof, the condition that the segments AA_1 and BB_1 are internal bisectors of the angles based at AB in $\triangle ABC$ is necessary.

- Using equalities (5) we compute

$$\alpha - \beta = \frac{|AB|}{(|AB|+|AC|)(|AB|+|BC|)}\,(|BC| - |AC|)$$

and obtain

$$A_1B_1 \parallel AB \quad\Leftrightarrow\quad \alpha = \beta \quad\Leftrightarrow\quad |AC| = |BC|.$$

The following statement is easily proved directly.

Proposition 3.11. *Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in $\triangle\,ABC$. Then $\triangle\,ABC$ is isosceles if and only if $A_1B_1 \parallel AB$.*

Proof. Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be the internal bisectors of $\angle CAB$ and $\angle CBA$, respectively, in $\triangle\,ABC$.

- Let $A_1B_1 \parallel AB$ (Figure 5).

  It follows that $\triangle\,AA_1B_1$ and $\triangle\,BB_1A_1$ are isosceles and the quadrilateral ABA_1B_1 is a trapezium with $|AB_1| = |BA_1|\,(= |A_1B_1|)$.

  Hence, $\triangle\,ABC$ is isosceles.

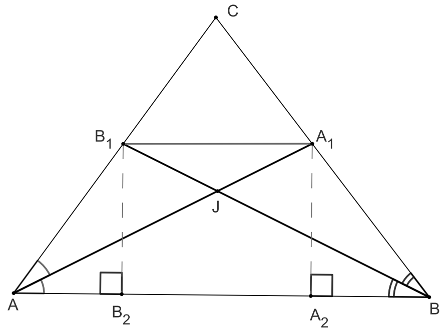

Figure 5.

- Let now $\triangle\,ABC$ be isosceles and $B_1B_2 \perp AB$ $(B_2 \in AB)$, $A_1A_2 \perp AB$ $(A_2 \in AB)$.

  Since $\triangle\,AA_1B \cong \triangle\,BB_1A$ (Figure 5), then $|AA_1| = |BB_1|$.

  Hence, $\triangle\,AA_1A_2 \cong \triangle\,BB_1B_2$, $|A_1A_2| = |B_1B_2|$ and $A_1B_1 \parallel AB$.

□

In view of this proposition we can reformulate the Lehmus-Steiner theorem in the following form:

Theorem 3.12. *Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in $\triangle\,ABC$. If $|AA_1| = |BB_1|$, then $A_1B_1 \parallel AB$.*

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 103

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FACULTY OF MATHEMATICS AND INFORMATICS
Volume 103

SMALL MINIMAL (3, 3)-RAMSEY GRAPHS

ALEKSANDAR BIKOV

We say that G is a $(3, 3)$-Ramsey graph if every 2-coloring of the edges of G forces a monochromatic triangle. The $(3, 3)$-Ramsey graph G is minimal if G does not contain a proper $(3, 3)$-Ramsey subgraph. In this work we find all minimal $(3, 3)$-Ramsey graphs with up to 13 vertices with the help of a computer, and we obtain some new results for these graphs. We also obtain new upper bounds for the independence number and new lower bounds for the minimum degree of arbitrary $(3, 3)$-Ramsey graphs.

Keywords: Ramsey graph, clique number, independence number, chromatic number

2000 Math. Subject Classification: 05C55

1. INTRODUCTION

In this work only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:

$\mathrm{V}(G)$ - the vertex set of G;

$\mathrm{E}(G)$ - the edge set of G;

$\overline{G}$ - the complement of G;

$\omega(G)$ - the clique number of G;

$\alpha(G)$ - the independence number of G;

$\chi(G)$ - the chromatic number of G;

$N_G(v), v \in \mathrm{V}(G)$ - the set of all vertices of G adjacent to v;

$d(v), v \in \mathrm{V}(G)$ - the degree of the vertex v, i.e. $d(v) = |N_G(v)|$;

$G(v), v \in \mathrm{V}(G)$ - subgraph of G induced by $N_G(v)$;

$G - v, v \in \mathrm{V}(G)$ - subgraph of G obtained from G by deleting the vertex v and all edges incident to v;

$G - e, e \in \mathrm{E}(G)$ - subgraph of G obtained from G by deleting the edge e;

$\Delta(G)$ - the maximum degree of G;

$\delta(G)$ - the minimum degree of G;

K_n - complete graph on n vertices;

C_n - simple cycle on n vertices;

$G_1 + G_2$ - graph G for which $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x, y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e. G is obtained by connecting every vertex of G_1 to every vertex of G_2.

All undefined terms can be found in [13].

Each partition

$$\mathrm{E}(G) = E_1 \cup \cdots \cup E_r, \quad E_i \cap E_j = \emptyset, \quad i \neq j \tag{1.1}$$

is called an r-coloring of the edges of G. We say that $H \subseteq G$ is a monochromatic subgraph of color i in the r-coloring (1.1), if $\mathrm{E}(H) \subseteq E_i$.

Let p and q be positive integers, $p \geq 2$ and $q \geq 2$. The notation $G \to (p, q)$ means that for every 2-coloring of $\mathrm{E}(G)$ there exists a p-clique of the first color or a q-clique of the second color. If $G \to (p, q)$, we say that G is a (p, q)-Ramsey graph. Similarly, the expression $G \to (p_1, \ldots, p_r)$ is defined for the r-colorings of $\mathrm{E}(G)$.

The smallest possible integer n for which $K_n \to (p, q)$ is called a Ramsey number and is denoted by $R(p, q)$. The Ramsey numbers $R(p_1, p_2, \ldots, p_r)$ are defined similarly. The existence of Ramsey numbers was proved by Ramsey in [32]. Only a few exact values of Ramsey numbers are known (see [30]). In this work we shall use the equality $R(3, 3) = 6$, which means that $K_6 \to (3, 3)$ and $K_5 \not\to (3, 3)$. Clearly, if $\omega(G) \geq 6$, then $G \to (3, 3)$. In [6] Erdös and Hajnal posed the problem:

Is there a graph $G \to (3, 3)$ with $\omega(G) < 6$?

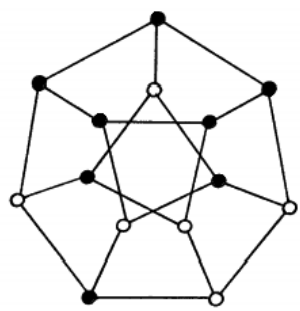


Figure 1: The complement of the van Lint graph from [12]

The first example of a graph which gives an affirmative answer to this question was given by van Lint. The complement of this graph is presented in Figure 1. Van Lint did not publish this result himself, but the graph was included in [12]. Later, Graham [11] constructed the smallest possible example of such a graph, namely $K_3 + C_5$. It is easy to see that the van Lint graph contains $K_3 + C_5$ (it is the subgraph induced by the black vertices in Figure 1).

There exist $(3,3)$-Ramsey graphs which do not contain K_5. These graphs have at least 15 vertices [29]. The first 15-vertex $(3,3)$-Ramsey graph which does not contain K_5 was constructed by Nenov [25]. This graph is obtained from the graph Γ presented in Figure 2 by adding a new vertex which is adjacent to all vertices of Γ.

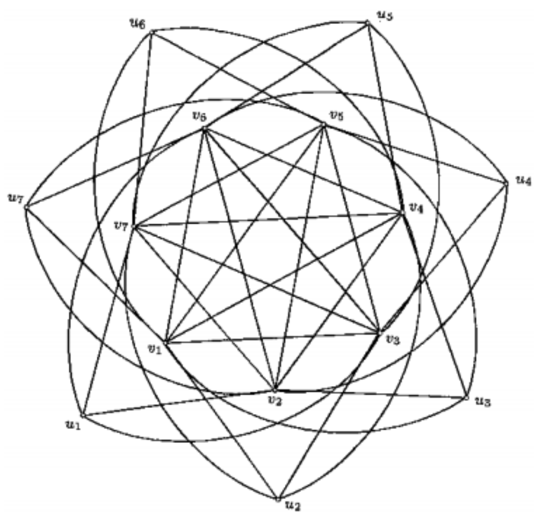


Figure 2: The Nenov graph Γ from [25]

Folkman [7] constructed a graph $G \to (3,3)$ with $\omega(G) = 3$. The minimum number of vertices of such graphs is not known. To date, we only know ([31] and [18]) that this minimum is between 19 and 786.

Obviously, if H is a (p,q)-Ramsey graph, then its every supergraph G is also a (p,q)-Ramsey graph.

Definition 1.1. *We say that G is a minimal (p,q)-Ramsey graph if $G \to (p,q)$ and $H \not\to (p,q)$ for each proper subgraph H of G.*

It is easy to see that K_6 is a minimal $(3,3)$-Ramsey graph and there are no minimal $(3,3)$-Ramsey graphs with 7 vertices. The only such 8-vertex graph is the Graham graph $K_3 + C_5$, and there is only one such 9-vertex graph, Nenov [22] (see Figure 3).

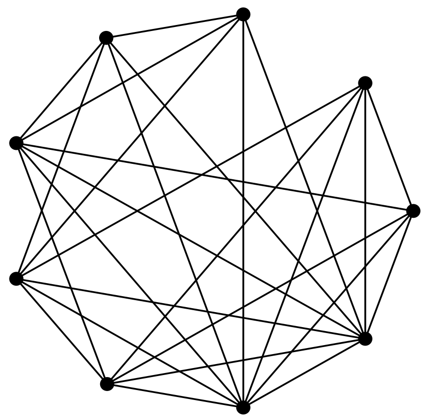


Figure 3: 9-vertex minimal $(3,3)$-Ramsey graph

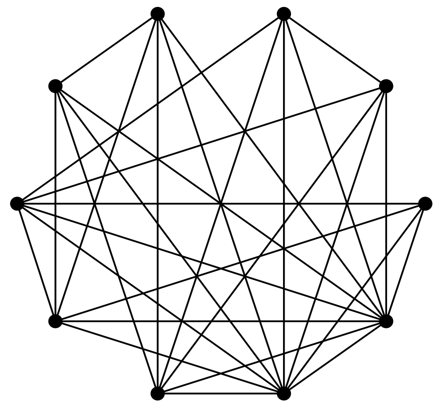


Figure 4: 10-vertex minimal $(3,3)$-Ramsey graph

For each pair of positive integers $p \geq 3$, $q \geq 3$ there exist infinitely many minimal (p,q)-Ramsey graphs [2], [8]. The simplest infinite sequence of minimal $(3,3)$-Ramsey graphs is formed by the graphs $K_3 + C_{2r+1}, r \geq 1$. This sequence contains the already mentioned graphs K_6 and $K_3 + C_5$. This sequence was found by Nenov and Khadzhiivanov in [27]. Later, it was rediscovered in [3], [9], [35].

Three 10-vertex minimal $(3,3)$-Ramsey graphs are known. One of them is $K_3 + C_7$ from the sequence $K_3 + C_{2r+1}, r \geq 1$. The other two were obtained by Nenov in [24] (the second graph is presented in Figure 4 and the third is a subgraph of $K_1 + \overline{C_9}$).

The main goal of this work is to find new minimal $(3,3)$-Ramsey graphs. To achieve this, we develop computer algorithms which are presented in Section 3. Using Algorithm 3.1, in Section 4 we find all minimal $(3,3)$-Ramsey graphs with up to 12 vertices. In the next Section 5 we find all 13-vertex minimal $(3,3)$-Ramsey graphs using Algorithm 3.11. From the graphs found in Section 4 and Section 5 we obtain interesting corollaries, which are presented in Section 6. With the help of Algorithm 3.8, in Section 7 and Section 8, respectively, we obtain new upper bounds for the independence number and new lower bounds for the minimum degree of minimal $(3,3)$-Ramsey graphs with an arbitrary number of vertices.

Similar computer aided research is made in [17], [29], [4], [5], [31], [36], [18] and [34]. We note that the algorithms from [29] were very useful to us.

This work is an extended version of the author Master Thesis written under the supervision of Prof. Nedyalko Nenov. The most essential new element is Algorithm 3.8, which is obtained jointly with Prof. Nenov.

We will need the following results:

Theorem 2.1. ([2],[8]) *Let* G *be a minimal* (p,p)*-Ramsey graph. Then,* $\delta(G) \geq (p-1)^2$*. In particular, when* $p=3$*, we have* $\delta(G) \geq 4$*.*

Definition 2.2. *We say that* G *is a Sperner graph if* $N_G(u) \subseteq N_G(v)$ *for some pair of vertices* $u, v \in \mathrm{V}(G)$*.*

Proposition 2.3. *If* G *is a minimal* (p,q)*-Ramsey graph, then* G *is not a Sperner graph.*

Proof. Suppose the opposite is true, and let $u, v \in \mathrm{V}(G)$ be such that $N_G(u) \subseteq N_G(v)$. We color the edges of $G-u$ with two colors in such a way that there is no monochromatic p-clique of the first color and no monochromatic q-clique of the second color. After that, for each vertex $w \in N_G(u)$ we color the edge $[u,w]$ with the same color as the edge $[v,w]$. We obtain a 2-coloring of the edges of G with no monochromatic p-cliques of the first color and no monochromatic q-cliques of the second color. □

Theorem 2.4. ([29]) *Let* G *be a* $(3,3)$ *- Ramsey graph and* $G \neq K_6$*. If* $|V(G)| \leq 14$*, then* $\omega(G) = 5$*.*

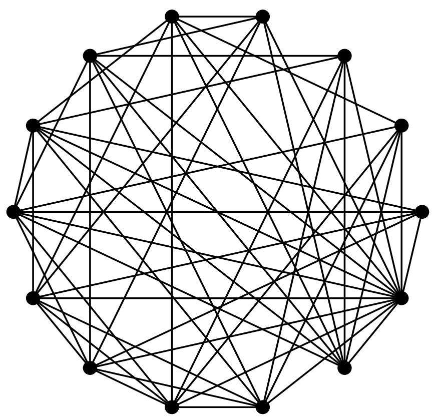


Figure 5: 14-vertex minimal $(3,3)$-Ramsey graph with a single 5 clique

According to Theorem 2.4, every $(3,3)$-Ramsey graph G with at most 14 vertices contains a 5-clique. There exist 14-vertex $(3,3)$-Ramsey graphs containing only a single 5-clique, an example of such a graph is presented in Figure 5. The graph in Figure 5 is obtained with the help of the only 15-vertex bicritical $(3,3)$-Ramsey graph with clique number 4 from [29]. First, by removing a vertex from the bicritical graph, we obtain 14-vertex graphs without 5 cliques. After that, by adding edges to the obtained graphs, we find a 14-vertex $(3,3)$-Ramsey graph with a single 5-clique whose subgraph is the minimal $(3,3)$-Ramsey graph in Figure 5. Let us note that in [29] the authors obtain all 15-vertex $(3,3)$-Ramsey graphs with clique number 4, and with the help of these graphs, one can find more examples of 14-vertex $(3,3)$-Ramsey graphs.

Theorem 2.5. ([19]) *Let G be a graph and $G \to (p,q)$. Then $\chi(G) \geq R(p,q)$. In particular, if $G \to (3,3)$, then $\chi(G) \geq 6$.*

Corollary 2.6. *Let $G \to (3,3)$, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $\chi(H) \geq 5$.*

Theorem 2.7. *Let G be a minimal $(3,3)$-Ramsey graph. Then, for each vertex $v \in \mathrm{V}(G)$ we have $\alpha(G(v)) \leq d(v) - 3$.*

Proof. Suppose the opposite is true, and let $A \subseteq N_G(v)$ be an independent set in $G(v)$ such that $|A| = d(v) - 2$. Let $a, b \in N_G(v) \setminus A$. Consider a 2-coloring of the edges of $G - v$ in which there are no monochromatic triangles. We color the edges $[v,a]$ and $[v,b]$ with the same color in such a way that there is no monochromatic triangle (if a and b are adjacent, we chose the color of $[v,a]$ and $[v,b]$ to be different from the color of $[a,b]$, and if a and b are not adjacent, then we chose an arbitrary color for $[v,a]$ and $[v,b]$). We color the remaining edges incident to v with the other color, which is different from the color of $[v,a]$ and $[v,b]$. Since $N_G(v) \setminus \{a,b\} = A$ is an independent set, we obtain a 2-coloring of the edges of G without monochromatic triangles, which is a contradiction. □

Corollary 2.8. *Let G be a minimal $(3,3)$-Ramsey graph and $d(v) = 4$ for some vertex $v \in \mathrm{V}(G)$. Then, $G(v) = K_4$.*

3. ALGORITHMS

In this section, the computer algorithms used in this work are presented.

The first algorithm is appropriate for finding all minimal $(3,3)$-Ramsey graphs with a small number of vertices.

Algorithm 3.1. *Finding all minimal $(3,3)$-Ramsey graphs with n vertices, where n is fixed and $7 \leq n \leq 14$.*

1. Generate all n-vertex non-isomorphic graphs with minimum degree at least 4, and denote the obtained set by $\mathcal{B}$.

2. Remove from $\mathcal{B}$ all Sperner graphs.

3. Remove from $\mathcal{B}$ all graphs with clique number not equal to 5.

4. Remove from $\mathcal{B}$ all graphs with chromatic number less than 6.

5. Remove from $\mathcal{B}$ all graphs which are not $(3,3)$-Ramsey graphs.

6. Remove from $\mathcal{B}$ all graphs which are not minimal $(3,3)$-Ramsey graphs.

Theorem 3.2. *Fix $n \in \{7, \dots, 14\}$. Then, after executing Algorithm 3.1, $\mathcal{B}$ consists of all n-vertex minimal $(3,3)$-Ramsey graphs.*

Proof. Step 6 guaranties that $\mathcal{B}$ contains only minimal $(3,3)$-Ramsey graphs with n vertices. Let G be an arbitrary n-vertex minimal $(3,3)$-Ramsey graph. We will prove that $G \in \mathcal{B}$. By Theorem 2.1, $\delta(G) \geq 4$, and by Theorem 2.3, G is not a Sperner graph. Since $|V(G)| \leq 14$, by Theorem 2.4 we have $\omega(G) = 5$. By Theorem 2.5, $\chi(G) \geq 6$. Therefore, after step 4, $G \in \mathcal{B}$. □

In Section 4 we apply Algorithm 3.1 to obtain all $(3,3)$-Ramsey graphs with up to 12 vertices. Algorithm 3.1 is not appropriate in the case $n \geq 13$, because the number of graphs generated in step 1 is too big. To find the 13-vertex minimal $(3,3)$-Ramsey graphs, we will apply Algorithm 3.11, which is given below.

In order to present the next algorithms, we shall need the following definitions and auxiliary propositions:

We say that a 2-coloring of the edges of a graph is $(3,3)$-free if it has no monochromatic triangles.

Definition 3.3. *Let G be a graph and $M \subseteq \mathrm{V}(G)$. Let G_1 be a graph which is obtained by adding a new vertex v to G such that $N_{G_1}(v) = M$. We say that M is a marked vertex set in G if there exists a $(3,3)$-free 2-coloring of the edges of G which cannot be extended to a $(3,3)$-free 2-coloring of the edges of G_1.*

It is clear that if $G \to (3,3)$, then there are no marked vertex sets in G. The following proposition is true:

Proposition 3.4. *Let G be a minimal $(3,3)$-Ramsey graph, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $N_G(v_i), i = 1, \dots, s$, are marked vertex sets in H.*

Proof. Suppose the opposite is true, i.e. $N_G(v_i)$ is not a marked vertex set in H for some $i \in \{1, \dots, s\}$. Since G is a minimal $(3,3)$-Ramsey graph, there exists a $(3,3)$-free 2-coloring of the edges of $G - v_i$, which induces a $(3,3)$-free 2-coloring of the edges of H. By supposition, we can extend this 2-coloring to a $(3,3)$-free 2-coloring of the edges of the graph $H_i = G - \{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_s\}$. Thus, we obtain a $(3,3)$-free 2-coloring of the edges of G, which is a contradiction. □

Definition 3.5. *Let $\{M_1, \dots, M_s\}$ be a family of marked vertex sets in the graph G. Let G_i be a graph which is obtained by adding a new vertex v_i to G such that $N_{G_i}(v_i) = M_i, i = 1, \dots, s$. We say that $\{M_1, \dots, M_s\}$ is a complete family of marked vertex sets in G, if for each $(3,3)$-free 2-coloring of the edges of G there exists $i \in \{1, \dots, s\}$ such that this 2-coloring can not be extended to a $(3,3)$-free 2-coloring of the edges of G_i.*

Proposition 3.6. *Let $v_1, \dots, v_s$ be independent vertices of the graph G and $H = G - \{v_1, \dots, v_s\}$. If $\{N_G(v_1), \dots, N_G(v_s)\}$ is a complete family of marked vertex sets in H, then $G \to (3,3)$.*

Proof. Consider a 2-coloring of the edges of G which induces a 2-coloring with no monochromatic triangles in H. According to Definition 3.5, this 2-coloring of the edges of H can not be extended in G without forming a monochromatic triangle. □

It is easy to prove the following strengthening of Proposition 3.4:

Proposition 3.7. *Let G be a minimal $(3,3)$-Ramsey graph, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $\{N_G(v_1), \dots, N_G(v_s)\}$ is a complete family of marked vertex sets in H. What is more, this family is a minimal complete family, in the sense that it does not contain a proper complete subfamily.*

Let G be a minimal $(3,3)$-Ramsey graph and $\alpha(G) \geq |\operatorname{V}(G)| - k \geq 1$. Let A be an independent set in G such that $|A| = |\operatorname{V}(G)| - k$. Then, $|\operatorname{V}(G - A)| = k$, and therefore the graph G is obtained by adding an independent set of vertices to the k-vertex graph $G - A$. From Proposition 2.3 it is easy to see that for a fixed k there are a finite number of minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) \geq |\operatorname{V}(G)| - k \geq 1$. Below we give an algorithm for finding all minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) \geq |\operatorname{V}(G)| - k \geq 1$, where k is fixed (but $\operatorname{V}(G)$ is not fixed).

Algorithm 3.8. (A. Bikov and N. Nenov) *Finding all minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < q$ and $\alpha(G) \geq |\operatorname{V}(G)| - k \geq 1$, where q and k are fixed positive integers.*

1. Denote by $\mathcal{A}$ the set of all k-vertex graphs H for which $\omega(H) < q$ and $\chi(H) \geq 5$. The obtained minimal $(3,3)$-Ramsey graphs will be output in the set $\mathcal{B}$, let $\mathcal{B} = \emptyset$.

2. For each graph $H \in \mathcal{A}$:

2.1. Find all subsets M of $\operatorname{V}(H)$ which have the properties:

(a) $K_{q-1} \not\subseteq H[M]$, i.e. M is a $K_{(q-1)}$-free subset.

(b) $M \not\subseteq N_H(v), \forall v \in \operatorname{V}(H)$.

(c) M is a marked vertex set in H (see Definition 3.3).

Denote by $\mathcal{M}(H)$ the family of subsets of $\operatorname{V}(H)$ which have the properties (a), (b) and (c). Enumerate the elements of $\mathcal{M}(H)$: $\mathcal{M}(H) = \{M_1, \dots, M_t\}$.

2.2. Find all minimal complete subfamilies of $\mathcal{M}(H)$ (see Definition 3.5). For each such found subfamily $\{M_{i_1}, \dots, M_{i_s}\}$ construct the graph $G = G(M_{i_1}, \dots, M_{i_s})$ by adding new independent vertices $v_1, v_2, \dots, v_s$ to $\operatorname{V}(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \dots, s$. Add G to $\mathcal{B}$. If there are no complete subfamilies of $\mathcal{M}(H)$, then no supergraphs of H are added to $\mathcal{B}$.

3. Remove the isomorphic copies of the graphs from $\mathcal{B}$.

4. Remove from $\mathcal{B}$ all non-minimal $(3,3)$-Ramsey graphs.

Remark 3.9. *It is clear that if G is a minimal $(3,3)$-Ramsey graph and $\omega(G) \geq 6$, then $G = K_6$. Obviously there are no $(3,3)$-Ramsey graphs with clique number less than 3. Therefore, we shall use Algorithm 3.8 only for $q \in \{4,5,6\}$.*

Theorem 3.10. *After executing Algorithm 3.8, the set* $\mathcal{B}$ *coincides with the set of all minimal* (3,3)*-Ramsey graphs* G *for which* $\omega(G) < q$ *and* $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$.

Proof. From step 2.2 it becomes clear that every graph G which is added to $\mathcal{B}$ is obtained by adding independent vertices $v_1, \dots, v_s$ to a graph $H \in \mathcal{A}$. Therefore, $\alpha(G) \geq s = |\mathrm{V}(G)| - |\mathrm{V}(H)| = |\mathrm{V}(G)| - k$. From $\omega(H) < q$ and $K_{q-1} \not\subseteq H[N_G(v_i)], i = 1, \dots, s$, it follows that $\omega(G) < q$. According to Proposition 3.6, after step 2.2 $\mathcal{B}$ contains only $(3,3)$-Ramsey graphs, and after step 4 $\mathcal{B}$ contains only minimal $(3,3)$-Ramsey graphs.

