ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

Том 95

ANNUAIRE

DE

L'UNIVERSITE DE SOFIA
"ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

Tome 95

СОФИЯ • 2004 • SOFIA

УНИВЕРСИТЕТСКО ИЗДАТЕЛСТВО „СВ. КЛИМЕНТ ОХРИДСКИ“

PRESSES UNIVERSITAIRES "ST. KLIMENT OHRIDSKI"

Annuaire de l' Université de Sofia "St. Kliment Ohridski"
Faculté de Mathématiques et Informatique

Годишник на Софийския университет „Св. Климент Охридски"
Факултет по математика и информатика

Address for correspondence:

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5, J. Bourchier blvd, P.O. Box 48
BG-1164 Sofia, Bulgaria

Fax xx(359 2) 8687 180
Electronic mail:
annuaire@fmi.uni-sofia.bg

Aims and Scope. The *Annuaire* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer sciences. It is reviewed by *Zentralblatt für Mathematik*, *Mathematical Reviews* and the Russian *Referativnii Jurnal.* The *Annuaire* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, i.e., algebra, geometry and topology, analysis, mathematical logic, theory of approximations, numerical methods, computer sciences, classical, fluid and solid mechanics, and their fundamental applications.

ISSN 0205–0808

проф. дмн Тодор Георгиев Генчев

1932 - 1998

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СЕДЕМДЕСЕТ ГОДИНИ ОТ РОЖДЕНИЕТО НА ТОДОР ГЕОРГИЕВ ГЕНЧЕВ (1932 – 1998)

ЕМИЛ ХОРОЗОВ

A short review of the scientific interest, main results and the broad educational activity of the Bulgarian mathematician Professor Todor Gentchev (1932 – 1998) is given. His most notable results concern the theory of partial differential equations and the theory of holomorphic and entire functions. The deep and fruitful influence of Professor Todor Gentchev on the development of the Bulgarian mathematics is also underlined.

На 18 февруари 2002 г. се навършиха 70 години от рождението на видния български математик проф. дмн Тодор Г. Генчев.

Проф. Т. Генчев е роден през 1932 г. в с. Чавдар, Софийско. Завършил е Софийския университет през 1955 г. и същата година става асистент във Факултета по математика. През 1965 г. става доцент, а през 1978 г. — професор. До края на живота си проф. Т. Генчев продължава да работи в Софийския университет.

С неговото име се свързва създаването на българската школа по частни диференциални уравнения, както и множество блестящи резултати в други научни области, сред които особено изпъкват трудовете му по комплексен анализ.

Първите научни изследвания на проф. Т. Генчев, както и на много други математици от неговото поколение, са направени под благотворното влияние на проф. Я. Тагамлицки. Постепенно проф. Т. Генчев намира своя тематика в областта на частните диференциални уравнения. Той е един от първите в света разработили и приложили метода на елиптичната и параболичната регуляризация към гранични задачи за частни диференциални уравнения. Идеята на метода е да се апроксимира разглежданото уравнение с подходяща редица от елиптични (или параболични) уравнения и съществуването на решение или на други свойства да се получи чрез граничен преход. Основната трудност, която той преодолява, е да намери подходяща апроксимираща редица, за която съответните гранични задачи са достатъчно добри и позволяват да се получат подходящи априорни оценки. За тази цел коефициентите на изходното уравнение се продължават в по-широка област, за да се елиминира влиянието на допълнителните гранични условия в апроксимиращите уравнения.

Работите на проф. Т. Генчев в тази област са високо оценени и многократно цитирани от най-видни световни специалисти - Ж. Л. Лионс, О. Олейник, Е. Радкевич и др. С тях той извежда българската математика в областта на частните диференциални уравнения на международно ниво.

Трудовете му в това направление са основа на кандидатската му дисертация, защитена през 1969 г.

След защитата на дисертацията си проф. Т. Генчев се насочва към други области на изследване, главно свързани с комплексния анализ. Едно класическо направление, към което се насочва, е изследването на екстремални свойства на цели функции. В работа, публикувана в Доклади на Академията на науките на СССР през 1975 г., проф. Т. Генчев обобщава класическото неравенство на С. Бернщайн, което дава оценка на максимума на производната на произволен тригонометричен полином чрез максимума на самия полином. Проф. Т. Генчев доказва аналогичен резултат за цели функции от експоненциален тип. Неравенствата, получени от него, обобщават и други класически резултати - на Ердьош-Лакс, Сегьо, Боас, Туран и Рахман и др. Заслужава да се отбележи, че от тези резултати следва и точен аналог на теоремата на Гаус–Люка за тригонометрични полиноми — един резултат, който остава незабелязан от класиците, макар че има класическо звучене. Коментирайки този резултат, А. Гудман (*Rocky Mountain J. Math.*, **27**, No 1, 1992) отбелязва остроумието на автора. Друг резултат, отнасящ се до неравенства за полиномите на Чебишев, дава възможност на проф. Т. Генчев да получи просто доказателство на класическото неравенство на А. Марков, за което съществуват серия различни доказателства от класици като А. Марков, М. Рис, Де ла Вале Пусен и др. Резултатите от този цикъл са високо оценени и от нашата научна общност — през 1977 г. проф. Т. Генчев получава наградата „Акад. Н.Обрешков“ за своите изследвания върху цели функции от експоненциален тип .

Едно друго направление, което е свързано със споменатите вече научни области, е теореми от идейния кръг на Пели-Винер. В тази област той получава редица нови резултати за интегрални представяния, като с това усилва класически теореми и опростява съществуващи доказателства. Един от най-интересните резултати в това направление е получаването на интегрални представяния за функциите от пространствата на Бергман в тръбовидни области. Тези резултати установяват изоморфизъм между бергмановите пространства и сравнително просто устроените пространства от реални функции, които са удобни за работа. Този изоморфизъм играе важна роля в пресмятането на някои бихоломорфни инварианти. За да подчертаят това обстоятелство, Скваржински и неговите ученици системно си служат с понятието „трансформация на Генчев“. В този кръг въпроси проф. Т. Генчев системно използва техника от теорията на разпределенията, която му позволява значително да опрости класическите резултати, вкл. и тези на Винер-Пели.

Наред с научната си работа проф. Т. Генчев посвещава много от силите си на преподавателската работа и възпитанието на висококвалифицирани кадри, особено в областта на диференциалните уравнения. Практически той е създател на съвременния курс по частни диференциални уравнения в Софийския университет, а оттам и в цяла България. Заедно с основния курс той подготвя за първи път и серия от специални курсове. Проф. Т. Генчев е автор на няколко учебника. Особено впечатляващ е неговият учебник по частни диференциални уравнения, претърпял четири издания и служещ повече от 25 години като основно учебно помагало за студенти и преподаватели.

Много от нас са изпитали благотворното влияние на проф. Т. Генчев. Той е възпитал няколко поколения математици, като им е предал своя вкус към дълбоки и важни проблеми. При това той ценеше и пропагандираше не само областите, в които извършваше своите научни изследвания, но и такива, които са малко или повече отдалечени от тях. Много от учениците му са се посветили на академичното поприще и работят в СУ, БАН или в чужбина.

Професор Т. Генчев ни напусна преждевременно през 1998 г. Това, което ни остави не само в науката и преподаването, но и във високите критерии на оценяването им, продължава да живее в нас.

Получена на 29.03.2002

Емил Хорозов
Факултет по математика и информатика
Софийски университет „Св. Климент Охридски"
1164 София, п.к. 64, БЪЛГАРИЯ
E-mail: horozov@fmi.uni-sofia.bg

СПИСЪК НА НАУЧНИТЕ ПУБЛИКАЦИИ НА ПРОФ. Д-Р ТОДОР ГЕОРГИЕВ ГЕНЧЕВ

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ПРОГРАМА

на научната сесия, посветена на 70 години от рождението на проф. Тодор Г. Генчев (1932-1998), София, 16 март 2002

14^{00} h	Елиптична регуляризация и субелиптичност *чл.кор. Петър Попиванов*
14^{20} h	Едно кратко доказателство за алгебрична затвореност на полето от изчислимите комплексни числа *проф. Димитър Скордев*
14^{40} h	Задача на Протер, сингулярности и решения *проф. Недю Попиванов*
15^{00} h	Хармонични почти-комплексни структури върху туисторни пространства *ст.н.с. Йохан Давидов и ст.н.с.* I *ст. Олег Мушкаров*
15^{20} h	Уравнение на неизотропната дифузия и приложение в теория на образа *ст.н.с. Николай Кутев*
15^{40} h - 16^{00} h	Почивка
16^{00} h	Корени на полиномите от шеста степен *доц. Ангел Живков*
16^{20} h	Двумерни фуксови системи и свойство на Чебишев *ст.н.с. Илия Илиев*
16^{40} h	Алгебра на Вайл и биспектрални оператори *проф. Емил Хорозов*
17^{00} h	За нулите на решенията на нелинейни хиперболични уравнения с постоянни коефициенти *ас. Зорница Петрова и доц. Димитър Мишев*

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

A SEMANTICS OF LOGIC PROGRAMS WITH PARAMETERS

VESSELA BALEVA

A generalized version of the declarative semantics of Horn clause programs on abstract structures with parameters is presented. The parameters are subsets of the domain of the structure. They are treated as effectively enumerable sets. The main feature of the semantics is that it does not admit searching in the domain of the structure. The obtained programming language is closed under recursion and has the greatest expressive power among the languages satisfying certain natural model-theoretic properties. It is shown that the obtained notion of computability is transitive.

Keywords: semantics, logic programming, parameters, abstract structures

2000 MSC: 03D75, 68Q05, 68Q55

1. INTRODUCTION

In this paper we present a semantics of logic programming on abstract structures with parameters. It is typical for this semantics that it does not admit searching in the domain of the structure. A semantics with searching in the domain of the structure is studied in [4].

In structural programming every subroutine of the relevant language may be joined as a function to the structure and the class of the computable functions in the extended structure will remain unchanged. One may suppose that logic programming has the same properties. Consider a structure $\mathfrak{A}$ and a subset A of the domain of $\mathfrak{A}$. It seems suitable to define a semantics of a logic program P, using A as a parameter, as the usual semantics of P on the extended structure $\langle \mathfrak{A}, A' \rangle$, where A' is the semicharacteristic predicate of the set A. Unfortunately, this approach is not satisfactory, because the obtained notion of computability is not transitive, unless the equality relation is an underlined predicate of $\mathfrak{A}$. It turns out that in order to obtain an appropriate semantics, parameters should be interpreted as "oracles", enumerating their elements, rather than predicates.

This idea is formalized in the paper by introducing first order structures with parameters, which are treated as effectively enumerable subsets of the domain of the structure. A semantics of logic programs on such structures is introduced and studied. The programming language obtained in this way has some interesting properties. First of all, it has greater expressive power compared to all programming languages that have certain natural properties. This fact helps us to show the transitivity of the obtained notion of computability and also to prove that the programming language is closed under recursion.

For the sake of simplicity, we consider only structures with unary functions, predicates and parameters. All definitions and results can be easily generalized for functions, predicates and parameters of arbitrary finite arity.

2. PRELIMINARIES

Let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_0, \ldots, \Sigma_k; A_1, \ldots, A_m)$ be a partial structure, where the domain of the structure B is a denumerable set, $\theta_1, \ldots, \theta_n$ are partial functions of one argument on B, $\Sigma_0, \ldots, \Sigma_k$ are partial predicates of one argument on B, $\Sigma_0 = \lambda s.true$, the parameters $A_1, \ldots, A_m$ are subsets of B, and $n, k, m \geq 0$. Moreover, we assume that the predicates $\Sigma_1, \ldots, \Sigma_k$ obtain only the value "true" whenever they are defined. The last assumption is made for the following reasons. First, it is not restrictive for our considerations (if Σ obtains the value "false", it can be represented by two predicates $\Sigma^+(t) \Leftrightarrow \Sigma(t)$ and $\Sigma^-(t) \Leftrightarrow \neg\Sigma(t)$). And second, logic programs cannot use the negative part of the predicates of the structure because of their syntax.

Let $\mathfrak{B} = (N; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ be a partial structure over the set N of the natural numbers. A subset W of N is called *recursively enumerable* (r. e.) in $\mathfrak{B}$ iff $W = \Gamma(\varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ for some enumeration operator Γ (see [1]).

An *enumeration* of the structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = (N; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ is a partial structure, $\sigma_0 = \lambda s.true$, the predicates $\sigma_1, \ldots, \sigma_k$ obtain only the value "true" whenever they are defined, and α is a partial surjective mapping from N onto B, such that the following conditions hold:

(i) The domain of α ($Dom(\alpha)$) is closed with respect to the partial operations $\varphi_1, \ldots, \varphi_n$;

(ii) $\alpha(\varphi_i(x)) \simeq \theta_i(\alpha(x))$ for all x of $Dom(\alpha)$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \Leftrightarrow \Sigma_j(\alpha(x))$ for all x of $Dom(\alpha)$, $1 \leq j \leq k$;

(iv) $\alpha(\xi_s) = \{\alpha(y) : y \in \xi_s\} = A_s$, $1 \leq s \leq m$;

(v) $\xi_s \subseteq Dom(\alpha)$, $1 \leq s \leq m$.

We shall suppose that an effective monotonic coding of finite sequences and sets of natural numbers is fixed. If $a_0, \dots, a_m$ is a sequence of natural numbers, by $\langle a_0, \dots, a_m \rangle$ we shall denote the code of the sequence $a_0, \dots, a_m$, and by E_v – the finite set with code v. We write $\xi(x)$ to denote that $x \in \xi$.

Let $(q)_i = \mu z[p_i^z/q \;\&\; \neg(p_i^{z+1}/q)]$, where p_i is the i-th prime number.

Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of $\mathfrak{A}$. We shall call the set

$$\begin{aligned} D(\mathfrak{B}) \;\; = & \;\; \{\langle i, x, y \rangle : 1 \leq i \leq n \;\&\; \varphi_i(x) \simeq y\} \\ \cup & \;\; \{\langle j, x \rangle : n+1 \leq j \leq n+k \;\;\&\; \sigma_{j-n}(x) \simeq true\} \\ \cup & \;\; \{\langle s, x \rangle : n+k+1 \leq s \leq n+k+m \;\&\; \xi_{s-n-k}(x)\} \end{aligned}$$

a *code* of the structure $\mathfrak{B}$. It is clear that for every $W \subseteq N$, W is r. e. in $\mathfrak{B}$ iff W is r. e. in $D(\mathfrak{B})$.

Let $A \subseteq B$. The set A is called *weak-admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ iff for some r. e. in $\mathfrak{B}$ subset W of N the following conditions hold:

(i) $W \subseteq Dom(\alpha)$;

(ii) $\alpha(W) = A$.

A subset A of B is called $\forall$-*weak-admissible* in $\mathfrak{A}$ iff it is weak-admissible in each enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$.

The equivalence between the $\forall$-weak-admissible sets and the sets definable by logic programs will be considered. The $\forall$-weak- admissible sets have an explicit characterization, which simplifies the considerations.

We shall use the following notation. The letters t, p will denote elements of B; x, y, z, u, v will be elements of N. We shall identify the predicates with partial mappings which takes values 0 (for "true") and 1 (for "false").

Formulas of the form $F^1 \& \dots \& F^l$, where each F^i is an universal closure of Horn clause, i. e. F^i is a formula of the form $\forall X_1 \dots \forall X_r(\Pi \vee \neg\Pi_1 \vee \dots \vee \neg\Pi_n)$, where $n \geq 0$ and $\Pi, \Pi_1, \dots, \Pi_n$ are atomic predicates, are called *logic programs.* We shall use the usual notation of the Horn clauses:

$$\Pi : -\Pi_1, \dots, \Pi_n.$$

Let $\mathfrak{L} = (f_1, \dots, f_n; T_0, \dots, T_k, T'; S_1, \dots, S_m)$ be the first-order language corresponding to the structure $\mathfrak{A}$, where $f_1, \dots, f_n$ are functional symbols, $T_0, \dots, T_k$ are symbols for predicates, T_0 represents the total predicate $\Sigma_0 = \lambda s.0$, T' represents the nowhere defined predicate, and $S_1, \dots, S_m$ are symbols for parameters.

Let $\{Z_1, Z_2, \dots\}$ be a denumerable set of variables and $\{X_0^S, X_1^S, \dots\}$ be a special set of variables for the elements of parameter S_s, $1 \leq s \leq m$. We shall use the capital letters X, Y, Z to denote the variables.

If τ is a term of the language $\mathfrak{L}$, then we shall write $\tau(\overline{Z})$ to denote that all of the variables in τ are among $\overline{Z} = (Z_1, \dots, Z_a)$. If $\tau(\overline{Z})$ is a term and $\bar{t} = t_1, \dots, t_a$ are arbitrary elements of B, then by $\tau_{\mathfrak{A}}(\overline{Z}/\bar{t})$ we shall denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements $t_1, \dots, t_a$.

Termal predicates in the language $\mathfrak{L}$ are defined by the following inductive clauses:

(i) $T_j(\tau)$, $0 \leq j \leq k$, and $T'(\tau)$, where τ is a term, are termal predicates;

(ii) $S_s(X_i^s)$, where $1 \leq s \leq m$ and i is an arbitrary natural number, is a termal predicate;

(iii) if Π^1 and Π^2 are termal predicates, then $\Pi^1 \& \Pi^2$ is a termal predicate.

Let $\Pi(\overline{Z})$ be a termal predicate and $t_1, \ldots, t_a$ be arbitrary elements of B. The value $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t})$ is defined as follows:

(i) if $\Pi = T_j(\tau)$, $0 \leq j \leq k$, then $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq \Sigma_j(\tau_{\mathfrak{A}}(\overline{Z}/\overline{t})$;

(ii) if $\Pi = T'(\tau)$, then $\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t})$ is undefined;

(iii) if $\Pi = S_s(X_i^s)$, $1 \leq s \leq m$, then $(S_s(X_i/t))_{\mathfrak{A}} \simeq 0 \Leftrightarrow t \in A_s$;

(iv) if $\Pi = \Pi^1 \& \Pi^2$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq \begin{cases} \Pi^2_{\mathfrak{A}}(\overline{Z}/\overline{t}), & \text{if } \Pi^1_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We shall call the expression $\exists X_1 \ldots \exists X_p \Pi$ an *existential termal predicate*, where Π is a termal predicate and $X_1, \ldots, X_n$ are all special variables of Π. If $\Sigma = \exists X_1 \ldots \exists X_p \Pi$ is an existential termal predicate with free variables $Z_1, \ldots, Z_a$, then the value of Σ is defined as follows:

$$\Sigma_{\mathfrak{A}}(\overline{Z}/\overline{t}) \simeq 0 \Leftrightarrow \exists e_1 \ldots \exists e_p (e_1, \ldots, e_p \in B \;\&\; \Pi(\overline{Z}/\overline{t}, \overline{X}/\overline{e}) \simeq 0).$$

We shall call the expression $\exists X_1 \ldots \exists X_p (\Pi \supset \tau)$ a *conditional term*, where Π is a termal predicate, τ is a term and $X_1, \ldots, X_p$ are all of the special variables in Π and τ. The value of the conditional term $Q = \exists X_1 \ldots \exists X_p (\Pi \supset \tau)$ with free variables among $Z_1, \ldots, Z_a$ is defined as follows:

$$\begin{aligned} Q_{\mathfrak{A}}(\overline{Z}/\overline{t}) \ni t \quad \Leftrightarrow \quad & \exists e_1 \ldots \exists e_p (e_1, \ldots, e_p \in B \;\&\; \Pi(\overline{Z}/\overline{t}, \overline{X}/\overline{e}) \simeq 0 \;\& \\ & \tau(\overline{Z}/\overline{t}, \overline{X}/\overline{t}) \simeq t). \end{aligned}$$

We shall assume that an effective coding of the language $\mathfrak{L}$ is fixed.

Let A be a subset of B. The set A is said to be *weak-computable* in the structure $\mathfrak{A}$ iff for some r. e. set V of codes of conditional terms $\{Q^v\}_{v \in V}$ with free variables $Z_1, \ldots, Z_a$ and for some fixed elements $t_1, \ldots, t_a$ of B the following equivalence is true:

$$p \in A \Leftrightarrow \exists v (v \in V \;\&\; Q^v_{\mathfrak{A}}(\overline{Z}/\overline{t}) \ni p).$$

3. ∀-WEAK-ADMISSIBILITY

In this section we shall give an explicit characterization of ∀-weak-admissible sets. The constructions and proves in this section will be used for the logic programs in the next section. The main tool in the proofs will be the set theoretic forcing. It is sufficient to use only special enumerations for our purposes.

The enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *special* iff the following conditions are true:

(i) if $\varphi_i(x) \simeq y$, then $y = \langle i, j, x \rangle$, where $1 \leq i \leq n$ and j is an arbitrary natural number;

(ii) if $x \in \xi_s$, then $x = \langle n+s, j \rangle$, where $1 \leq s \leq m$ and j is an arbitrary natural number.

In the sequel all enumerations will be special, unless something else is assumed.

We shall call

$$\Delta = \langle \alpha_1; H_1; \varphi'_1, \ldots, \varphi'_n; \sigma'_1, \ldots, \sigma'_k; \xi'_1, \ldots, \xi'_m \rangle$$

a *finite part*, where:

(i) H_1 and α_1 are respectively a finite set of natural numbers and a finite mapping from N in B and $H_1 \cap Dom(\alpha_1) = \emptyset$;

(ii) $\varphi'_1, \ldots, \varphi'_n$ are partial functions from $H_1 \cup Dom(\alpha_1)$ in $H_1 \cup Dom(\alpha_1)$;

(iii) if $\varphi'_i(x) \simeq y$, then $y = \langle i, j, x \rangle$, where $1 \leq i \leq n$ and j is an arbitrary natural number;

(iv) $Dom(\alpha_1)$ is closed with respect to $\varphi'_1, \ldots, \varphi'_n$;

(v) if $x \in Dom(\alpha_1)$ and $\varphi'_i(x) \simeq y$, then $\theta_i(\alpha_1(x)) \simeq \alpha_1(y)$, $1 \leq i \leq n$;

(vi) $\sigma'_1, \ldots, \sigma'_k$ are partial predicates on H_1 and obtain only the value "true" whenever they are defined;

(vii) $\xi'_s \subseteq Dom(\alpha_1)$, $1 \leq s \leq m$;

(viii) if $x \in \xi'_s$, then $x = \langle n+s, j \rangle$, where $1 \leq s \leq m$ and j is an arbitrary natural number;

(ix) $\alpha_1(\xi'_s) \subseteq A_s$, $1 \leq s \leq m$.

We shall denote finite parts by Δ and δ. We shall introduce relations "$\subseteq$" between finite parts and between a finite part and an enumeration and consider some of their properties.

Let $\Delta_i = \langle \alpha_i; H_i; \varphi^i_1, \ldots, \varphi^i_n; \sigma^i_1, \ldots, \sigma^i_k; \xi^i_1, \ldots, \xi^i_m \rangle$, $i = 1, 2$, be finite parts. We say that $\Delta_1 \subseteq \Delta_2$ (Δ_1 is included in Δ_2 or Δ_2 extends Δ_1) iff:

(i) $H_1 \subseteq H_2$; $\alpha_1 \leq \alpha_2$; $\varphi^1_i \leq \varphi^2_i$, $1 \leq i \leq n$; $\sigma^1_j \leq \sigma^2_j$, $1 \leq j \leq k$; $\xi^1_s \subseteq \xi^2_s$, $1 \leq s \leq m$;

(ii) if $\varphi^2_i(x) \simeq y$ and $y \in Dom(\alpha_1)$, then $\varphi^1_i(x) \simeq y$, $1 \leq i \leq n$;

(iii) if $\xi^2_s(x)$ and $x \in Dom(\alpha_1)$, then $\xi^1_s(x)$, $1 \leq s \leq m$.

Let Δ be a finite part and $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration. We say that $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ iff:

(i) $H_1 \cap Dom(\alpha) = \emptyset$ and $\alpha_1 \leq \alpha$;

(ii) $\varphi'_i \leq \varphi_i$, $1 \leq i \leq n$; $\sigma'_j \leq \sigma_j$, $1 \leq j \leq k$; $\xi'_s \subseteq \xi_s$, $1 \leq s \leq m$;

(iii) if $\varphi_i(x) \simeq y$ and $y \in Dom(\alpha_1)$, then $\varphi'_i(x) \simeq y$, $1 \leq i \leq n$;

(iv) if $\xi_s(y)$ and $y \in Dom(\alpha_1)$, then $\xi'_s(y)$, $1 \leq s \leq m$.

From the definitions of the relations "$\subseteq$" we get immediately:

1. $\Delta \subseteq \Delta$;
2. if $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \Delta_3$, then $\Delta_1 \subseteq \Delta_3$;
3. if $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$;

4. if $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then there exists a finite part Δ such that $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$, $\Delta_1 \subseteq \Delta$ and $\Delta_2 \subseteq \Delta$.

The structure $\mathfrak{B}$ *models* $F_e(y)$ (we write $\mathfrak{B} \models F_e(y)$) iff $y \in \Gamma_e(D(\mathfrak{B}))$, where Γ_e is the e-*th* enumeration operator.

By $W_e^{\mathfrak{B}}$ we denote the set $\Gamma_e(D(\mathfrak{B}))$, i. e.

$$y \in W_e^{\mathfrak{B}} \Leftrightarrow \exists v(\langle v, y \rangle \in W_e \;\&\; E_v \subseteq D(\mathfrak{B})).$$

We say that the enumeration $\langle \alpha, \mathfrak{B} \rangle$ *models* $F_e(y)$ ($\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$) iff $\mathfrak{B} \models F_e(y)$.

We define the relation "forces" ($\Vdash$) by the following clauses:

1. $\Delta \Vdash u$, where $u \in N$, if one of the following conditions is true:

(i) $u = \langle i, x, z \rangle$, $1 \leq i \leq n$, and $\varphi_i(x) \simeq z$;

(ii) $u = \langle n + j, x \rangle$, $1 \leq j \leq k$, and ($x \in H_1$ and $\sigma'_j(x) \simeq$ *true*) or ($x \in Dom(\alpha_1)$ and $\Sigma_j(\alpha_1(x)) \simeq$ *true*);

(iii) $u = \langle n + k + s, x \rangle$, $1 \leq s \leq m$, and $\xi_s(x)$;

2. $\Delta \Vdash E = \{u_1, \ldots, u_r\}$ if $\Delta \Vdash u_i$, $1 \leq i \leq r$;

3. $\Delta \Vdash F_e(y)$ if $\exists v(\langle v, y \rangle \in W_e \;\&\; \Delta \Vdash E_v)$.

The following properties of forcing are easily obtained:

1. if $\Delta \Vdash F_e(y)$ and $\Delta \subseteq \delta$, then $\delta \Vdash F_e(y)$;
2. if $\Delta \Vdash F_e(y)$ and $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$;
3. if $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$, then there exists a finite part $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ such that $\Delta \Vdash F_e(y)$.

Most of the proofs in this paper use stepwise constructions. On each step we construct a finite part of a certain enumeration. The finite parts constructed on later steps keep the forcing properties of the former steps. In this way we ensure the modelling property of the constructed enumerations, which is a weak admissibility indicator.

The following proposition gives a characterization of $\forall$-weak-admissible sets by means of finite parts and the relation "$\Vdash$".

Proposition 3.1. *Let D be a $\forall$-weak-admissible set. Then there exist a finite part Δ and a natural number e such that the following conditions hold:*

(i) $\forall \delta \supseteq \Delta \forall y(\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta)$;

(ii) $t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$.

Proof. Assume that a finite part Δ and a natural number e, satisfying the properties (i) and (ii), do not exist. We shall construct a special enumeration $\langle \alpha, \mathfrak{B} \rangle$ for which D is not weak-admissible. The construction will be made by steps. On the q-th step we shall construct a finite part Δ_q that extends Δ_{q-1}. On the steps q for which $(q)_0 = 4n, 4n+1, 4n+2$ we shall ensure some properties of the enumerations, while on steps for which $(q)_0 = 4n + 3$ and $(q)_1 = e$ we shall ensure non-admissibility of D with respect to Γ_e. We assume that an arbitrary enumeration of B is fixed.

1. Let $(q)_0 = 4n$. Let x be the first natural number which is not in $Dom(\alpha_{q-1}) \cup H_{q-1}$, and let t be the first element of B which is not in $Range(\alpha_{q-1})$ (if such t does not exist, then let t be an arbitrary element of B). We define:

$$\begin{aligned}
&\alpha_q(x) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;\\
&H_q \equiv H_{q-1}; \varphi_i^q \equiv \varphi_i^{q-1}, \quad 1 \leq i \leq n;\\
&\sigma_j^q \equiv \sigma_j^{q-1}, 1 \leq j \leq k; \quad \xi_s^q \equiv \xi_s^{q-1}, \quad 1 \leq s \leq m.
\end{aligned}$$

2. Let $(q)_0 = 4n+1$ and $(q)_1 = \langle i, x \rangle$, where $x \in Dom(\alpha_{q-1})$, $\varphi_i^{q-1}(x)$ is undefined and $\theta_i((\alpha_{q-1}(x)) \simeq t$. Let $y = \langle i, j, x \rangle$ and $y \notin Dom(\alpha_{q-1}) \cup H_{q-1}$ (such y exists, because j is an arbitrary natural number). We define:

$$\begin{aligned}
&\varphi_i^q(x) \simeq y \text{ and } \varphi_i^q(z) \simeq \varphi_i^{q-1}(z) \text{ for all other } z;\\
&\alpha_q(y) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;\\
&H_q \equiv H_{q-1}; \quad \varphi_l^q \equiv \varphi_l^{q-1}, \quad 1 \leq l \leq n, \quad l \neq i;\\
&\sigma_j^q \equiv \sigma_j^{q-1}, \quad 1 \leq j \leq k; \quad \xi_s^q \equiv \xi_s^{q-1}, \quad 1 \leq s \leq m.
\end{aligned}$$

3. Let $(q)_0 = 4n+2$ and $(q)_1 = \langle s, x' \rangle$, where $x' \in Dom(\alpha_{q-1})$, $1 \leq s \leq m$; $\alpha_{q-1}(x') \simeq t$, $t \in A_s$ and $t \notin \alpha_{q-1}(\xi_s)$. Let $x = \langle s+n, j \rangle$, where j is an arbitrary natural number such that $x \notin Dom(\alpha_{q-1}) \cup H_{q-1}$. We define:

$$\begin{aligned}
&\alpha_q(x) \simeq t \text{ and } \alpha_q(z) \simeq \alpha_{q-1}(z) \text{ for all other } z;\\
&H_q \equiv H_{q-1}; \quad \varphi_i^q \equiv \varphi_i^{q-1}, \quad 1 \leq i \leq n; \quad \sigma_j^q \equiv \sigma_j^{q-1}, \quad 1 \leq j \leq k;\\
&\xi_s^q \equiv \xi_s^{q-1} \cup \{x\} \text{ and } \xi_r^q \equiv \xi_r^{q-1}, \quad 1 \leq r \leq m, \quad r \neq m.
\end{aligned}$$

4. Let $(q)_0 = 4n+3$ and $(q)_1 = e$. We shall construct Δ_q such that if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_q$, then for $W_e^{\mathfrak{B}}$ and D one of the following conditions is false:

(a) $W_e^{\mathfrak{B}} \subseteq Dom(\alpha)$;

(b) $\alpha(W_e^{\mathfrak{B}}) = D$.

From the assumptions it follows that for Δ_{q-1} and e at least one of the conditions (i) or (ii) is violated:

A) Let (i) be false, i. e. $\exists \delta \supseteq \Delta \exists y (\delta \Vdash F_e(y) \;\&\; y \in H_\delta)$. Let $\Delta_q \equiv \delta$ and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_q$. Due to property 2 of the relation "$\Vdash$", it follows that $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$, i. e. $y \in W_e^{\mathfrak{B}}$. But $y \in H_q$, hence $y \notin Dom(\alpha)$. We have obtained that $W_e^{\mathfrak{B}} \not\subseteq Dom(\alpha)$.

B) Let (ii) be false. Then there exists some t such that one of the following is true:

$$t \in D \text{ and } \forall \delta \supseteq \Delta_{q-1} \forall y (\alpha_\delta(y) \simeq t \Rightarrow \delta \not\Vdash F_e(y)); \tag{3.1}$$

$$t \notin D \text{ and } \exists \delta \supseteq \Delta_{q-1} \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y)). \tag{3.2}$$

If (1) is true, then $\Delta_q \equiv \Delta_{q-1}$. Suppose that for some $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_q$ the conditions (a) and (b) are true. Then there exists $y \in W_e^{\mathfrak{B}}$ such that $\alpha(y) \simeq t$, hence

$\langle\alpha, \mathfrak{B}\rangle \models F_e(y)$. Due to property 3 of the "$\Vdash$" relation and property 4 of the "$\subseteq$" relation, there exists a finite part δ such that $\alpha_\delta(y) \simeq t$, $\delta \Vdash F_e(y)$ and $\delta \supseteq \Delta_q$. This contradiction proves that, for all $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_q$, at least one of (a) and (b) is violated.

Let (2) be true and let $\Delta_q \equiv \delta$ and $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_q$. Then $\alpha(y) \simeq t$ and due to property 2 of the "$\Vdash$" relation, $\langle\alpha, \mathfrak{B}\rangle \models F_e(y)$, i. e. $y \in W_e^{\mathfrak{B}}$, therefore $\alpha(W_e^{\mathfrak{B}}) \neq D$.

Now we define $\langle\alpha, \mathfrak{B}\rangle$ in the following way:

$$\alpha = \bigcup_q \alpha_q; \quad H = \bigcup_q H_q; \varphi_i = \bigcup_q \varphi_i^q, \quad 1 \leq i \leq n;$$

$$\xi_s = \bigcup_q \xi_s^q, 1 \leq s \leq m; \quad \sigma_j^* = \bigcup_q \sigma_j^q, \quad 1 \leq j \leq k;$$

$$\sigma_j(x) \simeq \begin{cases} \Sigma_j(\alpha(x)), & \text{if } x \in Dom(\alpha), \\ \sigma_j^*(x), & \text{if } x \in H. \end{cases}$$

It is easy to see that all constructions are correct. We have obtained an enumeration $\langle\alpha, \mathfrak{B}\rangle$ for which D is not weak-admissible. The last proves the proposition.□

Let $var = \{Y_0, Y_1, \ldots\}$ be the set of all non-special variables and val be a bijection of N onto var. Let Δ be a fixed finite part such that $Dom(\alpha_1) = \{w_1, \ldots, w_r\}$ and $\alpha_1(w_i) = t_i$, $1 \leq i \leq r$.

Proposition 3.2. *There exists an effective way to define, for every finite set E of natural numbers and for every natural y, a conditional term $\lambda(\overline{W})$ with free variables $\overline{W} = (W_1, \ldots, W_r)$, where $W_i = val(w_i)$, $1 \leq i \leq r$, such that the following conditions hold:*

(i) *if $t \in \lambda_{\mathfrak{A}}(\overline{W}/\overline{t})$, then $\exists\delta \supseteq \Delta(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash E)$;*

(ii) *if $t \notin \lambda_{\mathfrak{A}}(\overline{W}/\overline{t})$, then at least one of the following conditions is true:*

(a) $\exists\delta \supseteq \Delta(\delta \Vdash E \;\&\; y \in H_\delta)$;

(b) $\forall\delta \supseteq \Delta(\delta \Vdash E \Rightarrow \alpha_\delta(y) \not\simeq t)$.

Proof. The set E is said to be *consistent* iff the following conditions hold:

1. If $u \in E$, then $u = \langle i, x, z\rangle$, $1 \leq i \leq n$ or $u = \langle i, x\rangle$, $n+1 \leq i \leq n+k+m$.
2. If $\langle i, x, z\rangle \in E$ and $\langle i, x, z_1\rangle \in E$, then $z = z_1$.
3. If $\langle i, x, z\rangle \in E$ and $z \in Dom(\alpha_1)$, then $\varphi_i'(x) \simeq z$.
4. If $\langle i, x, z\rangle \in E$ and $\varphi'(x)$ is defined, then $\varphi_i'(x) \simeq z$.
5. If $\langle i, z\rangle \in E$, $n+k+1 \leq n+k+m$ and $z \in Dom(\alpha_1)$, then $\xi_{i-n-k}'(z)$.
6. If $\langle i, x, z\rangle \in E$, $1 \leq i \leq n$, then $x = \langle i, j, z\rangle$.
7. If $\langle i, z\rangle \in E$, $n+k+1 \leq i \leq n+k+m$, then $z = \langle i-k, j\rangle$.

Immediately from the definitions it follows that:

Lemma 3.1. *If there exists a finite part $\delta \supseteq \Delta$ such that $\delta \Vdash E$, then E is consistent.*

If E is not consistent, let $\lambda = T'(W_1) \supset W_1$. Now let E be consistent and let

$$E_1 = E \setminus (\{\langle i, x, z\rangle | z \in Dom(\alpha_1) \ \&\ 1 \le i \le n\} \cup \{\langle i, z\rangle | z \in Dom(\alpha_1) \ \&\ n + k + 1 \le i \le n + k + m\}).$$

It is easy to show that:

Lemma 3.2. *If $\delta \supseteq \Delta$, then the following equivalence is true:*

$$\delta \Vdash E \Leftrightarrow \delta \Vdash E_1.$$

Let

$$P = \{z | \langle i, z\rangle \in E_1 \ \&\ n + k + 1 \le i \le n + k + m\},$$
$$K = \{z | \langle i, x, z\rangle \in E_1 \ \&\ 1 \le i \le n\} \cup \{z | \langle j, z\rangle \in E_1 \ \&\ n + 1 \le j \le n + k\} \cup \{w_1, \ldots, w_r\} \cup P.$$

We define the relation "$\to$" (follows) between natural numbers as follows:

$$z_1 \to z_2 \text{ iff } \langle i, z_1, z_2\rangle \in E_1 \text{ and } 1 \le i \le n.$$

Here are some simple properties of this relation:

1. If $z_1 \to z$ and $z_2 \to z$, then $z_1 = z_2$.

2. If $z_1 \to z$, then there exists only one number i such that $\langle i, z_1, z\rangle \in E_1$ ($z = \langle i, j, z_1\rangle$).

3. If $z_1 \to z$, then $z_1 < z$ (the coding is monotonic).

Note that if $z \in P$, then z has no predecessor, because $z = \langle s, j\rangle$, where $n + 1 \le s \le n + m$, i. e. z cannot be a value of a function.

We define sets $K_0, K_1, \ldots$ as follows:

$$K_0 = \{w_1, \ldots, w_r\} \cup P,$$
$$K_{l+1} = \{z | \exists x (x \in K_l \ \&\ x \to z)\}, \quad l = 0, 1, \ldots$$

It is easy to show by induction that if $m_1 < m_2$, then $K_{m_1} \cap K_{m_2} = \emptyset$. Then there exists p such that $K_{p+1} = \emptyset$. Let $K^* = \bigcup_{l=0}^{p} K_l$. It is clear that K^* is a finite set.

For every $z \in K^*$ we define τ^z in the following way:

1. If $z \in K_0$, then:

(a) if $z \in \{w_1, \ldots, w_r\}$, then $\tau^z = val(z)$;

(b) if $z = \langle s, j\rangle \in P$, then $\tau^z = X_z^{s-n}$.

2. If $z \in K_{l+1}$, $x \to z$ and $x \in K_l$, then $\tau^z = f_i(\tau^x)$.

Let $E^* \subseteq E_1$ be such that

$$u \in E^* \stackrel{def}{\Leftrightarrow} ((u = \langle i, z_1, z_2\rangle \ \&\ 1 \le i \le n) \vee (u = \langle j, z_1\rangle \ \&\ n + 1 \le j \le n + k + m)) \ \&\ z_1 \in K^*.$$

Let $\pi = \{X_z^{s-n} | z = \langle s, j\rangle \in P\}$ and let $\pi = \{X_1^{s_1}, \ldots X_p^{s_p}\}$.
For every $u \in E^*$ we define L^u as follows:
(a) if $u = \langle i, z_1, z_2\rangle$ and $1 \leq i \leq n$, then $L^u = T_0(\tau^{z_2})$;
(b) if $u = \langle n+j, z\rangle$ and $1 \leq j \leq k$, then $L^u = T_j(\tau^z))$;
(c) if $u = \langle n+k+s, z\rangle$ and $1 \leq s \leq m$, then $L^u = S_s(X_z^s)$.
Now let $\Sigma = \&_{u\in E^*} L^u$ and $\Pi = \exists X_1^{s_1} \ldots \exists X_p^{s_p} \&_{u \in E^*} L^u$.
The next two lemmas follow immediately from the above constructions.