In order to prove that $\mathcal{B}$ contains all minimal $(3,3)$-Ramsey graphs which fulfill the conditions, consider an arbitrary minimal $(3,3)$-Ramsey graph G for which $\omega(G) < q$ and $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$. We will prove that $G \in \mathcal{B}$.

Denote $s = |\mathrm{V}(G)| - k \geq 1$. Let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. By 2.6, $\chi(H) \geq 5$. Therefore, after executing step 1, $H \in \mathcal{A}$.

From $\omega(G) < q$ it follows $\omega(G(v_i)) < q - 1$. By Proposition 2.3, G is not a Sperner graph, and therefore $N_G(v_i) \not\subseteq N_H(v), \forall v \in \mathrm{V}(H)$. According to Proposition 3.4, $N_G(v_i)$ are marked vertex sets in H. Therefore, after executing step 2.1, $N_G(v_i) \in \mathcal{M}(H), i = 1, \dots, s$.

From Proposition 3.7 it becomes clear that $\{N_G(v_1), ..., N_G(v_s)\}$ is a minimal complete subfamily of $\mathcal{M}(H)$. Therefore, in step 2.2 the graph G is added to $\mathcal{B}$.

Thus, the theorem is proved. □

In order to find the 13-vertex minimal $(3,3)$-Ramsey graphs we shall use the following modification of Algorithm 3.8 in which $n = |\mathrm{V}(G)|$ is fixed:

Algorithm 3.11. *Modification of Algorithm 3.8 for finding all* n*-vertex minimal* $(3,3)$*-Ramsey graphs* G *for which* $\omega(G) < q$ *and* $\alpha(G) \geq n - k \geq 1$*, where* q*,* k *and* n *are fixed positive integers.*

In step 2.2 of Algorithm 3.8 add the condition to consider only minimal complete subfamilies $\{M_{i_1}, ..., M_{i_s}\}$ *of* $\mathcal{M}(H)$ *in which* $s = n - k$.

4. MINIMAL (3, 3)-RAMSEY GRAPHS WITH UP TO 12 VERTICES

We execute Algorithm 3.1 for $n = 7, 8, 9, 10, 11, 12$, and we find all minimal $(3,3)$-Ramsey graphs with up to 12 vertices except K_6. In this way, we obtain the known results: there is no minimal $(3,3)$-Ramsey graph with 7 vertices, the Graham graph $K_3 + C_5$ is the only such 8-vertex graph, and there exists only one such 9-vertex graph, the Nenov graph from [22] (see Figure 3). We also obtain the following new results:

Theorem 4.1. *There are exactly 6 minimal 10-vertex* $(3,3)$*-Ramsey graphs. These graphs are given in Figure 14, and some of their properties are listed in Table 2.*

Theorem 4.2. *There are exactly 73 minimal 11-vertex* $(3,3)$*-Ramsey graphs. Some of their properties are listed in Table 3. Examples of 11-vertex minimal* $(3,3)$*-Ramsey graphs are given in Figure 15 and Figure 16.*

Theorem 4.3. *There are exactly 3041 minimal 12-vertex* $(3,3)$*-Ramsey graphs. Some of their properties are listed in Table 4. Examples of 12-vertex minimal* $(3,3)$*-Ramsey graphs are given in Figure 17 and Figure 18.*

We will use the following enumeration for the obtained minimal $(3,3)$-Ramsey graphs:

- $G_{10.1}, \ldots, G_{10.6}$ are the 10-vertex graphs;
- $G_{11.1}, \ldots, G_{11.73}$ are the 11-vertex graphs;
- $G_{12.1}, \ldots, G_{12.3041}$ are the 12-vertex graphs;

The indices correspond to the order of the graphs' canonical labels defined in *nauty* [20].

Detailed data for the number of graphs obtained at each step of the execution of Algorithm 3.1 is given in Table 1.

Step of Algorithm 3.1	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
1	424	15 471	1 249 973	187 095 840	48 211 096 031
2	59	2 365	206 288	33 128 053	9 148 907 379
3	9	380	41 296	8 093 890	2 763 460 021
4	1	7	356	78 738	44 904 195
5	1	3	126	23 429	11 670 079
6	1	1	6	73	3041

Table 1: Steps in finding all minimal $(3,3)$-Ramsey graphs with up to 12 vertices

5. MINIMAL $(3,3)$-RAMSEY GRAPHS WITH 13 VERTICES

The method we apply for findning all 13-vertex minimal $(3,3)$-Ramsey graphs consists of two parts:

1. First, we find the 13-vertex minimal $(3,3)$-Ramsey graphs with independence number 2. We use that (see [30]) $R(3,6) = 18$, and that all graphs G for which $\alpha(G) < 3$ and $\omega(G) < 6$ are known [21]. Among them, the 13-vertex graphs are 275 086. By computer check, we find that exactly 13 of these graphs are minimal $(3,3)$-Ramsey graphs.

2. It remains to find the 13-vertex minimal $(3,3)$-Ramsey graphs with independence number at least 3. To do this, we execute Algorithm 3.11 with $n = 13$; $k = 10$; $q = 6$. First, in step 1 of Algorithm 3.11 we find all 1 923 103 graphs H with 10 vertices for which $\omega(H) \leq 5$ and $\chi(H) \geq 5$. After that, in step 2 of Algorithm 3.11 we add 3 independent vertices to the obtained 10-vertex graphs,

$\|E(G)\|$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\|Aut(G)\|$	#
30	1	4	1	9	6	2	3	6	6	4	2
31	1	5	4			3	3			8	2
32	2	6	1							16	1
33	1									84	1
34	1										

Table 2: Some properties of the 10-vertex minimal $(3,3)$-Ramsey graphs

$\|E(G)\|$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\|Aut(G)\|$	#
35	6	4	5	8	1	2	4	6	73	1	20
36	13	5	58	10	72	3	66			2	29
37	23	6	10			4	3			4	14
38	25									6	1
39	5									8	4
41	1									12	1
										16	3
										24	1

Table 3: Some properties of the 11-vertex minimal $(3,3)$-Ramsey graphs

$\|E(G)\|$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\|Aut(G)\|$	#
38	5	4	129	8	43	2	124	6	3 041	1	1 792
39	27	5	2 178	9	1 196	3	2 431			2	851
40	144	6	611	11	1 802	4	485			4	286
41	418	7	123			5	1			6	1
42	1 014									8	67
43	459									12	16
44	224									16	18
45	351									24	6
46	299									32	1
47	84									36	1
48	16									96	1
										108	1

Table 4: Some properties of the 12-vertex minimal $(3,3)$-Ramsey graphs

$\|E(G)\|$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\|Aut(G)\|$	#
41	4	4	13 725	8	16	2	13	6	306 622	1	251 976
42	44	5	191 504	9	61 678	3	218 802	7	13	2	46 487
43	220	6	85 932	10	175 108	4	86 721			3	10
44	1 475	7	15 391	12	69 833	5	1 097			4	6 851
45	7 838	8	83			6	2			6	83
46	28 805									8	916
47	33 810									12	129
48	26 262									16	106
49	39 718									24	44
50	62 390									32	12
51	59 291									36	3
52	34 132									40	1
53	10 878									48	11
54	1 680									72	3
55	86									96	2
56	2									144	1

Table 5: Some properties of the 13-vertex minimal $(3,3)$-Ramsey graphs

and thus, we obtain all 306 622 minimal $(3, 3)$-Ramsey graphs with 13-vertices and independence number at least 3.

Finally, we obtain the following

Theorem 5.1. *There are exactly 306 635 minimal 13-vertex* $(3, 3)$*-Ramsey graphs. Some of their properties are listed in Table 5. Examples of 13-vertex minimal* $(3, 3)$*-Ramsey graphs are given in Figure 6, Figure 20 and Figure 21.*

We denote the obtained 13-vertex $(3, 3)$-Ramsey graphs by $G_{13.1}, \ldots, G_{13.306635}$.

As was noted, all graphs G for which $\alpha(G) < 3$ and $\omega(G) < 6$ are known and from $R(3, 6) = 18$ it follows that these graphs have at most 17 vertices. By computer check we find that there are no minimal $(3, 3)$-Ramsey graphs with independence number 2 and more than 13 vertices. Thus, we prove the following

Theorem 5.2. *Let G be a minimal* $(3, 3)$*-Ramsey graph and* $\alpha(G) = 2$*. Then,* $|\mathrm{V}(G)| \leq 13$*. There are exactly 145 minimal* $(3, 3)$*-Ramsey graphs for which* $\alpha(G) = 2$*:*

- 8-vertex: 1 ($K_3 + C_5$);
- 9-vertex: 1 (see Figure 3);
- 10-vertex: 3 ($G_{10.3}$, $G_{10.5}$, $G_{10.6}$, see Figure 14);
- 11-vertex: 4 ($G_{11.46}$, $G_{11.47}$, $G_{11.54}$, $G_{11.69}$, see Figure 16);
- 12-vertex: 124;
- 13-vertex: 13 (see Figure 21);

By executing Algorithm 3.11($n = 10, 11, 12; k = 7, 8, 9; q = 6$), we find all minimal $(3, 3)$-Ramsey graphs with 10, 11 and 12 vertices and independence number greater than 2. In this way, with the help of Theorem 5.2, we obtain a new proof of Theorem 4.1, Theorem 4.2 and Theorem 4.3.

6. COROLLARIES FROM THE OBTAINED RESULTS

6.1. MINIMUM AND MAXIMUM DEGREE

By Theorem 2.1, if G is a minimal $(3, 3)$-Ramsey graph, then $\delta(G) \geq 4$. Via very elegant constructions, in [2] and [8] it is proved that the bound $\delta(G) \geq (p-1)^2$ in Theorem 2.1 is exact. However, these constructions are not very economical in the case $p = 3$. For example, the minimal $(3, 3)$-Ramsey graph G from [8] with $\delta(G) = 4$ is not presented explicitly, but it is proved that it is a subgraph of a graph with 17577 vertices. From the next theorem we see that the smallest minimal $(3, 3)$-Ramsey graph G with $\delta(G) = 4$ has 10 vertices:

Theorem 6.1. *Let G be a minimal* $(3, 3)$*-Ramsey graph and* $\delta(G) = 4$*. Then,* $|\mathrm{V}(G)| \geq 10$*. There is only one 10-vertex minimal* $(3, 3)$*-Ramsey graph G with* $\delta(G) = 4$*, namely $G_{10.2}$ (see Figure 14). What is more, G has only a single vertex of degree 4. For all other 10-vertex minimal* $(3, 3)$*-Ramsey graphs G,* $\delta(G) = 5$*.*

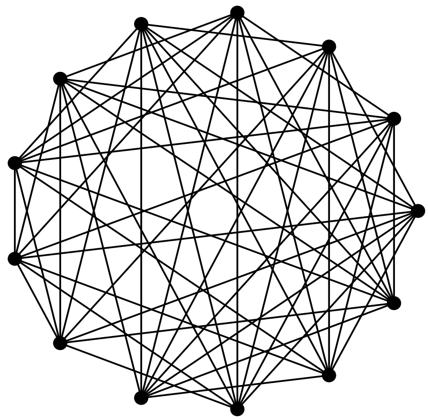


Figure 6: 8-regular 13-vertex minimal $(3,3)$-Ramsey graph

Let G be a $(3,3)$-Ramsey graph. By Theorem 2.5, $\chi(G) \geq 6$ and from the inequality $\chi(G) \leq \Delta(G)+1$ (see [13]) we obtain $\Delta(G) \geq 5$. From the Brooks' Theorem (see [13]) it follows that if $G \neq K_6$, then $\Delta(G) \geq 6$. The following related question arises naturally:

Are there minimal $(3,3)$-Ramsey graphs which are 6-regular?
(i.e. $d(v) = 6, \forall v \in \mathrm{V}(G)$)

From the obtained minimal $(3,3)$-Ramsey graphs we see that the following theorem is true:

Theorem 6.2. *Let G be a regular minimal $(3,3)$-Ramsey graph and $G \neq K_6$. Then, $|\mathrm{V}(G)| \geq 13$. There is only one regular minimal $(3,3)$-Ramsey with 13 vertices, and this is the graph presented in Figure 6, which is 8-regular.*

Regarding the maximum degree of the minimal $(3,3)$-Ramsey graphs, we obtain the following result:

Theorem 6.3. *Let G be a minimal $(3,3)$-Ramsey graph. Then:*
(a) $\Delta(G) = |\mathrm{V}(G)| - 1$, *if* $|\mathrm{V}(G)| \leq 10$.
(b) $\Delta(G) \geq 8$, *if* $|\mathrm{V}(G)| = 11$, 12 *or* 13.

6.2. CHROMATIC NUMBER

By Theorem 2.5, if G is a $(3,3)$-Ramsey graph, then $\chi(G) \geq 6$.

From the obtained minimal $(3,3)$-Ramsey graphs we derive the following results:

Theorem 6.4. *Let G be a minimal $(3,3)$-Ramsey graph and $|\mathrm{V}(G)| \leq 12$. Then $\chi(G) = 6$.*

Theorem 6.5. *Let G be a minimal $(3,3)$-Ramsey graph and $|V(G)| \leq 14$. Then $\chi(G) \leq 7$. The smallest 7-chromatic minimal $(3,3)$-Ramsey graphs are the 13 minimal $(3,3)$-Ramsey graph with 13 vertices and independence number 2, given in Figure 21.*

Proof. Suppose the opposite is true, i.e. $\chi(G) \geq 8$. Then, according to [26], $G = K_1 + Q$, where $\overline{Q}$ is the graph presented in Figure 7. The graph $K_1 + Q$ is a $(3,3)$-Ramsey graph, but it is not minimal. By Theorem 6.4, there are no 7-chromatic minimal $(3,3)$-Ramsey graphs with less than 13 vertices. The graphs in Figure 21 are 13-vertex minimal $(3,3)$-Ramsey graphs with independence number 2, and therefore these graphs are 7-chromatic. By computer check, we find that among the 13-vertex $(3,3)$-Ramsey graphs with independence number greater than 2 there are no 7-chromatic graphs. □

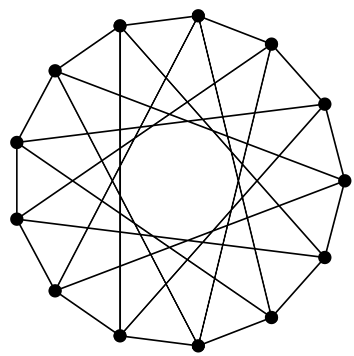


Figure 7: Graph $\overline{Q}$

6.3. MULTIPLICITIES

Definition 6.6. *Denote by $M(G)$ the minimum number of monochromatic triangles in all 2-colorings of $\mathrm{E}(G)$. The number $M(G)$ is called a K_3-multiplicity of the graph G.*

In [10] the K_3-multiplicities of all complete graphs are computed, i.e. $M(K_n)$ is computed for all positive integers n. Similarly, the K_p-multiplicity of a graph is defined [14]. The following papers are dedicated to the computation of the multiplicities of some concrete graphs: [15], [16], [33], [1], [28].

With the help of a computer, we check the K_3-multiplicities of the obtained minimal $(3,3)$-Ramsey graphs and we derive the following results:

Theorem 6.7. *If G is a minimal $(3,3)$-Ramsey graph, $|\mathrm{V}(G)| \le 13$ and $G \ne K_6$, then $M(G) = 1$.*

We suppose that the following hypothesis is true:

Hypothesis 6.8. *If G is a minimal $(3,3)$-Ramsey graph and $G \ne K_6$, then $M(G) = 1$.*

In support to this hypothesis we prove the following:

Proposition 6.9. *If G is a minimal $(3,3)$-Ramsey graph, $G \ne K_6$ and $\delta(G) \le 5$, then $M(G) = 1$.*

Proof. Let $v \in \mathrm{V}(G)$ and $d(v) \le 5$. Consider a 2-coloring of $\mathrm{E}(G - v)$ without monochromatic triangles. We will color the edges incident to v with two colors in such a way that we will obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle. To achieve this, we consider two cases:

Case 1: $d(v) = 4$. By Corollary 2.8, $G(v) = K_4$. Let $N_v = \{a, b, c, d\}$ and suppose that $[a, b]$ is colored with the first color. Then, $[c, d]$ is also colored with the first color (otherwise, by coloring $[v, a]$ and $[v, b]$ with the second color and $[v, c]$ and $[v, d]$ with the fist color, we would obtain a 2-coloring of $\mathrm{E}(G)$ without monochromatic triangles). Thus, $[a, b]$ and $[c, d]$ are colored in the first color. We color $[v, a]$ and $[v, b]$ with the first color and $[v, c]$ and $[v, d]$ with the second color. We obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle $[v, a, b]$.

Case 2: $d(v) = 5$. Since $\omega(G) \le 5$, in $N_G(v)$ there are two non-adjacent vertices a and b. From $G \to (3, 3)$ it follows easily that in $G(v) - \{a, b\}$ there is an edge of the first color and an edge of the second color. Therefore, we may assume that in $G(v) - \{a, b\}$ there is exactly one edge of one of the colors, say the first color. We color $[v, a]$ and $[v, b]$ with the second color and the other three edges incident to v with the first color. We obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle. □

In the end, we note that, according to [27], $M(K_3 + C_{2r+1}) = 1,\ r \ge 2$, which also supports our hypothesis.

6.4. AUTOMORPHISM GROUPS

Denote by $Aut(G)$ the automorphism group of the graph G. We use the *nauty* programs [20] to find the number of automorphisms of the obtained minimal $(3,3)$-Ramsey graphs with 10, 11, 12 and 13 vertices. Most of the obtained graphs have small automorphism groups (see Table 2, Table 3, Table 4 and Table 5). We list the graphs with at least 60 automorphisms:

- The graphs of the form $K_3 + C_{2r+1}$: $|Aut(K_3 + C_5)| = 60$. $|Aut(K_3 + C_7)| = 84$, $|Aut(K_3 + C_9)| = 108$;
- $|Aut(G_{12.2240})| = 96$ (see Figure 18);

- $|Aut(G_{13.255653})| = 144$, $|Aut(G_{13.248305})| = 96$, $|Aut(G_{13.304826})| = 96$, $|Aut(G_{13.113198})| = 72$, $|Aut(G_{13.175639})| = 72$, $|Aut(G_{13.302168})| = 72$ (see Figure 20);

7. UPPER BOUNDS ON THE INDEPENDENCE NUMBER OF THE MINIMAL $(3,3)$-RAMSEY GRAPHS

In regard to the maximal possible value of the independence number of the minimal $(3,3)$-Ramsey graphs, the following theorem holds:

Theorem 7.1. ([23]) *If G is a minimal $(3,3)$-Ramsey graph, $G \neq K_6$ and $G \neq K_3 + C_5$, then $\alpha(G) \leq |V(G)| - 7$. There is a finite number of graphs for which equality is reached.*

From Theorem 7.1 it follows that by executing Algorithm 3.8($q = 6; k = 8$) we obtain all minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) = |\mathrm{V}(G)| - 7$ or $\alpha(G) = |\mathrm{V}(G)| - 8$. Hence, we derive the following supplements to Theorem 7.1:

Theorem 7.2. *There are exactly 11 minimal $(3,3)$-Ramsey graphs G, for which $\alpha(G) = |\mathrm{V}(G)| - 7$:*

- 9-vertex: 1 (Figure 3);
- 10-vertex: 3 ($G_{10.1}$, $G_{10.2}$, $G_{10.4}$, see Figure 14);
- 11-vertex: 3 ($G_{11.1}$, $G_{11.2}$, $G_{11.21}$, see Figure 15);
- 12-vertex: 1 ($G_{12.163}$, see Figure 17);
- 13-vertex: 2 ($G_{13.}$, $G_{13.}$, see Figure 19);
- 14-vertex: 1 (see Figure 8).

Theorem 7.3. *There are exactly 8633 minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) = |\mathrm{V}(G)| - 8$. The largest of these graphs has 26 vertices, and it is given in Figure 9. There is only one minimal $(3,3)$-Ramsey graph G for which $\alpha(G) = |\mathrm{V}(G)| - 8$ and $\omega(G) < 5$, and it is the 15-vertex graph $K_1 + \Gamma$ from [25] (see Figure 2).*

Corollary 7.4. *Let G be a minimal $(3,3)$-Ramsey graph and $|\mathrm{V}(G)| \geq 27$. Then, $\alpha(G) \leq |\mathrm{V}(G)| - 9$.*

According to Theorem 7.3, if G is a minimal $(3,3)$-Ramsey graph, $\omega(G) < 5$, and $G \neq K_1 + \Gamma$, then $\alpha(G) \leq |\mathrm{V}(G)| - 9$. From Theorem 2.4 it follows that by executing Algorithm 3.8($q = 5; k = 9$) we obtain all minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < 5$ and $\alpha(G) = |\mathrm{V}(G)| - 9$, and the graph $K_1 + \Gamma$. As a result of the execution of this algorithm we derive:

Theorem 7.5. *There are exactly 8903 minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < 5$ and $\alpha(G) = |\mathrm{V}(G)| - 9$. The largest of these graphs has 29 vertices, and it is given in Figure 10.*

```
0 0 0 0 0 0 0 1 1 0 0 0 1 1
0 0 0 0 0 0 0 1 0 1 1 0 1 1
0 0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 0 0 0 1 1 1 0 1 1
1 1 0 0 1 1 0 0 1 0 0 0 1 1
1 0 1 1 0 0 1 1 0 0 0 0 1 1
0 1 1 0 1 0 1 0 0 0 1 1 1 1
0 1 0 1 0 1 1 0 0 1 0 1 1 1
0 0 1 1 1 1 0 0 0 1 1 0 1 1
1 1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 0
```

Figure 8: 14-vertex minimal (3, 3)-Ramsey graph with independence number 7

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 1
1 1 1 1 0 0 1 1 0 0 1 0 0 0 0 1 1 0 0 1 1 1 0 0 0 1
1 1 1 0 1 1 0 0 1 1 0 0 0 1 1 0 0 0 1 0 1 1 0 0 0 1
0 0 0 1 0 1 0 1 0 1 0 1 1 0 1 0 1 1 1 1 0 1 0 0 0 1
0 0 0 0 1 0 1 0 1 0 1 1 1 1 0 1 0 1 1 1 1 0 0 0 0 1
1 0 1 1 0 0 0 0 1 1 1 0 1 1 1 1 1 1 0 0 0 0 0 1 1 1
1 1 0 0 1 1 1 1 0 0 0 1 0 1 1 1 1 1 0 0 0 0 1 0 1 1
0 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
```

Figure 9: 26-vertex minimal (3, 3)-Ramsey graph with independence number 18

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 0 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0 0 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 0 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 1 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 1 0
1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 0 1 0
1 1 1 0 1 0 1 1 1 1 0 0 0 1 1 0 0 0 0 1 1 0 0 1 1 1 1 0 1
0 1 0 1 0 1 0 0 0 1 0 1 1 0 1 0 1 1 1 1 1 0 0 0 1 1 1 1 1
1 0 0 1 1 0 0 0 1 0 1 1 1 1 0 1 1 0 1 0 1 1 0 0 0 1 1 1 1
1 1 1 0 0 1 1 1 1 1 1 0 0 0 0 1 0 1 0 0 1 1 1 0 0 0 1 1 1
0 0 0 0 0 1 1 1 0 0 1 1 1 1 1 0 0 1 1 1 1 1 1 1 0 0 1 1 0
0 0 0 0 0 1 1 0 1 1 0 1 0 0 1 1 1 1 1 1 0 1 1 1 1 1 0 0 1
0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 0 1
1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 1 1 1 1 0 1 1 0
```

Figure 10: 29-vertex minimal (3, 3)-Ramsey graph with clique number 4 and independence number 20

Corollary 7.6. *Let G be a minimal $(3,3)$-Ramsey graph such that $\omega(G) < 5$ and $|\operatorname{V}(G)| \geq 30$. Then $\alpha(G) \leq |\operatorname{V}(G)| - 10$.*

8. LOWER BOUNDS ON THE MINIMUM DEGREE OF THE MINIMAL $(3,3)$-RAMSEY GRAPHS

According to Proposition 3.4, if G is a minimal $(3,3)$-Ramsey graph, then for each vertex v of G, $N_G(v)$ is a marked vertex set in $G - v$, and therefore $N_G(v)$ is a marked vertex set in $G(v)$.

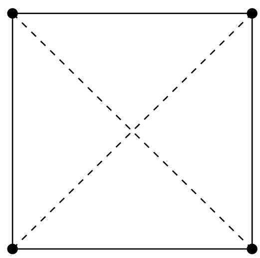
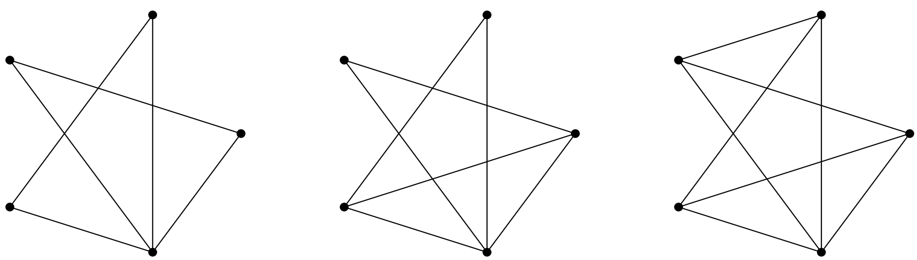


Figure 11: $(3,3)$-free 2-coloring of the edges of K_4

It is easy to see that if $W \subseteq \operatorname{V}(G)$ and $|W| \leq 3$, or $|W| = 4$ and $G[W] \neq K_4$, then W is not a marked vertex set in G. A $(3,3)$-free 2-coloring of K_4 which cannot be extended to a $(3,3)$-free 2-coloring of K_5 is shown in Figure 11. Therefore, the only 4-vertex graph N such that $\operatorname{V}(N)$ is a marked vertex set in N is K_4.



$N_{5.1}$ $N_{5.2}$ $N_{5.3}$

Figure 12: The graphs $N_{5.1}$, $N_{5.2}$, $N_{5.3}$

With the help of a computer, we obtain that there are exactly 3 graphs N with 5 vertices such that $K_4 \not\subset N$ and $\operatorname{V}(N)$ is a marked vertex set in N. Namely, they are the graphs $N_{5.1}$, $N_{5.2}$ and $N_{5.3}$ given in Figure 12. Note that $N_{5.1} \subset N_{5.2} \subset N_{5.3}$. From these results we derive

Theorem 8.1. *Let G be a minimal $(3,3)$-Ramsey graph and $\omega(G) \leq 4$. Then $\delta(G) \geq 5$. If $v \in \mathrm{V}(G)$ and $d(v) = 5$, then $G(v) = N_{5.i}$ for some $i \in \{1,2,3\}$ (see Figure 12).*

The bound $\delta(G) \geq 5$ in Theorem 8.1 is exact. For example, the graph $G = K_1 + \Gamma$ from [25] (see Figure 2) has 7 vertices v such that $d(v) = 5$ and $G(v) = N_{5.3}$.

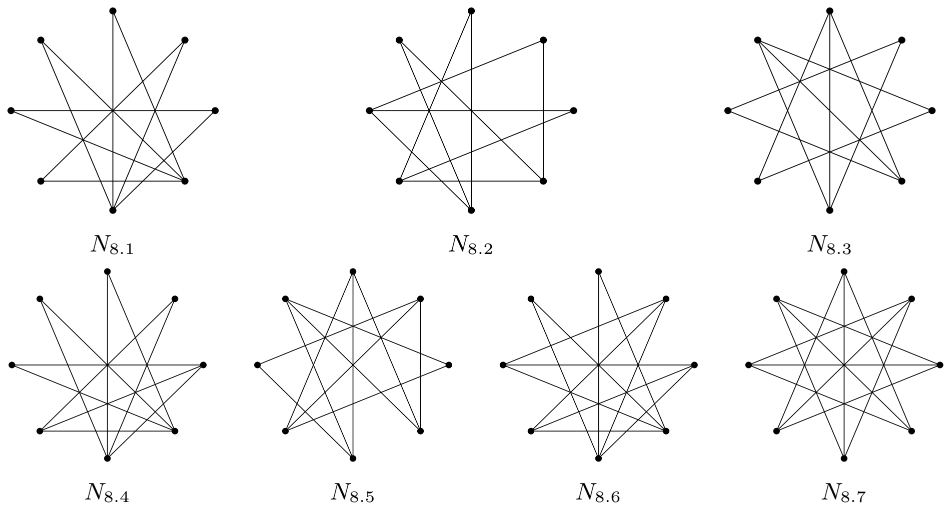


Figure 13: The graphs $N_{8.i}$, $\quad i = 1, \dots, 7$

With the help of a computer, we also obtain that the smallest graphs N such that $K_3 \not\subset N$ and $\mathrm{V}(N)$ is a marked vertex set in N have 8 vertices, and there are exactly 7 such graphs. Namely, they are the graphs $N_{8.i}, i = 1, \dots, 7$ presented in Figure 13. Among them, the minimal graphs are $N_{8.1}$, $N_{8.2}$ and $N_{8.3}$, and the remaining 4 graphs are their supergraphs. Thus, we derive the following

Theorem 8.2. *Let G be a minimal $(3,3)$-Ramsey graph and $\omega(G) = 3$. Then, $\delta(G) \geq 8$. If $v \in \mathrm{V}(G)$ and $d(v) = 8$, then $G(v) = N_{8.i}$ for some $i \in \{1, \dots, 7\}$ (see Figure 13).*

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```
0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 1 1 1 1
0 0 0 1 1 1 0 1 1 1
0 0 1 0 1 1 1 0 1 1
1 0 1 1 0 0 1 0 1 1
0 1 1 1 0 0 0 1 1 1
1 1 0 1 1 0 0 0 1 1
1 1 1 0 0 1 0 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.1}$

```
0 0 0 0 0 1 1 0 1 1
0 0 1 0 0 1 0 1 1 1
0 1 0 0 0 1 0 1 1 1
0 0 0 0 1 0 1 1 1 1
0 0 0 1 0 0 1 1 1 1
1 1 1 0 0 0 1 0 1 1
1 0 0 1 1 1 0 0 1 1
0 1 1 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.2}$

```
0 0 1 1 0 0 1 0 1 1
0 0 0 0 1 1 0 1 1 1
1 0 0 1 1 0 1 1 1 1
1 0 1 0 0 1 1 1 1 1
0 1 1 0 0 1 1 1 1 1
0 1 0 1 1 0 1 1 1 1
1 0 1 1 1 1 0 1 0 1
0 1 1 1 1 1 1 0 0 1
1 1 1 1 1 1 0 0 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.3}$

```
0 1 1 0 0 0 0 1 1 1
1 0 0 0 0 1 0 1 1 1
1 0 0 0 0 0 1 1 1 1
0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 1 1 1 1
0 1 0 1 0 0 0 1 1 1
0 0 1 0 1 0 0 1 1 1
1 1 1 1 1 1 1 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.4}$

```
0 1 1 0 0 0 0 1 1 1
1 0 0 1 0 0 1 1 1 1
1 0 0 0 1 1 0 1 1 1
0 1 0 0 1 1 1 1 0 1
0 0 1 1 0 1 1 0 1 1
0 0 1 1 1 0 1 0 1 1
0 1 0 1 1 1 0 1 0 1
1 1 1 1 0 0 1 0 1 1
1 1 1 0 1 1 0 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.5}$

```
0 1 1 1 0 0 1 1 0 1
1 0 0 0 1 1 1 0 1 1
1 0 0 1 0 0 1 1 1 1
1 0 1 0 0 0 1 1 1 1
0 1 0 0 0 1 1 1 1 1
0 1 0 0 1 0 1 1 1 1
1 1 1 1 1 1 0 0 0 1
1 0 1 1 1 1 0 0 1 1
0 1 1 1 1 1 0 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.6}$

Figure 14: 10-vertex minimal $(3,3)$-Ramsey graphs

```
0 0 0 0 0 0 1 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 1 1 1 0 0 1 1
0 1 1 1 0 0 0 1 0 1 1
0 0 1 1 0 0 1 0 1 1 1
1 0 0 1 0 1 0 0 1 1 1
1 1 1 0 1 0 0 0 1 1 1
1 1 0 0 0 1 1 1 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.1}$

```
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 1 0 1 1
1 0 0 0 0 1 1 0 1 1 1
0 1 0 1 1 0 1 0 0 1 1
0 0 1 1 1 1 0 0 0 1 1
1 1 1 1 0 0 0 0 1 1 1
1 1 1 0 1 0 0 1 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.2}$

```
0 0 0 1 0 0 0 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1
1 0 0 0 0 0 1 1 0 1 1
0 1 0 0 0 1 1 0 1 1 1
0 0 1 0 1 0 1 0 1 1 1
0 0 0 1 1 1 0 1 0 1 1
1 1 1 1 0 0 1 0 0 1 1
1 1 1 0 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.21}$

Figure 15: 11-vertex minimal $(3,3)$-Ramsey graphs
with independence number 4

```
0 0 1 0 1 1 1 0 1 0 1
0 0 0 1 1 1 0 1 0 1 1
1 0 0 1 0 0 1 0 1 1 1
0 1 1 0 0 0 0 1 1 1 1
1 1 0 0 0 1 1 1 0 1 1
1 1 0 0 1 0 1 1 1 0 1
1 0 1 0 1 1 0 0 1 1 1
0 1 0 1 1 1 0 0 1 1 1
1 0 1 1 0 1 1 1 0 0 1
0 1 1 1 1 0 1 1 0 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.46}$

```
0 0 1 1 1 0 0 0 1 0 1
0 0 0 0 0 1 1 1 0 1 1
1 0 0 1 1 1 1 0 1 1 1
1 0 1 0 1 1 0 1 1 1 1
1 0 1 1 0 0 1 1 1 1 1
0 1 1 1 0 0 1 1 1 1 1
0 1 1 0 1 1 0 1 1 1 1
0 1 0 1 1 1 1 0 1 1 1
1 0 1 1 1 1 1 1 0 1 0
0 1 1 1 1 1 1 1 1 0 0
1 1 1 1 1 1 1 1 0 0 0
```

$G_{11.47}$

```
0 1 0 0 1 1 0 0 1 1 1
1 0 0 0 1 1 0 0 1 1 1
0 0 0 1 0 0 1 1 1 1 1
0 0 1 0 0 0 1 1 1 1 1
1 1 0 0 0 1 1 1 1 0 1
1 1 0 0 1 0 1 1 0 1 1
0 0 1 1 1 1 0 1 0 1 1
0 0 1 1 1 1 1 0 1 0 1
1 1 1 1 1 0 0 1 0 0 1
1 1 1 1 0 1 1 0 0 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.54}$

```
0 1 1 0 1 1 1 0 0 0 1
1 0 0 1 0 1 1 0 0 1 1
1 0 0 1 1 0 0 1 1 0 1
0 1 1 0 0 0 0 1 1 1 1
1 0 1 0 0 1 1 1 1 0 1
1 1 0 0 1 0 1 1 0 1 1
1 1 0 0 1 1 0 0 1 1 1
0 0 1 1 1 1 0 0 1 1 1
0 0 1 1 1 0 1 1 0 1 1
0 1 0 1 0 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.69}$

Figure 16: 11-vertex minimal $(3, 3)$-Ramsey graphs
with independence number 2

```
0 0 0 0 1 0 0 1 0 1 1 1
0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 1 0 0 0 1 1 1 1
0 0 0 0 0 1 0 1 0 1 1 1
1 0 1 0 0 1 1 0 0 0 1 1
0 1 0 1 1 0 1 0 0 0 1 1
0 0 0 0 1 1 0 1 1 0 1 1
1 0 0 1 0 0 1 0 1 1 1 1
0 1 1 0 0 0 1 1 0 1 1 1
1 1 1 1 0 0 0 1 1 0 1 1
1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{12.163}$

Figure 17: 12-vertex minimal
$(3, 3)$-Ramsey graph
with independence number 5

```
0 1 0 0 0 0 1 1 0 1 1 1
1 0 0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 1 0 1 1 1 1
0 0 1 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1 1
1 1 1 1 0 0 0 1 1 0 0 1
1 1 0 0 1 1 1 0 1 0 0 1
0 0 1 1 1 1 1 1 0 0 0 1
1 1 1 1 1 1 0 0 0 0 1 1
1 1 1 1 1 1 0 0 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{12.2240}$

Figure 18: 12-vertex minimal
$(3, 3)$-Ramsey graph
with 96 automorphisms

```
0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 0 0 1 0 1 1 0 1 1
1 0 1 0 0 1 0 1 0 0 1 1 1
0 1 0 1 1 0 1 0 0 0 1 1 1
1 0 0 1 1 1 0 0 0 1 0 1 1
0 1 1 0 1 1 0 0 1 0 0 1 1
1 1 1 1 0 0 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.1}$

```
0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 0 0 1 1 1 1 1
1 1 1 0 0 0 0 0 0 0 1 1 1
1 0 0 1 1 0 0 0 1 1 0 1 1
0 1 0 1 0 1 0 1 0 1 0 1 1
0 0 1 0 1 1 0 1 1 0 0 1 1
1 1 1 1 1 1 1 0 0 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.2}$

Figure 19: 13-vertex minimal $(3,3)$-Ramsey graphs
with independence number 6

```
0 0 0 1 0 0 1 0 0 1 1 1 1
0 0 0 0 1 0 1 0 0 1 1 1 1
0 0 0 0 0 1 1 0 0 1 1 1 1
1 0 0 0 1 1 1 1 1 0 0 0 1
0 1 0 1 0 1 1 1 1 0 0 0 1
0 0 1 1 1 0 1 1 1 0 0 0 1
1 1 1 1 1 1 0 0 0 0 0 0 1
0 0 0 1 1 1 0 0 1 1 1 1 0
0 0 0 1 1 1 0 1 0 1 1 1 0
1 1 1 0 0 0 0 1 1 0 1 1 1
1 1 1 0 0 0 0 1 1 1 0 1 1
1 1 1 0 0 0 0 1 1 1 1 0 1
1 1 1 1 1 1 1 0 0 1 1 1 0
```

$G_{13.113198}$

```
0 0 0 0 0 0 1 1 1 0 0 0 1
0 0 1 1 0 0 0 0 0 1 1 1 1
0 1 0 1 0 0 0 0 0 1 1 1 1
0 1 1 0 0 0 0 0 0 1 1 1 1
0 0 0 0 0 1 1 1 1 1 1 1 0
0 0 0 0 1 0 1 1 1 1 1 1 0
1 0 0 0 1 1 0 1 1 1 1 0 1
1 0 0 0 1 1 1 0 1 1 0 1 1
1 0 0 0 1 1 1 1 0 0 1 1 1
0 1 1 1 1 1 1 1 0 0 0 0 1
0 1 1 1 1 1 1 0 1 0 0 0 1
0 1 1 1 1 1 0 1 1 0 0 0 1
1 1 1 1 0 0 1 1 1 1 1 1 0
```

$G_{13.175639}$

```
0 1 0 0 0 0 0 1 1 0 1 1 1
1 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 0 1 0 1 1 1 1
0 0 1 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 1 0 1 1 0 1 1
0 0 0 0 1 0 1 0 1 1 0 1 1
0 0 0 0 1 1 0 0 1 1 0 1 1
1 1 1 1 0 0 0 0 1 1 1 0 0
1 1 0 0 1 1 1 1 0 1 1 0 0
0 0 1 1 1 1 1 1 1 0 1 0 0
1 1 1 1 0 0 0 1 1 1 0 1 1
1 1 1 1 1 1 1 0 0 0 1 0 1
1 1 1 1 1 1 1 0 0 0 1 1 0
```

$G_{13.248305}$

```
0 1 0 0 0 0 0 0 1 1 1 1 0
1 0 0 0 0 0 0 0 1 1 1 0 1
0 0 0 1 1 0 0 0 0 1 0 1 1
0 0 1 0 1 0 0 0 0 1 0 1 1
0 0 1 1 0 0 0 0 0 1 0 1 1
0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 1 0 1 0 0 1 1 1
0 0 0 0 0 1 1 0 0 0 1 1 1
1 1 0 0 0 0 0 0 0 1 1 1 1
1 1 1 1 1 0 0 0 1 0 1 1 1
1 1 0 0 0 1 1 1 1 1 0 1 1
1 0 1 1 1 1 1 1 1 1 1 0 0
0 1 1 1 1 1 1 1 1 1 1 0 0
```

$G_{13.255653}$

```
0 1 1 0 0 0 1 1 1 1 0 0 1
1 0 1 0 0 0 0 1 1 1 1 0 1
1 1 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 1 1 1 1 0 0 1 1 1
0 0 0 1 0 1 1 0 1 0 1 1 1
0 0 0 1 1 0 1 0 0 1 1 1 1
1 0 0 1 1 1 0 1 1 1 0 0 1
1 1 1 1 0 0 1 0 0 0 1 1 1
1 1 1 0 1 0 1 0 0 0 1 1 1
1 1 1 0 0 1 1 0 0 0 1 1 1
0 1 0 1 1 1 0 1 1 1 0 0 1
0 0 1 1 1 1 0 1 1 1 0 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.302168}$

```
0 1 1 0 0 0 0 1 1 1 1 0 0
1 0 1 1 1 0 0 0 0 0 1 1 1
1 1 0 0 0 1 1 0 0 0 1 1 1
0 1 0 0 1 0 0 1 1 1 0 1 1
0 1 0 1 0 0 0 1 1 1 0 1 1
0 0 1 0 0 0 1 1 1 1 0 1 1
0 0 1 0 0 1 0 1 1 1 0 1 1
1 0 0 1 1 1 1 0 1 1 1 0 0
1 0 0 1 1 1 1 1 0 1 1 0 0
1 0 0 1 1 1 1 1 1 0 1 0 0
1 1 1 0 0 0 0 1 1 1 0 1 1
0 1 1 1 1 1 1 0 0 0 1 0 1
0 1 1 1 1 1 1 0 0 0 1 1 0
```

$G_{13.304826}$

Figure 20: 13-vertex minimal $(3,3)$-Ramsey graphs
with a large number of automorphisms

```
0 0 0 1 1 1 0 0 1 1 1 1 0
0 0 1 0 0 0 1 1 1 1 0 1 1
0 1 0 0 0 0 1 1 1 1 1 0 1
1 0 0 0 1 1 0 0 1 1 1 1 1
1 0 0 1 0 1 0 1 0 1 1 1 1
1 0 0 1 1 0 1 1 0 0 1 1 1
0 1 1 0 0 1 0 1 1 0 1 1 1
0 1 1 0 1 1 1 0 0 0 1 1 1
1 1 1 1 0 0 1 0 0 1 0 1 1
1 1 1 1 1 0 0 0 1 0 1 1 0
1 0 1 1 1 1 1 1 0 1 0 0 1
1 1 0 1 1 1 1 1 1 1 0 0 0
0 1 1 1 1 1 1 1 1 0 1 0 0
```

$G_{13.193684}$

```
0 0 0 1 1 1 1 1 0 0 0 1 1
0 0 1 0 0 0 1 1 1 1 1 1 0
0 1 0 0 0 0 1 1 1 1 1 1 0
1 0 0 0 1 1 0 1 1 1 0 1 1
1 0 0 1 0 1 1 0 1 0 1 1 1
1 0 0 1 1 0 1 0 0 1 1 1 1
1 1 1 0 1 1 0 0 0 0 1 1 1
1 1 1 1 0 0 0 0 1 1 0 1 1
0 1 1 1 1 0 0 1 0 1 1 0 1
0 1 1 1 0 1 0 1 1 0 1 0 1
0 1 1 0 1 1 1 0 1 1 0 0 1
1 1 1 1 1 1 1 1 0 0 0 0 0
1 0 0 1 1 1 1 1 1 1 1 0 0
```

$G_{13.193760}$

```
0 0 0 1 1 1 1 0 1 1 1 0 0
0 0 1 0 0 0 1 1 1 1 0 1 1
0 1 0 0 0 1 0 1 1 1 0 1 1
1 0 0 0 1 1 1 1 0 0 1 1 1
1 0 0 1 0 1 1 0 0 1 1 1 1
1 0 1 1 1 0 0 1 1 0 1 0 1
1 1 0 1 1 0 0 0 0 1 1 1 1
0 1 1 1 0 1 0 0 1 0 1 1 1
1 1 1 0 0 1 0 1 0 1 1 1 0
1 1 1 0 1 0 1 0 1 0 1 1 0
1 0 0 1 1 1 1 1 1 1 0 0 1
0 1 1 1 1 0 1 1 1 1 0 0 1
0 1 1 1 1 1 1 1 0 0 1 1 0
```

$G_{13.193988}$

```
0 1 0 0 1 0 0 1 1 1 1 0 1
1 0 0 0 0 1 1 0 1 1 1 0 1
0 0 0 1 1 1 1 1 1 0 1 1 0
0 0 1 0 1 1 1 1 0 1 1 1 0
1 0 1 1 0 1 0 1 1 0 0 1 1
0 1 1 1 1 0 1 0 1 0 0 1 1
0 1 1 1 0 1 0 0 0 1 1 1 1
1 0 1 1 1 0 0 0 1 1 1 1 0
1 1 1 0 1 1 0 1 0 0 0 1 1
1 1 0 1 0 0 1 1 0 0 1 1 1
1 1 1 1 0 0 1 1 0 1 0 0 1
0 0 1 1 1 1 1 1 1 1 0 0 1
1 1 0 0 1 1 1 0 1 1 1 1 0
```

$G_{13.265221}$

```
0 1 0 0 1 1 0 0 1 1 1 0 1
1 0 0 0 0 0 1 1 1 1 1 0 1
0 0 0 1 1 1 1 1 0 1 0 1 1
0 0 1 0 1 1 1 1 0 0 1 1 1
1 0 1 1 0 1 1 0 0 1 0 1 1
1 0 1 1 1 0 0 1 0 0 1 1 1
0 1 1 1 1 0 0 1 1 1 0 1 0
0 1 1 1 0 1 1 0 1 0 1 1 0
1 1 0 0 0 0 1 1 0 1 1 1 1
1 1 1 0 1 0 1 0 1 0 0 1 1
1 1 0 1 0 1 0 1 1 0 0 1 1
0 0 1 1 1 1 1 1 1 1 1 0 0
1 1 1 1 1 1 0 0 1 1 1 0 0
```

$G_{13.265299}$

```
0 1 1 0 1 1 0 0 1 1 1 1 0
1 0 0 1 0 1 0 1 1 1 1 0 1
1 0 0 1 1 0 1 0 1 1 1 1 0
0 1 1 0 0 0 1 1 1 1 1 0 1
1 0 1 0 0 1 1 1 0 1 0 1 1
1 1 0 0 1 0 1 1 0 0 1 1 1
0 0 1 1 1 1 0 1 0 0 1 1 1
0 1 0 1 1 1 1 0 0 1 0 1 1
1 1 1 1 0 0 0 0 0 1 1 1 1
1 1 1 1 1 0 0 1 1 0 0 1 1
1 1 1 1 0 1 1 0 1 0 0 1 1
1 0 1 0 1 1 1 1 1 1 1 0 0
0 1 0 1 1 1 1 1 1 1 1 0 0
```

$G_{13.299797}$

```
0 1 1 1 1 1 0 0 0 0 1 1 0
1 0 0 1 1 0 0 1 1 0 1 1 1
1 0 0 1 0 1 1 0 1 1 1 0 1
1 1 1 0 1 1 0 0 0 1 0 1 1
1 1 0 1 0 1 0 1 1 1 0 1 0
1 0 1 1 1 0 1 0 0 1 1 1 0
0 0 1 0 0 1 0 1 1 1 1 1 1
0 1 0 0 1 0 1 0 1 1 1 1 1
0 1 1 0 1 0 1 1 0 1 1 0 1
0 0 1 1 1 1 1 1 1 0 0 0 1
1 1 1 0 0 1 1 1 1 0 0 1 1
1 1 0 1 1 1 1 1 0 0 1 0 1
0 1 1 1 0 0 1 1 1 1 1 1 0
```

$G_{13.301368}$

```
0 1 1 0 0 0 1 0 1 1 0 1 1
1 0 1 0 0 0 0 0 1 1 1 1 1
1 1 0 0 0 0 0 1 1 1 1 0 1
0 0 0 0 1 1 1 1 1 0 1 1 1
0 0 0 1 0 1 1 1 0 1 1 1 1
0 0 0 1 1 0 1 1 1 1 1 1 0
1 0 0 1 1 1 0 1 0 1 0 1 1
0 0 1 1 1 1 1 0 0 1 1 0 1
1 1 1 1 0 1 0 0 0 0 1 1 1
1 1 1 0 1 1 1 1 0 0 0 1 0
0 1 1 1 1 1 0 1 1 0 0 1 1
1 1 0 1 1 1 1 0 1 1 1 0 0
1 1 1 1 1 0 1 1 1 0 1 0 0
```