Lemma 3.3. *Let $\delta \supseteq \Delta$ and $\delta \Vdash E_1$. Then $P \subseteq Dom(\alpha_\delta)$.*

Lemma 3.4. *Let $\delta \supseteq \Delta$, $\delta \Vdash E_1$ and $\alpha_\delta(p_i) \simeq e_i$, $1 \leq i \leq p$. Then the following conditions are true:*
(i) $K^* \subseteq Dom(\alpha_\delta)$;
(ii) $\forall z \in K^*(\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}))$;
(iii) $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. *($\overline{X}$ stands for $X_1^{s_1}, \ldots, X_p^{s_p}$).*

Lemma 3.5. *Let $H_1 \cap K^* = \emptyset$ and $e_1, \ldots, e_q \in B$ be such that $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e}) \simeq 0$. Then there exists a finite part $\delta \supseteq \Delta$ with the following properties:*
(i) $\alpha_\delta(p_i) \simeq e_i$, $1 \leq i \leq q$;
(ii) $Dom(\alpha_\delta) = K^*$ *and* $H_\delta = H_1 \cup (K/K^*)$;
(iii) *if* $z \in K^*$, *then* $\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$;
(iv) $\delta \Vdash E_1$.

Proof. For $z \in K^*$ we define α_δ as follows:
1. If $z \in K_0$, then:
(a) if $z = w_i$, then $\alpha_\delta(z) \simeq t_i \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$ $(\tau^z = w_i)$;
(b) if $z = p_i$, then $\alpha_\delta(z) \simeq e_i \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$ $(\tau^z = X_{p_i}^{s_i})$.
2. If $z \in K_{l+1}$, then $\alpha_\delta(z) \simeq \tau^z_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{e})$.
The other components of δ we define as follows:

$H_\delta = H_1 \cup (K/K^*)$;

$\varphi_i^\delta(z_1) \simeq z_2 \overset{def}{\Leftrightarrow} \langle i, z_1, z_2\rangle \in E_1 \vee \varphi_i'(z_1) \simeq z_2$ for $1 \leq i \leq n$ and $z_1 \in H_\delta \cup Dom(\alpha_\delta)$;

$\xi_s^\delta(z) \overset{def}{\Leftrightarrow} \langle s+n+k, z\rangle \in E_1 \vee \xi_s'(z)$ for $1 \leq s \leq m$ and $z \in Dom(\alpha_\delta)$;

$\sigma_j^\delta(z) \simeq true \overset{def}{\Leftrightarrow} \langle n+j, z\rangle \in E_1 \vee \sigma_j'(z) \simeq true$ for $1 \leq j \leq k$ and $z \in H_\delta$.

It is easy to show that the finite part δ defined above satisfies (i)-(iv). □

Now we can continue the *proof* of Proposition 3.2. Let

$$\lambda = \begin{cases} T'(W_1) \supset W_1, & \text{if } y \notin K^* \text{ or } K^* \cap H_1 \neq \emptyset, \\ \Pi \supset \tau^y, & \text{otherwise.} \end{cases}$$

Let $\lambda_{\mathfrak{A}}(\overline{W}/\bar{t}) \ni t$. Due to Lemma 3.5, there exists $\delta \supseteq \Delta$ such that $\alpha_\delta(y) \simeq \tau^y(\overline{x}/\overline{e}, \overline{W}/\bar{t}) \simeq t$ and $\delta \Vdash E_1$, and hence $\delta \Vdash E$.

Let $\lambda_{\mathfrak{A}}(\overline{W}/\bar{t}) \not\ni t$ and E be consistent. There exist three possibilities:

1. $K^* \cap H_1 \neq \emptyset$. Suppose that $\delta \supseteq \Delta$ and $\delta \Vdash E$. Then due to Lemma 3.3 and Lemma 3.4, $K^* \subseteq Dom(\alpha_\delta)$. This contradicts the fact that $Dom(\alpha_\delta) \cap H_1 = \emptyset$, hence for all $\delta \supseteq \Delta$ it is true that $\delta \not\Vdash E$, i. e. (ii)(a) is satisfied.

2. $K^* \cap H_1 = \emptyset$ and $y \in K^*$. Suppose that $\delta \supseteq \Delta$ and $\delta \Vdash E$. Hence $\delta \Vdash E_1$ and, due to Lemma 3.4, $P \subseteq Dom(\alpha_\delta)$, $\alpha_\delta(y) \simeq \tau^y(\overline{W}/\bar{t}, \overline{X}/\overline{e})$ and $\Sigma_{\mathfrak{A}}(\overline{W}/\bar{t}, \overline{X}/\overline{e}) \simeq 0$, where $e_i = \alpha_\delta(p_i)$, $1 \le i \le p$. If $\alpha_\delta(y) \simeq t$, then $\lambda(\overline{W}/\bar{t}) \simeq t$. This is a contradiction, hence $\alpha_\delta(y) \not\simeq t$ and (ii)(b) holds.

3. $K^* \cap H_1 = \emptyset$ and $y \notin K^*$:

(a) If $\Pi(\overline{W}/\bar{t}) \simeq 0$, then there exist $e_1, \ldots, e_q \in B$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\bar{t}, \overline{X}/\overline{e}) \simeq 0$. Due to Lemma 3.5 and the properties of the relation "$\Vdash$", there exists $\delta \supseteq \Delta$ such that $\delta \Vdash E$ and $y \in H_\delta$. Then (ii)(a) is true.

(b) Let $\Pi(\overline{W}/\bar{t}) \not\simeq 0$. Let $\delta \supseteq \Delta$ and suppose that $\delta \Vdash E$. Due to Lemma 3.3, there exist $e_1, \ldots, e_q \in B$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\bar{t}, \overline{X}/\overline{e}) \simeq 0$. It follows from this contradiction that (ii)(b) is true.

Let $\lambda_{\mathfrak{A}}(\overline{W}/\bar{t}) \not\ni t$ and let E be not a consistent set. Then for all $\delta \supseteq \Delta$, $\delta \not\Vdash E$ and (ii)(b) is true. That proves Proposition 3.2. □

Theorem 3.1. *If D is a $\forall$-weak-admissible set, then D is weak-computable.*

Proof. Let D be $\forall$-weak-admissible. Due to Proposition 3.1, there exist a finite part Δ and a natural number e such that:

(i) $\forall \delta \supseteq \Delta \forall y (\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta)$;

(ii) $t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$.

Let $t \in D$. It follows from (ii) that there exists a natural number v such that $\langle v, y \rangle \in W_e$ and $\delta \Vdash E_v$. Consider the conditional term $\lambda^{v,y}$ for E_v and y from Proposition 3.2. Suppose that $\lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\bar{t}) \not\ni t$. There exist two possibilities:

1. There exists $\delta' \supseteq \Delta$ such that $\delta' \Vdash E_v$ and $y \in H_{\delta'}$. Hence $\delta' \Vdash F_e(y)$ and $y \in H_{\delta'}$, which contradicts (i).

2. For all $\delta' \supseteq \Delta(\delta' \Vdash E_v \Rightarrow \alpha_{\delta'}(y) \not\simeq t)$. This case is also impossible, because $\delta \supseteq \Delta$, $\delta \Vdash E_v$ and $\alpha_\delta(y) \simeq t$.

So we have that $t \in \lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\bar{t})$.

Now let $t \in \lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\bar{t})$. Then $\exists \delta \supseteq \Delta(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash E_v)$ and, due to (ii), $t \in D$.

Finally, we obtain that

$$t \in D \Leftrightarrow \exists \langle v, y \rangle \in W_e(\lambda^{v,y}_{\mathfrak{A}}(\overline{W}/\overline{T}) \ni t),$$

which proves the theorem. □

Now we are ready to introduce our semantics of the logic programs. Let $\mathfrak{L} = (f_1, \ldots, f_n; T_0, \ldots, T_k; S_1, \ldots, S_m)$ be a first-order language corresponding to the partial structure $\mathfrak{A}$. Let $C = \{c_1, \ldots, c_r\}$ be a set of constants. For every $t \in B$ we introduce new constants k_t^s as names for t, $1 \leq s \leq m$. We define the sets $K_s = \{k_t^s | t \in A_s\}$, $1 \leq s \leq m$, and $K = \bigcup_{s=1}^m K_s$. Let $\mathfrak{L}_K = \mathfrak{L} \cup C \cup K$ and let $\mathfrak{A}^*$ be the enrichment of $\mathfrak{A}$ to the extended language $\mathfrak{L}_K$. Let $\mathfrak{T}_K$ be the set of all ground terms of $\mathfrak{L}_K$. The set

$$\partial_1^C(\mathfrak{A}) = \{T_j(\tau) | 0 \leq j \leq k \;\&\; \tau \in \mathfrak{T}_k \;\&\; \Sigma_j(\tau_{\mathfrak{A}^*}) \simeq 0\}$$

is called *a diagram* without parameters of the structure $\mathfrak{A}$. For all parameters we also introduce a diagram

$$\partial(A_s) = \{S_s(k_t^s) | k_t^s \in K_s\}, \quad 1 \leq s \leq m.$$

Now we define a *diagram* of the whole structure $\mathfrak{A}$:

$$\partial^C(\mathfrak{A}) = \partial_1^C(\mathfrak{A}) \cup \partial(A_1) \cup \ldots \cup \partial(A_m).$$

A subset D of B is called *definable by logic programs (LP-definable)* in the structure $\mathfrak{A}$ iff there exist an ordered pair $\langle P, H \rangle$ (P is a logic program and H is a new predicate symbol) and a set of constants $C = \{c_1, \ldots, c_r\}$ such that the following equivalence is true:

$$t \in D \Leftrightarrow \exists \tau (\tau \in \mathfrak{T}_K \;\&\; \partial^C(\mathfrak{A}) \cup P \vdash H(\tau) \;\&\; \tau_{\mathfrak{A}^*} \simeq t)$$

(the sign "$\vdash$" means derivability in the sense of the first-order predicate calculus).

Notice that in the definition of $\partial^C(\mathfrak{A})$ the underlined predicates and the parameters are not treated in equal manner. For example, suppose that $\theta_i(z) = t$. Suppose that $\Sigma_j(t) \simeq 0$ and $t \in A_s$. Then both $T_j(k_t^s)$ and $T_j(f_i(k_z^s))$ are elements of $\partial^C(\mathfrak{A})$. On the other hand, $S_s(k_t^s) \in \partial^C(\mathfrak{A})$, but $S_s(f_i(k_z^s)) \notin \partial^C(\mathfrak{A})$. The picture changes if the equality relation is among the underlined predicates. In such case, we have $f_i(k_z^s) = k_t^s \in \partial^C(\mathfrak{A})$ and hence $\partial^C(\mathfrak{A}) \vdash S_m(f_i(k_z^s))$.

Now we shall consider the relation between the LP-definable and $\forall$-weak-admissible sets. For this purpose, we shall translate the constructions from Proposition 3.2 into logic programs. We shall introduce some auxiliary terms.

A natural number e and a finite part Δ are called *compatible* iff

$$\forall \delta \supseteq \Delta \forall y (\delta \Vdash F_e(y) \Rightarrow y \notin H_\delta).$$

A subset D of B is said to be *sufficient* for the finite part Δ and the natural number e iff the following equivalence is true:

$$t \in D \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y)).$$

A family of sets $\mathfrak{P}$ is called *sufficient* iff, for every compatible finite part Δ and natural number e, there exists $D \in \mathfrak{P}$ such that D is sufficient for Δ and e.

Note that if D and D_1 are sufficient for Δ and e, then $D \equiv D_1$. It follows from Proposition 3.1 that if D is $\forall$-weak-admissible, then it belongs to every sufficient family.

Let fix a finite part Δ. For every natural number e compatible with Δ we shall construct a logic program $\langle P', F\rangle$ such that the set defined by $\langle P', F\rangle$ is sufficient for Δ and e.

Let $\underline{0}$ and nil be new constant symbols, let f_0 be a new unary functional symbol, and h be a new binary functional symbol.

For every natural n by $\underline{n}$ we note the term $f_0^n(\underline{0})$. Let $\underline{N}$ denote the set $\{\underline{n}|n \in N\}$.

The following proposition is a reformulation of a well-known result.

Proposition 4.3. *For every r.e. subset W of N^k and for every k-ary predicate symbol Q, there exists a logic program P with the following properties:*

(i) *if $(x_1, \ldots, x_k) \in W$, then $P \vdash Q(\underline{x}_1, \ldots, \underline{x}_k)$;*

(ii) *there exists a Herbrand interpretation I of P, which is a model of P and*

$$I(Q)(\alpha_1, \ldots, \alpha_k) = 0 \Leftrightarrow$$
$$\exists x_1 \ldots \exists x_k((x_1, \ldots, x_k) \in W \;\&\; \alpha_1 = \underline{x}_1 \;\&\; \ldots \;\&\; \alpha_k = \underline{x}_k).$$

Such interpretations of P we call *standard.*

We define *lists* in the following inductive way:

(i) *nil* is a list;

(ii) if α is a list and β is a term, then $g(\alpha, \beta)$ is a list.

Let $\Delta = \langle \alpha_1; H_1; \varphi'_1, \ldots, \varphi'_n; \sigma'_1, \ldots, \sigma'_k; \xi'_1, \ldots, \xi'_m\rangle$; $Dom(\alpha_1) = \{w_1, \ldots, w_r\}$; $\alpha_1(w_i) \simeq t_i$, $1 \leq i \leq r$, and let $c_1, \ldots, c_r$ be new constant symbols which are interpreted in $\mathfrak{A}$ as $t_1, \ldots, t_r$. Let $R = \{c_1, \ldots, c_r, \underline{0}, nil, f_0, \ldots, f_n, h\}$ and $X_0^s, X_1^s, \ldots$ be special variables. Let $\mathfrak{T}$ be the set of all terms constructed by means of R and the special variables. We shall denote the elements of $\mathfrak{T}$ by $a, b, c, \ldots$ Let $var(a)$ be the set of the variables of a. We consider Herbrand interpretations of $\mathfrak{T}$. For a consistent set E, we shall use the sets P, K, E_1, K^* and E^*, constructed in the proof of Proposition 3.2.

We consider substitutions of the form $\{X_{p_1}^{s_1}/\mu_1, \ldots, X_{p_q}^{s_q}/\mu_q\}$, where $\mu_1, \ldots, \mu_q \in \mathfrak{T}$. If $\mu_i = \underline{p_i}$ and $p_i = \langle s_i + n, j\rangle$, $1 \leq i \leq q$, the substitution is called a *correspondence* and the list $[[X_{p_1}^{s_1}|\underline{p}_1], \ldots, [X_{p_q}^{s_q}|\underline{p}_q]]$ is called a *representation* of the correspondence. For a substitution κ and $a \in \mathfrak{T}$, by $a\kappa$ we denote the term, provided by applying κ over a. If l is a representation of a correspondence, $l\kappa$ is called a *pseudocorrespondence.*

If l and f are correspondences, then we shall write $l \leq_1 f$ to denote that $l = f$ or $l = append(l, [X, \underline{m}])$ for some special variable X and a natural m. We use the sign "$\leq$" to denote the reflexive and transitive closure of "$\leq$".

Consider the sets:

$$\begin{aligned}
Neq &= \{(\underline{x}, \underline{y}) | \underline{x}, \underline{y} \in \underline{N} \ \& \ x \neq y\}, \\
Cod2 &= \{(\underline{x}, \underline{i}, \underline{y}) | \underline{x}, \underline{i}, \underline{y} \in \underline{N} \ \& \ x = \langle i, y \rangle\}, \\
Cod3 &= \{(\underline{x}, \underline{i}, \underline{y}, \underline{z}) | \underline{x}, \underline{i}, \underline{y}, \underline{z} \in \underline{N} \ \& \ x = \langle i, y, z \rangle\}, \\
Dalpha &= \{\underline{w}_1, \ldots, \underline{w}_r\}, \\
NDalpha &= \underline{N}/Dalpha, \\
NCod2 &= \underline{N}/L, \text{ where } L = \{\underline{x} | x = \langle i, z \rangle \ \& \ i, z \in N\}, \\
NCod3 &= \underline{N}/L', \text{ where } L' = \{\underline{x} | x = \langle i, j, z \rangle \ \& \ 1 \leq i \leq n \ \& \ j, z \in N\}.
\end{aligned}$$

All of them are recursively enumerable. Let the logic programs P_{neq}, P_{cod2}, P_{cod3}, P_{dalpha}, $P_{ndalpha}$, P_{ncod2}, P_{ncod3} represent the above sets with predicate symbols *neq*, *cod*2, *cod*3, *dalpha*, *ndalpha*, *ncod*2 and *ncod*3 and suppose that they have no common predicate symbols.

We shall identify the finite set of atoms with their conjunction if the set is not empty, and with true if it is empty.

We shall consider several programs, needed in the construction of P'. When a program uses already defined predicates, we shall suppose that the texts of the corresponding programs are appended to the text of this program. For example, we shall suppose that in the next program P_0 the programs P_{neq}, P_{cod2}, P_{cod3} and $P_{ndalpha}$ are included.

P_0

$e_1(Y, [Y|R])$:-$cod3(Y, \underline{i}, Z, X), ndalpha(X)$. $1 \leq i \leq n$

$e_1(Y, [Y|R])$:-$cod2(Y, \underline{j}, X)$. $n + 1 \leq j \leq n + k$

$e_1(Y, [Y|R])$:-$cod2(Y, \underline{s}, X), ndalpha(X)$. $n + k + 1 \leq s \leq n + k + m$

$e_1(X, [Y|R])$:-$neq(X, Y), e_1(X, R)$.

The next proposition is a verification of the program P_0. The method used is developed in [3].

Proposition 4.4. *If $\underline{x} \in \underline{N}$ and $E = \{u_1, \ldots, u_l\}$ is a consistent set, then*

$$P_0 \vdash e_1(\underline{x}, [\underline{u}_1, \ldots, \underline{u}_l]) \textit{ iff } x \in E_1.$$

Proof. The "if" part is proved by induction on l. To prove the "only if" part, we shall define a special Herbrand interpretation of P_0. Let take a special Herbrand interpretation I of the predicates that occur in P_{neq}, P_{cod2}, P_{cod3} and $P_{ndalpha}$. We define the predicate $I(e_1)$ as follows:

(a) $I(e_1) \simeq 0$ if $a \notin N$ or τ is not a list representing a consistent set;

(b) $I(e_1) \simeq 0$ if $a = \underline{x} \in \underline{N}$ and τ is a list representing a consistent set E such that $x \in E_1$.

A straightforward proof shows that I is a model for P_0. This fact together with the definition of $I(e_1)$ proves the proposition. □

The following programs are verified in a similar way.