$G_{13.302151}$

```
0 1 1 0 0 1 0 0 1 1 1 0 1
1 0 1 0 0 0 0 0 1 1 1 1 1
1 1 0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 1 1 1 1 1 0 1 1 1
0 0 0 1 0 1 1 1 1 1 1 1 0
1 0 0 1 1 0 1 1 1 0 1 0 1
0 0 1 1 1 1 0 1 0 1 0 1 1
0 0 1 1 1 1 1 0 0 0 1 1 1
1 1 0 1 1 1 0 0 0 1 1 1 0
1 1 1 0 1 0 1 0 1 0 1 1 0
1 1 0 1 1 1 0 1 1 1 0 0 1
0 1 1 1 1 0 1 1 1 1 0 0 1
1 1 1 1 0 1 1 1 0 0 1 1 0
```

$G_{13.302764}$

```
0 1 1 1 1 1 0 0 0 0 1 1 0
1 0 1 0 0 0 1 1 0 0 1 1 1
1 1 0 0 0 0 1 1 0 0 1 1 1
1 0 0 0 1 1 1 0 1 1 1 0 0
1 0 0 1 0 1 0 1 1 1 0 1 0
1 0 0 1 1 0 0 0 1 1 1 1 1
0 1 1 1 0 0 0 1 1 1 1 0 1
0 1 1 0 1 0 1 0 1 1 0 1 1
0 0 0 1 1 1 1 1 0 1 1 0 1
0 0 0 1 1 1 1 1 1 0 0 1 1
1 1 1 1 0 1 1 0 1 0 0 1 1
1 1 1 0 1 1 0 1 0 1 1 0 1
0 1 1 0 0 1 1 1 1 1 1 1 0
```

$G_{13.305857}$

```
0 1 1 1 1 1 1 0 0 0 1 0 0
1 0 1 1 1 0 1 0 1 0 0 1 1
1 1 0 0 0 1 1 0 0 1 1 1 1
1 1 0 0 1 0 1 1 1 0 1 1 0
1 1 0 1 0 1 0 1 1 0 1 0 1
1 0 1 0 1 0 0 1 0 1 1 1 1
1 1 1 1 0 0 0 0 1 1 1 1 0
0 0 0 1 1 1 0 0 1 1 1 1 1
0 1 0 1 1 0 1 1 0 1 0 1 1
0 0 1 0 0 1 1 1 1 0 1 1 1
1 0 1 1 1 1 1 1 0 1 0 0 1
0 1 1 1 0 1 1 1 1 1 0 0 1
0 1 1 0 1 1 0 1 1 1 1 1 0
```

$G_{13.306448}$

```
0 1 1 1 1 1 1 0 0 0 1 0 0
1 0 1 1 1 1 0 0 0 0 1 1 1
1 1 0 0 1 0 1 1 0 0 1 1 1
1 1 0 0 0 1 1 0 1 1 0 1 1
1 1 1 0 0 1 0 1 0 1 1 0 1
1 1 0 1 1 0 0 0 1 1 1 0 1
1 0 1 1 0 0 0 1 1 1 1 1 0
0 0 1 0 1 0 1 0 1 1 1 1 1
0 0 0 1 0 1 1 1 0 1 1 1 1
0 0 0 1 1 1 1 1 1 0 0 1 1
1 1 1 0 1 1 1 1 1 0 0 1 0
0 1 1 1 0 0 1 1 1 1 1 0 1
0 1 1 1 1 1 0 1 1 1 0 1 0
```

$G_{13.306460}$

```
0 1 1 1 0 0 0 0 0 1 1 1 1
1 0 1 1 0 0 0 0 0 1 1 1 1
1 1 0 1 0 0 0 1 0 1 1 0 1
1 1 1 0 0 0 0 0 1 1 0 1 1
0 0 0 0 0 1 1 1 1 1 1 1 0
0 0 0 0 1 0 1 1 1 1 1 1 0
0 0 0 0 1 1 0 1 1 0 1 1 1
0 0 1 0 1 1 1 0 1 0 1 0 1
0 0 0 1 1 1 1 1 0 0 0 1 1
1 1 1 1 1 1 0 0 0 0 1 1 0
1 1 1 0 1 1 1 1 0 1 0 0 1
1 1 0 1 1 1 1 0 1 1 0 0 1
1 1 1 1 0 0 1 1 1 0 1 1 0
```

$G_{13.306470}$

Figure 21: 13-vertex minimal $(3, 3)$-Ramsey graphs with independence number 2

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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DETERMINISTIC SQEMA AND APPLICATION FOR PRE-CONTACT LOGIC

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SQEMA is a set of rules for finding first-order correspondents of modal formulas, and can be used for proving axiomatic completeness. SQEMA succeeds for the Sahlqvist and Inductive formulas.

A deterministic, terminating, but sometimes failing algorithm based on SQEMA for a modal language with nominals, reversed modalities and the universal modality - $\mathrm{ML}(T, U)$ - is presented. Deterministic SQEMA finds first-order correspondents, and it can be used to prove di-persistence. It succeeds for the Sahlqvist and Inductive formulas.

The axiomatic system for $\mathrm{ML}(T, U)$ is shown and its strong completeness is proven. It is shown that adding di-persistent formulas as axioms preserves strong completeness.

Deterministic SQEMA is extended for the language of pre-contact logics using a modified translation into $\mathrm{ML}(T, U)$. Deterministic SQEMA succeeds for the Sahlqvist class of pre-contact formulas.

Keywords: SQEMA, correspondence, Sahlqvist, modal, logic

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1. INTRODUCTION

The problem of the existence of first-order correspondent formulas for modal formulas was proposed by van Benthem. This problem is not computable, as shown by Chagrova in her PhD thesis in 1989, see [4]. However, there have been solutions for some modal formulas. The most famous class of formulas for which there is a first-order correspondent is the Sahlqvist class, shown in [19], where one can use the Sahlqvist-van Benthem algorithm as described in [23] and [3] to obtain first-order correspondents.

There are other algorithms for finding first-order correspondents, for example in [11] Gabbay and Ohlbach introduced the SCAN algorithm, and in [20], Szalas introduced DLS. SCAN is based on a resolution procedure applied on a Skolemized translation of the modal formula into the second-order logic, while DLS works on the same translation, but is based on a transformation procedure using a lemma by Ackermann. Both algorithms use a procedure of unskolemization, which is not always successful.

In [6, 7, 8, 10, 9] another algorithm, called SQEMA, for computing first-order correspondents in modal logic is introduced. It is based on a modal version of the Ackermann Lemma. SQEMA works directly on the modal formulas without translating them into the second-order logic and without using Skolemization. SQEMA succeeds not only on all Sahlqvist formulas, but also on the extended class of *inductive formulas* introduced in [5, 16]. There are examples of modal formulas on which SQEMA succeeds, while both SCAN and DLS fail, e.g.: $(\Box(\Box p \leftrightarrow q) \to p)$.

As proved in [6, 7, 8] SQEMA only succeeds on d-persistent (for languages without nominals) or di-persistent (for reversive languages with nominals) — and hence, by [3, 5, 15, 16], canonical formulas, i.e., whenever successful, it not only computes a local first-order correspondent of the input modal formula, but also proves its canonicity and therefore the canonical completeness of the modal logic axiomatized with that formula. This extends to any set of modal formulas on which SQEMA succeeds. Thus, SQEMA can also be used as an automated prover of canonical model completeness of modal logics.

An implementation of SQEMA in Java was given in [13]. Some additional simplifications were added to the implementation thanks to a suggestion by Renate Schmidt, which helps the implementation to succeed on formulas such as $((\Box\Diamond p \to \Diamond\Box p) \vee (\Box p \to \Diamond p))$.

The universal modality and nominals were introduced in [17].

In [14], SQEMA was augmented to $\mathrm{ML}(\Box, [U])$, the basic modal language extended by adding the universal modality. In [7], SQEMA for a reversive language with nominals is discussed, promising an extension with $[U]$. In [10], SQEMA with downwards monotonicity for Ackermann's rule is presented. In [22, 9], an extension of SQEMA for a reversive language with $[U]$ and nominals is introduced, with the output being in the first-order μ-calculus.

In this paper, we define a deterministic and terminating strategy for using the SQEMA rules for the language with universal modality, countably infinitely many couples of converse modalities, and nominals, $\mathrm{ML}(T, U)$. We show that Deterministic SQEMA always succeeds on Sahlqvist and inductive formulas. We show, like in [7], that Deterministic SQEMA succeeds only on di-persistent formulas. We show the axiomatic system for $\mathrm{ML}(T, U)$ and its strong completeness, following closely [17, 18, 12, 3]. Like in [15, 16, 5, 21], we show strong completeness of di-persistent formulas. Therefore, Deterministic SQEMA can be used to prove strong axiomatic completeness of a formula. We extend Deterministic SQEMA to the language of pre-contact logics, using a modified form of the translation from [1] as to obtain Sahlqvist formulas from Sahlqvist formulas of the pre-contact language, as defined in [2], so that Deterministic SQEMA succeeds on them. Completeness of all pre-contact formulas is shown in [1].

2. PRELIMINARIES

We use $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{n}$ for natural numbers. If a and b are words, we write $a \nearrow b$ iff a occurs in b. If a is a word and b is a sequence or a set of words, $a \nearrow b$ means that a occurs in some of the words of b. The negation of $a \nearrow b$ is denoted by $a \not\nearrow b$.

Definition 1. (Formulas of $\mathrm{ML}(T, U)$) Formulas of $\mathrm{ML}(T, U)$ are:

$$\boldsymbol{\phi} ::= \bot|\top|p_{\boldsymbol{i}}|c_{\boldsymbol{i}}|\neg\boldsymbol{\phi}|(\boldsymbol{\phi} \vee \boldsymbol{\phi})|(\boldsymbol{\phi} \wedge \boldsymbol{\phi})|\Diamond_{\boldsymbol{i}}\boldsymbol{\phi}|\Diamond_{\boldsymbol{i}}^{-1}\boldsymbol{\phi}|\Box_{\boldsymbol{i}}\boldsymbol{\phi}|\Box_{\boldsymbol{i}}^{-1}\boldsymbol{\phi}$$

where $c_0, c_1, \ldots$ are *nominals*, $p_0, p_1, \ldots$ are *propositional variables*, and there are at most countably many pairs of mutually converse *boxes* and *diamonds*. We denote "any box" by $\boldsymbol{\Box}$, its converse by $\boldsymbol{\Box}^{-1}$, "any diamond" by $\boldsymbol{\Diamond}$, its converse by $\boldsymbol{\Diamond}^{-1}$. $\langle U \rangle$ means $\Diamond_0$, $[U]$ means $\Box_0$. $\mathrm{PROP}(\boldsymbol{\phi})$ is the set of propositional variables, occurring in $\boldsymbol{\phi}$. $\mathrm{NOM}(\boldsymbol{\phi})$ is the set of nominals, occurring in $\boldsymbol{\phi}$. $\boldsymbol{\phi}$ is *pure* iff $\mathrm{PROP}(\boldsymbol{\phi}) = \emptyset$. $(\boldsymbol{\phi_1} \to \boldsymbol{\phi_2})$ stands for $(\neg\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$, $\bigwedge(\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n})$ for $\boldsymbol{n} \geq 0$ and different $\boldsymbol{\phi_i}$ stands for $(\boldsymbol{\phi_1} \wedge \ldots (\boldsymbol{\phi_{n-1}} \wedge \boldsymbol{\phi_n}) \ldots)$ if $\boldsymbol{n} > 0$, and $\top$ otherwise. $\bigvee(\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n})$ for $\boldsymbol{n} \geq 0$ and different $\boldsymbol{\phi_i}$ stands for $(\boldsymbol{\phi_1} \vee \ldots (\boldsymbol{\phi_{n-1}} \vee \boldsymbol{\phi_n}) \ldots)$ if $\boldsymbol{n} > 0$, and $\bot$ otherwise. We emphasize a disjunction in a formula by using $\veebar$ instead of $\vee$. We also use $\boldsymbol{\gamma}$ for formulas. We use the standard definitions for a *positive/negative occurrence* of $\boldsymbol{p}$ in $\boldsymbol{\phi}$, for $\boldsymbol{\phi}$ being *positive/negative* in $\boldsymbol{p}$, and for $\boldsymbol{\phi}$ being *positive/negative.*

Definition 2. (New Nominal) We denote by $c_{\boldsymbol{k}} \not\nearrow^{\infty} \boldsymbol{\phi}$ iff $c_{\boldsymbol{k}}$ is the first nominal, such that for all $\boldsymbol{n} \geq \boldsymbol{k}$: $c_{\boldsymbol{n}} \not\nearrow \boldsymbol{\phi}$. We denote by $c_{\boldsymbol{k}} \not\nearrow^{\infty} \Gamma$ for a set of modal formulas Γ iff for all $\boldsymbol{\phi} \in \Gamma$: $c_{\boldsymbol{k}} \not\nearrow^{\infty} \boldsymbol{\phi}$.

Definition 3. (Kripke Frame) A *Kripke frame* for $\mathrm{ML}(T, U)$, or just a *frame*, is a tuple $\langle W, \mathcal{R} \rangle$, where W is a non-empty set of *possible worlds*, also a *universe*, and for all $\boldsymbol{i}$, $\mathcal{R}(\boldsymbol{i}) \subseteq W \times W$ are *accessibility relations*, where $\mathcal{R}(0) = W \times W$. We use $\mathbf{F}$ for frames, $\boldsymbol{w}$, $\boldsymbol{u}$, $\boldsymbol{v}$ for possible worlds, and $\boldsymbol{s}$ for sets of possible worlds. If $\boldsymbol{w} \in \boldsymbol{W}$, we say that $\boldsymbol{w}$ is *in* $\mathbf{F}$.

Definition 4. (Kripke Model) Let $\mathbf{F} = \langle \boldsymbol{W}, \mathcal{R} \rangle$. A *Kripke model* for $\mathrm{ML}(T, U)$, or just a *model*, is a tuple $\langle \mathbf{F}, V, A \rangle$, where $V : \mathrm{PROP} \to \mathbb{P}(\boldsymbol{W})$ is a *valuation*, and $A : \mathrm{NOM} \to \boldsymbol{W}$ is an *assignment*. We say that the model thus defined is *based on* $\boldsymbol{F}$. We use $\mathbf{M}$ for models, $\boldsymbol{V}$ for valuations, $\boldsymbol{A}$ for assignments. If $\boldsymbol{w} \in \boldsymbol{W}$, we say that $\boldsymbol{w}$ is *in* $\mathbf{M}$. $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$ is *named* iff $\boldsymbol{A}$ is surjective.

Definition 5. (Modal Truth and Validity). Let $\mathbf{F} = \langle \boldsymbol{W}, \mathcal{R} \rangle$, $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$, and $\boldsymbol{w} \in \boldsymbol{W}$. We say that, by induction on $\boldsymbol{\phi}$, $\boldsymbol{\phi}$ is *true in* $\mathbf{M}$ *at* $\boldsymbol{w}$, denoted by $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff:
- $\mathbf{M}, \boldsymbol{w} \Vdash \top$
- $\mathbf{M}, \boldsymbol{w} \nVdash \bot$
- $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{p}$ iff $\boldsymbol{w} \in \boldsymbol{V}(\boldsymbol{p})$
- $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{c}$ iff $\boldsymbol{w} = \boldsymbol{A}(\boldsymbol{c})$
- $\mathbf{M}, \boldsymbol{w} \Vdash \neg\boldsymbol{\phi_1}$ iff $\mathbf{M}, \boldsymbol{w} \nVdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$ iff $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ or $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$
- $\mathbf{M}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$ iff $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ and $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Diamond_{\boldsymbol{i}}\boldsymbol{\phi_1}$ iff for some $v \in \boldsymbol{W}$: $\boldsymbol{w}\ \mathcal{R}(\boldsymbol{i})\ v$ and $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Diamond_{\boldsymbol{i}}^{-1}\boldsymbol{\phi_1}$ iff for some $v \in \boldsymbol{W}$: $v\ \mathcal{R}(\boldsymbol{i})\ \boldsymbol{w}$ and $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Box_{\boldsymbol{i}}\boldsymbol{\phi_1}$ iff for all $v \in \boldsymbol{W}$: $\boldsymbol{w}\ \mathcal{R}(\boldsymbol{i})\ v$ implies $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Box_{\boldsymbol{i}}^{-1}\boldsymbol{\phi_1}$ iff for all $v \in \boldsymbol{W}$: $v\ \mathcal{R}(\boldsymbol{i})\ \boldsymbol{w}$ implies $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$

We say that $\boldsymbol{\phi}$ is *true in* $\mathbf{M}$ iff for all $\boldsymbol{w} \in \mathbf{M}$: $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is *valid in* $\mathbf{F}$ *at* $\boldsymbol{w}$ (*local validity*), denoted by $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$, iff for every model $\mathbf{M}$ based on $\mathbf{F}$, $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is *valid in* $\mathbf{F}$ (*frame validity*) iff $\boldsymbol{\phi}$ is true in every model based on $\mathbf{F}$ iff for all $\boldsymbol{w}$ in $\mathbf{F}$, $\boldsymbol{\phi}$ is valid in $\mathbf{F}$ at $\boldsymbol{w}$. We say that $\boldsymbol{\phi}$ is *valid*, denoted by $\Vdash \boldsymbol{\phi}$, iff it is valid in all frames. The *extension* of $\boldsymbol{\phi}$ in $\mathbf{M}$, denoted by $[\![\boldsymbol{\phi}]\!]_{\mathbf{M}}$, is the set of all $w \in \boldsymbol{W}$ such that $\mathbf{M}, w \Vdash \boldsymbol{\phi}$. It is clear that, if $\mathbf{M_1} = \langle \mathbf{F}, \boldsymbol{V_1}, \boldsymbol{A_1} \rangle$ and $\mathbf{M_2} = \langle \mathbf{F}, \boldsymbol{V_2}, \boldsymbol{A_2} \rangle$ *agree on* $\mathrm{NOM}(\boldsymbol{\phi}) \cup \mathrm{PROP}(\boldsymbol{\phi})$, meaning that $\boldsymbol{V_1} \upharpoonright \mathrm{PROP}(\boldsymbol{\phi}) = \boldsymbol{V_2} \upharpoonright \mathrm{PROP}(\boldsymbol{\phi})$ and $\boldsymbol{A_1} \upharpoonright \mathrm{NOM}(\boldsymbol{\phi}) = \boldsymbol{A_2} \upharpoonright \mathrm{NOM}(\boldsymbol{\phi})$, then $[\![\boldsymbol{\phi}]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi}]\!]_{\mathbf{M_2}}$. We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *semantically equivalent*, denoted by $\boldsymbol{\phi_1} \equiv \boldsymbol{\phi_2}$, iff for every model $\mathbf{M}$, their extensions in $\mathbf{M}$ are equal. We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *opposite* iff $\boldsymbol{\phi_1} \equiv \neg\boldsymbol{\phi_2}$ We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally frame-equivalent*, denoted by $\boldsymbol{\phi_1} \sim \boldsymbol{\phi_2}$, iff for every frame, $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$: $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ iff $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$. We define a *modified assignment* as follows: $\boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}](\boldsymbol{c}) := \boldsymbol{w}$ and $\boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}](\boldsymbol{c'}) := \boldsymbol{A}(\boldsymbol{c'})$ for all $\boldsymbol{c'}$ distinct from $\boldsymbol{c}$. We define a *modified model* as follows: $\mathbf{M}[\boldsymbol{c} \to \boldsymbol{w}] := \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}] \rangle$.

If for a $\mathbf{F}$ and for a $\boldsymbol{w}$ in $\mathbf{F}$, we have that for every $\mathbf{M}$ over $\mathbf{F}$, there is a $\boldsymbol{w_2}$ in $\mathbf{F}$: $\mathbf{M}[c_{\boldsymbol{k}} \to \boldsymbol{w}], \boldsymbol{w_2} \Vdash \boldsymbol{\phi}$, then $\boldsymbol{\phi}$ is called *relatively* $\boldsymbol{k}$*-true in* $\mathbf{F}$ *at* $\boldsymbol{w}$, $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi}$.

For a given $\boldsymbol{k}$, two formulas $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally frame-equivalent with respect to* $c_{\boldsymbol{k}}$, denoted by $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\phi_2}$, iff for every frame $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$, we have that $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_1}$ iff $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_2}$.

Definition 6. (Uniform Substitution) We denote by $\boldsymbol{\phi_1}[\boldsymbol{p}/\boldsymbol{\phi'}]$ the word obtained from $\boldsymbol{\phi_1}$, where each occurrence of $\boldsymbol{p}$ (if any) has been replaced with $\boldsymbol{\phi'}$. According to Definition 1, the word thus constructed is also a formula, $\boldsymbol{\phi_2}$. We call the rule for obtaining $\boldsymbol{\phi_2}$ from $\boldsymbol{\phi_1}$ *uniform substitution* of $\boldsymbol{p}$ by $\boldsymbol{\phi'}$ in $\boldsymbol{\phi_1}$.

Proposition 7. (Properties of the Uniform Substitution)

1. Let $\mathbf{M_1}$ and $\mathbf{M_2}$ be based on $\mathbf{F}$ and be such that they agree on the nominals and variables, occurring in $\boldsymbol{\phi}$, except for $\boldsymbol{p}$. If $[\![\boldsymbol{\phi'}]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi''}]\!]_{\mathbf{M_2}}$, then $[\![\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi''}]]\!]_{\mathbf{M_2}}$.

2. If $\mathbf{F} \Vdash \boldsymbol{\phi}$, then $\mathbf{F} \Vdash \boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]$.

3. If $\phi' \equiv \phi''$, then $\phi[p/\phi'] = \phi[p/\phi'']$.

Proof. Follows directly from the definitions. □

If the elements of PROP(ϕ) are, in left-to-right order of initial occurrence in ϕ: $p_1, \ldots, p_n$ with $n \geq 0$, and the elements of NOM(ϕ) are, in left-to-right order of initial occurrence in ϕ: $c_1, \ldots, c_m$ with $m \geq 0$, then $[\![\phi]\!]$ is an operator from n,m-tuples of n sets of states and m states to a set of states, defined thus: if $\mathbf{M} = \langle \mathbf{F}, V, A \rangle$ is a model, $V(p_1) = s_1, \ldots, V(p_n) = s_n$, $A(c_1) = w_1, \ldots, A(c_m) = w_m$, then $[\![\phi]\!](s_1, \ldots, s_n, w_1, \ldots, w_m)$ is $[\![\phi]\!]_{\mathbf{M}}$.

Definition 8. (General Discrete Frame) Let $\mathbf{F} = \langle W, \mathcal{R} \rangle$. We say that $\langle \mathbf{F}, \mathbb{W} \rangle$ is a *general discrete frame for* ML(T, U), or just a *general discrete frame*, iff $\mathbb{W} \subseteq \mathbb{P}(W)$ is non-empty and the following conditions hold:
- for every $w \in W$, $\{w\} \in \mathbb{W}$
- $\mathbb{W}$ is closed under $[\![\neg p_0]\!]$
- $\mathbb{W}$ is closed under $[\![(p_0 \vee p_1)]\!]$
- $\mathbb{W}$ is closed under $[\![\Diamond p_0]\!]$ for all diamonds $\Diamond$.

$\mathbb{W}$ is the set of *admissible valuations.* It is clear that $W, \emptyset \in \mathbb{W}$. It is clear that $\mathbf{F}$ with universe W is also the *full general discrete frame* $\langle \mathbf{F}, \mathbb{P}(W) \rangle$. We use $\mathfrak{g}$ for general discrete frames, $\mathbb{W}$ for sets of admissible valuations. If $\mathfrak{g} = \langle \mathbf{F}, \mathbb{W} \rangle$, then we denote by $\mathfrak{g}_{\#}$ the *underlying frame* of $\mathfrak{g}$, $\mathbf{F}$. w is in $\mathfrak{g}$ iff it is in $\mathfrak{g}_{\#}$. $\mathbf{M} = \langle \mathfrak{g}_{\#}, V, A \rangle$ is a *model over* $\mathfrak{g}$ iff for each propositional variable p, $V(p) \in \mathbb{W}$. An induction on ϕ shows that if $\mathbf{M}$ is a model over $\mathfrak{g}$, then $[\![\phi]\!]_{\mathbf{M}} \in \mathbb{W}$ for any formula. ϕ is *valid* in $\mathfrak{g}$, denoted with $\mathfrak{g} \Vdash \phi$, iff it is true in all models over $\mathfrak{g}$.

ϕ is *di-persistent* iff for every $\mathfrak{g}$, $\mathfrak{g} \Vdash \phi$ iff $\mathfrak{g}_{\#} \Vdash \phi$. As we show later, di-persistence is a sufficient condition for strong axiomatic completeness of a formula.

For every named $\mathbf{M}$ over $\mathbf{F}$, there is a $\mathfrak{g} = \langle \mathbf{F}, \{[\![\phi]\!]_{\mathbf{M}} \mid \phi \in \mathrm{ML}(T, U)\} \rangle$. This helps to prove strong axiomatic completeness of di-persistent formulas.

Definition 9. (Local Equivalence for General Discrete Frames) We say that ϕ_1 and ϕ_2 are *locally di-equivalent*, denoted by $\phi_1 \approx \phi_2$, iff for every $\mathfrak{g}$ and every w in $\mathfrak{g}$: $\mathfrak{g}, w \Vdash \phi_1$ iff $\mathfrak{g}, w \Vdash \phi_2$.

If for a $\mathfrak{g}$ and for a w in $\mathfrak{g}$, we have that for every $\mathbf{M}$ over $\mathfrak{g}$, there is a w_2 in $\mathfrak{g}$: $\mathbf{M}[c_k \to w], w_2 \Vdash \phi$, then ϕ is called *relatively* k*-true in* $\mathfrak{g}$ *at* w, $\mathfrak{g}, w \Vdash_k \phi$.

For a given k, we say that ϕ_1 and ϕ_2 are *locally di-equivalent with respect to* c_k, denoted by $\phi_1 \approx_k \phi_2$, iff for every general discrete frame $\mathfrak{g}$ and every w in $\mathfrak{g}$, we have that $\mathfrak{g}, w \Vdash_k \phi_1$ iff $\mathfrak{g}, w \Vdash_k \phi_2$.

Because every frame is also a full general discrete frame, if $\phi_1 \approx_k \phi_2$, then $\phi_1 \sim_k \phi_2$, and if $\phi_1 \approx \phi_2$, then $\phi_1 \sim \phi_2$.

Proposition 10. (Sufficient Condition for Di-Persistence) Let ϕ be a modal formula, let c_k be such that $c_k \not\nearrow \phi$, let ϕ' be such that ϕ' is a pure formula and $(c_k \wedge \phi) \approx_k \phi'$. Then ϕ is di-persistent.

Proof. Let $\mathfrak{g}$ be a general discrete frame and let w be in $\mathfrak{g}$. Then:

$\mathfrak{g}, w \Vdash \phi$ iff (because $c_k \not\nearrow \phi$)

$\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} (c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$ iff (because of di-equivalence)

$\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi'}$ iff (because $\boldsymbol{\phi'}$ is pure)

$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi'}$ iff (because of di-equivalence)

$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} (c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$ iff (because $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$)

$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. □

A direct corollary to the above is that every pure formula is di-persistent.

3. FIRST-ORDER CORRESPONDENCE PROBLEM

We define a first-order language with equality and binary predicate symbols. The language is called FOL. We use $\boldsymbol{\psi}$ for *FOL formulas*.

Definition 11. (First-Order formulas) FOL formulas are:

$$\boldsymbol{\psi} ::= \bot | \top | (\boldsymbol{x'} = \boldsymbol{x''}) | (\boldsymbol{x'}\, r_{\boldsymbol{i}}\, \boldsymbol{x''}) | \neg\boldsymbol{\psi} | (\boldsymbol{\psi} \vee \boldsymbol{\psi}) | (\boldsymbol{\psi} \wedge \boldsymbol{\psi}) | \exists \boldsymbol{x} \boldsymbol{\psi} | \forall \boldsymbol{x} \boldsymbol{\psi},$$

where $x_0, x_1, \ldots$ are *individual variables*, $r_1, r_2, \ldots$ are *binary relational predicate symbols*, $=$ is *equality*, $\exists$ and $\forall$ are *quantifiers*. An occurrence of $\boldsymbol{x}$ in $\boldsymbol{\psi}$ is *bound* iff it occurs in an occurrence of $\exists \boldsymbol{x} \boldsymbol{\psi_1}$ or of $\forall \boldsymbol{x} \boldsymbol{\psi_1}$ in $\boldsymbol{\psi}$. Any occurrence of $\boldsymbol{x}$ in $\boldsymbol{\psi}$ that is not bound is *free*. We say that $\boldsymbol{x}$ is a *free variable of* $\boldsymbol{\psi}$ iff $\boldsymbol{\psi}$ contains a free occurrence of $\boldsymbol{x}$. We say that $\boldsymbol{\psi}$ is *closed*, or that $\boldsymbol{\psi}$ is a *sentence* iff it has no free variables. We denote by FREE($\boldsymbol{\psi}$) the set of all free variables of $\boldsymbol{\psi}$. If the elements of FREE($\boldsymbol{\psi}$) are, in left-to-right order of initial occurrence in $\boldsymbol{\psi}$, $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$ for some $\boldsymbol{k} \geq 0$, then we denote $\boldsymbol{\psi}$ by $\boldsymbol{\psi}(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n})$, where $\boldsymbol{n} > 0$ and $\boldsymbol{n} \geq \boldsymbol{k}$.

Definition 12. (Semantics of FOL formulas) Let $\mathbf{F}$ be a Kripke frame and let $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$ be a Kripke model over $\mathbf{F}$. We extend $\boldsymbol{A}$ to all individual variables as follows: $\boldsymbol{A}(x_{\boldsymbol{i}}) := \boldsymbol{A}(c_{\boldsymbol{i}})$. We use the usual semantics of $\mathbf{M} \vDash \boldsymbol{\psi}$. We say that $\boldsymbol{\psi}(\boldsymbol{x})$ is *true in* $\mathbf{F}$ *at* $\boldsymbol{w}$, $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$ iff for some model $\mathbf{M}$ over $\mathbf{F}$: $\mathbf{M}[c_{\boldsymbol{i}} \to \boldsymbol{w}] \vDash \boldsymbol{\psi}(x_{\boldsymbol{i}})$. We say that $\boldsymbol{\psi}$ is *valid* in $\mathbf{F}$ (*Kripke frame validity*) iff $\boldsymbol{\psi}$ is true in every model based on $\mathbf{F}$. Thus, $\boldsymbol{\psi}(\boldsymbol{x})$ is valid in $\mathbf{F}$ iff $\boldsymbol{\phi}(\boldsymbol{x})$ is true in $\mathbf{F}$ at every state in $\mathbf{F}$.

Definition 13. (First-order Correspondence) We say that a modal formula $\boldsymbol{\phi}$ and FOL formula $\boldsymbol{\psi}(\boldsymbol{x})$ are *locally correspondent*, denoted $\boldsymbol{\phi} \sim \boldsymbol{\psi}(\boldsymbol{x})$, iff for every frame $\mathbf{F}$ and every state $\boldsymbol{w}$ in $\mathbf{F}$: $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$. We say that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are *globally correspondent* iff for every frame $\mathbf{F}$: $\mathbf{F} \Vdash \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}$. It is clear that, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}(\boldsymbol{x})$ are locally correspondent, then they are globally correspondent.

For a given $\boldsymbol{k}$, we say that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}(x_{\boldsymbol{k}})$ are *locally correspondent with respect to* $c_{\boldsymbol{k}}$, denoted by $\boldsymbol{\phi} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, iff for every frame $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$, we have that: $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$

An easy argument shows that if $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, and $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi}) \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, then $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Also, if $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\phi_2}$, and if $\boldsymbol{\phi_2} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, then $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$.

Combined with the properties of local frame equivalence, we now have a sufficient condition for the existence of a local first-order correspondent:

Proposition 14. (Sufficient Condition for First-order Correspondence) Let $\boldsymbol{\phi}$ be a modal formula, let $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, let $\boldsymbol{\phi'}$ be such that $\boldsymbol{\phi'}$ is a pure formula, $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi}) \sim_{\boldsymbol{k}} \boldsymbol{\phi'}$, and $\boldsymbol{\phi'} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Then $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$. □

An immediate corollary of propositions 14 and 10 is the following:

Proposition 15. (Sufficient Condition for Di-Persistence and First-order Correspondence) Let $\boldsymbol{\phi}$ be a modal formula, let $c_{\boldsymbol{k}}$ be such that $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, let $\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n}$ be a sequence, such that $\boldsymbol{\phi_1}$ is $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$, $\boldsymbol{\phi_i} \approx_{\boldsymbol{k}} \boldsymbol{\phi_j}$ for any $1 \leq i \leq j \leq n$, and $\boldsymbol{\phi_n}$ is a pure formula, such that $\boldsymbol{\phi_n} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Then it follows that:

1. $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$.
2. $\boldsymbol{\phi}$ is di-persistent. □

As we show later, it is enough to have $c_{\boldsymbol{k}}$ and $\boldsymbol{\phi_n}$ to find a $\boldsymbol{\psi}(x_{\boldsymbol{k}})$, and also $c_{\boldsymbol{k}}$ is uniquely defined for $\boldsymbol{\phi}$. Therefore, if such a sequence for $\boldsymbol{\phi}$ exists, we call $\boldsymbol{\psi}(x_{\boldsymbol{k}})$ a *solution for* $\boldsymbol{\phi}$.

Therefore, a good approach for both finding first-order correspondents and for proving that a formula is di-persistent is to have rules for elimination of propositional variables that replace formulas with formulas that are locally di-equivalent with respect to a given nominal.

For reducing the size of the problem, we need a lemma for conjunctions.

Lemma 16. (Conjunction Lemma)

1. Let $\boldsymbol{\phi_1} \sim \boldsymbol{\psi_1}(x_{\boldsymbol{k}})$ and $\boldsymbol{\phi_2} \sim \boldsymbol{\psi_2}(x_{\boldsymbol{k}})$. Then $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \sim (\boldsymbol{\psi_1}(x_{\boldsymbol{k}}) \wedge \boldsymbol{\psi_2}(x_{\boldsymbol{k}}))$.
2. If $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are di-persistent, then so is $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$.

Proof. For 1, let $\boldsymbol{w}$ be a world in $\mathbf{F}$. Then, by the hypothesis, $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ iff $\mathbf{F} \Vdash \boldsymbol{\psi_1}[\boldsymbol{w}]$ and $\mathbf{F}, \boldsymbol{w} \vDash \boldsymbol{\phi_2}$ iff $\mathbf{F} \vDash \boldsymbol{\psi_2}[\boldsymbol{w}]$. Let $\mathbf{F}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$. Then, $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ and $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$. Therefore, $\mathbf{F} \vDash \boldsymbol{\psi_1}[\boldsymbol{w}]$ and $\mathbf{F} \vDash \boldsymbol{\psi_2}[\boldsymbol{w}]$, so $\mathbf{F} \vDash (\boldsymbol{\psi_1}(x_{\boldsymbol{k}}) \wedge \boldsymbol{\psi_2}(x_{\boldsymbol{k}}))[\boldsymbol{w}]$. The converse direction is analogous.

For 2, it follows directly from the definition of di-persistence. □

Therefore, to find a solution for $\bigwedge(\gamma_1, \ldots, \gamma_n)$, it is enough to find solutions for each of $\gamma_1, \ldots, \gamma_n$ with respect to the same $c_{\boldsymbol{k}}$, such that $c_{\boldsymbol{k}} \not\nearrow \bigwedge(\gamma_1, \ldots, \gamma_n)$, and to take the conjunction of the solutions, then this becomes a solution for the whole formula.

4. DETERMINISTIC SQEMA

A formula $\boldsymbol{\phi}$ is in *negation normal form* iff $\neg$ occurs only in front of atomic formulas.

We follow [6, 10]. First, we give a simplified informal definition of the algorithm. Let $\boldsymbol{\phi}$ be the input modal formula. The goal is to obtain a nominal $c_{\boldsymbol{k}}$, and a pure formula $\boldsymbol{\phi'}$, such that $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$ and $\boldsymbol{\phi} \approx_{\boldsymbol{k}} \boldsymbol{\phi'}$. Then it is very easy, as we show below, to obtain a local first-order correspondent for $\boldsymbol{\phi}$.

First, we negate $\boldsymbol{\phi}$ and rewrite it in negation normal form, obtaining $\boldsymbol{\gamma}$. We start eliminating variables by a process similar to Gaussian elimination. Thus, we solve a *system of equations* (actually a conjunction of disjunctions), starting with a system with the single equation $(\neg c_{\boldsymbol{k}} \vee \boldsymbol{\gamma})$. We eliminate each variable separately, so let $\boldsymbol{p}$ be the current variable to eliminate. The elimination is carried out by applying the following rules:

Ackermann rule:

$$\begin{cases} \bigwedge((\boldsymbol{\alpha}_1 \vee \boldsymbol{p}), \dots, (\boldsymbol{\alpha}_{n_a} \vee \boldsymbol{p})) \wedge \\ \bigwedge(\boldsymbol{\beta}_1(\neg \boldsymbol{p}), \dots, \boldsymbol{\beta}_{n_b}(\neg \boldsymbol{p})) \wedge \\ \bigwedge(\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t}) \end{cases} \Rightarrow \begin{cases} \bigwedge(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b})[\boldsymbol{p}/\neg \bigwedge(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a})] \wedge \\ \bigwedge(\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t}) \end{cases}$$

where $\boldsymbol{p} \not\curlyvee \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t}\}$ and $\bigwedge(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b})$ is negative in $\boldsymbol{p}$.

$\Box$-rule: $(\boldsymbol{\phi_1} \vee \Box \boldsymbol{\phi_2}) \Rightarrow (\Box^{-1} \boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$

$\Diamond$-rule: $(\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{\phi}) \Rightarrow (\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{c''}) \wedge (\neg \boldsymbol{c''} \vee \boldsymbol{\phi})$, where $\boldsymbol{c''}$ is a new nominal.

Now we are ready to formalize the algorithm.

Proposition 17. (SQEMA rules)

1. *Equivalence rule.*
   If $\boldsymbol{\phi_1} \equiv \boldsymbol{\phi_2}$, then $\boldsymbol{\phi_1} \approx_{\boldsymbol{k}} \boldsymbol{\phi_2}$. As per Proposition 7, we can also replace (occurrences of) subformulas with semantically equivalent ones.
2. *Polarity reversing rule.*
   $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\neg \boldsymbol{p}]$.
3. *Positive elimination rule.*
   Let $\boldsymbol{\phi}$ be positive in $\boldsymbol{p}$. Then $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\top]$.
4. *Negative elimination rule.*
   Let $\boldsymbol{\phi}$ be negative in $\boldsymbol{p}$. Then $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\bot]$.
5. $\Box$*-rule.*
   $\neg(\boldsymbol{\phi'} \wedge (\boldsymbol{\phi_1} \vee \Box \boldsymbol{\phi_2})) \approx_{\boldsymbol{k}} \neg(\boldsymbol{\phi'} \wedge (\Box^{-1} \boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}))$.
6. $\Diamond$*-rule.*
   Let $\boldsymbol{c''}$ be such that $\boldsymbol{c''} \not\curlyvee \{c_{\boldsymbol{k}}, \boldsymbol{c'}, \boldsymbol{\phi'}, \boldsymbol{\phi}\}$. Then:
   $\neg(\boldsymbol{\phi'} \wedge (\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{\phi})) \approx_{\boldsymbol{k}} \neg(\boldsymbol{\phi'} \wedge ((\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{c''}) \wedge (\neg \boldsymbol{c''} \vee \boldsymbol{\phi})))$.
7. *The Ackermann rule.* Let $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t}$ be formulas which contain no occurrences of $\boldsymbol{p}$, let $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b}$ be formulas which are either *negative* or *downwards monotone* in $\boldsymbol{p}$. Let:
   $\boldsymbol{\gamma'} := \neg \bigwedge((\boldsymbol{\alpha}_1 \vee \boldsymbol{p}), \dots, (\boldsymbol{\alpha}_{n_a} \vee \boldsymbol{p}), \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t})$
   $\boldsymbol{\gamma''} := \neg \bigwedge(\bigwedge(\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b})[\boldsymbol{p}/\neg \bigwedge(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a})], \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t})$
   Then: $\boldsymbol{\gamma'} \approx_{\boldsymbol{k}} \boldsymbol{\gamma''}$.

Proof. For the equivalence rule, the result follows immediately.

The rest of the rules are in the form $\neg \boldsymbol{\phi'} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi''}$ for some formulas $\boldsymbol{\phi'}$ and $\boldsymbol{\phi''}$. Let $\mathfrak{g}$ be a general discrete frame, and let $\boldsymbol{w}$ be a world in $\mathfrak{g}$. To prove that $\neg \boldsymbol{\phi'} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi''}$, it is enough to prove that for every model $\mathbf{M}$ over $\mathfrak{g}$, such that

$[\![c_k]\!]_{\mathbf{M}} = \{w\}$ and $\mathbf{M} \Vdash \phi'$, there is a model $\mathbf{M'}$ over $\mathfrak{g}$, such that $[\![c_k]\!]_{\mathbf{M'}} = \{w\}$ and $\mathbf{M'} \Vdash \phi''$, and vice versa.

Polarity reversing rule: Because negations of admissible valuations are admissible, we set $\mathbf{M'}$ to be equal to $\mathbf{M}$, except $[\![p]\!]_{\mathbf{M'}}$ is set to be the complement of $[\![p]\!]_{\mathbf{M}}$. The implication follows by Definition 5. The converse follows analogously.

Positive elimination rule: Let W be the universe of $\mathbf{M}$. By induction on ϕ, we get that $[\![\phi]\!]_{\mathbf{M}} \subseteq [\![\phi[p/\top]]\!]_{\mathbf{M}}$.

First, let $[\![c_k]\!]_{\mathbf{M}} = \{w\}$ and $\mathbf{M} \Vdash \phi$. We set $\mathbf{M'}$ to be equal to $\mathbf{M}$, except $[\![p]\!]_{\mathbf{M'}}$ is set to be W, which is admissible. We have that $W = [\![\phi]\!]_{\mathbf{M}} \subseteq [\![\phi[p/\top]]\!]_{\mathbf{M}} = [\![\phi[p/\top]]\!]_{\mathbf{M'}}$, by Proposition 7. Therefore, $[\![\phi[p/\top]]\!]_{\mathbf{M'}} = W$.

Now, let $[\![c_k]\!]_{\mathbf{M}} = \{w\}$ and $\mathbf{M} \Vdash \phi[p/\top]$. We construct $\mathbf{M'}$ in the same way, and it is straightforward to prove that $[\![\phi]\!]_{\mathbf{M'}} = W$.

Negative elimination rule: Follows from the polarity reversing rule and the positive elimination rule.

$\Box$-rule: Let $R_\Box$ be the (converse) relation of $\mathbf{M}$, which corresponds to $\Box$.

First, let $\mathbf{M} \Vdash (\phi' \wedge (\phi_1 \vee \Box\phi_2))$, suppose that $\mathbf{M} \nVdash (\phi' \wedge (\Box^{-1}\phi_1 \vee \phi_2))$. Then, there is a $w_1 \in W$: $\mathbf{M}, w_1 \nVdash (\Box^{-1}\phi_1 \vee \phi_2)$. Then, $\mathbf{M}, w_1 \nVdash \Box^{-1}\phi_1$ and $\mathbf{M}, w_1 \nVdash \phi_2$. Therefore, there is a $w_2 \in W$: $\mathbf{M}, w_2 \nVdash \phi_1$ and $w_2 \; R_\Box \; w_1$. However, $\mathbf{M} \Vdash (\phi_1 \vee \Box\phi_2)$, therefore $\mathbf{M}, w_2 \Vdash \Box\phi_2$, so $\mathbf{M}, w_1 \Vdash \phi_2$, contradiction.

Now, let $\mathbf{M} \Vdash (\phi' \wedge (\Box^{-1}\phi_1 \vee \phi_2))$, suppose that $\mathbf{M} \nVdash (\phi' \wedge (\phi_1 \vee \Box\phi_2))$. Analogously to the above, we derive a contradiction.

$\Diamond$-rule: Let $\mathbf{M} = \langle \mathfrak{g}_\#, V, A \rangle$. Let $R_\Diamond$ be the relation or converse relation of $\mathbf{M}$, corresponding to $\Diamond$.

First, let $\mathbf{M} \Vdash (\phi' \wedge (\neg c' \vee \Diamond\phi))$, and let $w_1 := A(c')$. Then, $\mathbf{M}, w_1 \Vdash \Diamond\phi$. So, there is a $w_2 \in W$: $w_1 \; R_\Diamond \; w_2$ and $\mathbf{M}, w_2 \Vdash \phi$. We set $\mathbf{M'} := \mathbf{M}[c'' \to w_2]$. By Proposition 7, and by the hypothesis on c'', the condition holds.

Now, let $\mathbf{M} \Vdash (\phi' \wedge ((\neg c' \vee \Diamond c'') \wedge (\neg c'' \vee \phi)))$. Then, $\mathbf{M} \Vdash (\phi' \wedge (\neg c' \vee \Diamond\phi))$.

The Ackermann rule: It is easy to show that if β is negative in p, then it is downwards monotone in p. Let α be $\bigwedge(\alpha_1, \ldots, \alpha_{n_a})$, β be $\bigwedge(\beta_1, \ldots, \beta_{n_b})$, and β be downwards monotone in p. First, let $\mathbf{M} \Vdash \neg\gamma'$, so $\mathbf{M} \Vdash (\alpha \vee p)$ and $\mathbf{M} \Vdash \beta$. Then, $[\![\neg\alpha]\!]_{\mathbf{M}} \subseteq [\![p]\!]_{\mathbf{M}}$, therefore $W = [\![\beta]\!]_{\mathbf{M}} \subseteq [\![\beta[p/\neg\alpha]]\!]_{\mathbf{M}}$, so $\mathbf{M} \Vdash \neg\gamma''$. Now, let $\mathbf{M} \Vdash \neg\gamma''$, and let $\mathbf{M} = \langle \mathbf{F}, V, A \rangle$. Let $V'(p) := [\![\neg\alpha]\!]_{\mathbf{M}}$, and let $V'(p') := V(p')$ for other variables p'. Let $\mathbf{M'} := \langle \mathbf{F}, V', A \rangle$. Then, $\mathbf{M'} \Vdash \neg\gamma'$. $\Box$

Definition 18. (Standard Translation) In the function definition below, $st(n, x, \phi)$ stands for *special standard translation for pure formulas*, or simply *standard translation*. For $st(n, x, \phi)$, we assume that ϕ is pure, that x is x_i, such that c_i does not occur in ϕ, and that n is such that $c_n \not\nearrow^{\infty} \{c_i, \phi\}$.

$st(n, x, \bot) := \bot$
$st(n, x, \top) := \top$
$st(n, x, c_i) := (x = x_i)$
$st(n, x, \neg\phi) := \neg st(n, x, \phi)$

$st(n, x_i, (\phi_1 \vee \phi_2)) := (st(n, x_i, \phi_1) \vee st(n', x_i, \phi_2))$, where n' is the least number such that $n' \geq n$, $n' > i$ and for all x_j, occurring in $st(n, x_i, \phi_1)$, $n' > j$.
$st(n, x_i, (\phi_1 \wedge \phi_2)) := (st(n, x_i, \phi_1) \wedge st(n', x_i, \phi_2))$, where n' is the least number such that $n' \geq n$, $n' > i$ and for all x_j, occurring in $st(n, x_i, \phi_1)$, $n' > j$.
$st(n, x, \langle U \rangle \phi) := \exists x_n st(n+1, x_n, \phi)$
$st(n, x, \Diamond_0{}^{-1} \phi) := \exists x_n st(n+1, x_n, \phi)$
$st(n, x, [U] \phi) := \forall x_n st(n+1, x_n, \phi)$
$st(n, x, \Box_0{}^{-1} \phi) := \forall x_n st(n+1, x_n, \phi)$
$st(n, x, \Diamond_i \phi) := \exists x_n ((x\ r_i\ x_n) \wedge st(n+1, x_n, \phi))$
$st(n, x, \Diamond_i{}^{-1} \phi) := \exists x_n ((x_n\ r_i\ x) \wedge st(n+1, x_n, \phi))$
$st(n, x, \Box_i \phi) := \forall x_n (\neg (x\ r_i\ x_n) \vee st(n+1, x_n, \phi))$
$st(n, x, \Box_i{}^{-1} \phi) := \forall x_n (\neg (x_n\ r_i\ x) \vee st(n+1, x_n, \phi))$
It is immediate that *st* defines a unique function if the conditions for it hold. It is also clear that the result of *st* can be effectively obtained.

An easy, but somewhat tedious, induction on pure formulas ϕ shows that, under the above assumptions for n and x_i, for any model $\mathbf{M}$ and any world w in $\mathbf{M}$, it is the case that $\mathbf{M}, w \Vdash \phi$ iff $\mathbf{M}[c_i \to w] \vDash st(n, x_i, \phi)$. We call this the *main property of st*.

Lemma 19. (Standard Translation Lemma) Let c_k be a nominal, and let ϕ be a pure formula. Then for ϕ there can be effectively obtained a first-order formula $\psi(x_k)$, such that $\phi \sim_k \psi(x_k)$.