P_1

$$p(x,z)\text{:-}cod2(Y,\underline{s},X),e_1(Y,Z).$$

Proposition 4.5. *If* $\underline{x} \in \underline{N}$ *and* $E = \{u_1,\ldots,u_r\}$ *is a consistent set, then*

$$P_1 \vdash p(\underline{x},[\underline{u}_1,\ldots,\underline{u}_r]) \text{ iff } x \in P.$$

P_2

$p_1([\,],[\,])\text{:-}.$
$p_1(X,[H|T])\text{:-}ncod2(H),p_1(X,T).$
$p_1(X,[H|T])\text{:-}cod2(H,\underline{j},Z),p_1(X,T).\quad n+1 \leq j \leq n+k$
$p_1(X,[H|T])\text{:-}cod2(H,\underline{s},Z),dalpha(Z),p_1(X,T).$
$\qquad n+k+1 \leq s \leq n+k+m$
$p_1([H|Y],[H|T])\text{:-}cod2(H,\underline{s},X),ndalpha(X),p_1(Y,T).$
$\qquad n+k+1 \leq s \leq n+k+m.$

Proposition 4.6. *If* e *is a list representing a consistent set* E*, then*

$$P_2 \vdash p_1(f,e) \text{ iff } f \text{ represents the set } P.$$

P_3

$e_{11}([\,],[\,])\text{:-}.$
$e_{11}(X,[H|T])\text{:-}cod3(H,\underline{i},Z,Y),dalpha(Y),e_{11}(X,T).\quad 1 \leq i \leq n$
$e_{11}(X,[H|T])\text{:-}cod2(H,\underline{s},Z),dalpha(Z),e_{11}(X,T).$
$\qquad n+k+1 \leq s \leq n+k+m$
$e_{11}([H|X],[H|T])\text{:-}cod3(H,\underline{i},Z,Y),ndalpha(Y),e_{11}(X,T).\quad 1 \leq i \leq n$
$e_{11}([H|X],[H|T])\text{:-}cod2(H,\underline{j},Z),e_{11}(X,T).\quad n+1 \leq j \leq n+k$
$e_{11}([H|X],[H|T])\text{:-}cod2(H,\underline{s},Z),ndalpha(z),e_{11}(X,T).$
$\qquad n+k+1 \leq s \leq n+k+m.$

Proposition 4.7. *If e is a list representing a consistent set E, then*

$$P_3 \vdash e_{11}(a, e) \text{ iff } a \text{ represents the set } E_1.$$

P_4

$$nel([\,], X)\text{:-}.$$
$$nel([X1|Y], X)\text{:-}neq(X1, X), nel(Y, X).$$

Proposition 4.8. *If a is a list of elements of $\underline{N}$ and b is an element of $\underline{N}$, then*

$$P_4 \vdash nel(a, b) \text{ iff } b \text{ is not an element of } a.$$

P_5

$$k(\underline{w}_i, Z)\text{:-}.\ 1 \le i \le r$$
$$k(X, Z)\text{:-}p(X, Z).$$
$$k(X, Z)\text{:-}cod3(X, \underline{i}, J, Z1), cod3(Y, \underline{i}, Z1, X), e_1(Y, Z), k(Z1, Z).$$
$$1 \le i \le n.$$

Proposition 4.9. *If $\underline{x} \in \underline{N}$ and e is a list representing a consistent set E, then*

$$P_5 \vdash k(\underline{x}, e) \text{ iff } x \in K^*.$$

P_6

$$\overline{k}(X, Z)\text{:-}ncod3(X), p_1(Y, Z), nel(Y, X), ndalpha(X).$$
$$\overline{k}(X, Z)\text{:-}cod3(X, \underline{i}, J, Y), cod3(R, \underline{i}, Y, X), e_{11}(X1, Z), nel(X1, R),$$
$$p1(Y1, Z), nel(Y1, X), ndalpha(X).\ 1 \le i \le n$$
$$\overline{k}(X, Z)\text{:-}cod3(X, \underline{i}, J, Y), p_1(Y1, Z), nel(Y1, X), ndalpha(X),$$
$$cod3(R, \underline{i}, Y, X), e_1(R, Z), \overline{k}(Y, Z).\ 1 \le i \le n.$$

Proposition 4.10. *If $\underline{x} \in \underline{N}$ and e is a list representing a consistent set E, then*

$$P_6 \vdash \overline{k}(\underline{x}, e) \text{ iff } x \notin K^*$$

P_7

$$tau(\underline{w}_i, c_i, X, X, E)\text{:-}.\ 1 \le i \le r$$
$$tau(X, Y, [\,], [[Y, X]], E)\text{:-}p(X, E).$$
$$tau(X, Y, [[Y1, X1]|Z1], [[Y1, X1]|Z2], E)\text{:-}p(X, E), p_1(X1, E), neq(X, X1), tau(X, Y, Z1, Z2, E).$$
$$tau(X, f_i(V), S, Q, E)\text{:-}cod3(X, \underline{i}, J, X1), k(X1, E), cod3(R, \underline{i}, X1, X), e_1(R, E), tau(X1, V, S, Q, E).\ 1 \le i \le n.$$

Proposition 4.11. *Let the list e represent the consistent set E,* $\underline{x} \in \underline{N}$, $x \in K^*$, *and c is a pseudocorrespondence. Let b and d be elements of* $\mathfrak{T}$. *Then:*

$P_7 \vdash tau(\underline{x}, b, c, d, e)$ *iff there exist a substitution* κ, *a term* τ *of* $\mathfrak{T}$ *and correspondences l and f such that* $\tau\kappa = b$, $l\kappa = c$, $f\kappa = d$ *and*

$$l \le_1 f, var(\tau) \cup var(l) = var(f) \text{ and } \tau f = \tau^x(\overline{W}/\overline{c}, \overline{X}/\overline{p}),$$

where τ^x *is the term constructed for x in the proof of Proposition* 3.2, $\overline{c} = (c_1, \ldots, c_r)$ *and* $\overline{p} = (p_1, \ldots, p_q)$.

If l_1 and l_2 are lists, we shall write $l_1 \le_1^* l_2$ to denote that $l_1 \equiv l_2$ or there exists a term a of $\mathfrak{T}$ such that $l_2 = [a, l_1]$. By "$\le^*$" we denote the transitive closure of "$\le_1^*$".

Let $\mathfrak{L}_C = (c_1, \ldots, c_r; f_1, \ldots, f_n, T_0, \ldots, T_k; S_1, \ldots, S_m)$ be a first-order language, where special variables are also available. Atoms in $\mathfrak{L}$ are atoms in which may occur $T_0, \ldots, T_n$ and $S_1, \ldots, S_m$. Let $\mathfrak{L}'_C$ be an enrichment of $\mathfrak{L}_C$ with the constants $\underline{0}$, *nil* and the functional symbols f_0 and h.

P_8

$$pi([\,], E, Z)\text{:-}.$$
$$pi([X|Y], E, Z)\text{:-}cod3(X, \underline{i}, X1, Y1), \overline{k}(Y1, E), pi(Y, E, Z).\ 1 \le i \le n$$
$$pi([X|Y], E, Z)\text{:-}cod3(X, \underline{i}, X2, X1), k(X1, E), tau(X1, Y1, Z, Z1, E), T_0(Y1), pi(Y, E, Z1).\ 1 \le i \le n$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{j}, X1), k(X1, E), tau(X1, Y1, Z, Z1, E), T_{n-j}(Y1), pi(Y, E, Z1).\ n+1 \le j \le n+k$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{j}, X1), \overline{k}(X1, E), pi(Y, E, Z).\ n+1 \le j \le n+k$$
$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{s}, X1), p(X1, E), tau(X1, Y1, Z, Z1, E), S_{s-n-k}(Y1), pi(Y, E, Z1).\ n+k+1 \le s \le n+k+m$$

$$pi([X|Y], E, Z)\text{:-}cod2(X, \underline{s}, X1), dalpha(X1), pi(Y, E, Z1).$$
$$n + k + 1 \leq s \leq n + k + m.$$

Proposition 4.12. *Let e and e_1 be lists such that $e_1 \leq^* e$, let e represent a consistent set E and let b be a pseudocorrespondence. Then for every finite set G of atoms in the language $\mathfrak{L}'_C$ holds $P_8 \vdash G \Rightarrow pi(e_1, e, b)$ iff there exist a substitution κ, a finite set G^0 of atoms in $\mathfrak{L}'_C$ and correspondences l and f such that $G = G^0\kappa$, $b = l\kappa$, $l \leq f$, e_1 represents the set E' and $G^0 f \supset \tilde{\Sigma}_1(\overline{W}/\overline{c}, \overline{X}/\overline{p})$, where $\tilde{\Sigma}_1 = \bigcup_{u \in E^* \cap E'} \{L_u\}$ and $P = \{p_1, \ldots, p_q\}$ is the set corresponding to E constructed in Proposition 3.2.*

Proof. The "if" part is similar to the previous propositions. To prove the "only if" part, we define a class $\mathfrak{K}$ of Herbrand interpretations of P_8. A Herbrand interpretation I of P_8 belongs to $\mathfrak{K}$ if the following conditions are satisfied:

(i) I is standard for the already defined predicates.

(ii) If e_1, e and b belong to $\mathfrak{T}$, then:

(a) $I(pi)(e_1, e, b) \simeq 0$ if e_1 or e are not lists or $e_1 \not\leq^* e$ or e does not represent a consistent set or b is not a pseudocorrespondence;

(b) $I(pi)(e_1, e, b) \simeq 0$ if e_1 and e are lists; $e_1 \leq^* e$; e represents a consistent set; b is a pseudocorrespondence; there exists a finite set $G = \{\beta_1, \ldots, \beta_h\}$ of atoms such that $I(\beta_1) \simeq \cdots \simeq I(\beta_h) \simeq 0$ and there exist a finite set G^0 of atoms in $\mathfrak{L}'_C$, a substitution κ and correspondences l and f such that $b = l\kappa$, $l \leq f$ and $G^0 f \supseteq \tilde{\Sigma}_1(\overline{W}/\overline{c}, \overline{X}/\overline{p})$.

It is easy to show that every interpretation in $\mathfrak{K}$ is a model of P_8.

Let $G = \{\beta_1, \ldots, \beta_h\}$ be a finite set of atoms in $\mathfrak{L}'_C$, e_1 and e be lists, $e_1 \leq^* e$, let e represent a consistent set E, b be a pseudocorrespondence and $P_8 \vdash G \Rightarrow pi(e_1, e, b)$. Consider $I \in \mathfrak{K}$ such that $I(\beta_1) \simeq 0$ iff $\beta_1 \in G$. I is a model of $\mathfrak{K}$, hence $I(pi)(e_1, e, b) \simeq 0$. The latter together with the definition of I proves the proposition. □

Proposition 4.13. *For every natural e compatible with the finite part Δ, there exists a logic program $\langle P', F \rangle$ such that the set definable by means of $\langle P', F \rangle$ and the constants $c_1, \ldots, c_r$ is sufficient for Δ and e.*

Proof. Consider the set

$$W_1 = \{\langle v, y \rangle \in W_e | E_v \text{ is consistent}, y \in K^* \text{ and } K^* \cap H_1 = \emptyset\}.$$

It is clear that W_1 is a r. e. set. Let Q be a new unary predicate symbol and P_9 be a logic program that represents W_1 by means of Q. Let *list* be a new binary predicate symbol and P_{10} be a logic program that has no common predicate symbols with the other programs and satisfies the following conditions:

(i) if u is a code of the finite set $\{v_1, \ldots, v_l\}$, then $P_{10} \vdash list(\underline{u}, [\underline{v}_1, \ldots, \underline{v}_l])$;

(ii) there exists a Herbrand interpretation I of P_{10} such that if $u \in N$ and $E_u = \{v_1, \ldots, v_l\}$, then $I(list)(\underline{u}, b) \simeq 0$ iff $b = [\underline{v}_1, \ldots, \underline{v}_l]$.

Consider the following logic program:

P'

$$F(Y)\text{:-}Q(Z), cod2(Z, U, X), list(U, U1), tau(X, Y, [\,], F, U1), \\ pi(U1, U1, F).$$

As in the previous propositions, it may be proved that for every finite set G of atoms in $\mathfrak{L}'_C$ and for every term τ, $P' \vdash G \Rightarrow H(\tau)$ iff there exist a substitution κ, an ordered pair $\langle v, y\rangle \in W_1$, a finite set G^0 of atoms in $\mathfrak{L}'_C$, a term τ^0 in $\mathfrak{L}'_C$ and a correspondence l such that $G = G^0\kappa$, $\tau = \tau^0\kappa$, $G^0 l \supseteq \tilde{\Sigma}(\overline{W}/\overline{c}, \overline{X}/\overline{p})$ and $\tau^0 l \equiv \tau^y(\overline{W}/\overline{c}, \overline{X}/\overline{p})$, where $\overline{c} = (c_1, \ldots, c_r)$, $\overline{p} = (p_1, \ldots, p_q)$, E_v is the finite set with code v, $P = \{p_1, \ldots, p_q\}$ and E^* are its corresponding sets constructed in Proposition 3.2, $\tilde{\Sigma} = \bigcup_{u \in E^*} L_u$ and τ^y is the term corresponding to y.

Let the subset D of B be LP-definable by $\langle P', F\rangle$ and $c_1, \ldots, c_r$. We shall prove that D is sufficient. Let $\exists\delta \supseteq \Delta\exists y(\alpha_\delta(y) \simeq t \;\&\; \delta \Vdash F_e(y))$. It follows from $\delta \Vdash F_e(y)$ that there exists $\langle v, y\rangle \in W_1$ such that $\delta \Vdash E_v$. Let K^* be the set corresponding to E_v, constructed in the proof of Proposition 3.2. It is easy to prove that $\langle v, y\rangle \in W_1$.

From $\delta \supseteq \Delta$ and $\delta \Vdash E_v$ it follows that $K^* \subseteq Dom(\alpha_\delta)$, i.e. there exist $l_1, \ldots, l_q$ such that $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq 0$. Let k_i be the name of l_i with respect to the parameter with number s_i (there exists such a name, because $p_i = \langle s_i + n, j\rangle \in \xi_{s_i}$). Let $l = [[X^{s_1}_{p_1}, \underline{p}_1], \ldots, [X^{s_q}_{p_q}, \underline{p}_q]]$, $\tau^0 = \tau^y(\overline{W}/\overline{c})$ and $G^0 = \tilde{\Sigma}(\overline{W}/\overline{c})$. Then for the empty substitution κ, $P \vdash G^0 \Rightarrow F(\tau^0)$ holds. From $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{k}) \simeq 0$ it follows that $G^0(\overline{X}/\overline{k}) \subseteq \partial^C(\mathfrak{A})$, where $\overline{k} = (k_1, \ldots, k_q)$. From the Theorem of constants and the Deduction theorem it follows that $P \cup \partial^C(\mathfrak{A}) \vdash F(\tau)$, where $\tau = \tau^0(\overline{X}/\overline{k})$. In addition, $\tau_{\mathfrak{A}} \simeq \tau^y_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{k}) \simeq \tau^y_{\mathfrak{A}}(\overline{W}/\overline{t}, \overline{X}/\overline{l}) \simeq \alpha_\delta(y) \simeq t$. We obtained that $t \in D$.

Now let $t \in D$. From the Theorem of constants and the Reduction theorem it follows that there exists a finite set G of atoms in $\mathfrak{L}_C$ such that $P \vdash G \Rightarrow H(\tau)$. Let $X_1, \ldots, X_{q'}$ be the set of variables occurring in the formula $G \Rightarrow H(\tau)$. Then there exist $d_1, \ldots, d_{q'} \in B$ such that $G_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq 0$ and $\tau_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq \tau$.

From the characterization of P, there exist $\langle v, y\rangle \in W_1$, a substitution κ, a finite set G^0 of atoms in $\mathfrak{l}'_C$ and a correspondence f such that $G = G^0\kappa$, $\tau = \tau^0\kappa$, $G^0 f \supseteq \tilde{\Sigma}(\overline{W}/\overline{c}, \overline{X}/\overline{p})$ and $\tau^0 f = \tau^y(\overline{W}/\overline{c}, \overline{X}/\overline{p})$. Let $X^{s_1}_{p_1}, \ldots, X^{s_j}_{p_j}$ be the variables occurring in G^0 and τ^0, $f = \{X^{s_1}_{l_1}/l_1, \ldots, X^{s_j}_{l_j}/l_j\}$ and $\kappa = \{X^{s_1}_{i_1}/\mu^1, \ldots, X^{s_j}_{i_j}/\mu_j\}$. Let $\mu^s_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq l_i$ and let the first q variables in f and κ be $X^{s_1}_{p_1}, \ldots, X^{s_q}_{p_q}$. Then:

$$G_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq G^0_{\mathfrak{A}}(X^{s_1}_{i_1}/l_1, \ldots, X^{s_j}_{i_j}/l_j) \simeq 0,$$
$$\tau_{\mathfrak{A}}(X_1/d_1, \ldots, X_{q'}/d_{q'}) \simeq \tau^0_{\mathfrak{A}}(X^{s_1}_{i_1}/l_1, \ldots, X^{s_j}_{i_j}/l_j) \simeq t.$$

Hence $\Sigma_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq 0$ and $\tau^v_{\mathfrak{A}}(\overline{W}/\overline{c}, \overline{X}/\overline{l}) \simeq t$, where $\overline{X} = (X^{s_1}_{p_1}, \ldots, X^{s_q}_{p_q})$. It follows from $\langle v, y\rangle \in W_1$ that $H_1 \cap K^* = \emptyset$, $y \in K^*$ and E_v is consistent. This, together with Lemma 3.5, implies $\delta \Vdash F_e(y)$ and $\alpha_\delta(y) \simeq t$, which proves the proposition. □

From the previous considerations follows:

Theorem 4.2. *If D is $\forall$-weak-admissible, then D is LP-definable.*

It is interesting to note that LP-definability implies $\forall$-weak-admissibility, i. e. the classes of LP-definable and $\forall$-weak-admissible sets coincide. The interested reader is referred to [4], where the proof of this fact is given in the case where the searching in the domain of the structure is allowed.

5. PROGRAMMING LANGUAGES

In this section we shall consider the computational power and shall prove the transitivity of the new semantics. Consider the first order language $\mathfrak{L} = (c_1, \ldots, c_r; f_1, \ldots, f_n; T_0, \ldots, T_k; S_1, \ldots, S_m)$. Let $\mathfrak{K}$ be the class of all structures corresponding to $\mathfrak{L}$ such that

$$\mathfrak{A} \in \mathfrak{K} \Leftrightarrow \mathfrak{A} = (B; t_1, \ldots, t_r; \theta_1, \ldots, \theta_n; \Sigma_0, \ldots, \Sigma_k; A_1, \ldots, A_m)$$

and Σ_j be true whenever be defined, $1 \leq j \leq n$, $\Sigma_0 = \lambda s.true$ and A_s are subsets of B.

A *programming language* on $\mathfrak{K}$ (see [4]) is an ordered triple $L = \langle \mathfrak{D}, \rho, \mathfrak{S}\rangle$, where $\mathfrak{D}$ is a denumerable set of objects – the syntactic descriptions of the programs of L, ρ – the arity function – is a mapping of $\mathfrak{D}$ into $N \setminus \{0\}$, and $\mathfrak{S}$ – the semantics of the programs in L – is a mapping of $\mathfrak{D} \times \mathfrak{K}$ such that if $d \in \mathfrak{D}$ and $\mathfrak{A} \in \mathfrak{K}$, then $\mathfrak{S}(d, \mathfrak{A})$ is equal to the object computable by means of the program d on the structure $\mathfrak{A}$. This object is typically a partial function or a set. Here we shall suppose that $\mathfrak{S}(d, \mathfrak{A})$ is a subset of $|\mathfrak{A}|^{\rho(d)}$ (by $|\mathfrak{A}|$ we denote the universe of the structure $\mathfrak{A}$).

There are at least two natural conditions that should satisfy each programming language L on $\mathfrak{K}$, cf. [4].

First of all, it should be effective in some sense. A language L is called *effective* if for all $p \in \mathfrak{D}$ there exists an enumeration operator Γ such that, for all $\mathfrak{B} \in \mathfrak{K}$ for which $|\mathfrak{B}| = N$, it holds $\Gamma(D'(\mathfrak{B})) = \mathfrak{S}(p, \mathfrak{B})$, where $D'(\mathfrak{B}) = D(\mathfrak{B}) \cup \{\langle n+k+m+i, t_i\rangle : 1 \leq i \leq r\}$.

The second condition is related to the implementation independence of L.

Let $\mathfrak{A}_i = (B_i; t^i{}_1, \ldots, t^i{}_r; \theta^i{}_1, \ldots, \theta^i{}_n; \Sigma^i{}_0, \ldots, \Sigma^i{}_k; A^i{}_1, \ldots, A^i{}_m) \in \mathfrak{K}$, $i = 1, 2$. A surjective mapping κ of B_1 onto B_2 is called a *strong homomorphism* iff the following conditions are true:

(i) $\kappa(t^1_i) \simeq t^2_i$, $1 \leq i \leq r$;

(ii) $\theta^2_i(\kappa(s)) \simeq \kappa(\theta^1_i(s))$ for each $s \in B$, $1 \leq i \leq n$;

(iii) $\Sigma_j^2(\kappa(s)) \simeq \Sigma_j^1(s)$ for each $s \in B$, $1 \le j \le k$;

(iv) $\kappa(A_s^1) \equiv A_s^2$, $1 \le s \le m$.

The language L is called *invariant* if for all structures $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$, such that there exists a strong homomorphism κ from $|\mathfrak{A}|$ onto $|\mathfrak{B}|$, and for all $p \in \mathfrak{D}$, $\kappa(\mathfrak{S}(p, \mathfrak{A})) \equiv \mathfrak{S}(p, \mathfrak{B})$.

We introduce an extra third condition, which is related to the fact that searching of the domain of the structure is not allowed in our semantics. This condition means that the programs use no external information about the structure, in other words, they ask only questions concerning the parameters during the execution. That is why the structures of the next definition have the same parameters.

Let $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{K}$. We shall call that $\mathfrak{A}_1$ is a *substructure* of $\mathfrak{A}_2$ (we denote $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$) if:

(i) $B_1 \subseteq B_2$;

(ii) $t_i^1 = t_i^2$, $1 \le i \le r$;

(iii) $A_s^1 \equiv A_s^2$, $1 \le s \le m$;

(iv) $\theta_i^1(t) \simeq \theta_i^2(t)$ for all $t \in B_1$, $1 \le i \le n$;

(v) $\Sigma_j^1(t) \simeq \Sigma_j^2(t)$ for all $t \in B_1$, $1 \le j \le k$.

We say that the language L has a *substructure property* if for all $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathfrak{K}$, such that $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$, $\mathfrak{S}(p, \mathfrak{A}_1) \equiv \mathfrak{S}(p, \mathfrak{A}_2)$.