Proof. Let i be such that $c_i \not\nearrow \phi$. Consider ψ: $\forall x_{j_1} \ldots \forall x_{j_m} \exists x_i st(n, x_i, \phi)$, where $c_n \not\nearrow^{\infty} \{c_i, \phi\}$, and $[j_1, \ldots, j_m]$ are such that $[c_{j_1}, \ldots, c_{j_m}]$ is the list of members of $\mathrm{NOM}(\phi) \setminus \{c_k\}$ in left-to-right order of initial occurrence in ϕ. Note that ψ can be denoted by $\psi(x_k)$, because the only free variable, if any, is x_k. We show that $\phi \sim_k \psi(x_k)$. For convenience, denote $\neg\phi$ by ϕ'.

For given $\mathbf{F}$ and w in $\mathbf{F}$, let $\mathbf{M} = \langle \mathbf{F}, V, A \rangle$ be a model over $\mathbf{F}$ such that $\mathbf{M}, w \Vdash c_k$ and $\mathbf{M} \Vdash \phi'$. By the main property of *st*, for every w_1 in $\mathbf{F}$: $\mathbf{M}[c_i \to w_1] \vDash st(n, x_i, \phi')$ iff $\mathbf{M}, w_1 \Vdash \phi'$. Therefore, $\mathbf{M} \vDash \forall x_i st(n, x_i, \phi')$. Because A assigns every nominal, $\mathbf{M} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, \phi')$. Because $A(c_k) = w$, and because x_k is the only free variable in $\exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, \phi')$, if any, we have that $\mathbf{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, \phi')[w]$.

Now, for given $\mathbf{F}$ and w in $\mathbf{F}$, let $\mathbf{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, \phi')[w]$. Let $\mathbf{M}$ be any model over $\mathbf{F}$, then $\mathbf{M}[c_k \to w] \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, \phi')$. We define the model $\mathbf{M'}$ over $\mathbf{F}$, such that $\mathbf{M'}, w \Vdash c_k$ and $\mathbf{M'} \Vdash \phi'$. Because there are states $v_{j_1}, \ldots, v_{j_m}$ in $\mathbf{F}$, such that $\mathbf{M}[c_k \to w][c_{j_1} \to v_{j_1}] \ldots [c_{j_m} \to v_{j_m}] \vDash \forall x_i st(n, x_i, \phi')$, we set $\mathbf{M'}$ to the above modification of $\mathbf{M}$. We show that $\mathbf{M'} \vDash \forall x_i st(n, x_i, \phi')$ iff $\mathbf{M'} \Vdash \phi'$. But this follows by the main property of *st* and Definition 12. $\Box$

Now, we need a deterministic and terminating strategy for applying the SQEMA rules. *Equations* are formulas of the kind $(c' \to \Diamond c'')$ or of the kind $(\phi' \veebar \phi'')$, such that ϕ' and ϕ'' are in negation normal form. A *system* is a formula of the kind $\neg \bigwedge(\chi_1, \ldots, \chi_n)$ for some $n \geq 0$, where $\chi_1, \ldots, \chi_n$ are equations. We use σ for systems of equations and χ for equations. σ is *solved for* p iff there are no occurrences of $\boldsymbol{p}$ in $\boldsymbol{\sigma}$. $\boldsymbol{\sigma}$ is *solved* iff it is pure. The algorithm first splits the input formula, by the conjunction lemma, into several systems of equations, trying to solve each of them in sequence. Below, we say that $\boldsymbol{c}$ is a *new nominal*, if $\boldsymbol{c}$ is such that: if $\boldsymbol{\gamma_1}, \ldots, \boldsymbol{\gamma_n}$ are all formulas that have occurred as input or during the execution of any branch of the algorithm so far, it is the case that $\boldsymbol{c} \not\nearrow^{\infty} \{\boldsymbol{\gamma_1}, \ldots, \boldsymbol{\gamma_n}\}$ (see Definition 2).

If $\boldsymbol{\sigma}$ is $\neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_m})$, we denote by $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//\boldsymbol{\chi_1'}, \ldots, \boldsymbol{\chi_m'}]$: $\neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_{j-1}}, \boldsymbol{\chi_1'}, \ldots, \boldsymbol{\chi_m'}, \boldsymbol{\chi_{j+1}}, \ldots, \boldsymbol{\chi_n})$. We denote by $\boldsymbol{\sigma}[\boldsymbol{p}//\neg\boldsymbol{p}]$ the system of equations, produced from $\boldsymbol{\sigma}$, where, simultaneously, every occurrence of $\boldsymbol{p}$ has been replaced with $\neg\boldsymbol{p}$ and every occurrence of $\neg\boldsymbol{p}$ has been replaced with $\boldsymbol{p}$.

We now describe a deterministic version of the SQEMA algorithm from [6].

The algorithm Deterministic SQEMA

INPUT: $\phi \in \mathrm{ML}(T, U)$

OUTPUT: $\langle success, fol(\phi)\rangle$ or $\langle failure\rangle$

STEP 1: Rewrite ϕ in negation normal form. Then, distribute all boxes, which are not in the scope of a diamond, and all disjunctions, over conjunctions as much as possible, using the semantic equivalences:

Rule 1.1: $\Box(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \equiv (\Box\boldsymbol{\phi_1} \wedge \Box\boldsymbol{\phi_2})$

Rule 1.2: $((\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \vee \boldsymbol{\phi_3}) \equiv ((\boldsymbol{\phi_1} \vee \boldsymbol{\phi_3}) \wedge (\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3}))$

Rule 1.3: $(\boldsymbol{\phi_1} \vee (\boldsymbol{\phi_2} \wedge \boldsymbol{\phi_3})) \equiv ((\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}) \wedge (\boldsymbol{\phi_1} \vee \boldsymbol{\phi_3}))$

Thus, obtain $\phi \equiv \bigwedge(\phi_1, \ldots, \phi_n)$ where no further applications of rules 1.1, 1.2 or 1.3 are possible on any $\phi_{\boldsymbol{i}}$. Now reserve the nominal c_k, such that $c_k \not\nearrow^{\infty} \phi$ (see Definition 2), and use it throughout the steps. Proceed with STEP 2, applied separately on each of the subformulas $\phi_{\boldsymbol{i}}$, and if it succeeds for all $\phi_{\boldsymbol{i}}$, proceed to STEP 5. Otherwise, if anyone of the branches for a single $\boldsymbol{i}$ fails, then return $\langle failure\rangle$ as output and stop.

STEP 2: Let $\phi_{\boldsymbol{i}}$ be one of the conjuncts from STEP 1. Let ϕ' be the *normalized* form, of $\neg\phi_{\boldsymbol{i}}$, which we define below, but for now it suffices to know that it means that ϕ' is in negation normal form, and any variable, that occurs only positively or negatively in $\neg\phi_{\boldsymbol{i}}$ has been replaced, by the positive or negative elimination rules, with $\top$, or $\bot$, respectively. Now, construct the equation $(\neg c_k \veebar \phi')$, where c_k is the nominal from STEP 1. By the sufficient condition and the equivalence rule, try solving σ: $\neg \bigwedge((\neg c_k \vee \phi'))$ by proceeding to STEP 3.

STEP 3: Let the current system be σ. For every permutation of $\mathrm{PROP}(\sigma)$, try it as the *variable elimination order*, trying to eliminate each variable in that order by proceeding to STEP 4 with a new, empty *backtracking stack*. If a permutation succeeds, and thus, all propositional variables have been eliminated from the current system, proceed to STEP 5. If all elimination orders fail, report failure for the current system and go back to executing STEP 2.

STEP 4: Take the propositional variable p that has to be eliminated and the system σ_0 as input. Save a backtracking context $\langle p, \sigma_0\rangle$, to the stack for the application of the polarity reversing rule, but only if the input hasn't come out of the stack. Deterministically apply the SQEMA rules in order to try eliminating all occurrences of p, converting σ_0 to σ_1. Use the deterministic strategy for SQEMA rules application which is shown below. If p has been eliminated, report success and return the normalized form of σ_1 (defined below) to STEP 3 to try eliminating the remaining variables. If this fails, check if the backtracking stack is empty. If it is empty, report failure to eliminate p and resume executing STEP 3 to try other permutations. Otherwise, backtrack to the context $\langle p', \sigma'_0 \rangle$ from the top of the stack, which may apply to a previous variable, then execute STEP 4 with p' and $\sigma'_0[p'//\neg p']$, skipping the saving of backtracking context.

STEP 5: If this step is reached by all branches of the execution, then all propositional variables have been eliminated from all systems resulting from the input formula. Let all pure systems be $\sigma_1, \ldots, \sigma_{\boldsymbol{n}}$. For each pure system $\sigma_{\boldsymbol{i}}$, let $\mathrm{NOM}(\sigma_{\boldsymbol{i}}) \backslash \{c_k\} = \{c_{\boldsymbol{j^i_1}}, \ldots, c_{\boldsymbol{j^i_{l_i}}}\}$, and let $c_{\boldsymbol{m_i}} \not\nearrow^{\infty} \{c_k, \sigma_{\boldsymbol{i}}\}$ (see Definition 2). Using the standard translation lemma, let $fol_{\boldsymbol{i}}(\phi)$ be: $\forall x_{\boldsymbol{j^i_1}} \ldots \forall x_{\boldsymbol{j^i_{l_i}}} \exists x_{\boldsymbol{m_i}} st(\boldsymbol{m_i}+1, x_{\boldsymbol{m_i}}, \sigma_{\boldsymbol{i}})$. Let $fol(\phi)$ be $\bigwedge(fol_1(\phi), \ldots, fol_{\boldsymbol{n}}(\phi))$, by the conjunction lemma, Lemma 19. Return the result $\langle success,\ fol(\phi) \rangle$.

Now, we define: a) the normalization of a formula used in STEP 2 with diamond extraction, b) the normalization of a system of equations used in STEP 4, and c) the deterministic SQEMA rules application strategy.

a) It is clear how we can obtain a formula in negation normal form for a given $\boldsymbol{\gamma}$, such that $\Box_0^{-1}$ and $\Diamond_0^{-1}$ do not occur, because these are semantically equivalent to $[U]$ and $\langle U \rangle$. We use this procedure to reduce the number of subformulas in the output, by applying the equivalence rule for some obvious boolean and modal laws, as well as the following rules for the universal modality:

For $j \in \{1, 2\}$, we use U_j for either $[U]$ or $\langle U \rangle$, we use $\overset{\wedge}{\vee}$ for either $\vee$ or $\wedge$.

Replace	with	Replace	with
$(c_1 \to \langle U \rangle c_2)$	$\top$	$(\langle U \rangle \gamma_1 \vee \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\top$
$U_1 U_2 \gamma$	$U_2 \gamma$	$(\langle U \rangle \gamma \vee \Diamond\gamma)$	$\langle U \rangle \gamma$
$\Box U_1 \gamma$	$(U_1 \gamma \vee \Box\bot)$	$(\langle U \rangle \gamma \vee \gamma)$	$\langle U \rangle \gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(\langle U \rangle \gamma \wedge \Diamond\gamma)$	$\Diamond\gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} [U]\gamma_2)$	$(\langle U \rangle \gamma \wedge \gamma)$	γ
$[U]\neg c$	$\bot$	$([U]\gamma_1 \wedge \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\bot$
$\Diamond U_1 \gamma$	$(U_1\gamma \wedge \Diamond\top)$	$([U]\gamma \wedge \Box\gamma)$	$[U]\gamma$
$\langle U \rangle (U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$([U]\gamma \wedge \gamma)$	$[U]\gamma$
$\langle U \rangle (U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} \langle U \rangle\gamma_2)$	$([U]\gamma \vee \Box\gamma)$	$\Box\gamma$
$\langle U \rangle c$	$\top$	$([U]\gamma \vee \gamma)$	γ

Then, we define a procedure for constructing a *conjunctive normal form*, using the standard definition of this notion. It is clear how this normal form can be constructed. During this construction, also perform *diamond extraction*, applying the rule $(\Diamond\boldsymbol{\phi'} \vee \Diamond\boldsymbol{\phi''}) \equiv \Diamond(\boldsymbol{\phi'} \vee \boldsymbol{\phi''})$. Attempt to eliminate semantically equivalent or opposite members of any disjunction, by comparing their normal forms. The output must not have subformulas of the kind $(\bot \overset{\wedge}{\vee} \gamma)$ or $(\gamma \overset{\wedge}{\vee} \bot)$.

Two improvements can be made: during the elimination, a tableaux method for $\mathrm{ML}(T, U)$ could be used to prove an equivalence, instead of comparing normal forms. Also, in the conjunction construction phase, modal resolution can be performed, as in example 6.14 of [7].

This is the normalization procedure for $\boldsymbol{\gamma}$, which produces the *normal form of* $\boldsymbol{\gamma}$: First, convert $\boldsymbol{\gamma}$ to negation normal form, then convert the result to conjunctive normal form simultaneously performing diamond extraction, by the equivalence rule, then perform box extraction using the semantic equivalence $(\Box\boldsymbol{\phi_1} \wedge \Box\boldsymbol{\phi_2}) \equiv \Box(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$, and finally replace any variables that occur only positively or negatively in $\boldsymbol{\gamma}$ with $\top$, or $\bot$, respectively. Repeat the whole process until no further changes to the formula can be made.

b) Now, we normalize a system of equations $\boldsymbol{\sigma}$. Let $\boldsymbol{\sigma}$ be $\neg\bigwedge(\boldsymbol{\chi_1}, \dots, \boldsymbol{\chi_n})$. Let $\boldsymbol{\phi'}$ be the normal form of $\bigwedge(\boldsymbol{\chi_1}, \dots, \boldsymbol{\chi_n})$. If $\boldsymbol{\phi'}$ is of the kind $(\neg\boldsymbol{c} \vee \boldsymbol{\phi''})$, then the *normal form of* $\boldsymbol{\sigma}$ is $\neg\bigwedge((\neg\boldsymbol{c} \veebar \boldsymbol{\phi''}))$; otherwise, it is $\neg\bigwedge((\bot \veebar \boldsymbol{\phi'}))$.

c) The deterministic strategy for applying the SQEMA rules for a given variable $\boldsymbol{p}$ is to use the *step* function (given below) repeatedly until either a formula without occurrences of $\boldsymbol{p}$ is reached, or *failure* is obtained.

Definition 20. (Deterministic SQEMA Step) We describe a single step of the strategy, which is uniquely defined for $\boldsymbol{\sigma}$ and $\boldsymbol{p}$.

(1) If $\boldsymbol{p} \not\nearrow \boldsymbol{\sigma}$, then the result is $\boldsymbol{\sigma}$.

(2) Else, if $\boldsymbol{\sigma}$ is $\neg\bigwedge((\boldsymbol{\alpha}_1 \veebar \boldsymbol{p}), \dots, (\boldsymbol{\alpha}_{n_a} \veebar \boldsymbol{p}), \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t})$, where $n_a \geq 0$, $n_b \geq 0$, $n_t \geq 0$, $\boldsymbol{p} \not\nearrow \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t}\}$, and $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b}$ are formulas which are *negative* in $\boldsymbol{p}$, then we can apply the Ackermann rule for $\boldsymbol{p}$ and $\boldsymbol{\sigma}$. Let for $1 \leq \boldsymbol{l} \leq n_b$, $\boldsymbol{\beta'_l}$ be obtained from $\boldsymbol{\beta_l}$ by replacing all occurrences of $\neg\boldsymbol{p}$ with $\bigwedge(\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n_a})$. Then the result for $\boldsymbol{\sigma}$ is $\neg\bigwedge(\boldsymbol{\beta}'_1, \dots, \boldsymbol{\beta}'_{n_b}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n_t})$.
This can be improved by allowing $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n_b}$ to be *downwards monotone* in $\boldsymbol{p}$. This can be tested by proving $[U]([U](\boldsymbol{p''} \rightarrow \boldsymbol{p'}) \rightarrow [U](\boldsymbol{\beta_l}[\boldsymbol{p}/\boldsymbol{p'}] \rightarrow \boldsymbol{\beta_l}[\boldsymbol{p}/\boldsymbol{p''}]))$, such that $\boldsymbol{p'} \not\nearrow \boldsymbol{\beta_l}$ and $\boldsymbol{p''} \not\nearrow \boldsymbol{\beta_l}$, using a tableaux method.

(3) If we are not in any of the above two cases, then there is at least one positive occurrence of $\boldsymbol{p}$ in $\boldsymbol{\sigma}$, which is not in an equation of the kind $(\boldsymbol{\alpha} \veebar \boldsymbol{p})$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}$. For convenience, let $\boldsymbol{\sigma}$ be $\neg\bigwedge(\boldsymbol{\chi_1}, \dots, \boldsymbol{\chi_m})$, let $\boldsymbol{j}$ be the least number, such that $\boldsymbol{p}$ occurs positively in $\boldsymbol{\chi_j}$, $\boldsymbol{\chi_j}$ is not as described, and let let $\boldsymbol{\chi_j}$ be $(\boldsymbol{\phi'} \veebar \boldsymbol{\phi_1})$.

(3.1) If $\boldsymbol{\phi_1}$ is $(\boldsymbol{\phi_2} \wedge \boldsymbol{\phi_3})$, then, by the equivalence rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j} // (\boldsymbol{\phi'} \veebar \boldsymbol{\phi_1}), (\boldsymbol{\phi'} \veebar \boldsymbol{\phi_2})]$.

(3.2) If $\boldsymbol{\phi_1}$ is $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$, then there are three cases. If $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$, then by the equivalence rule the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j} // ((\boldsymbol{\phi'} \vee \boldsymbol{\phi_2}) \veebar \boldsymbol{\phi_3})]$. Otherwise, if

$\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$, then, by the equivalence rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//((\boldsymbol{\phi'} \vee \boldsymbol{\phi_3}) \veebar \boldsymbol{\phi_2})]$. Otherwise, the result for $\boldsymbol{\sigma}$ is *failure*.

(3.3) If $\boldsymbol{\phi_1}$ is $\Box\boldsymbol{\phi_2}$, by the box rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//(\Box^{-1}\boldsymbol{\phi'} \veebar \boldsymbol{\phi_1})]$.

(3.4) If $\boldsymbol{\phi_1}$ is $\Diamond\boldsymbol{\phi_2}$ and $\boldsymbol{\phi'}$ is either $\neg\boldsymbol{c'}$ or $(\bot \vee \neg\boldsymbol{c'})$, then, by the diamond rule, let $\boldsymbol{c''}$ be a new nominal, then the result for $\boldsymbol{\sigma'}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//(\boldsymbol{c'} \to \Diamond\boldsymbol{c''}), (\neg\boldsymbol{c''} \veebar \boldsymbol{\phi_1})]$.

(3.5) If we are not in any of the above four cases, the result for $\boldsymbol{\sigma}$ is *failure*.

It is immediate that the above describes a uniquely defined effective function over the systems of equations and propositional variables. We denote the function by *step*.

Immediately by the definition of *step*, we have that $\boldsymbol{\sigma} \approx_{\boldsymbol{k}} \textit{step}(\boldsymbol{\sigma}, \boldsymbol{p})$ by the SQEMA rules, Proposition 17.

We prove that the application of *step* can be composed only finitely many times for $\boldsymbol{\sigma}$ and $\boldsymbol{p}$, before reaching either a $\boldsymbol{\sigma'}$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\sigma'}$, or *failure*.

Indeed, if the result is ever obtained by (1), (2), (3.5), or the failing condition of (3.2), it is clear that this is the final application of *step*. Therefore, suppose there is an infinite sequence of results, obtained by (3.1), (3.3), (3.4), or the non-failing conditions of (3.2). Then, there is an infinite sequence $\boldsymbol{\sigma_0}, \boldsymbol{\sigma_1}, \dots$, and let $S_0, S_1, \dots$ be the sum of lengths of right-hand sides of equations in the corresponding $\boldsymbol{\sigma}$-s. It is clear that $S_0 > 0$ and for $i < j$, $S_i > S_j$, which is impossible. Therefore, we can only apply *step* a finite number of times. □

This concludes our definition of Deterministic SQEMA and the proof for its soundness and termination.

5. SAHLQVIST AND INDUCTIVE FORMULAS

We now examine some famous classes of elementary formulas and we prove that Deterministic SQEMA succeeds for them.

We use ideas from the proofs in [6].

Definition 21. (Sahlqvist formulas) A *boxed atom* is a formula $\boldsymbol{\phi}$, which is either a propositional variable $\boldsymbol{p}$ or $\Box\boldsymbol{\phi'}$, where $\boldsymbol{\phi'}$ is a boxed atom. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, boxed atoms and negative formulas, using $\wedge$, $\vee$ and $\Diamond$. A *Sahlqvist implication* is of the form $(\boldsymbol{\phi'} \to \boldsymbol{\phi''})$, where $\boldsymbol{\phi'}$ is a Sahlqvist antecedent and $\boldsymbol{\phi''}$ is positive. A *Sahlqvist formula* (in the classical definition) is built up from Sahlqvist implications by using boxes and conjunctions, and by applying disjunctions only between formulas which do not share propositional variables. An *extended Sahlqvist formula* is built up from Sahlqvist implications by using boxes, conjunctions, and disjunctions. From now on, we simply say *Sahlqvist formula* instead of *extended Sahlqvist formula*.

A *boxed piece* is a formula $\boldsymbol{\phi}$ which is either $\boldsymbol{p}$, $\Box\boldsymbol{\phi'}$, $\boldsymbol{Neg}$, $(\boldsymbol{pure} \vee \boldsymbol{\phi'})$, $(\boldsymbol{\phi'} \vee \boldsymbol{pure})$ or $(\boldsymbol{\phi'_1} \wedge \boldsymbol{\phi'_2})$, where $\boldsymbol{\phi'}$, $\boldsymbol{\phi'_1}$, and $\boldsymbol{\phi'_2}$ are boxed pieces, $\boldsymbol{Neg}$ is a negative formula,

$\boldsymbol{pure}$ is a pure formula, $\boldsymbol{\phi}$ is in negation normal form, disjunction over conjunction distribution may only apply to negative or pure subformulas of $\boldsymbol{\phi}$.

A *good piece* is a formula $\boldsymbol{\phi}$ which is built up from boxed pieces using $\wedge$ and $\Diamond$ such that $\boldsymbol{\phi}$ is in negation normal form, disjunction over conjunction distribution may only apply to negative or pure subformulas of $\boldsymbol{\phi}$, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas.

We denote by $\boldsymbol{\delta}$ a formula which is either a boxed piece, or of the form $(\neg\boldsymbol{c} \vee \boldsymbol{\phi})$ where $\boldsymbol{\phi}$ is a good piece. We denote by $\boldsymbol{\delta'}$ a formula which is either a $\boldsymbol{\delta}$, or of the kind $((\bot \vee \neg\boldsymbol{c}) \vee \boldsymbol{\phi})$ where $\boldsymbol{\phi}$ is a good piece. We denote by $\boldsymbol{\delta''}$ a formula which is either a $\boldsymbol{\delta}$, or of the kind $(\boldsymbol{\phi} \vee \neg\boldsymbol{c})$ where $\boldsymbol{\phi}$ is a good piece.

Proposition 22. If $\boldsymbol{\sigma}$ is a system of equations, where each equation $\boldsymbol{\chi}$ of $\boldsymbol{\sigma}$ is such that either $\boldsymbol{\chi}$ is a $\boldsymbol{\delta'}$, or $\boldsymbol{\chi}$ is of the form $(\bot \veebar \bigwedge(\boldsymbol{\delta''_1}, \ldots, \boldsymbol{\delta''_n}))$, then

1) Applying *step* gives a system of the same kind, and never *failure*.

2) The result of a system normalization procedure on $\boldsymbol{\sigma}$ is also a system of the same kind.

3) On Sahlqvist input formulas, Deterministic SQEMA only works on systems of the above kind. □

Corollary 23. Deterministic SQEMA succeeds on every Sahlqvist formula at the first permutation of its variables, without backtracking. □

Definition 24. (Inductive formulas) Let # be a symbol, which is not in the alphabet of ML(T, U). # is a *box-form* of #. If $B(\#)$ is a box-form of #, then $\Box B(\#)$ is a box-form of # for any $\Box$, and $(\boldsymbol{\phi} \to B(\#))$ is a box-form of # for any *positive* formula $\boldsymbol{\phi}$. Replacing all occurrences of # in $B(\#)$ with $\boldsymbol{p}$, we get $B(\boldsymbol{p})$, a *box-formula* of $\boldsymbol{p}$. The only positive occurrence of $\boldsymbol{p}$ in $B(\boldsymbol{p})$ is the *head of* $B(\boldsymbol{p})$, and any other occurrence of a propositional variable in $B(\boldsymbol{p})$ is *inessential*. For convenience, we also say that $\boldsymbol{p}$ is the *head of* $B(\boldsymbol{p})$ and the variables which have inessential occurrences in $B(\boldsymbol{p})$ are *inessential*. A *monadic regular formula (MRF)* is a modal formula built up from $\top$, $\bot$, positive formulas and negated box-formulas by applying $\wedge$, $\vee$ and $\Box$. The *dependency graph* of a set of box-formulas $\mathfrak{B} = \{B_1(\boldsymbol{p_1}), \ldots, B_n(\boldsymbol{p_n})\}$ is a directed graph $G(\mathfrak{B}) = \langle V, E\rangle$ where $V = \{\boldsymbol{p_1}, \ldots, \boldsymbol{p_n}\}$ is the set of heads in $\mathfrak{B}$ and E is the set of edges, such that $\langle \boldsymbol{p_i}, \boldsymbol{p_j}\rangle \in E$ iff $\boldsymbol{p_i}$ occurs as an inessential variable in a box-formula from $\mathfrak{B}$ with head $\boldsymbol{p_j}$. A directed graph is *acyclic* iff it does not contain directed cycles. The dependency graph of an MRF $\boldsymbol{\phi}$ is the dependency graph of the set of box-formulas which occur in the construction of $\boldsymbol{\phi}$ as an MRF. A *monadic inductive formula (MIF)* is a monadic regular formula with an acyclic dependency graph. We say that a conjunction of MIFs is an *inductive formula*.

We extend the definitions to negation normal forms of the above.

We define an *extended box-formula of* $\boldsymbol{p}$ thusly: $\boldsymbol{p}$ is an $\boldsymbol{EB}(\boldsymbol{p})$, $\Box\boldsymbol{EB}(\boldsymbol{p})$ is an $\boldsymbol{EB}(\boldsymbol{p})$, $(\boldsymbol{EB_1}(\boldsymbol{p}) \wedge \boldsymbol{EB_2}(\boldsymbol{p}))$ is an $\boldsymbol{EB}(\boldsymbol{p})$, if $\boldsymbol{Neg'}$ and $\boldsymbol{Neg''}$ are negative formulas, then each of $(\boldsymbol{Neg'} \vee \boldsymbol{EB}(\boldsymbol{p}))$, $(\boldsymbol{EB}(\boldsymbol{p}) \vee \boldsymbol{Neg'})$, $(\boldsymbol{Neg''} \wedge \boldsymbol{EB}(\boldsymbol{p}))$ and $(\boldsymbol{EB}(\boldsymbol{p}) \wedge \boldsymbol{Neg''})$ is an $\boldsymbol{EB}(\boldsymbol{p})$, and also $\boldsymbol{EB}(\boldsymbol{p})$ is in negation normal form. Here, $\boldsymbol{p}$ is the *head* of the extended box-formula, any occurrences of propositional variables in any of the $\boldsymbol{Neg'}$ formulas is *inessential*. The dependency graph of $\boldsymbol{EB}(\boldsymbol{p})$ is defined analogously to the above, but note that the variables of any $\boldsymbol{Neg''}$ do not count as inessential.

PureBox is a pure formula built up from negated nominals, $\bot$, $\vee$ and $\Box$.

We say that a formula $\boldsymbol{\phi}$ is a ***Good*** formula if it is such that $\boldsymbol{\phi}$ is in negation normal form, and $\boldsymbol{\phi}$ is either $\boldsymbol{EB}(\boldsymbol{p})$, $\boldsymbol{Neg}$, $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$, $\Diamond\boldsymbol{\phi'}$ outside the scope of boxes and disjunctions, $\Box\boldsymbol{\phi'}$, $(\boldsymbol{\phi'} \vee \boldsymbol{PureBox})$, or $(\boldsymbol{PureBox} \vee \boldsymbol{\phi'})$, where $\boldsymbol{Neg}$ is a negative formula, $\boldsymbol{\phi'}$, $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are ***Good*** formulas, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas. The *dependency graph of* ***Good*** is the union of the dependency graphs of the occurring formulas of kind $\boldsymbol{EB}(\boldsymbol{p})$, and we require that all ***Good*** formulas have an acyclic dependency graph.

A *good system* is a system of equations $\boldsymbol{\sigma} = \neg\bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n})$, such that every $\boldsymbol{\chi_i}$ is a *good equation* with an acyclic *dependency graph* $G(\boldsymbol{\chi_i})$ defined below, $G(\boldsymbol{\sigma}) = \bigcup\{G(\chi_1), \ldots, G(\chi_n)\}$, $G(\boldsymbol{\sigma})$ is acyclic, where exactly one of the following holds for each $\boldsymbol{\chi_i}$:

good.1. $\boldsymbol{\chi_i}$ is either $(\boldsymbol{Neg_{i_1}} \veebar \boldsymbol{Neg_{i_2}})$ or $(\boldsymbol{c'_i} \rightarrow \Diamond\boldsymbol{c''_i})$, with $G(\boldsymbol{\chi_i}) = \langle\emptyset, \emptyset\rangle$,

good.2.1. $\boldsymbol{\chi_i}$ is not of kind good.1, but is either $(\neg\boldsymbol{c_i} \veebar \boldsymbol{Good_i})$ or $((\bot \vee \neg\boldsymbol{c_i}) \veebar \boldsymbol{Good_i})$, with $G(\boldsymbol{\chi_i}) = G(\boldsymbol{Good_i})$,

good.2.2. $\boldsymbol{\chi_i}$ is not of the kind good.1 or good.2.1., but is $(\boldsymbol{PureBox} \veebar \boldsymbol{Good'_i})$, such that 1. there are no diamonds in $\boldsymbol{Good'_i}$ outside of box-formulas or negative subformulas, 2. $G(\boldsymbol{\chi_i}) = G(\boldsymbol{Good'_i})$,

good.3. $\boldsymbol{\chi_i}$ is not of the above kinds, but $\boldsymbol{\chi_i}$ is $(\boldsymbol{Neg'_i} \veebar \boldsymbol{EB'_i}(\boldsymbol{p_i}))$, such that $\boldsymbol{\chi_i}$ is some $\boldsymbol{EB_i}(\boldsymbol{p_i})$ with an acyclic graph, and $G(\boldsymbol{\chi_i}) = G(\boldsymbol{EB_i}(\boldsymbol{p_i}))$,

good.4. $\boldsymbol{\chi_i}$ is not of the above kinds, but $\boldsymbol{\chi_i}$ is $(\bot \veebar \bigwedge(\delta_1, \ldots, \delta_m))$, where each δ_j is either 1. negative with $G(\delta_j) = \langle\emptyset, \emptyset\rangle$, 2. $(\neg\boldsymbol{c} \vee \boldsymbol{Good})$ or $(\boldsymbol{Good} \vee \neg\boldsymbol{c})$ with $G(\delta_j) = G(\boldsymbol{Good})$, 3. $(\boldsymbol{PureBox} \vee \boldsymbol{Good'})$ or $(\boldsymbol{Good'} \vee \boldsymbol{PureBox})$ with $G(\delta_j) = G(\boldsymbol{Good'})$, such that there are no diamonds in $\boldsymbol{Good'}$ outside of box-formulas or negative formulas, or 4. an $\boldsymbol{EB}$ with $G(\delta_j) = G(\boldsymbol{EB})$ - an acyclic graph. The graph $G(\boldsymbol{\chi_i}) = \bigcup\{G(\delta_1), \ldots, G(\delta_m)\}$ is acyclic.

Claim 25. Every output of *step*, where the input is a good system, is a good system.

Proof. Consider ***result***, which is $step(\boldsymbol{\sigma}, \boldsymbol{p})$, where for $\boldsymbol{\sigma}$ the invariant holds. We show that ***result*** is not *failure* and that the invariant holds for ***result***.

If ***result*** is obtained from (1), then the invariant holds.

If ***result*** is obtained from (2), the Ackermann rule, then ***result*** is $\boldsymbol{\sigma'}$, $\boldsymbol{\sigma}$ is $\neg\bigwedge((\boldsymbol{\alpha}_1 \veebar \boldsymbol{p}), \ldots, (\boldsymbol{\alpha}_{n_a} \veebar \boldsymbol{p}), \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}$. Then, each $(\boldsymbol{\alpha} \veebar \boldsymbol{p})$ is of the form good.2.1, good.2.2, good.3, or good.4, so each $\boldsymbol{\alpha}$ is a $\boldsymbol{Neg}$, and the occurrences of $\neg\boldsymbol{p}$ within every $\boldsymbol{\beta}$ are in occurrences of a $\boldsymbol{Neg}$ within $\boldsymbol{\beta}$.

It remains to prove that $G(\boldsymbol{\sigma'})$ is acyclic. It would follow that the graph of every resulting equation is acyclic and that each of the resulting equations are in some of the good equation forms.

Because of the replacement, for every edge $\langle q_1, q_2\rangle$ of $G(\boldsymbol{\sigma'})$ either $\langle q_1, q_2\rangle$ is an edge of $G(\boldsymbol{\sigma})$ or there are edges $\langle q_1, \boldsymbol{p}\rangle$ and $\langle \boldsymbol{p}, q_2\rangle$ of $G(\boldsymbol{\sigma})$. Then for every cycle in $G(\boldsymbol{\sigma'})$ there is a corresponding cycle in $G(\boldsymbol{\sigma})$. Hence $G(\boldsymbol{\sigma'})$ is acyclic.

Then $\boldsymbol{\sigma'}$ is a good system.

If ***result*** is obtained from (3.1), then ***result*** is $\boldsymbol{\sigma'}$. We have split on $\wedge$ an equation of type good.2.1, good.2.2, good.3, or good.4. Equations of type good.2.1 split into two equations of type good.2.1, or one of type good.1 and one of type good.2.1. Equations of type good.2.2 split into two equations of type good.2.2, or one of type good.1 and one of type good.2.2. Equations of type good.3 split into two equations of the same kind, or one of kind good.1 and one of kind good.3. Equations of kind good.4 split into two equations, each of them of type either good.1, good.2.2, good.3, or good.4. All resulting equations are good equations, because the resulting equations have graphs that are subgraphs of the original ones. Hence $\boldsymbol{\sigma'}$ is a good system.

If ***result*** is obtained from (3.2), then let the changed equation of $\boldsymbol{\sigma}$ be $\boldsymbol{\chi}$, which is $(\boldsymbol{\phi'} \veebar (\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3}))$. We have that $\boldsymbol{\chi}$ is not negative, and because of the invariant for $\boldsymbol{\sigma}$ and the definition of ***Good***, we have that $\boldsymbol{\chi}$ is either of type good.2.1, good.2.2, good.3, or good.4 with $m = 1$.

First, let $\boldsymbol{\chi}$ be of type good.2.1, good.2.2 or good.3. Then $\boldsymbol{\phi'}$ is negative. Because the graph of $\boldsymbol{\chi}$ is acyclic, either $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$, with $\boldsymbol{\phi_2}$ negative or pure and $\boldsymbol{\phi_3}$ a ***Good*** formula, or vice versa. So ***result*** is $\boldsymbol{\sigma'}$, not ***failure***, and the invariant holds for $\boldsymbol{\sigma'}$ because we have converted $\boldsymbol{\chi}$ to an equation of type good 2.1, good 2.2 or good.3 with a graph that is the same.

Now, let $\boldsymbol{\chi}$ be of type good.4 with $m = 1$. Then $\boldsymbol{\phi'}$ is $\bot$. Then, because $\boldsymbol{p}$ occurs positively in $\boldsymbol{\chi}$, there are three cases for $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is $(\neg \boldsymbol{c} \vee \boldsymbol{Good})$ or $(\boldsymbol{PureBox} \vee \boldsymbol{Good'})$, then $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is $(\boldsymbol{Good} \vee \neg \boldsymbol{c})$ or $(\boldsymbol{Good'} \vee \boldsymbol{PureBox})$, then $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$. In these two cases we have converted $\boldsymbol{\chi}$ into an equation of type good.2.1 or good.2.2. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is an $\boldsymbol{EB}(\boldsymbol{p'})$ with an acyclic graph, then clearly $\boldsymbol{p}$ is $\boldsymbol{p'}$. Either $\boldsymbol{\phi_2}$ is negative and $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$ or $\boldsymbol{\phi_3}$ is negative and $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$. In this case we have converted $\boldsymbol{\chi}$ into an equation of type good.3.

In either case, ***result*** is $\boldsymbol{\sigma'}$ and the invariant holds for $\boldsymbol{\sigma'}$.

If ***result*** is obtained from (3.3), then ***result*** is $\boldsymbol{\sigma'}$. Suppose for the sake of contradiction that we have changed an equation $\boldsymbol{\chi}$ of kind good.4. Then either $\boldsymbol{\chi}$ is a negative formula, which contradicts the fact that $\boldsymbol{\chi}$ is not of kind good.1, or the right-hand side of $\boldsymbol{\chi}$ is a box, which contradicts the fact that $\boldsymbol{\chi}$ is not of kind good.3. Now, because $\boldsymbol{p}$ occurs positively in the changed equation of σ, there are three cases. First, an equation of type good.3 was changed, then we have converted the equation into another one of type good.3 with a graph that is the same. Second, we have converted an equation of type good.2.2 into another one of the same kind, with a graph that is the same. Third, we have converted an equation of type good.2.1 into an equation of type good.2.2 with a graph that is the same. Therefore, the invariant holds.

If ***result*** is obtained from (3.4) or from (3.5), let the first equation of $\boldsymbol{\sigma}$ where $\boldsymbol{p}$ occurs positively and which is not of kind $(\boldsymbol{\alpha} \veebar \boldsymbol{p})$ such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}$, be $\boldsymbol{\chi}$, which is $(\boldsymbol{\phi'} \veebar \Diamond\boldsymbol{\phi_2})$. Because $\boldsymbol{\chi}$ is not negative, $\boldsymbol{\chi}$ can only be of type good.2.1, and the result can only have been obtained from (3.4). The invariant holds because we have converted $\boldsymbol{\chi}$ into an equation of type good.1 and an equation of type good.2.1. $\Box$

Claim 26. Every output result of the system normalization procedure, where the input is a good system, is a good system.

Proof. It can be verified that the negation normal form procedure, followed by the conjunctive normal form procedure with diamond extraction, followed by the box extraction procedure, output a good system with a graph which is a subgraph of a graph of the original. $\Box$

Claim 27. On inductive input formulas, Deterministic SQEMA only works on good systems, with the starting equation being either of kind good.1 or of kind good.2.1.

Proof. By Claim 25 and Claim 26, it is enough to show that every initial equation is one of the kinds good.1, good.2.1, good.2.2, good.3, or good.4.

It can be verified that every initial equation on inductive formula inputs is of kind good.1 or good.2.1. $\Box$

Corollary 28. Deterministic SQEMA succeeds on every inductive formula at the first permutation of its variables, without backtracking. $\Box$

6. EXAMPLES

Let us consider the formula $(\Box_1 p_0 \to [U]p_0)$. After negation and normalization, the initial equation is $(\neg c_0 \veebar (\langle U\rangle\neg p_0 \wedge \Box_1 p_0))$. The system is split into two equations using the Equivalence Rule: $(\neg c_0 \veebar \langle U\rangle\neg p_0), (\neg c_0 \veebar \Box_1 p_0)$. Then, the Box-Rule is applied: $(\neg c_0 \veebar \langle U\rangle\neg p_0), (\Box_1^{-1}\neg c_0 \veebar p_0)$. After that, the Ackermann Rule is applied: $(\neg c_0 \veebar \langle U\rangle\Box_1^{-1}\neg c_0)$. The final result is: $\langle \mathit{success}, \forall x_1(x_0\ r_1\ x_1)\rangle$.

Now, let us take Löb's formula $(\Box_1(\Box_1 p_0 \to p_0) \to \Box_1 p_0)$. The initial equation is $(\neg c_0 \veebar (\Diamond_1\neg p_0 \wedge \Box_1(\Diamond_1\neg p_0 \vee p_0)))$. The Equivalence Rule is applied: $(\neg c_0 \veebar \Diamond_1\neg p_0), (\neg c_0 \veebar \Box_1(\Diamond_1\neg p_0 \vee p_0))$. The Box-Rule is applied: $(\neg c_0 \veebar \Diamond_1\neg p_0), (\Box_1^{-1}\neg c_0 \veebar (\Diamond_1\neg p_0 \vee p_0))$. This is where we have our first failure to eliminate p_0, so backtracking occurs. We backtrack to the initial equation, reversing the polarity of p_0: $(\neg c_0 \veebar (\Diamond_1 p_0 \wedge \Box_1(\Diamond_1 p_0 \vee \neg p_0)))$. The Equivalence Rule is applied: $(\neg c_0 \veebar \Diamond_1 p_0), (\neg c_0 \veebar \Box_1(\Diamond_1 p_0 \vee \neg p_0))$. The Box-Rule is applied: $(\neg c_0 \veebar \Diamond_1 p_0), (\Box_1^{-1}\neg c_0 \veebar (\Diamond_1 p_0 \vee \neg p_0))$. Here we fail again. The backtracking stack is empty, so the result is $\langle \mathit{failure}\rangle$.

Here, we follow the axiomatic system for nominals and universal modality, described in [17, 18, 12], with some differences in the proofs.

We show an axiomatic system for the valid formulas from the language ML(T, U). For simplicity of the axiomatic system, we use implications and we only use $\Diamond$, $\wedge$, $\vee$, $\neg$ and $\top$ as *defined symbols*. We use p and q for variables. Therefore, our language for this section becomes:

$$\boldsymbol{\phi} ::= \bot | \boldsymbol{p} | \boldsymbol{c} | (\boldsymbol{\phi} \to \boldsymbol{\phi}) | \Box_{\boldsymbol{i}} \boldsymbol{\phi} | \Box_{\boldsymbol{i}}^{-1} \boldsymbol{\phi}$$

Definition 29. (Admissible Form) Let $\#$ be a symbol, which is not in the alphabet of ML(T, U). $\#$ is an *admissible form.* If $AF(\#)$ is an admissible form, then so are $\Box AF(\#)$ and $(\boldsymbol{\phi} \to AF(\#))$. The formula, obtained by replacing all occurrences of $\#$ with $\boldsymbol{\phi}$ in $AF(\#)$ is denoted by $AF(\boldsymbol{\phi})$.

We use the same notation for *nominal substitution*, replacing a nominal with another nominal, as the notation for uniform substitution.

Axioms:
The axioms of propositional calculus.
(K) $(\Box(p \to q) \to (\Box p \to \Box q))$ for every box $\Box$
(T for U) $([U]p \to p)$
(B for U) $(p \to [U]\langle U \rangle p)$
(4 for U) $([U]p \to [U][U]p)$
(U) $([U]p \to \Box p)$ for every box $\Box$
(GP) $(p \to \Box_{\boldsymbol{i}} \Diamond_{\boldsymbol{i}}^{-1} p)$ for every number $\boldsymbol{i}$
(HF) $(p \to \Box_{\boldsymbol{i}}^{-1} \Diamond_{\boldsymbol{i}} p)$ for every number $\boldsymbol{i}$
(Nom1) $\langle U \rangle c$
(Nom2) $(\langle U \rangle (c \wedge p) \to [U](c \to p))$

Rules:

$$\text{Modus Ponens (MP): } \frac{\boldsymbol{\phi_1}, (\boldsymbol{\phi_1} \to \boldsymbol{\phi_2})}{\boldsymbol{\phi_2}}, \text{ Gen: } \frac{\boldsymbol{\phi}}{\Box \boldsymbol{\phi}},$$

$$\text{Uniform Substitution: } \frac{\boldsymbol{\phi}}{\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]}, \text{ Nominal Substitution: } \frac{\boldsymbol{\phi}}{\boldsymbol{\phi}[\boldsymbol{c'}/\boldsymbol{c''}]},$$

$$\text{Cov*: } \frac{AF(\neg \boldsymbol{c}) \text{ for some } \boldsymbol{c} \not\nearrow AF(\#)}{AF(\bot)}.$$

A *normal modal logic*, or just *logic*, is a set of formulas Λ such that Λ contains all axioms and is closed under applications of the five rules.

$K_{(T,U)}$ is the smallest logic. Let $\boldsymbol{\phi}$ be a formula. We denote the smallest logic, which contains $\boldsymbol{\phi}$ by $K_{(T,U)} + \boldsymbol{\phi}$, and $\boldsymbol{\phi}$ is called *the axiom of* $K_{(T,U)} + \boldsymbol{\phi}$. We denote by $\vdash_\Lambda \boldsymbol{\phi}$ iff $\boldsymbol{\phi} \in \Lambda$. We use the capital greek letters Γ, Δ, Σ for sets of formulas. A

Λ-*theory* Γ is a set of formulas Γ such that $\Lambda \subseteq \Gamma$ and Γ is closed under applications of MP and the infinitary rule Cov:

$$\text{Cov: } \frac{AF(\neg \boldsymbol{c}) \text{ for all } \boldsymbol{c}}{AF(\bot)}.$$

The Λ-*theory of a set of formulas* Γ, $Th_\Lambda(\Gamma)$, is the smallest Λ-theory such that $\Gamma \subseteq Th_\Lambda(\Gamma)$. Despite the infinitary rule, the deduction lemma holds:

Lemma 30. (Deduction Lemma) $(\phi_1 \to \phi_2) \in Th_\Lambda(\Gamma)$ iff $\phi_2 \in Th_\Lambda(\Gamma \cup \{\phi_1\})$.

Proof. The left to right direction is obvious. Let $\phi_2 \in Th_\Lambda(\Gamma \cup \{\phi_1\})$ and let $\Gamma' := \{\phi' \mid (\phi_1 \to \phi') \in Th_\Lambda(\Gamma)\}$. Easily, $\phi_1 \in \Gamma'$ and $\Lambda \subseteq Th_\Lambda(\Gamma) \subseteq \Gamma'$. Also, Γ' is closed under applications of MP. To see that Γ' is closed under applications of Cov, let $AF(\#)$ be an admissible form, and suppose that for each nominal $\boldsymbol{c}$, $AF(\neg \boldsymbol{c}) \in \Gamma'$. Then, by propositional reasoning, for each nominal $\boldsymbol{c}$: $(\phi_1 \to AF(\neg \boldsymbol{c})) \in Th_\Lambda(\Gamma)$. Applying Cov to $(\phi_1 \to AF(\#))$, we get that $(\phi_1 \to AF(\bot)) \in Th_\Lambda(\Gamma)$, therefore $AF(\bot) \in \Gamma'$, so Γ' is closed under applications of Cov. Therefore, $Th_\Lambda(\Gamma \cup \{\phi_1\}) \subseteq \Gamma'$, so by the definition of Γ', $(\phi_1 \to \phi_2) \in Th_\Lambda(\Gamma)$. □

A set Γ is Λ-*consistent* iff $\bot \notin Th_\Lambda(\Gamma)$, and is Λ-*inconsistent*, otherwise. Γ is a *complete* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for every formula $\boldsymbol{\phi}$, it is the case that either $\boldsymbol{\phi} \in \Gamma$ or $\neg\boldsymbol{\phi} \in \Gamma$. Γ is a *maximal* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for any set Σ such that $\Gamma \subsetneq \Sigma$, Σ is Λ-inconsistent.

Corollary 31. A theory is maximal iff it is complete.

Proof. First, let Γ be a complete Λ-theory and let for some set Σ such that $\Gamma \subseteq \Sigma$, $\boldsymbol{\phi} \in \Sigma \setminus \Gamma$. Then $\neg\boldsymbol{\phi} \in \Gamma$, so by propositional reasoning $\bot \in Th_\Lambda(\Sigma)$. Now, let Γ be a maximal Λ-theory and let $\phi \notin \Gamma$. Then, $\bot \in Th_\Lambda(\Gamma \cup \{\phi\})$, so by the deduction lemma, $(\phi \to \bot) \in \Gamma$, therefore $\neg\phi \in \Gamma$. □

Note that the classical Lindenbaum lemma here has the following form:

Lemma 32. (Lindenbaum Lemma) Let Γ be Λ-consistent. Then Γ can be extended to a complete Λ-theory.

Proof. Let $\phi_1, \phi_2, \ldots$ be an enumeration of all formulas of $\mathrm{ML}(T, U)$. We construct by induction an infinite chain of Λ-consistent Λ-theories $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ with the property that for every $i \geq 1$, either $\phi_i \in \Gamma_i$ or $\neg\phi_i \in \Gamma_i$ in the following way. Let Γ_0 be $Th_\Lambda(\Gamma)$. Thus Γ_0 is Λ-consistent. Suppose that Γ_i is defined.