Consider two programming languages $L = \langle \mathfrak{D}, \rho, \mathfrak{S} \rangle$ and $L' = \langle \mathfrak{D}', \rho', \mathfrak{S}' \rangle$. Let $\mathfrak{D}$ be the set of the ordered pairs $\langle P, H \rangle$, where P is a logic program and H is an unary predicate symbol. Let $\mathfrak{D}'$ be the set of the ordered pairs $\langle F, H \rangle$, where F is an arbitrary first order formula. Let ρ and ρ' be the constant 1. Let K, $\mathfrak{L}_K$, $\mathfrak{T}_K$ and $\partial^C(\mathfrak{A})$ for $\mathfrak{A} \in \mathfrak{K}$ be the same as in the previous section. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be defined as follows:

$$t \in \mathfrak{P}(\langle P, H \rangle, \mathfrak{A}) \Leftrightarrow \exists \tau (\tau \in \mathfrak{T}_K \;\&\; \partial^C(\mathfrak{A}) \cup P \vdash H(\tau) \;\&\; \tau_{\mathfrak{A}^*} \simeq t),$$
$$t \in \mathfrak{P}'(\langle F, H \rangle, \mathfrak{A}) \Leftrightarrow \exists \tau (\tau \in \mathfrak{T}_K \;\&\; \partial^C(\mathfrak{A}) \cup F \vdash H(\tau) \;\&\; \tau_{\mathfrak{A}^*} \simeq t).$$

Let $\mathfrak{S}'$ coincide with $\mathfrak{P}'$, and $\mathfrak{S}$ – with $\mathfrak{P}$.

It is easy to prove that the languages L and L' are effective, invariant and have the substructure property.

Now we shall prove that the language L is maximal among the effective, invariant languages with substructure property, i. e. every set, computable by a language with these properties, is also computable by L.

We say that the language $L_1 = \langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1 \rangle$ is translatable into the language $L_2 = \langle \mathfrak{D}_2, \rho_2, \mathfrak{S}_2 \rangle$ (see [4]) (we denote $L_1 \le_{\mathfrak{K}} L_2$) iff

$$\forall p_1 \in \mathfrak{D}_1 \exists p_2 \in \mathfrak{D}_2 ((\rho_1(p_1) = \rho_2(p_2)) \;\&\; \forall \mathfrak{A} \in \mathfrak{K} (\mathfrak{S}_1(p_1, \mathfrak{A}) \equiv \mathfrak{S}_2(p_2, \mathfrak{A}))).$$

Theorem 5.3. *Let $L_1 = \langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1 \rangle$ be an arbitrary programming language on $\mathfrak{K}$, which is effective, invariant and has substructure property. Then $L_1 \le_{\mathfrak{K}} L$.*

Proof. Let $p_1 \in \mathfrak{D}_1$. Consider an arbitrary structure $\mathfrak{A} \in \mathfrak{K}$. Let $\langle \alpha, \mathfrak{B} \rangle$ be its enumeration and let $\mathfrak{A} = (B; t_1, \ldots, t_r; \theta_1, \ldots, \theta_n; \Sigma_0, \ldots, \Sigma_k; A_1, \ldots, A_m)$, $\mathfrak{B} = (N; x_1, \ldots, x_r; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ and $\alpha(x_i) \simeq t_i$, $1 \leq i \leq r$. Then $\mathfrak{B} \in \mathfrak{K}$, and if φ_i^* and σ_j^* are the restrictions of φ_i and σ_j on $Dom(\alpha)$, respectively, then α is a strong homomorphism from $\mathfrak{B} = (N; x_1, \ldots, x_r; \varphi_1^*, \ldots, \varphi_n^*; \sigma^*{}_0, \ldots, \sigma^*{}_k; \xi_1, \ldots, \xi_m)$ onto $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq \mathfrak{B}$ ($Dom(\alpha)$ is closed with respect to $\varphi_1, \ldots, \varphi_n$). Due to the properties of L_1, we obtain $\mathfrak{S}_1(p_1, \mathfrak{A}) = \alpha(\mathfrak{S}_1(p_1, \mathfrak{B}^*))$ $= \alpha(\mathfrak{S}_1(p_1, \mathfrak{B})) = \alpha(\Gamma_{p_1}(D'_{\mathfrak{B}}))$ and $\Gamma_{p_1}(D'_{\mathfrak{B}}) \subseteq Dom(\alpha)$, i. e. for every $\mathfrak{A} \in \mathfrak{K}$ and for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ it is true that

$$\mathfrak{S}_1(p_1, \mathfrak{A}) \equiv \alpha(\Gamma_{p_1}(D'(\mathfrak{B}))), \tag{5.1}$$

$$\Gamma_{p_1}(D'(\mathfrak{B}) \subseteq Dom(\alpha). \tag{5.2}$$

Let us fix natural numbers $w_1, \ldots, w_r$. Let e_1 be the number of Γ_{p_1} and let

$$\begin{aligned} W = \; & \{\langle x, v\rangle | \exists v' \exists x (\langle x, v'\rangle \in W_{e_1} \text{ and } v \text{ be the code of the set } E_v, \\ & \text{obtained from } E_{v'} \text{ by removing elements} \\ & \text{of the form } \langle n+k+m+i, w_i\rangle, 1 \leq i \leq r)\}. \end{aligned}$$

This set is r.e. Let e be its Gödel code and let us fix an arbitrary $\mathfrak{A} \in \mathfrak{K}$. Consider a finite part Δ of $\mathfrak{A}$ such that $H_1 = \xi_1' = \ldots = \xi_m' = \emptyset$, $\varphi_1' = \ldots = \varphi_n' = \emptyset$, $\sigma_1' = \ldots = \sigma_k' = \emptyset$, and $\alpha_1(w_i) \simeq (c_i)_{\mathfrak{A}}$. Let $\delta \supseteq \Delta$ and $\delta \Vdash F_e(y)$. Then there exists $\langle v, y\rangle \in W_e$ such that $\delta \Vdash E_v$. For all $\langle \alpha, \mathfrak{B} \rangle \supseteq \delta$ it is true that $\Gamma_{p_1}(D'(\mathfrak{B})) \equiv \Gamma_e(D(\mathfrak{B}))$ and $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$ (there exists at least one such enumeration). Then $y \in \Gamma_{p_1}(D'(\mathfrak{B})) \Leftrightarrow y \in \Gamma_e(D(\mathfrak{B}))$ and from (5.2) it follows that $y \in Dom(\alpha)$, i. e. $y \notin H_\delta$. We obtained that for every $\mathfrak{A} \in \mathfrak{K}$, the finite part Δ, constructed above, and e are compatible. Consider the definition of the consistent set E_v for the fixed Δ. It can be seen that the consistency of E_v depends only on $w_1, \ldots, w_r$. The same is true for K^* and P. Then the set

$$\begin{aligned} W = \; & \{\langle x, v\rangle | \langle x, v\rangle \in W_e \text{ and } E_v \text{ is consistent} \\ & \& \; K_* \cap H_1 = \emptyset \; \& \; y \in K_*\} \end{aligned}$$

depends only on $w_1, \ldots, w_r$. Consider the program $\langle P', F\rangle$ from Proposition 4.13. It is true that $\forall \mathfrak{A} \in (\mathfrak{S}(\langle P', F\rangle, \mathfrak{A}) = D_{\mathfrak{A}})$, where $D_{\mathfrak{A}}$ is sufficient in $\mathfrak{A}$ for e and Δ, i. e. the condition $t \in D_{\mathfrak{A}} \Leftrightarrow \exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \; \& \; \delta \Vdash F_e(y))$ is true.

We shall prove that $D_{\mathfrak{A}} = \mathfrak{S}_1(p_1, \mathfrak{A})$ for every $\mathfrak{A} \in \mathfrak{K}$. Let $s \in \mathfrak{S}_1(p_1, \mathfrak{A})$ and let fix an enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $\mathfrak{B} = (N; w_1, \ldots, w_r; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)$ and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$. Then there exists y such that $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$ and $\alpha(y) \simeq s$. Hence there exists $\delta \supseteq \Delta$ and $\delta \Vdash F_e(y)$ and $\alpha_\delta(y) \simeq s$. It follows that $s \in D_{\mathfrak{A}}$.

Now let $t \in D_{\mathfrak{A}}$, then $\exists \delta \supseteq \Delta \exists y (\alpha_\delta(y) \simeq t \; \& \; \delta \Vdash F_e(y))$. Let fix an arbitrary enumeration $\langle \alpha, \mathfrak{B} = (N; w_1, \ldots, w_r; \varphi_1, \ldots, \varphi_n; \sigma_0, \ldots, \sigma_k; \xi_1, \ldots, \xi_m)\rangle$ of $\mathfrak{A}$ such that $\langle \alpha, \mathfrak{B} \rangle \supseteq \delta$, then $\Gamma_{p_1}(D'(\mathfrak{B})) \equiv \Gamma_e(D(\mathfrak{B}))$ and due to (5.2) and the fact that $\delta \Vdash F_e(y)$, $\langle \alpha, \mathfrak{B} \rangle \models F_e(y)$ and $\alpha(y) \simeq t$, it follows that $t \in \mathfrak{S}_1(p_1, \mathfrak{A})$.

Finally, we obtain that $D_{\mathfrak{A}} = \mathfrak{S}_1(p_1, \mathfrak{A})$. □

From the theorem it follows that $L' \leq_{\mathfrak{K}} L$. This means that the Horn clause programs are at least as strong as any other language using arbitrary first order formulas as programs. For $m = 0$ we obtain the same result for logic programs without parameters.

Let $\langle P_0, H_0 \rangle \in \mathfrak{D}$ and for every $\mathfrak{A} \in \mathfrak{K}$ we denote $W_{\mathfrak{A}} = \mathfrak{S}(\langle P_0, H_0 \rangle, \mathfrak{A})$.

Proposition 5.14. *For every program $\langle P, H \rangle$ there exists a program $\langle Q, R \rangle$ such that for every $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}})) \equiv \mathfrak{S}(\langle Q, R \rangle, \mathfrak{A})$, where $(\mathfrak{A}, W_{\mathfrak{A}})$ is a structure obtained from $\mathfrak{A}$ by adding the parameter $W_{\mathfrak{A}}$.*

Proof. Let $L^* = \langle \mathfrak{D}^*, \rho^*, \mathfrak{S}^* \rangle$ be a new programming language, where $\mathfrak{D}^* \equiv \mathfrak{D}$; $\rho^* \equiv \rho$ and $\mathfrak{S}^*(\langle P, H \rangle, \mathfrak{A}) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}}))$. We shall show that L^* is effective, invariant and has the substructure property.

Effectiveness follows from the effectiveness of L and the fact that enumeration operators are closed with respect to composition.

Let κ be a strong homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. Then $\kappa(W_{\mathfrak{A}}) \equiv W_{\mathfrak{B}}$ and from the invariance of L it follows that

$$\kappa(\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}}))) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{B}, W_{\mathfrak{B}})),$$

i. e. L^* is invariant.

Let $\mathfrak{A} \subseteq \mathfrak{B}$. L has the substructure property, hence $W_{\mathfrak{A}} \equiv W_{\mathfrak{B}}$ and

$$\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W_{\mathfrak{A}})) \equiv \mathfrak{P}(\langle P, H \rangle, (\mathfrak{B}, W_{\mathfrak{B}})),$$

i. e. L^* has the substructure property.

Now applying Teorem 5.3 to L^*, we obtain the proposition. □

6. HORN CLAUSE OPERATORS

Let $\mathfrak{A} \in \mathfrak{K}$ and let $\langle P, H \rangle$ be a Horn clause program, where H is an unary predicate. We define a mapping $\Gamma_{P,H}$ from the subsets of $|\mathfrak{A}|$ onto the subsets of $|\mathfrak{A}|$ by

$$\Gamma_{P,H}(W) = \mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W)).$$

It follows from the definition of $\mathfrak{P}$ that the operator $\Gamma_{P,H}$ is compact, i. e.

$$s \in \Gamma_{P,H}(W) \Leftrightarrow \exists D(D \subseteq W \ \&\ D \text{ is finite } \&\ s \in \Gamma_{P,H}(D)).$$

Applying the Knaster–Tarski theorem, we obtain that $\Gamma_{P,H}$ has a least fixed point $W_0 = \bigcup_{k=0}^{\infty} \Gamma^k_{P,H}(\emptyset)$. We denote this fixed point by

$$\mu W.\mathfrak{P}(\langle P, H \rangle, (\mathfrak{A}, W)).$$

Now we shall show that the least fixed point of each Horn clause operator is computable by means of Horn clause programs. In fact, we even have

Theorem 6.4. *For each Horn clause program $\langle P, H\rangle$ there exists a Horn clause program $\langle P^*, H^*\rangle$ such that for all $\mathfrak{A} \in \mathfrak{K}$*

$$\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W)) = \mathfrak{P}(\langle P^*, H^*\rangle, \mathfrak{A}).$$

Proof. Let L_1 be the programming language $\langle \mathfrak{D}_1, \rho_1, \mathfrak{S}_1\rangle$ on $\mathfrak{K}$, where $\mathfrak{D}_1 \equiv \mathfrak{D}$, $\rho_1 \equiv \rho$ and $\mathfrak{S}_1(\langle P, H\rangle, \mathfrak{A}) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W))$. To prove the theorem, it is sufficient to show that $L_1 \leq_{\mathfrak{K}} L$.

We shall prove that L_1 is effective, invariant and has the substructure property. Indeed, the effectiveness of L_1 follows from the uniform version of the First recursion theorem for enumeration operators. To prove the invariance of L_1, suppose that $\langle P, H\rangle \in \mathfrak{D}_1$, let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$ and κ be a strong homomorphism form $\mathfrak{A}$ to $\mathfrak{B}$. Let us define the sequences of sets $W^n_{\mathfrak{A}}$ and $W^n_{\mathfrak{B}}$ in the following way:

$$W^0_{\mathfrak{A}} = W^0_{\mathfrak{B}} = \emptyset,$$
$$W^{n+1}_{\mathfrak{A}} = \mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W^n_{\mathfrak{A}})) \text{ and } W^{n+1}_{\mathfrak{B}} = \mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W^n_{\mathfrak{B}})).$$

Now using the invariance of L, we obtain by induction on n that $\kappa(W^n_{\mathfrak{A}}) = W^n_{\mathfrak{B}}$, $n = 0, 1, \ldots$ Hence,

$$\kappa(\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W))) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W)).$$

By this the invariance of L_1 is proved.

Now let $\mathfrak{A} \subseteq \mathfrak{B}$. Using the above sequences and the substructure property of L, we obtain by induction on n that $W^n_{\mathfrak{A}} = W^n_{\mathfrak{B}}$, $n = 0, 1, \ldots$ Hence,

$$\mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{A}, W)) = \mu W.\mathfrak{P}(\langle P, H\rangle, (\mathfrak{B}, W)).$$

We obtained that L_1 is invariant, effective and has the substructure property. Applying Theorem 5.3 to L_1, we prove that $L_1 \leq_{\mathfrak{K}} L$. □

Acknowledgments. The author would like to thank Prof. Soskov for introducing her to the matter and for his inestimable help, and also the anonymous referee for his or her thoughtful remarks on the representation of the paper.

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Received on March 1, 2002

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: vbaleva@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

GENERICITY IN ABSTRACT STRUCTURE DEGREES

VERA BOUTCHKOVA

The generalized notion of genericity in the theory of abstract structure degrees is used to obtain a characterization of abstractly generic predicate of natural numbers as the preimage of some predicate of the denumerable set N and generic regular enumeration.

Keywords: genericity, enumerations
2000 MSC: 03D75

1. INTRODUCTION

In this paper we deal with *search-computability,* defined by Moschovakis in [2], though for the proofs of most of the propositions we have used the Skordev's definition of search-computability, in [3] Skordev has proved both are equivalent.

The idea of considering two-sort structures was presented by I. N. Soskov during the cycle of lectures at the Seminar on Computability Theory at Sofia University, 1998. The abstract structure degrees were defined also by him during the same seminar, as well as their regular enumerations.

The first sort of the mentioned two-sort abstract structures is an arbitrary denumerable set and the other one is the set of natural numbers. The presence of the equality among the basic predicates of the structure is required.

In these terms we present an analogue of some notions from the theory of the enumeration degrees, namely the *set genericity* and the related results, applying the techniques used by Copestake in [1]. We generalize the characterization obtained in [6], stating that a set of natural numbers is generic relatively a set B if and only if it is the preimage of some set A, using a B-generic B-regular enumeration such that both A and its complement are e-reducible to B.

Here we introduce the notion of *genericity* for abstract predicates. Using the enumerations of two-sort abstract structures (in the way they are used in [4]), we obtain a characterization of this type of abstract genericity, which claims that a predicate A of natural numbers is generic relatively the two-sort abstract structure $\mathfrak{B}$ with one predicate if and only if there exist a predicate Σ on the first sort, which is search computable in $\mathfrak{B}$, and a $\mathfrak{B}$-generic regular enumeration f, such that $A = f_N^{-1}(\Sigma)$.

This paper is a part of the author's Master's Thesis, supervised by I. Soskov.

2. PRELIMINARIES

We use some standard definitions and notations: $\leq_e$ denotes the enumeration reducibility between sets and Ψ_e denotes the e-th enumeration operator, i.e. $\Psi_e(B) = \{x \mid \exists v(\langle x, v\rangle \in W_e \,\&\, D_v \subseteq B)\}$, where W_e is the recursively enumerable set with Gödel code e, B is a set of natural numbers and D_v is the finite set with code v. Recall the *join* operation for sets of naturals: $A \oplus B$ is the set $\{2x | x \in A\} \cup \{2x+1 | x \in B\}$, used to induce the least upper bound of the e-degrees of A and B.

Given a countable set N and $0^* \notin N$, N^* denotes the *Moschovakis' extension* of N, i.e. the smallest extension of $N \cup \{0^*\}$ closed under the operation *ordered pair* $\langle \cdot, \cdot \rangle$ (we will use the same notation for effective coding of pairs of natural numbers); ω denotes the set of the natural numbers and $\omega^* \subseteq N^*$ is the set of elements $0^*, \ldots, (n+1)^*, \ldots$, such that $(n+1)^* = \langle 0^*, n^* \rangle \in \omega^*$. By $\mathcal{F}$ we denote the set of one-argument partial functions $\varphi : N^* \multimap N^*$. We write $\varphi \in \mathbf{SC}(\varphi_1, \ldots, \varphi_n)$ to say that φ is *search computable* in the set of functions $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{F}$ (see [3]).

From now on, we consider the abstract partial two-sort structures:

$$\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \ldots, \Sigma_k \rangle,$$

with two fixed basic predicates in N^2: $=_N$ (equality) and $\neq_N$ (inequality), and partial predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$ such that $a_i, b_i \geq 0$, but not both being zero. This kind of structures will be denoted by $\mathfrak{A}(\Sigma_1, \ldots, \Sigma_k)$.

The notation $\Sigma_0 \leq_{\mathrm{SC}} \mathfrak{A}$ says that Σ_0 is search computable in the set of $\mathfrak{A}$'s predicates, including the equality and inequality, i.e. $\widehat{\Sigma}_0 \in \mathbf{SC}(\widehat{\Sigma}_1^{\mathfrak{A}}, \ldots, \widehat{\Sigma}_k^{\mathfrak{A}}, \widehat{=}_N, \widehat{\neq}_N)$ (we also write $\widehat{\Sigma}_0 \in \mathbf{SC}(\mathfrak{A})$), where $\widehat{\Sigma} : N^* \multimap N^*$ is the *semi-characteristic* function of the predicate.

Soskov has defined $\mathfrak{A} \oplus \mathfrak{B}$ to be the two-sort structure with predicates $=_N$, $\neq_N$, $\Sigma_1^{\mathfrak{A}}, \ldots, \Sigma_{k_{\mathfrak{A}}}^{\mathfrak{A}}, \Sigma_1^{\mathfrak{B}}, \ldots, \Sigma_{k_{\mathfrak{B}}}^{\mathfrak{B}}$; $\mathfrak{A} \leq_{\mathrm{SC}} \mathfrak{B}$ if and only if $\forall i_{(1 \leq i \leq k_{\mathfrak{A}})}$: $\Sigma_i^{\mathfrak{A}} \leq_{\mathrm{SC}} \mathfrak{B}$, and $\mathfrak{A} \equiv_{\mathrm{SC}} \mathfrak{B}$ if and only if $\mathfrak{A} \leq_{\mathrm{SC}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathrm{SC}} \mathfrak{A}$.

Definition 2.1 (Soskov). The *abstract structure degrees* are the equivalence classes induced by the relation $\equiv_{\mathrm{SC}}$ between structures. We denote them by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$... and for every $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{D}$, $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{D}_s(\mathfrak{A} \oplus \mathfrak{B})$ for some $\mathfrak{A} \in \mathfrak{a}$ and $\mathfrak{B} \in \mathfrak{b}$.

We write $\mathfrak{D}$ for the set of all abstract structure degrees with the partial ordering induced by $\leq_{SC}$. Thus the structure $\langle \mathfrak{D}, \leq_{SC}, \cup, \mathfrak{O} \rangle$ is an upper semi-lattice with a least element the empty structure $\mathfrak{O} = \langle N; \omega; =_N, \neq_N \rangle$.