1. If $\Gamma_i \cup \{\phi_i\}$ is Λ-consistent, let $\Gamma_{i+1} = Th_\Lambda(\Gamma_i \cup \{\phi_i\})$.

2. If $\Gamma_i \cup \{\phi_i\}$ is Λ-inconsistent, then $\neg\phi_i \in \Gamma_i$. There are two cases.

2.1. If ϕ_i is not in the form $AF(\bot)$, then let $\Gamma_{i+1} = \Gamma_i$.

2.2. If ϕ is $AF(\bot)$ for some admissible form $AF(\#)$, then we show that there is a nominal $\boldsymbol{c}_i$ such that $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\}$ is Λ-consistent. Suppose for the sake of contradiction that for all $\boldsymbol{c}$: $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c})\}$ is Λ-inconsistent. Then by the deduction lemma, for all $\boldsymbol{c}$: $(\neg AF(\neg \boldsymbol{c}) \to \bot) \in \Gamma_i$, hence for all $\boldsymbol{c}$: $AF(\neg \boldsymbol{c}) \in \Gamma_i$. Since Γ_i is a Λ-theory, by Cov, $AF(\bot) \in \Gamma_i$, so $\phi_i \in \Gamma_i$. Thus Γ_i is Λ-inconsistent, which contradicts the Λ-consistency of Γ_i. We conclude that there is a nominal $\boldsymbol{c}_i$ such that $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\}$ is Λ-consistent. Let Γ_{i+1} be $Th_\Lambda(\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\})$.

According to the construction, Γ_{i+1} is a Λ-consistent extension of Γ_i.

Let $\Gamma^+ := \bigcup_{i=0}^{\infty} \Gamma_i$.

First note that $\bot \notin \Gamma^+$ since for all $i \geq 0$, $\bot \notin \Gamma_i$.

Now we show that Γ^+ is closed under applications of Modus Ponens. Let $\phi_1, (\phi_1 \to \phi_2) \in \Gamma^+$. Then there is a step i such that $\phi_1, (\phi_1 \to \phi_2) \in \Gamma_i$. But Γ_i is closed under applications of MP, so $\phi_2 \in \Gamma_i \subseteq \Gamma^+$.

We now show that Γ^+ is closed under applications of Cov. Let there be an $AF(\#)$ such that for all $\boldsymbol{c}$: $AF(\neg \boldsymbol{c}) \in \Gamma^+$ and suppose for the sake of contradiction that $AF(\bot) \notin \Gamma^+$. There is an index i such that $AF(\bot)$ is ϕ_i, and by case 2.2 of the construction, there is a nominal c' such that $\neg AF(\neg c') \in \Gamma_{i+1} \subseteq \Gamma^+$. By propositional reasoning, $\bot \in \Gamma^+$, contradiction. Therefore, Γ^+ is closed under applications of Cov.

Since every formula is ϕ_i for some i, by the construction either $\phi_i \in \Gamma_{i+1}$ or $\neg\phi_i \in \Gamma_{i+1}$. Thus Γ^+ is a complete Λ-theory. $\square$

We denote by $\Gamma \vdash_\Lambda \boldsymbol{\phi}$ if $\boldsymbol{\phi} \in Th_\Lambda(\Gamma)$. Thus $\emptyset \vdash_\Lambda \boldsymbol{\phi}$ iff $\vdash_\Lambda \boldsymbol{\phi}$. $\mathrm{M}, w \Vdash \Gamma$ iff for all $\boldsymbol{\phi} \in \Gamma$, $\mathrm{M}, w \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is a *local semantic consequence* of Γ over the class S of frames, denoted by $\Gamma \Vdash_S \boldsymbol{\phi}$, or, if $\Gamma = \emptyset$, as $S \Vdash \boldsymbol{\phi}$, iff for every frame $\mathrm{F} \in S$, every model M over F and every state w from F, it is the case that if $\mathrm{M}, w \Vdash \Gamma$, then $\mathrm{M}, w \Vdash \boldsymbol{\phi}$. The *class of frames of* Λ, $Fr(\Lambda)$, is the class S of all frames F such that $\mathrm{F} \Vdash \Lambda$. Γ is *satisfiable on* S iff there is an $\mathrm{F} \in S$, an M over F and a w in F such that $\mathrm{M}, w \Vdash \Gamma$.

Our goal is to examine the relationship between $\vdash$ and $\Vdash$.

Soundness: If $\Gamma \vdash_\Lambda \boldsymbol{\phi}$, then $\Gamma \Vdash_{Fr(\Lambda)} \boldsymbol{\phi}$.

Proof. All axioms are valid. Every rule preserves validity on any given frame. The result follows in the usual way. $\square$

The converse, known as *strong completeness*, can be proven for some logics. Here, like in [21], we prove it for $K_{(T,U)}$ and $K_{(T,U)} + \boldsymbol{\phi}$ for di-persistent $\boldsymbol{\phi}$.

Strong Completeness, First Form: If $\Gamma \Vdash_{Fr(\Lambda)} \boldsymbol{\phi}$, then $\Gamma \vdash_\Lambda \boldsymbol{\phi}$.

Strong Completeness, Second Form: If Γ is Λ-consistent, then Γ is satisfiable on $Fr(\Lambda)$.

Proposition 33. The two forms of strong completeness are equivalent.

Proof. See [3]. Note that here we use the deduction lemma.

For every box $\Box$, we denote by $\Box\Gamma$ the set $\{\phi \mid \Box\phi \in \Gamma\}$.

Lemma 34. Let Γ, Σ and Δ be Λ-consistent Λ-theories. Then

1. The set $\Gamma' := \Box\Gamma$ is a Λ-theory and if for some formula ϕ, $\Box\phi \notin \Gamma$, then Γ' is Λ-consistent.
2. $[U]\Gamma$ is Λ-consistent, $[U]\Gamma \subseteq \Gamma$ and $[U]\Gamma \subseteq \Box\Gamma$ for every box.

3. If Γ is complete, then $\Box\phi \notin \Gamma$ iff there is a complete Λ-theory Σ such that $\Box\Gamma \subseteq \Sigma$ and $\phi \notin \Sigma$.
4. If Γ and Σ are complete, then $\Box_{\boldsymbol{i}}\Gamma \subseteq \Sigma$ iff $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Gamma$.
5. If Γ and Σ are complete, then $[U]\Gamma \subseteq \Sigma$ iff $[U]\Sigma \subseteq \Gamma$.
6. If Γ, Σ and Δ are complete, $[U]\Gamma \subseteq \Sigma$ and $[U]\Sigma \subseteq \Delta$, then $[U]\Delta \subseteq \Gamma$.
7. If Γ and Σ are complete and $[U]\Gamma \subseteq \Sigma$, then $[U]\Gamma = [U]\Sigma$.

Proof. We only show 1. The proofs for the rest are standard, and follow easily from the axioms, 1., the deduction lemma and the Lindenbaum lemma.

Let $(\phi_1 \to \phi_2), \phi_1 \in \Gamma'$, therefore $\Box(\phi_1 \to \phi_2), \Box\phi_1 \in \Gamma$. Because of (K), $\vdash_\Lambda (\Box(\phi_1 \to \phi_2) \to (\Box\phi_1 \to \Box\phi_2))$, therefore, by MP, $\Box\phi_2 \in \Gamma$, so $\phi_2 \in \Gamma'$.

Now, let for all $\boldsymbol{c}$, $AF(\neg\boldsymbol{c}) \in \Gamma'$. Then for all $\boldsymbol{c}$, $\Box AF(\neg\boldsymbol{c}) \in \Gamma$, so by Cov, $\Box AF(\bot) \in \Gamma$ and hence $AF(\bot) \in \Gamma'$.

Finally, if $\Box\phi \notin \Gamma$, then $\phi \notin \Gamma'$ and hence Γ' is Λ-consistent. □

For given Λ and a complete Λ-theory Γ, let F be $\langle W, \mathcal{R}\rangle$, where W is the set of all complete Λ-theories Σ, such that $[U]\Gamma \subseteq \Sigma$, $\mathcal{R}(0) = W \times W$ and for $i > 0$, $\langle \Sigma_1, \Sigma_2\rangle \in \mathcal{R}(i)$ iff $\Box_i\Sigma_1 \subseteq \Sigma_2$. Then F is called the Λ-*canonical frame for* Γ.

Proposition 35. Let Λ be a logic, Γ be a complete Λ-theory. If $\mathrm{F} = \langle W, \mathcal{R}\rangle$ is the Λ-canonical frame for Γ, then

1. for every $\Sigma \in W$ at least one $\boldsymbol{c} \in \Sigma$.
2. for every $\boldsymbol{c}$ there is exactly one $\Sigma \in W$ such that $\boldsymbol{c} \in \Sigma$.

Proof. 1. Let Σ be a complete Λ-theory. Suppose that for all $\boldsymbol{c}$, $\boldsymbol{c} \notin \Sigma$. Then, by the completeness of Σ, for all $\boldsymbol{c}$, $\neg\boldsymbol{c} \in \Sigma$. Therefore, by Cov, $\bot \in \Sigma$, contradiction.

2. First, we show that for every $\boldsymbol{c}$, there is a $\Sigma \in W$ such that $\boldsymbol{c} \in \Sigma$. Suppose this is not the case, so there is a $\boldsymbol{c}$ such that for all $\Sigma \in W$, $\boldsymbol{c} \notin \Sigma$. Then $\neg\boldsymbol{c} \in \Sigma$, therefore $[U]\neg\boldsymbol{c} \in \Gamma$, which contradicts axiom (Nom1). Second, let for some $\boldsymbol{c}$ there be $\Sigma_1, \Sigma_2 \in W$, such that $\boldsymbol{c} \in \Sigma_1 \cap \Sigma_2$. Let $\phi \in \Sigma_1$. Then, $(\boldsymbol{c} \wedge \phi) \in \Sigma_1$. Suppose $\phi \notin \Sigma_2$, then $(\boldsymbol{c} \to \neg\phi) \in \Sigma_2$. Now, there are two cases. First, if $[U](\boldsymbol{c} \to \neg\phi) \in \Sigma_2$, then because of the definition of W, $(\boldsymbol{c} \to \neg\phi) \in \Sigma_1$, contradiction. Second, if $[U](\boldsymbol{c} \to \neg\phi) \notin \Sigma_2$, then $\neg[U](\boldsymbol{c} \to \neg\phi) \in \Sigma_2$, so $\langle U\rangle(\boldsymbol{c} \wedge \phi) \in \Sigma_2$, then because of (Nom2), $[U](\boldsymbol{c} \to \phi) \in \Sigma_2$, but $[U]\Sigma_2 \subseteq \Sigma_2$, so $\phi \in \Sigma_2$, contradiction. So, we have that $\Sigma_1 \subseteq \Sigma_2$. The converse inclusion is proven similarly, so $\Sigma_1 = \Sigma_2$. □

It easily follows that all axioms of $K_{(T,U)}$ are valid in any Λ-canonical frame.

We are now ready to define the Λ-*canonical model* for a given complete Λ-theory Γ. Let $\mathrm{F} = \langle W, \mathcal{R}\rangle$ be the Λ-canonical frame for Γ, then we define $\mathrm{M} := \langle \mathrm{F}, V, A\rangle$, where $V(p) := \{\Sigma \in W \mid p \in \Sigma\}$, and $A(c) := \Sigma$, where Σ is the only element of W, such that $c \in \Sigma$. The definition of A is correct by Proposition 35. It follows that M is a *named* model.

Lemma 36. (Truth Lemma) Let $\mathrm{M} = \langle\langle W, \mathcal{R}\rangle, V, A\rangle$ be the Λ-canonical model for some complete Λ-theory Γ. Then for any formula ϕ and any Σ in M, $\phi \in \Sigma$ iff $\mathrm{M}, \Sigma \Vdash \phi$.

Proof. Induction on ϕ. For atomic ϕ and for $\bot$, the result follows by the definition of the canonical model. For $(\phi_1 \to \phi_2)$, the result follows by the induction hypothesis and propositional reasoning. For $\Box_{\boldsymbol{i}}^{-1}\phi$: first, let $\Box_{\boldsymbol{i}}^{-1}\phi \in \Sigma$. Let $W' := \{\Sigma' \in W \mid \Box_{\boldsymbol{i}}\Sigma' \subseteq \Sigma\}$. Because for any Λ-complete theory Σ', $[U]\Sigma' \subseteq \Box_{\boldsymbol{i}}\Sigma'$, we have that for all Σ', such that $[U]\Sigma' \subseteq \Box_{\boldsymbol{i}}\Sigma' \subseteq \Sigma$, it is the case that $[U]\Sigma' = [U]\Sigma = [U]\Gamma$, therefore $\Sigma' \in W$. Then, for all these Σ', $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Sigma'$, so $\phi \in \Sigma'$. Therefore, by the induction hypothesis, for all these Σ': $\mathrm{M}, \Sigma' \Vdash \phi$, so, by the definition of $\mathcal{R}(\boldsymbol{i})$ and Definition 5, $\mathrm{M}, \Sigma \Vdash \Box_{\boldsymbol{i}}^{-1}\phi$. Now, let $\mathrm{M}, \Sigma \Vdash \Box_{\boldsymbol{i}}^{-1}\phi$. Then, using the same definition of W', we have that for all such Σ', we can use the induction hypothesis and find that $\phi \in \Sigma'$. Because W' contains exactly all Σ', such that $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Sigma'$, then it follows that $\Box_{\boldsymbol{i}}^{-1}\phi \in \Sigma$. For $\Box_{\boldsymbol{i}}\phi$, the result follows by Lemma 34. □

Theorem 1 *(1) $K_{(T,U)}$ is strongly complete. (2) $K_{(T,U)} + \phi$ is strongly complete for any di-persistent modal formula ϕ.*

Proof. We use the second form of strong completeness. Like [15, 16, 5, 21]:

We show (1) and (2) together. Let Λ be either $K_{(T,U)}$ or $K_{(T,U)} + \phi$. Let Γ be a Λ-consistent set. By the Lindenbaum lemma, there is a complete Λ-theory Γ^+ extending Γ. We construct the canonical model M for Γ^+ and let its universe be W. By the truth lemma, Γ^+ is satisfiable in M at Γ^+, therefore Γ also is. The frame of M, F, also validates all axioms of $K_{(T,U)}$, which proves (1). For (2), it remains to prove that F validates ϕ. Because M is a named model, we construct $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$, where $\mathbb{W} = \{[\![\phi']\!]_{\mathrm{M}} \mid \phi' \in \mathrm{ML}(T, U)\}$. Because ϕ is di-persistent, it is enough to show that $\mathfrak{g} \Vdash \phi$. Clearly, M is a model over $\mathfrak{g}$ and $\mathrm{M} \Vdash \phi$, so $[\![\phi]\!]_{\mathrm{M}} = W$. If $\mathrm{PROP}(\phi) \cup \mathrm{NOM}(\phi) = \emptyset$, then we are done. Otherwise, let all propositional variables occurring in ϕ be, in left-to-right order of initial occurrence, $p_1, \ldots, p_n$, and let the nominals of ϕ be, in left-to-right order of initial occurrence, $c_1, \ldots, c_m$. Then, clearly, for any model M' over $\mathfrak{g}$, $[\![\phi]\!]_{\mathrm{M}'} = [\![\phi]\!](s_1, \ldots, s_n, w_1, \ldots, w_m)$ for some $s_1, \ldots, s_n \in \mathbb{W}$ and $w_1, \ldots, w_m \in W$, which, by the definition of $\mathbb{W}$ as the extensions in M of all possible formulas, and the fact that every w_i contains a nominal, is equal to the following set: $[\![\phi[p_1/\phi_1, \ldots, p_n/\phi_n, c_1/c'_1, \ldots, c_m/c'_m]]\!]_{\mathrm{M}}$ for some formulas $\phi_1, \ldots, \phi_n$ and some nominals $c'_1, \ldots, c'_m$. However, $\Lambda \subseteq \Sigma$ for any complete Λ-theory Σ, and Λ is closed under applications of uniform substitution and nominal substitution. Therefore, for all $\Sigma \in W$: $\phi[p_1/\phi_1, \ldots, p_n/\phi_n, c_1/c'_1, \ldots, c_m/c'_m] \in \Sigma$. So, by the truth lemma, the result follows. □

Corollary 37. For all formulas $\boldsymbol{\phi}$, for which Deterministic SQEMA succeeds, $K_{(T,U)} + \boldsymbol{\phi}$ is strongly complete.

Proof. All formulas, for which Deterministic SQEMA succeeds, are di-persistent, so the result follows by the above theorem. □

The language of pre-contact logics (PCL) is a first-order language with equality (=) and without quantifiers. It is intended to be a propositional language for point-free theories of space, as outlined in [1].

Boolean terms of PCL are: $\tau ::= p|0|1|-\tau|(\tau \cup \tau)|(\tau \cap \tau)$ where p is a *variable*, 0 and 1 are *boolean constants*. *Atomic formulas* are: $\alpha ::= \bot|\top|(\tau = \tau)|(\tau \leq \tau)|C(\tau, \tau)$ where part-of ($\leq$) and contact (C) are binary predicates. Pre-Contact formulas are: $\psi ::= \alpha|\neg\psi|(\psi \vee \psi)|(\psi \wedge \psi)$. We may use $\rightarrow$ and $\leftrightarrow$ as defined symbols with their usual meaning.

The usual definitions of Kripke frames and Kripke models are used.

If $\mathrm{M} = \langle \mathrm{F}, V \rangle$ is a model, where $\mathrm{F} = \langle W, \mathcal{R} \rangle$, then the valuation V can be extended to all boolean terms in the following way:

$V(0) = \emptyset$, $V(1) = W$

$V(-\tau_1) = W \setminus V(\tau_1)$

$V((\tau_1 \cup \tau_2)) = V(\tau_1) \cup V(\tau_2)$

$V((\tau_1 \cap \tau_2)) = V(\tau_1) \cap V(\tau_2)$

The definition of truth of atomic formulas in a Kripke model M is as follows:

$\mathrm{M} \vDash (\tau_1 = \tau_2)$ iff $V(\tau_1) = V(\tau_2)$

$\mathrm{M} \vDash (\tau_1 \leq \tau_2)$ iff $V(\tau_1) \subseteq V(\tau_2)$

$\mathrm{M} \vDash C(\tau_1, \tau_2)$ iff $\exists x \exists y (x \in V(\tau_1) \wedge y \in V(\tau_2) \wedge x \, \mathcal{R}(1) \, y)$

Truth of pre-contact formulas in M is defined in the standard way.

We say that ψ is *valid* in a frame F, $\mathrm{F} \vDash \psi$, iff ψ is true in all models over F.

It is shown in [1] that pre-contact formulas can be represented as formulas of $\mathrm{ML}(T, U)$. More precisely, there is a translation $\mathtt{t} : \mathrm{PCL} \rightarrow \mathrm{ML}(T, U)$ with the property that for every PCL formula ψ and every Kripke model M, $\mathrm{M} \vDash \psi$ iff $\mathrm{M} \Vdash \mathtt{t}(\psi)$. For describing the translation, we use the defined symbol $\leftrightarrow$ in the language $\mathrm{ML}(T, U)$ with its usual meaning.

This translation $\mathtt{t}$ maps variables to propositional variables. Function symbols map to the corresponding boolean connectives. $\mathtt{t}(0) = \bot \in \mathrm{ML}(T, U)$, $\mathtt{t}(1) = \top \in \mathrm{ML}(T, U)$. Let τ_1, τ_2 be terms. The predicate symbols translate as follows:

$\mathtt{t}((\tau_1 = \tau_2)) = [U](\mathtt{t}(\tau_1) \leftrightarrow \mathtt{t}(\tau_2))$

$\mathtt{t}((\tau_1 \leq \tau_2)) = [U](\mathtt{t}(\tau_1) \rightarrow \mathtt{t}(\tau_2))$

$\mathtt{t}(C(\tau_1, \tau_2)) = \langle U \rangle (\mathtt{t}(\tau_1) \wedge \Diamond_1 \mathtt{t}(\tau_2))$

The boolean connectives translate to themselves.

Now, we discuss Sahlqvist PCL formulas, as defined in [2].

A *positive* term is built up from variables, -0 and 1, using only $\cup$ and $\cap$.

A *negation-free* formula is built up from $\neg(\tau_1 = 0)$ and $C(\tau_1, \tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *positive* formula is built up from $\neg(\tau_1 = 0)$, $(-\tau_1 = 0)$, $(\tau_1 = 1)$, $C(\tau_1, \tau_2)$, and $\neg C(-\tau_1, -\tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *Sahlqvist* formula ψ is an implication $(\psi_1 \to \psi_2)$, where ψ_1 is negation-free, and ψ_2 is positive.

To translate Sahlqvist formulas, as defined in [2], into Sahlqvist formulas in $\mathrm{ML}(T, U)$, we define a *modified translation* $\mathtt{t}'$ as follows:

$\mathtt{t}'(p) := p \in \mathrm{ML}(T, U)$

$\mathtt{t}'(0) := \bot \in \mathrm{ML}(T, U)$

$\mathtt{t}'(1) := \top \in \mathrm{ML}(T, U)$

$\mathtt{t}'(-\tau) := \neg\mathtt{t}'(\tau)$ where τ is any term

$\mathtt{t}'((\tau_1 \cup \tau_2)) := (\mathtt{t}'(\tau_1) \vee \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((\tau_1 \cap \tau_2)) := (\mathtt{t}'(\tau_1) \wedge \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((-\tau = 0)) := [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau = 1)) := [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau_1 = \tau_2)) := [U](\mathtt{t}'(\tau_1) \leftrightarrow \mathtt{t}'(\tau_2))$, where $(\tau_1 = \tau_2)$ is not as in the above two cases

$\mathtt{t}'((\tau_1 \leq \tau_2)) := [U](\mathtt{t}'(\tau_1) \to \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(C(\tau_1, \tau_2)) := \langle U\rangle(\mathtt{t}'(\tau_1) \wedge \Diamond_1\mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg C(-\tau_1, -\tau_2)) := [U](\mathtt{t}'(\tau_1) \vee \Box_1\mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg(\tau = 0)) := \langle U\rangle\mathtt{t}'(\tau)$, where τ is any term

$\mathtt{t}'(\neg\psi) := \neg\mathtt{t}'(\psi)$, where $\neg\psi$ is not as in the above two cases

$\mathtt{t}'((\psi_1 \vee \psi_2)) := (\mathtt{t}'(\psi_1) \vee \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

$\mathtt{t}'((\psi_1 \wedge \psi_2)) := (\mathtt{t}'(\psi_1) \wedge \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

It is easy to see, by induction on PCL terms and PCL formulas, that for any PCL formula ψ, ψ and $\mathtt{t}'(\psi)$ are true in the same models.

We show now how to derive a result from [2] that Sahlqvist formulas have a first-order correspondent as a corollary to the fact that Deterministic SQEMA succeeds on all Sahlqvist $\mathrm{ML}(T, U)$ formulas.

Theorem 2 *The modified translation maps Sahlqvist PCL formulas to Sahlqvist implications from* $\mathrm{ML}(T, U)$.

Proof. An easy induction on PCL terms shows that $\mathtt{t}'(\tau)$ for a positive term τ is a positive $\mathrm{ML}(T, U)$ formula. Similarly, it is simple to show that $\mathtt{t}'(\psi)$ for a positive ψ is a positive $\mathrm{ML}(T, U)$ formula. It remains to show that $\mathtt{t}'$ maps negation-free PCL formulas to $\mathrm{ML}(T, U)$ Sahlqvist antecedents. This again follows from an easy induction, using the definition of $\mathtt{t}'$. □

We use Deterministic SQEMA for the language of Pre-Contact Logic, by translating a pre-contact formula to a formula of $\mathrm{ML}(T, U)$, using $\mathtt{t}'$, and running Deterministic SQEMA on the translation. It immediately follows that Deterministic SQEMA succeeds on the modified translation of any Sahlqvist PCL formula.

It was proved in [1] that: Every pre-contact formula is complete with respect to the class of finite frames defined by it. Hence, every pre-contact formula is complete.

Theorem 3 *Every PCL formula ψ, on whose modified translation Deterministic SQEMA succeeds and produces a FOL formula ψ', is complete on the class of frames defined by ψ'.*

Proof. By the properties of Deterministic SQEMA, $\mathtt{t}'(\psi)$ and ψ' are locally correspondent, therefore globally correspondent. By the properties of $\mathtt{t}'$, ψ and $\mathtt{t}'(\psi)$ define the same class of frames, so ψ and ψ' define the same class of frames, therefore they define the same class of finite frames. By the above-mentioned result in [1], ψ is complete in the class of finite frames, defined by ψ', and therefore is complete in the class of all frames, defined by ψ'. □

9. CONCLUSION

We have shown sufficient conditions for di-persistence and for the existence of first-order correspondents. We have shown that SQEMA can be reduced to a Deterministic SQEMA. We have proven that it always succeeds for Sahlqvist and inductive formulas, and that it always terminates. We have shown the strong completeness of all formulas, on which Deterministic SQEMA succeeds, in the language of ML(T, U). We have extended Deterministic SQEMA so that it succeeds on all Sahlqvist formulas of the pre-contact language. Deterministic SQEMA could be extended via a resolution procedure and a tableaux method in the normalization procedure and via a tableaux method in the *step* function.

It would be interesting to show how Deterministic SQEMA can be modified to succeed on all formulas having only the universal modality.

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