At the Seminar on Computability Theory in 1998, I. Soskov introduced the following definition of search computability and proved its equivalence with the standard ones (see [2] and [3]):

$$\mathfrak{A} \leq_{SC} \mathfrak{B} \text{ iff } \forall \alpha (\mathfrak{B} \leq \alpha \Rightarrow \mathfrak{A} \leq \alpha),$$

where $\alpha = (f, R)$ is an enumeration structure and $\mathfrak{A} \leq \alpha$ if and only if $f^{-1}(\mathfrak{A}) \leq_e R$. Here we shall use it for a single predicate $\Sigma \subseteq N^{a_i} \times \omega^{b_i}$ in the following particular form:

$$\Sigma \leq_{SC} \mathfrak{A} \text{ iff } \left(f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A}), \text{ for every } N\text{-enumeration } f_N \right), \tag{2.1}$$

where $f_N : \omega \to N$ is a total and surjective function that we shall call *N-enumeration*, $f_N^{-1}(\Sigma) = \{\langle x_1, \ldots, x_a, y_1, \ldots, y_b \rangle \in \omega \mid (f_N(x_1), \ldots, f_N(x_a), y_1, \ldots, y_b) \in \Sigma\}$, and for the structure $\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \ldots, \Sigma_k \rangle$ the preimage $f_N^{-1}(\mathfrak{A})$ is defined in such a way that it is e-equivalent to

$$f_N^{-1}(\Sigma_1) \oplus, \ldots, \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(=_N) \oplus f_N^{-1}(\neq_N).$$

3. ENUMERATIONS

Many of the definitions and the proofs from [4] concerning the enumeration approach and the normal form theorem are applicable in our case. We recall them in order to use them later in Section 4 and for the characterization in Section 5.

Definition 3.1. 1) N-string τ_N is a finite function $\tau_N : [0, \ldots, n-1] \to N$, with domain an initial segment of ω with length $lh(\tau_N) = n$.

We shall call the strings used in [6] *ω-strings*, i.e. an ω-string is a finite sequence of naturals meant to be an initial segment of ω.

2) $\tau_N \subseteq \sigma_N$ iff $\forall x (x < lh(\tau_N) \Rightarrow \tau_N(x) = \sigma_N(x))$.

3) Code of the N-string τ_N is defined to be $\ulcorner \tau_N \urcorner = \langle n^*, \tau_N(0), \ldots, \tau_N(n-1) \rangle$.

Definition 3.2 ([4]). For a structure $\mathfrak{A}(\Sigma_1, \ldots, \Sigma_k)$ with $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, an N-string τ_N and a formula $F_e(z)$ with $e, z \in \omega$, define the *forcing relation* $\tau_n \Vdash_{\mathfrak{A}} F_e(z)$ as follows:

(1) $\tau_N \Vdash_{\mathfrak{A}} F_e(z)$ iff $\exists v (\langle v, z \rangle \in W_e \;\&\; \tau_n \Vdash_{\mathfrak{A}} D_v)$;

(2) $\tau_N \Vdash_{\mathfrak{A}} D_v$ iff $\forall u \in D_v (u = \langle i, \langle x_1, \ldots, x_{a_i}, y_1, \ldots, y_{b_i} \rangle \rangle \;\&$ $1 \leq i \leq k \;\&\; x_1 \ldots x_{a_i} \in Dom(\tau_N) \;\&(\tau_N(x_1) \ldots \tau_N(x_{a_i}), y_1, \ldots, y_{b_i}) \in \Sigma_i \vee \; u = \langle 0, 2\langle x, y \rangle \rangle \;\&\; x, y \in Dom(\tau_N) \;\&\; \tau_N(x) = \tau_N(y) \;\&\; u = \langle 0, 2\langle x, y \rangle + 1 \rangle \;\&\; x, y \in Dom(\tau_N) \;\&\; \tau_N(x) \neq_N \tau_N(y))$.

Definition 3.3 ([4]). For an N-enumeration $f_N : \omega \to N$ and a structure $\mathfrak{A}(\Sigma_1, \ldots, \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, define

$$f_N \models_{\mathfrak{A}} F_e(z) \text{ if and only if } z \in \Psi_e(f_N^{-1}(\mathfrak{A})).$$

Definition 3.4 ([4]). We say that the predicate $\Sigma \subseteq N^a \times \omega^b$ has *a normal form* in the structure $\mathfrak{A}(\Sigma_1, \ldots, \Sigma_k)$ if there exist $e \in \omega$, an N-string δ_N and $x_1, \ldots, x_a \notin Dom(\delta_N)$ such that for all $s_1, \ldots, s_a \in N$ and for all $y_1, \ldots, y_b \in \omega$, $(s_1, \ldots, s_a, y_1, \ldots, y_b) \in \Sigma$ iff $\exists \tau_N \supseteq \delta_N$ such that $\forall_{1 \leq i \leq a} (\tau_N(x_i) = s_i)$ & $\tau_N \Vdash_{\mathfrak{A}} F_e(\langle x_1, \ldots, x_a, y_1, \ldots, y_b \rangle)$.

The next theorem is a corollary from the Normal Form Theorem in [4] for the case of two-sort structures.

Theorem 3.1 (Normal Form Theorem). *Let $\mathfrak{A}(\Sigma_1, \ldots, \Sigma_k)$ be a structure with predicates $\Sigma_i \leq N^{a_i} \times \omega^{b_i}$. Then every predicate $\Sigma \subseteq N^a \times \omega^b$, where Σ is search computable in $\mathfrak{A}$, has a normal form in $\mathfrak{A}$.*

4. GENERIC PREDICATES

Definition 4.1. 1) Let $\Sigma \subseteq N^a \times \omega^b$ be a predicate. We define the *characteristic function* of Σ to be the function $\chi_\Sigma : N^* \multimap N^*$, defined as follows:

$$\chi_\Sigma(s) = \begin{cases} 0^*, & \text{if } s = \langle s_1, \ldots, s_a, x_1^*, \ldots, x_b^* \rangle \ \& \ (s_1, \ldots, s_a, x_1, \ldots, x_b) \in \Sigma, \\ 1^*, & \text{if } s = \langle s_1, \ldots, s_a, x_1^*, \ldots, x_b^* \rangle \ \& \ (s_1, \ldots, s_a, x_1, \ldots, x_b) \notin \Sigma, \\ \uparrow, & \text{otherwise.} \end{cases}$$

2) Let $\mathcal{F}_{a,b}$, where $a + b \geq 1$, be the set of all partial functions $\varphi \in \mathcal{F}$ such that $Dom(\varphi) \subseteq \{\langle s_1, \ldots, s_a, x_1^*, \ldots, x_b^* \rangle \mid (s_1, \ldots, s_a, x_1, \ldots, x_b) \in N^a \times \omega^b\}$ and $Range(\varphi) \subseteq \omega^*$.

3) Define (a, b)-string to be a finite function $\alpha \in \mathcal{F}_{a,b}$ with $Range(\alpha) \subseteq \{0^*, 1^*\}$. We may define the code of the (a, b)-string α (denote $\ulcorner\alpha\urcorner$) to be $\langle k^*, \langle s_1, \alpha(s_1) \rangle, \ldots, \langle s_k, \alpha(s_k) \rangle \rangle \in N^*$ if $Dom(\alpha) = \{s_1, \ldots, s_k\}$; and $\ulcorner\varnothing\urcorner = 0^*$ for the empty function.

Remark. Since an (a, b)-string may have more than one (but only finitely many) different codes, by $\alpha \in S^* \subseteq N$ we mean that there exists a code of α, which belongs to the set S^*; respectively, $\alpha \notin S^*$ means there is no code of α that belongs to the set. We say that S^* is a set of codes of (a, b)-strings when each element is a code of some (a, b)-string, it is not necessary for S^* to contain all the codes of an (a, b)-string.

4) *Semi-characteristic function* of the set $S^* \subseteq N$ we call the function $C_{S^*} : N^* \multimap N^*$, defined as follows:

$$C_{S^*}(s) \cong \begin{cases} 0^*, & \text{if } s \in S^*, \\ \uparrow, & \text{otherwise.} \end{cases}$$

For a given set $S^* \subseteq N^*$ and structure $\mathfrak{B}(\Sigma_1, \dots, \Sigma_k)$, we write $S^* \in \mathbf{SC}(\mathfrak{B})$ when $C_{S^*} \in \mathbf{SC}(\hat{=}_N, \hat{\neq}_N, \hat{\Sigma}_1, \dots, \hat{\Sigma}_k)$.

5) For every a and b, which are not both zero, and every function $\varphi \in \mathcal{F}_{a,b}$, we define the *graph-predicate* of φ to be the predicate $\Sigma_\varphi \subseteq N^a \times \omega^{b+1}$ such that for all $s_1, \dots, s_a \in N$ and $x_1, \dots, x_b, y \in \omega$, $(s_1, \dots, s_a, x_1, \dots, x_b, y) \in \Sigma_\varphi$ iff $\varphi(\langle s_1, \dots, s_a, x_1^*, \dots, x_b^* \rangle) = y^*$.

Definition 4.2. Given a structure $\mathfrak{B}(\Sigma_1, \dots, \Sigma_k)$, we say that the predicate $\Sigma \subseteq N^a \times \omega^b$ is $\mathfrak{B}$-*generic* if for every set $S^* \subseteq N^*$ of codes of (a, b)-strings such that $S^* \in \mathbf{SC}(\mathfrak{B})$, the following holds:

$$\exists \alpha \subseteq \chi_\Sigma \big(\alpha \in S^* \vee \forall \beta \supseteq \alpha (\beta \notin S^*)\big).$$

Note. If we consider a structure $\mathfrak{B}(B)$ with one predicate of naturals and a predicate $\Sigma \subseteq \omega$, then Σ is $\mathfrak{B}$-generic in the sense of Definition 4.2 if and only if the set Σ is B-generic set of natural numbers in the classical sense. The proof uses the definition of $\mathbf{SC}$ via enumerations (2.1).

Proposition 4.1. *For every structure $\mathfrak{B} = \mathfrak{B}(\Sigma_1, \dots, \Sigma_k)$ and $a, b \in \omega$, such that $a + b \geq 1$, there exists a $\mathfrak{B}$-generic predicate $\Sigma \subseteq N^a \times \omega^b$.*

Proof. For brevity, in this proof and from now on $\overline{x}$ will denote a finite sequence of elements (an appropriate number of them).

We can find such Σ by building its characteristic function as a union of (a, b)-strings, that we build at stages, such that at even stages we satisfy the requirements $Dom(\chi_\Sigma)$ to be a domain of a predicate's characteristic function and at odd stages – the genericity.

Let us have some enumeration $S_0^*, \dots, S_n^*, \dots$ of the domains of the partial functions from $\mathbf{SC}(\mathfrak{B})$, i.e. $S_n^* = Dom(\varphi)$ for $\varphi \in \mathbf{SC}(\mathfrak{B})$.

Stage 0. Define $\alpha_0 = \varnothing$.

Stage $2n+1$. We have defined α_q for $q = 2n$. Let $\langle \overline{s}, \overline{x}^* \rangle \in N^*$ be such that $(\overline{s}, \overline{x})$ is *the least* according to some order in $N^a \times \omega^b$ element for which $\langle \overline{s}, \overline{x}^* \rangle \notin Dom(\alpha)$. Define α_{q+1} to extend α_q with one new argument, i.e. such that $\alpha_{q+1}(\langle s_1, \dots, s_a, x_1^*, \dots, x_b^* \rangle) = 0^*$.

Stage $2n+2$. We have defined α_q for $q = 2n + 1$.

Case 1. If there exists in S_n^* an (a, b)-string β, extending α_q, define α_{q+1} to be the first such β.

Case 2. Otherwise, define $\alpha_{q+1} = \alpha_q$.

Finally, we can define $\chi_\Sigma = \bigcup_{q=0}^{\infty} \alpha_q$, that is the characteristic function of some $\mathfrak{B}$-generic predicate. □

Proposition 4.2. *Let $\mathfrak{B}$ be an abstract structure and $\Sigma \subseteq N^a \times \omega^b$ be a $\mathfrak{B}$-generic predicate. Then the following holds:*

P1) *The predicate $\overline{\Sigma} \subseteq N^a \times \omega^b$ is $\mathfrak{B}$-generic.*

P2) *There is no infinite predicate* $C \subseteq N^a \times \omega^b$ *such that* $C \leq_{SC} \mathfrak{B}$ *and* $C \subseteq \Sigma$.
P3) Σ *is infinite.*
P4) $\Sigma \not\leq_{SC} \mathfrak{B}$.

Proof. Each of (P3) and (P4) follows directly from the previous properties. To prove (P1), we may assume it is false. Therefore there is a set of codes of (a, b)-strings, namely $P^* \in \mathbf{SC}(\mathfrak{B})$, such that:

(a) $\forall \alpha \subseteq \chi_{\overline{\Sigma}} (\alpha \notin P^* \ \& \ \exists \beta \supseteq \alpha(\beta \in P^*))$.

There is a recursive function translating (codes of) (a, b)-strings into their reverse, e.g. the reverse of α being the (a, b)-string $\overline{\alpha}$, such that $\forall s \in Dom(\alpha)$, $\alpha(x) = 0^*$ iff $\overline{\alpha}(x) = 1^*$. Thus the set $S^* = \{\alpha \mid \overline{\alpha} \in P^*\} \in \mathbf{SC}(\mathfrak{B})$ and therefore there exists an $\alpha \subseteq \chi_\Sigma$ (and therefore $\overline{\alpha} \subseteq \chi_{\overline{\Sigma}}$) such that the next (1) or (2) holds:

(1) $\alpha \in S^*$. Then $\overline{\alpha} \in P^*$ and $\overline{\alpha} \subseteq \chi_{\overline{\Sigma}}$, which is a contradiction with (a).

(2) $\forall \beta \supseteq \alpha \ (\beta \notin S^*)$. But from (a) for $\overline{\alpha}$ follows there exists an (a, b)-string $\beta \in P^*$ extending $\overline{\alpha}$. Since $\overline{\overline{\beta}} = \beta$, we have that $\beta \in S^*$ and $\beta \supseteq \alpha$, which is a contradiction.

In both cases we have found a contradiction, therefore Σ is $\mathfrak{B}$-generic.

To prove (P2), we may assume there exists such $C \subseteq N^a \times \omega^b$ and define a set $S^* = \{\alpha \mid \exists s_1, \ldots, s_a \in N, \exists y_1, \ldots, y_b \in \omega \ ((s_1, \ldots, s_a, y_1, \ldots, y_b) \in C \ \& \ \alpha(\langle s_1, \ldots, s_a, y_1^*, \ldots, y_b^* \rangle) = 1^*)\}$, that will lead to contradiction. □

Definition 4.3. Let us define the structure $\mathfrak{A}(\Sigma_1^{\mathfrak{A}}, \ldots, \Sigma_n^{\mathfrak{A}})$ to be total iff $\overline{\Sigma_i^{\mathfrak{A}}} \leq_{SC} \mathfrak{A}$ for $1 \leq i \leq n$. The generalization of the *quasi-minimal* and the *minimal-like* structure (see [1]) will have the following form:

1. $\mathfrak{A}$ is *quasi-minimal* over $\mathfrak{B}$ if the following two conditions hold:

- $\mathfrak{B} \leq_{SC} \mathfrak{A}$ and $\mathfrak{A} \not\leq_{SC} \mathfrak{B}$;
- For every total structure $\mathfrak{C}$, if $\mathfrak{C} \leq_{SC} \mathfrak{A}$, then $\mathfrak{C} \leq_{SC} \mathfrak{B}$.

2. $\mathfrak{A}$ is *minimal-like* over $\mathfrak{B}$ if the following two conditions hold:

- $\mathfrak{B} \leq_{SC} \mathfrak{A}$ and $\mathfrak{A} \not\leq_{SC} \mathfrak{B}$;
- For every function $\varphi \in \mathcal{F}_{a,b}$, if $\varphi \in \mathbf{SC}(\mathfrak{A})$, there exists a function $\psi \in \mathcal{F}_{a,b}$ such that $\varphi \subseteq \psi$ and $\psi \in \mathbf{SC}(\mathfrak{B})$.

For the (a, b)-string α we define a predicate α^+ to be the set

$$\{(s_1, \ldots, s_a, x_1, \ldots, x_b) \mid \alpha(\langle s_1, \ldots, s_a, x_1^*, \ldots, x_b^* \rangle) = 0^*\}.$$

If Σ_0 is a predicate and $\mathfrak{B}(\Sigma_1, \ldots, \Sigma_k)$ is a structure, we denote by $\Sigma_0 \oplus \mathfrak{B}$ the two-sort structure with predicates $\Sigma_0, \Sigma_1, \ldots, \Sigma_k$.

Proposition 4.3. *For given* $\mathfrak{B}(\Sigma_1, \ldots, \Sigma_k)$ *and* $\mathfrak{B}$*-generic predicate* Σ_0, *the structure* $\Sigma_0 \oplus \mathfrak{B}$ *is minimal-like over* $\mathfrak{B}$.

Proof. Since Σ_0 is $\mathfrak{B}$-generic, $\Sigma_0 \not\leq_{\mathbf{SC}} \mathfrak{B}$ and therefore $\mathfrak{B} \lneq_{\mathbf{SC}} \Sigma_0 \oplus \mathfrak{B}$. Let (a_i, b_i) be the arity of the predicate $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$.

For $\varphi \in \mathcal{F}_{a,b}$, such that $\varphi \in \mathbf{SC}(\Sigma_0 \oplus \mathfrak{B})$, we define its graph-predicate Σ_φ for which $\widehat{\Sigma}_\varphi \in \mathbf{SC}(\Sigma_0 \oplus \mathfrak{B})$, i.e. $\Sigma_\varphi \leq_{\mathbf{SC}} \Sigma_0 \oplus \mathfrak{B}$, and from the Normal Form Theorem 3.1 it follows that Σ_φ has a normal form in $\Sigma_0 \oplus \mathfrak{B}$, i.e. there are $e \in \omega$, an N-string δ_N and $z_1, \dots, z_a \notin Dom(\delta_N)$ such that for all $s_1, \dots, s_a \in N$ and $x_1, \dots, x_b, y \in \omega$, $(s_1, \dots, s_a, x_1, \dots, x_b, y) \in \Sigma_\varphi$ iff $\exists \tau_N \supseteq \delta_N$, where $(\tau_N(z_i) = s_i$ & $\tau_N \Vdash_{\Sigma_0 \oplus \mathfrak{B}} F_e(\langle z_1, \dots, z_a, x_1, \dots, x_b, y\rangle))$. If we denote by $P_{a,b}$ the set of codes of all (a,b)-strings and by P_N the set of all codes of N-strings, we may define the set S^* to be the set of all $\beta_0 \in P_{a_0,b_0}$ for which there exist $\beta_i \in P_{a_i,b_i}$ for $\forall_{1 \leq i \leq k}$, such that $\beta_i^+ \subseteq \Sigma_i$, and there exist $\tau_N^1, \tau_N^2 \in P_N$, both extending δ_N and such that $z_1, \dots, z_a \in Dom(\tau_N^1) \cap Dom(\tau_N^2)$, and there exist natural numbers $x_1, \dots, x_b \in \omega, y_1 \neq y_2 \in \omega$, such that $\tau_N^\varepsilon \Vdash_{\mathfrak{A}(\beta_0^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1, \dots, z_a, x_1, \dots, x_b, y_\varepsilon\rangle)$ for each $\varepsilon \in \{1, 2\}$, where $\mathfrak{A}(\beta_0^+, \beta_1^+, , \dots, , \beta_k^+)$ denotes the structure with finite predicates $\beta_i^+ \subseteq N^{a_i} \times \omega^{b_i}$. Therefore $S^* \in \mathfrak{B}$ and there is an (a_0, b_0)-string $\alpha \subseteq \chi_{\Sigma_0}$ such that $\alpha \in S^*$ or $\forall \beta \supseteq \alpha\ (\beta \notin S^*)$.

In the first case, since $\alpha \subseteq \chi_{\Sigma_0}$, then $\alpha^+ \subseteq \Sigma_0$, and from $\tau_N^\varepsilon \Vdash_{\mathfrak{A}(\alpha^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1, \dots, z_a, x_1, \dots, x_b, y_\varepsilon\rangle)$ follows that $\tau_N^\varepsilon \Vdash_{\Sigma_0 \oplus \mathfrak{B}} F_e(\langle z_1, \dots, z_a, x_1, \dots, x_b, y_\varepsilon\rangle)$, and using the normal form of Σ_φ we obtain a contradiction. So, it remains the second case $\forall \beta \supseteq \alpha (\beta \notin S^*)$ and now we can define a predicate Σ_ψ as follows: $\Sigma_\psi = \{(s_1, \dots, s_a, x_1, \dots, x_b, y) \mid (\exists \beta_0 \in P_{a_0,b_0}, \dots, \exists \beta_k \in P_{a_k,b_k}, \exists \tau_N \in P_N)$ such that $(\beta_0 \supseteq \alpha$ & $\forall_{1 \leq i \leq k}\ \beta_i^+ \subseteq \Sigma_i$ & $\tau_N \supseteq \delta_N$ & $\forall_{1 \leq j \leq a}\ \tau_N(z_j) = s_j$ & $\tau_N \Vdash_{\mathfrak{A}(\beta_0^+, \beta_1^+, \dots, \beta_k^+)} F_e(\langle z_1, \dots, z_a, x_1, \dots, x_b, y\rangle)$, which is the graph-predicate of some function ψ and it is search computable in $\mathfrak{B}$, therefore $\psi \in \mathbf{SC}(\mathfrak{B})$.

Using the above definition and the normal form of Σ_φ, it is not difficult to verify that $\Sigma_\varphi \subseteq \Sigma_\psi$, from which follows that $\varphi \subseteq \psi$, and this proves our proposition. □

Given a structure $\mathfrak{C}(\Sigma_1, \dots, \Sigma_k)$ and a predicate $\Sigma \subseteq N^a \times \omega^b$, if $\Sigma \leq_{\mathbf{SC}} \mathfrak{C}$ and $\overline{\Sigma} \leq_{\mathbf{SC}} \mathfrak{C}$, then its characteristic function $\chi_\Sigma \in \mathbf{SC}(\mathfrak{C})$. This fact can be used to prove the following:

Proposition 4.4. *Given a structure* $\mathfrak{B}(\Sigma_1, \dots, \Sigma_k)$ *and a* $\mathfrak{B}$*-generic predicate* Σ, *the structure* $\Sigma \oplus \mathfrak{B}$ *with predicates* $\Sigma, \Sigma_1, \dots, \Sigma_k$ *is quasi-minimal over* $\mathfrak{B}$.

The above is true for a single predicate, but not in the general case with multiple $\mathfrak{B}$-generic predicates. For example, for any total structure $\mathfrak{A}(\Sigma, \overline{\Sigma})$ with $\mathfrak{B}$-generic predicates Σ and $\overline{\Sigma} \subseteq N^a \times \omega^b$, the structure $\mathfrak{A} \oplus \mathfrak{B}$ is not quasi-minimal over $\mathfrak{B}$.

5. GENERIC REGULAR ENUMERATIONS

The regular enumerations are introduced by I. Soskov in [5] and here we shall use their modification for two-sort structures. An enumeration for two-sort structures is the pair $f = (f_N, f_\omega)$, where $f_N : \omega \to N$ and $f_\omega : \omega \to \omega$ are total surjective functions.

$Gr(f_N) = \{(s, x) \mid f_N(x) = s\} \subseteq N \times \omega$ is the graph of f_N.

$Gr(f_\omega) = \{\langle x, y\rangle \mid f_\omega(x) = y\} \subseteq \omega$ is the graph of f_ω.

The enumerations $f = (f_N, f_\omega)$ define a unique structure $\mathfrak{A}(Gr(f_N), Gr(f_\omega))$, denoted by $\mathfrak{A}_f$.

Since every two-sort structure (with finite number of predicates) is equivalent, in terms of search computability, to a structure with one predicate, in this section we consider only structures with one predicate.

Definition 5.1. Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that the enumeration $f = (f_N, f_\omega)$ is $\mathfrak{B}$-regular if the function f_ω is an $f_N^{-1}(\Sigma^{\mathfrak{B}})$-regular enumeration of ω in the sense of [5] and [6], i.e. f_ω is a total surjective mapping of ω onto ω such that $f_\omega(2\omega) = f_N^{-1}(\Sigma^{\mathfrak{B}})$.

Definition 5.2. 1) A *pair of strings* $\tau = (\tau_N, \tau_\omega)$ is the pair of an N-string $\tau_N : \omega \multimap N$ and an ω-string $\tau_\omega : \omega \multimap \omega$ (see Definition 3.1). The pair $\varnothing = (\varnothing_N, \varnothing_\omega)$ is referred as the empty pair of strings.

2) Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that the pair of strings $\tau = (\tau_N, \tau_\omega)$ is *$\mathfrak{B}$-regular* if $\tau_\omega(2\omega) \subseteq \tau_N^{-1}(\Sigma^{\mathfrak{B}})$, where $\tau_N^{-1}(\Sigma^{\mathfrak{B}}) = \{\langle x_1, \ldots, x_a, y_1, \ldots, y_b\rangle \in Dom(\tau_N)^a \times \omega^b \mid \& \ (\tau_N(x_1), \ldots, \tau_N(x_a), y_1, \ldots, y_b) \in \Sigma^{\mathfrak{B}}\}$ and $\tau_\omega(2\omega) = \{y \mid \exists x \ (\tau_\omega(2x) = y)\}$.

3) The N^*-code of $\tau = (\tau_N, \tau_\omega)$ is denoted by $\ulcorner\tau\urcorner^*$ and defined to be the pair of codes $\ulcorner\tau\urcorner^* = \langle\ulcorner\tau_N\urcorner^*, \ulcorner\tau_\omega\urcorner^*\rangle$, where $\ulcorner\tau_N\urcorner^* = \langle n^*, \langle 1^*, \tau_N(1)\rangle, \ldots, \langle n^*, \tau_N(n)\rangle\rangle$ and $\ulcorner\tau_\omega\urcorner^* = \langle m^*, \langle 1^*, (\tau_\omega(1))^*\rangle, \ldots, \langle m^*, (\tau_\omega(m))^*\rangle\rangle$, $n = lh(\tau_N)$ and $m = lh(\tau_\omega)$; define $\ulcorner\varnothing_N\urcorner^* = 0^*$ and $\ulcorner\varnothing_\omega\urcorner^* = 0^*$.

4) We say that τ extends σ, write $\sigma \subseteq \tau$, if both $\sigma_N \subseteq \tau_N$ and $\sigma_\omega \subseteq \tau_\omega$. For an enumeration $f = (f_N, f_\omega)$ and a pair of strings $\tau = (\tau_N, \tau_\omega)$ we say that $\tau \subseteq f$ when both $\tau_N \subseteq f_N$ and $\tau_\omega \subseteq f_\omega$.

Remark. Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$, let $Reg_{\mathfrak{B}}$ denote the set of codes of all $\mathfrak{B}$-regular pairs of strings. Thus $\tau \in Reg_{\mathfrak{B}} \Leftrightarrow \tau_\omega(2\omega) \subseteq \tau_N^{-1}(\Sigma^{\mathfrak{B}})$, and therefore $Reg_{\mathfrak{B}} \in \mathbf{SC}(\mathfrak{B})$.

Definition 5.3. Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, we say that $f = (f_N, f_\omega)$ is a $\mathfrak{B}$-generic regular enumeration if it is $\mathfrak{B}$-regular enumeration and for every set $S^* \subseteq N^*$ of codes of $\mathfrak{B}(\Sigma^{\mathfrak{B}})$-regular pairs of strings, for which $S^* \in \mathbf{SC}(\mathfrak{B})$, there exists a pair of strings $\tau \subseteq f$ such that $\tau \in S^*$ or $\forall \sigma \supseteq \tau \ (\sigma \notin S^*)$.

Proposition 5.1. *For every structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, there exists a $\mathfrak{B}$-generic regular enumeration $f = (f_N, f_\omega)$.*

Proof. Let $S_0^*, \ldots, S_n^*, \ldots$ be a sequence of all the sets $S^* \in \mathbf{SC}(\mathfrak{B})$ and $s_0, \ldots, s_n, \ldots$ be all the elements of N. We can build a $\mathfrak{B}$-generic regular enumeration in the standard way starting from the empty pair of strings and building an increasing sequence of $\mathfrak{B}$-regular pair of strings such that at even stages we will monitor the n-th set S_n^* and take care to satisfy the requirements for genericity. At odd stages we will satisfy $\tau_\omega^q(2\omega) \subseteq (\tau_\omega^q)^{-1}(\Sigma^{\mathfrak{B}})$ and in the same time $f_N^{-1} \subseteq f_\omega(2\omega)$, as follows:

Suppose at *Stage* $2n+1$ we have defined $\tau_q = (\tau_N^q, \tau_\omega^q)$ for $q = 2n$. We may define τ_N^{q+1} to extend τ_N^q, so that for $x = lh(\tau_N^q)$, $\tau_N^{q+1}(x) = s_n$. For the set $(\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}})$ we have two possibilities: if it is empty, define $\tau_\omega^{q+1} = \tau_\omega^q$; otherwise, $(\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}}) \neq \emptyset$. In this case we consider the set $A_q = (\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}}) \setminus \tau_\omega^q(2\omega)$ and define τ_ω^{q+1} to extend τ_ω^q such that in the first odd number $x_1 \notin Dom(\tau_\omega^q)$ define $\tau_\omega^{q+1}(x_1) = n$, and in the first even number $x_0 \notin Dom(\tau_\omega^q)$ define $\tau_\omega^{q+1}(x_0)$ to be the first $y \in A_q$ if $A_q \neq \emptyset$, or the first $y \in (\tau_N^{q+1})^{-1}(\Sigma^{\mathfrak{B}})$ if $A_q = \emptyset$.

In this way we obtain the desired enumeration. □

To prove the following proposition and the lemma, it may be convenient to define two notations for a $(0,1)$-string α and an N-string τ_N :

$cmp(\alpha, \tau_N)$ if and only if $\forall x \in \omega\ (x^* \in Dom(\alpha) \Leftrightarrow x \in Dom(\tau_N))$,

$\alpha \sim_\Sigma \tau_N$ if and only if $\forall x^* \in Dom(\alpha)\ \big(\alpha(x^*) = 0^* \Leftrightarrow \tau_N(x) \in \Sigma\big)$.

Proposition 5.2. *For a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with one predicate $\emptyset \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$ and a $\mathfrak{B}$-generic regular enumeration $f = (f_N, f_\omega)$ the following properties hold:*

(1) $\mathfrak{B} \leq_{sc} \mathfrak{A}_f$.

(2) $\mathfrak{A}_f \not\leq_{sc} \mathfrak{B}$.

(3) *For every predicate $\Sigma \subseteq N^a \times \omega^b$, if $\Sigma \leq_{sc} \mathfrak{B}$ and $\overline{\Sigma} \leq_{sc} \mathfrak{A}_f$, then $\overline{\Sigma} \leq_{sc} \mathfrak{B}$.*

(4) *For every predicate $\Sigma \subseteq N$, if $\emptyset \neq \Sigma \leq_{sc} \mathfrak{B}$ and $\emptyset \neq \overline{\Sigma} \leq_{sc} \mathfrak{B}$, then $f_N^{-1}(\Sigma)$ is a $\mathfrak{B}$-generic predicate.*

(5) *For every predicate $\Sigma \subseteq N$, if $\emptyset \neq \Sigma \leq_{sc} \mathfrak{B}$ and $\emptyset \neq \overline{\Sigma} \leq_{sc} \mathfrak{B}$, the structure $\mathfrak{A}(f_N^{-1}(\Sigma), \Sigma^{\mathfrak{B}})$ is quasi-minimal over $\mathfrak{B}$.*

Proof. These properties follow easily from the definitions and the properties of the enumerations. For example, for the proof of (4) we may assume that $f_N^{-1}(\Sigma)$ is not a $\mathfrak{B}$-generic predicate. Then there exists a set of $(0,1)$-strings S that fails the genericity, and consider the set of $\mathfrak{B}$-regular pairs of strings:

$$P^* = \{\tau \in Reg_{\mathfrak{B}} \mid \exists \alpha \in S(cmp(\alpha, \tau_N)\ \&\ \alpha \sim_\Sigma \tau_N)\}.$$

Since for each τ there is a unique α such that $cmp(\alpha, \tau_N)$ and $\alpha \sim_\Sigma \tau_N$, and for each α there is such τ_N, we can obtain a contradiction with the genericity of f. □

Lemma 5.1. *Given a structure $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ with $\emptyset \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$, a pair of strings δ, a $\mathfrak{B}$-generic predicate $A \subseteq \omega$ and a predicate $\Sigma \subseteq N$, such that $\emptyset \neq \Sigma \leq_{sc} \mathfrak{B}$ and $\emptyset \neq \overline{\Sigma} \leq_{sc} \mathfrak{B}$, for which the following two conditions hold:*

(1) *δ is $\mathfrak{B}$-regular;*

(2) $\forall x < lh(\delta_N)\ (x \in A \Leftrightarrow \delta_N(x) \in \Sigma)$,

if $S^* \subseteq N^*$ is a set of codes of $\mathfrak{B}$-regular pairs of strings and $S \leq_{sc} \mathfrak{B}$, then there exists a pair of strings σ with the following properties:

(a) $\sigma \supseteq \delta$;
(b) σ *is* $\mathfrak{B}$*-regular*;
(c) $\forall x < lh(\sigma_N)$ $(x \in A \Leftrightarrow \sigma_N(x) \in \Sigma)$ (*this is the property* (2) *for* σ);
(d) $\sigma \in S \vee \forall\tau(\tau \supseteq \sigma \Rightarrow \tau \notin S)$.

Proof. The proof is very similar to the one of the corresponding lemma in the classical case (Lemma 2.4. in [6]). □

Proposition 5.3. *Given a structure* $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ *with* $\varnothing \neq \Sigma^{\mathfrak{B}}$, *a* $\mathfrak{B}$*-generic predicate* $A \subseteq \omega$, *and a predicate* $\Sigma \subseteq N$, *such that* $\varnothing \neq \Sigma \leq_{SC} \mathfrak{B}$ *and* $\varnothing \neq \overline{\Sigma} \leq_{SC} \mathfrak{B}$, *there exists a* $\mathfrak{B}$*-generic regular enumeration* f *such that* $A = f_N^{-1}(\Sigma)$.

Proof. We can build f by the standard construction of increasing sequence of pairs of strings σ_q (starting from the empty pair of strings) with the properties (1) and (2) from the above lemma. Moreover, we want them to satisfy four additional properties:
(3) $\exists n \forall e \geq n\ \big(lh(\sigma_N^{2e+1}) \geq lh(\sigma_N^{2e})$ and $lh(\sigma_\omega^{2e+1}) \geq lh(\sigma_\omega^{2e})\big)$;
(4) $\forall s \in N\ \exists e\ (s \in Range(\sigma_N^{2e+1}))$ and $\forall y \in \omega\ \exists e\ (y \in Range(\sigma_\omega^{2e+1}))$;
(5) $\forall p\ \forall x \in (\sigma_N^p)^{-1}(\Sigma^{\mathfrak{B}})\ \exists e\ (x \in \sigma_\omega^{2e+1}(2\omega))$;
(6) $\forall e\ \big($ if $S_e \subseteq Reg_{\mathfrak{B}}$, then $(\sigma_{2e+2} \in S_e \vee \forall\tau \supseteq \sigma_{2e+2}\ (\tau \notin S_e))\big)$, where S_e is the e-th search computable in $\mathfrak{B}$ set in some given enumeration of all the sets from $\mathbf{SC}(\mathfrak{B})$, and $Reg_{\mathfrak{B}}$ is the set of the $\mathfrak{B}$-regular pair of strings.

These requirements guarantee that $f = \bigcup_{q=0}^{\infty} \sigma_q$ will be a $\mathfrak{B}$-generic regular enumeration and $A = f_N^{-1}(\Sigma)$.

Stage 2e+1. Suppose σ_q is defined for $q = 2e$. Define σ_N^{q+1} to extend σ_N^q with new elements and to have the property (2) defined in the previous lemma. If $(\sigma_N^q)^{-1}(\Sigma^{\mathfrak{B}})$ is empty, we define $\sigma_\omega^{q+1} = \sigma_\omega^q$, otherwise define σ_ω^{q+1} to extend σ_ω^q with the first two elements $x_0 \in 2\omega \setminus Dom(\sigma_\omega^q)$ and $x_1 \in (2\omega + 1) \setminus Dom(\sigma_\omega^q)$ for which:

- $\sigma_\omega^{q+1}(x_1)$ = the first y such that $y \notin Range(\sigma_\omega^q)$;
- $\sigma_\omega^{q+1}(x_0)$ = the first y such that $y \in (\sigma_\omega^q)^{-1}(\Sigma^{\mathfrak{B}}) \setminus \sigma_\omega^q(2\omega)$ if not empty, or the first $y \in (\sigma_\omega^q)^{-1}(\Sigma^{\mathfrak{B}})$ otherwise.

Stage 2e+2. Suppose σ_q is defined for $q = 2e + 1$. Let G be the set of all pairs of strings having the properties (1) and (2) from the previous lemma. We have two possibilities:

Case 1. $\exists\sigma \supseteq \sigma_q\ \big(\sigma \in G\ \&\ (\sigma \in S_e \vee \forall\tau \supseteq \sigma\ (\tau \notin S_e))\big)$. Define σ_{q+1} to be the first such σ.

Case 2. Otherwise, define $\sigma_{q+1} = \sigma_q$.

Now it can be verified that this construction meets the requirements (3) – (6), defined earlier in the current proof. For example, to verify (6), we can use the previous lemma to show that Case 2 never happens if S_e is a set of $\mathfrak{B}$-regular pair of strings. □

Theorem 5.1. *Let a structure* $\mathfrak{B}(\Sigma^{\mathfrak{B}})$ *with one predicate* $\varnothing \neq \Sigma^{\mathfrak{B}} \subseteq N^a \times \omega^b$ *be given. Then for any predicate* $A \subseteq \omega$, *A is* $\mathfrak{B}$*-generic if and only if there exists a predicate* $\Sigma \subseteq N$ *such that* $\varnothing \neq \Sigma \leq_{SC} \mathfrak{B}$ *and* $\varnothing \neq \overline{\Sigma} \leq_{SC} \mathfrak{B}$, *and there exists a* $\mathfrak{B}$*-generic regular enumeration f such that* $A = f_N^{-1}(\Sigma)$.

Proof. ($\Leftarrow$) The Proposition 5.2(4).

($\Rightarrow$) Consider the predicate $\Sigma = \{s\}$ for which it is clear that $\varnothing \neq \Sigma \leq_{\mathrm{SC}} \mathfrak{B}$ and $\varnothing \neq \overline{\Sigma} \leq_{\mathrm{SC}} \mathfrak{B}$. From the previous proposition it follows that there exists a $\mathfrak{B}$-generic regular enumeration f such that $A = f_N^{-1}(\Sigma)$. □

Acknowledgments. The author is deeply indebted to Tinko Tinchev for all the advice and encouragement in the realization of this paper.

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Received on March 1, 2002

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: vboutchkova@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

SPHERICAL 2-DISTANCE SETS WHICH ARE SPHERICAL 3-DESIGNS

PETER BOYVALENKOV, MAYA STOYANOVA

We classify spherical codes which admit exactly two different nonzero distances between their points and are spherical 3-designs. We prove that such codes have the maximal possible cardinality provided the dimension and the minimum distance are fixed.

Keywords: spherical designs, maximal spherical codes, 2-distance sets
2000 MSC: 05B30

1. INTRODUCTION

Let $\mathbf{S}^{n-1}$ be the n-dimensional unit sphere with the usual Euclidean metric and inner product. A spherical code C is a finite nonempty subset $\mathbf{S}^{n-1}$. Some characterizations of spherical codes are given by the dimension n, their cardinality $|C|$, the maximal inner product $s(C) = \max\{\langle x, y\rangle : x, y \in C, x \neq y\}$ (or, equivalently, the minimum distance $d(C) = \min\{d(x,y) : x, y \in C, x \neq y\} = \sqrt{2(1-s(C))}$). By (n, M, s) we denote any code $C \subset \mathbf{S}^{n-1}$ with $|C| = M$ and $s(C) = s$.

Denote by $\ell = \ell(C)$ the number of distinct inner products of different points of C. Then C is called an ℓ-distance spherical set. If $A(C)$ is the set of all distinct inner products, then $|A(C)| = \ell(C)$.

A spherical τ-design is a spherical code $C \subset \mathbf{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbf{S}^{n-1})} \int_{\mathbf{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \dots, x_n)$ of degree at most τ (i.e. the average of f over the set is equal to the average of f over $\mathbf{S}^{n-1}$). The number τ is called strength of C. The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [13] in analogy with the classical combinatorial designs.

Examples, constructions and classification results for spherical ℓ-distance sets can be found in [12, 13, 16, 6, 4]. However, a few ℓ-distance sets of large (with respect to n and ℓ) cardinality are known.

Many investigations of combinatorial objects start with an assumption of certain regularity. Since almost all known maximal ℓ-distance sets are spherical designs of suitable strength, we have decided to investigate further this connection.

We consider spherical 2-distance sets which are simultaneously spherical 3-designs. We prove that such codes have maximal possible cardinality for fixed dimension and the maximal inner product. This implies that the codes achieve the so-called Levenshtein bound which gives strong restrictions.

2. SOME PRELIMINARIES

Let $C \subset \mathbf{S}^{n-1}$ be a spherical code and $x \in C$. Then the system $\{A_t(x) : -1 \leq t < 1\}$ of integers

$$A_t(x) = |\{y \in C : \langle x, y \rangle = t\}|$$

is called distance distribution of C with respect to x. We take only the nonzero entries in the distance distribution.

A spherical code is called distance regular if its distance distributions do not depend on x. In this case we omit the point x in the notation.

Delsarte-Goethals-Seidel [13] give the following connection between the ℓ-distance sets and the spherical τ-designs.

Theorem 2.1. *Let $C \subset \mathbf{S}^{n-1}$ be an ℓ-distance spherical set and a spherical τ-design. Then:*

a) (*the absolute bound*) $\tau \leq 2\ell$ *and* $|C| \leq \binom{n+\ell-1}{\ell} + \binom{n+\ell-2}{\ell-1}$. *If one of these bounds is attained, then so does the another.*

b) $\tau \geq \ell - 1$ *implies that C is distance regular.*

c) (*Delsarte-Goethals-Seidel bound*)

$$|C| \geq \begin{cases} \binom{n+e-1}{e} + \binom{n+e-2}{e-1}, & \text{if } \tau = 2e; \\ 2\binom{n+e-2}{e-1}, & \text{if } \tau = 2e-1. \end{cases} \tag{2.1}$$

Let $M_n(\ell) = \max\{|C| : C \subset \mathbf{S}^{n-1} \text{ is an } \ell\text{-distance set}\}$ be the maximal possible cardinality of a spherical ℓ-distance set. Then the absolute bound and an easy lower bound state that

$$\binom{n+\ell-1}{\ell} \leq M_n(\ell) \leq \binom{n+\ell-1}{\ell} + \binom{n+\ell-2}{\ell-1}. \tag{2.2}$$

Despite this gives the asymptotic behaviour of $M_n(\ell)$, a few examples are known to attain the upper bound. Moreover, a few ℓ-distance sets are known to be close to this bound.

The following definition for spherical designs is crucial for our approach. If $C \subset \mathbf{S}^{n-1}$ is a spherical τ-design, then for every point $y \in C$ and for every real polynomial $f(t)$ of degree at most τ the equality

$$\sum_{x \in C \setminus \{y\}} f(\langle x, y \rangle) = f_0|C| - f(1) \tag{2.3}$$

holds, where

$$f_0 = c_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt, \quad c_n = \frac{\Gamma(n-1)}{2^{n-2}(\Gamma(\frac{n-1}{2}))^2}$$

(f_0 is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22]). In fact, for calculations of f_0 we use the following formula:

$$f_0 = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)}, \tag{2.4}$$

where $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$.

We also need the notion of maximal spherical codes. If the dimension n and the maximal inner product s are fixed, a classical problem in geometry and coding theory asks for finding exact values or bounds on

$$A(n,s) = \max\{|C| : C \subset \mathbf{S}^{n-1}, s(C) \leq s\}.$$

A spherical $(n, A(n,s), s)$-code is called maximal.

As usually in the coding theory, lower bounds on $A(n,s)$ are given by constructions (cf. [11] and references therein) and the best upper bounds are obtained by linear programming (cf. [14, 15, 11, 7]). We are especially interested in some of Levenstein's bounds.

The Levenshtein's bounds have somewhat complicated description. However, we need here only a particular case

$$A(n,s) \leq \frac{n(1-s)[(n+1)s+2]}{1-ns^2} \text{ for } 0 \leq s \leq \frac{\sqrt{n+3}-1}{n+2}. \tag{2.5}$$

Clearly, a code which attains (2.5), i.e. an $(n, L_3(n,s), s)$-code, is maximal. Our main result shows that every spherical 2-distance set which is a spherical 3-design is nothing but such a maximal spherical code.

Let $C \subset \mathbf{S}^{n-1}$ be a 2-distance set and a spherical 3-design. It follows from (2.2) for $\ell = 2$ and the Delsarte-Goethals-Seidel bound for $\tau = 3$ that

$$2n \leq |C| \leq \frac{n(n+3)}{2}.$$

If the upper bound is attained, then C is already 4-design. Since all feasible parameter sets of 4-designs are determined [5, 8], we assume that $|C| < n(n+3)/2-1$. Then we consider the whole range despite the feasible codes with fewer than $n(n+1)/2$ points would not be maximal 2-distance sets.

It is worth to note that the known constructions of spherical 3-designs (see [2, 3]) do not provide examples of 2-distance sets we are searching for.

Theorem 3.1. *A spherical code $C \subset \mathbf{S}^{n-1}$ is a 2-distance spherical set and a spherical 3-design if and only if C attains the Levenshtein's bound* (2.5).

Proof. "$\Leftarrow$" This direction is known. The necessary conditions for attaining the bound (2.5) show that C is a 2-distance set and a 3-design.

"$\Rightarrow$" Let the spherical code $C \subset \mathbf{S}^{n-1}$ be a 2-distance set and a 3-design. Then

$$2n \leq |C| \leq \frac{n(n+3)}{2}$$

and we set $|C| = 2n + k$, where $0 \leq k \leq n(n-1)/2$, k is integer.

It follows from Theorem 2.1b) that C is distance regular. Let $A(C) = \{t_1, t_2\}$, $A_{t_1}(x) = P$ and $A_{t_2}(x) = Q$ (the last two numbers do not depend on x). We assume that $t_1 < t_2$.

The equality (2.3) gives

$$Pf(t_1) + Qf(t_2) = (2n+k)f_0 - f(1) \tag{3.1}$$

for every real polynomial $f(t)$ of degree at most 3.

We first prove a Lloyd-type theorem by proving that t_1 and t_2 are roots of a quadratic equation with integer coefficients.

Using first degree polynomials $f(t) = t - t_2$ and $f(t) = t - t_1$ in (3.1), we obtain

$$P = \frac{(2n+k-1)t_2 + 1}{t_2 - t_1} \quad \text{and} \quad Q = \frac{(2n+k-1)t_1 + 1}{t_1 - t_2},$$

respectively.

By the second degree polynomial $f(t) = (t - t_1)(t - t_2)$ we obtain from (3.1) that

$$t_1 t_2 = \frac{n(t_1 + t_2) + n + k}{n(1 - 2n - k)}.$$

Using these three relations in

$$Pt_1^3 + Qt_2^3 = -1$$

(which is obtained from (3.1) by using $f(t) = t^3$), we derive

$$t_1 + t_2 = -\frac{n-1}{n+k-1} \quad \text{and} \quad t_1 t_2 = -\frac{k}{n(n+k-1)}.$$

Therefore t_1 and t_2 are the roots of the quadratic equation

$$n(n+k-1)t^2 + n(n-1)t - k = 0.$$

In particular, we also see that $t_1 < 0 < t_2$ and $|t_1| > t_2$.

As a short second step, we observe that t_1 and t_2 are in fact rational numbers. Indeed, setting $f(t) = t$ in (3.1), we obtain $Pt_1 + Qt_2 = -1$, which is equivalent to $P(t_1 + t_2) + (Q - P)t_2 = -1$. Since P, $t_1 + t_2$ and $Q - P$ are rationals, this implies that t_2 is rational as well. Analogously, we see that t_1 is rational.

In the third step we already prove that C is an $(n, L_3(n,s), s)$-code for $s = t_2$. Indeed, $s(C) = t_2$ and the equality

$$L_3(n, t_2) = \frac{n(1 - t_2)[(n+1)t_2 + 2]}{1 - nt_2^2}$$

is an identity (note that it is an identity for $s = t_1$ as well), which completes the proof. □

Theorem 3.1 shows that an examination of spherical 2-distance sets which are spherical 3-designs can be done via results on $(n, L_3(n,s), s)$-codes. The latter codes were studied by Boyvalenkov-Langjev in [10] and further by Boyvalenkov-Danev in [9]. In [9] all feasible parameters of $(n, L_3(n,s), s)$-codes in dimensions $n \leq 1600$ were found together with eleven infinite series.

Acknowledgments. This research was partially supported by the Framework Cultural Agreement between the University of Perugia (Italy) and the Institute of Mathematics at the Bulgarian Academy of Sciences (Bulgaria) and the Bulgarian NSF under Contract MM-901/99.

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Received on December 14, 2001

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
8, G. Bonchev str., 1113 Sofia,
BULGARIA
E-mail: peter@moi.math.bas.bg

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: stoyanova@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

A GENERALIZATION OF A RESULT OF DIRAC

NEDYALKO NENOV

Let G be a graph, $\chi(G) = r$ and $\mathrm{cl}(G) < r$. Dirac has proved in [2] that for such graph $|V(G)| \geq r+2$ and $|V(G)| = r+2$ only if $G = K_{r-3} + C_5$. The main result in the current article generalizes the proposition mentioned above (Theorem 2.1). As a consequence of Theorem 2.1, some results for Folkman graphs are obtained (Theorems 7.1–7.4, 8.1).

Keywords: chromatic number, Folkman graph, α-critical graph

2000 MSC: 05C15, 05C55

1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p, such that the graph G contains a p-clique, is denoted by $\mathrm{cl}(G)$ (the clique number of G).

If $X \subseteq V(G)$, then:

$G[X]$ is the subgraph of G induced by X;
$G - X$ is the subgraph of G induced by $V(G) \setminus X$;
$\Gamma_G(X)$ is the set of vertices in G, adjacent to at least one vertex of X.

In this paper we shall use also the following notations:

$\alpha(G)$ — the independence number of G;
$\chi(G)$ — the chromatic number of G;
$\pi(G)$ — the maximum number of independent edges in G
(the matching number of G);

$\overline{G}$ — the complement of G;
K_n — the complete graph of n vertices;
C_n — the simple cycle of n vertices.

By $G-e$, $e \in E(G)$, we denote the supergraph of G such that $V(G-e) = V(G)$, $E(G-e) = E(G) \setminus \{e\}$ and $G+e$, $e \in E(\overline{G})$, is the supergraph of G for which $V(G+e) = V(G)$, $E(G+e) = E(G) \cup \{e\}$.

Let G_1 and G_2 be graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[v_1, v_2],\ v_1 \in V(G_1),\ v_2 \in V(G_2)\}$.

2. THE MAIN RESULT

Definition 2.1. The partition $V(G) = V_1 \cup \ldots \cup V_r$ is p-saturated if the union of each p of the sets V_i, $i = 1, \ldots, r$, contains a p-clique of G.

Definition 2.2. The partition $V(G) = V_1 \cup \ldots \cup V_r$ is r-chromatic if the sets V_i, $i = 1, \ldots, r$, are independent.

Definition 2.3. A graph G is p-saturated if each $\chi(G)$-chromatic partition of $V(G)$ is p-saturated.

It is clear that if $\chi(G) \geq 2$, then G is 2-saturated. Dirac has proved in [2] the following proposition:

Let $\chi(G) = r$ *and* $\mathrm{cl}(G) < r$. *Then* $|V(G)| \geq r+2$ *and if* $|V(G)| = r+2$, *then* $G = K_{r-3} + C_5$.

The main result in this paper is the following generalization of the above mentioned proposition:

Theorem 2.1. *Let* $\chi(G) = r$, $\mathrm{cl}(G) < r$ *and* G *is* p*-saturated, but is not* $(p+1)$*-saturated. Then* $|V(G)| \geq r+p$ *and* $|V(G)| = r+p$ *only if* $G = K_{r-p-1} + \overline{C}_{2p+1}$.

We need the next propositions.

Proposition 2.1. *For any graph* G

$$\chi(G) + \pi(\overline{G}) \leq |V(G)|.$$

Proof. Let $|V(G)| = n$, $\pi(\overline{G}) = s$, and $\{x_1, y_1\}, \ldots, \{x_s, y_s\}$ be a matching of $\overline{G}$. If $v_1, \ldots, v_{n-2s}$ are the other vertices of G, then

$$\{x_1, y_1\} \cup \ldots \cup \{x_s, y_s\} \cup \{v_1\} \cup \ldots \cup \{v_{n-2s}\}$$

is an $(n-s)$-chromatic partition of G. Hence, $\chi(G) \leq n - s$. □

Proposition 2.2. *Let $\chi(G) = r$, G be a p-saturated, $2 \leq p \leq r$, and $V(G) = V_1 \cup \ldots \cup V_r$ be an r-chromatic partition of G. Then for any k, $p \leq k \leq r$, the graph $G[V_1 \cup \ldots \cup V_k]$ is p-saturated.*

Proof. We put $G[V_1 \cup \ldots \cup V_k] = G'$. It is clear that $\chi(G') = k$. Assume the opposite and let $V_1' \cup \ldots \cup V_k'$ be a k-chromatic partition of $V(G')$ which is not p-saturated. Then the r-chromatic partition $V_1' \cup \ldots \cup V_k' \cup V_{k+1} \cup \ldots \cup V_r$ of $V(G)$ is also not p-saturated, which is a contradiction. □

3. EXAMPLES OF p-SATURATED GRAPHS

Lemma 3.1. *Let $V' \subseteq V(\overline{C}_{2p+1})$, $|V'| = m < 2p + 1$ and $G = \overline{C}_{2p+1}[V']$. Then $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$.*

Proof. It follows from $m < 2p + 1$ that $\chi(\overline{G}) \leq 2$. Let $V(\overline{G}) = V_1 \cup V_2$, where V_1 and V_2 are independent sets of $\overline{G}$. Then $\alpha(\overline{G}) \geq \max\{|V_1|, |V_2|\}$. Hence $\alpha(\overline{G}) \geq \left\lceil \frac{m}{2} \right\rceil$, i.e. $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$. □

Proposition 3.1. *For any $p \geq 3$ the graph $\overline{C}_{2p+1}$ is p-saturated, but the graph $\overline{C}_{2p+1} - e$ is not p-saturated for any $e \in E(\overline{C}_{2p+1})$.*

Proof. It is clear that $\chi(\overline{C}_{2p+1}) = p + 1$. Let $V_1 \cup \ldots \cup V_{p+1}$ be $(p + 1)$-chromatic partition of $V(C_{2p+1})$ and let $V' = V(G) \setminus V_i$. We put $G' = \overline{C}_{2p+1}[V']$. From $\alpha(\overline{C}_{2p+1}) = 2$ it follows that $2p - 1 \leq |V'| \leq 2p$. By Lemma 3.1, $\mathrm{cl}(G') \geq p$. Hence $\overline{C}_{2p+1}$ is p-saturated.

Let $e \in E(\overline{C}_{2p+1})$ and $\widetilde{G} = \overline{C}_{2p+1} - e$. Assume that $V(C_{2p+1}) = \{v_1, \ldots, v_{2p+1}\}$ and $E(C_{2p+1}) = \{[v_i, v_{i+1}], i = 1, \ldots, 2p, [v_1, v_{2p+1}]\}$. We may assume that $e = [v_1, v_{2s+1}]$, $1 \leq s \leq p - 1$.

Case 1. $s = 1$. In this case $\mathrm{cl}(\widetilde{G}) = p - 1$ and hence $\widetilde{G}$ is not p-saturated.

Case 2. $2 \leq s \leq p - 1$. In this case $\alpha(\widetilde{G}) = 2$. Hence $\chi(\widetilde{G}) = p + 1$. It is clear that

$$\{v_1\} \cup \{v_2, v_3\} \cup \ldots \cup \{v_{2p}, v_{2p+1}\}$$

is a $(p + 1)$-chromatic partition of $V(\widetilde{G})$. Obviously, $\widetilde{G}[v_1, \ldots, v_{2s+1}] = \overline{C}_{2s+1}$. Hence $\{v_1, \ldots, v_{2s+1}\}$ contains no $(s+1)$-clique of $\widetilde{G}$. Thus $\{v_1, \ldots, v_{2p-1}\}$ contains no p-clique and $\widetilde{G}$ is not p-saturated. □

Proposition 3.2. *Let $2 \leq p < r$ and $G = K_{r-p-1} + \overline{C}_{2p+1}$. Then the graph G is p-saturated, but for any $e \in E(G)$ the graph $G - e$ is not p-saturated or $\chi(G - e) < r$.*

Proof. If $r = p + 1$, Proposition 3.2 follows from Proposition 3.1. Let $r \geq p + 2$. Obviously, $\chi(G) = r$. We put $V(K_{r-p-1}) = \{z_1, \ldots, z_{r-p-1}\}$. Let $V_1 \cup \ldots \cup V_{p+1}$ be a $(p+1)$-chromatic partition of $V(\overline{C}_{2p+1})$. Then $\{z_1\}\cup\ldots\cup\{z_{r-p-1}\}\cup V_1\cup\ldots\cup V_{p+1}$ is an r-chromatic partition of $V(G)$. It is clear that each r-chromatic partition of $V(G)$ has this form. Let V be the union of p subsets of this r-chromatic partition, $V' = V(K_{r-p-1}) \cap V$, $V'' = V(\overline{C}_{2p+1}) \cap V$ and $|V'| = q$. Then V' is a q-clique. Since $\overline{C}_{2p+1}$ is p-saturated (Proposition 3.1), V'' contains a $(p-q)$-clique. Hence V contains a p-clique. This proves that G is p-saturated.

Consider the graph $\widetilde{G} = G - e$, $e \in E(G)$.

Case 1. $e \notin E(\overline{C}_{2p+1})$. In this case obviously $\chi(\widetilde{G}) < r$.

Case 2. $e \in E(\overline{C}_{2p+1})$. By Proposition 3.1, the graph $\overline{C}_{2p+1} - e$ is not p-saturated. Hence $\widetilde{G} = K_{p-r-1} + (\overline{C}_{2p+1} - e)$ is also not p-saturated. □

4. α-CRITICAL GRAPHS

Definition 4.1. A graph G is said to be α-critical if $\alpha(G-e) > \alpha(G)$ for all $e \in E(G)$.

For the α-critical graphs the following facts are known:

Theorem A ([4], see also [1, Th. 8, p. 290]). *In an α-critical graph G without isolated vertices, each independent set A satisfies $|\Gamma_G(A)| \geq |A|$.*

Theorem B ([5, p. 58, exercise 25]). *Let G be a connected α-critical graph with $|V(G)| = 2\alpha(G)+1$. Then G is the simple cycle with $2\alpha(G)+1$ vertices.*

5. THE LEMMAS

Lemma 5.1. *Let G be a graph and $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$. If the graph H is such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(G)$ and $E(H) \supseteq E(G)$, then $\mathrm{cl}(H-v) = \mathrm{cl}(H)$ for all $v \in V(H)$.*

Proof. We have

$$\mathrm{cl}(H-v) \leq \mathrm{cl}(H) = \mathrm{cl}(G) = \mathrm{cl}(G-v) \leq \mathrm{cl}(H-v).$$

Hence $\mathrm{cl}(H) = \mathrm{cl}(H-v)$ for all $v \in V(H)$. □

Lemma 5.2. *Let G be a graph such that $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$. Then*:

(a) $|\Gamma_{\overline{G}}(Q)| \geq |Q|$ *for each clique Q of G*;

(b) $\pi(\overline{G}) \geq \mathrm{cl}(G)$;

(c) $|V(G)| \geq \chi(G) + \mathrm{cl}(G)$.

Proof. Let the graph H be such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(G)$, $E(H) \supseteq E(G)$ and $\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$. From Lemma 5.1, $\mathrm{cl}(H) = \mathrm{cl}(H-v)$ for all $v \in V(G)$. Hence $\overline{H}$ is a graph without isolated vertices. It follows from

$\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$ that $\alpha(\overline{H}-e) > \alpha(\overline{H})$ for all $e \in E(\overline{H})$. So, $\overline{H}$ is an α-critical graph without isolated vertices. By Theorem A, $|\Gamma_{\overline{H}}(Q)| \geq |Q|$ for each independent set Q of $\overline{H}$, i.e. for each clique Q of H. Since $\Gamma_{\overline{H}}(Q) \subseteq \Gamma_{\overline{G}}(Q)$, $|\Gamma_{\overline{G}}(Q)| \geq |Q|$.

Let Q be a clique of G such that $|Q| = \mathrm{cl}(G)$. From (a) and Hall's theorem it follows that $\pi(\overline{G}) \geq \mathrm{cl}(G)$. This inequality together with Proposition 2.1 imply (c). □

Remark. The proposition (a) of Lemma 5.2 is essentially the same as exercise 8, p. 302 in [1]. Another proof of (b) is obtained in [17].

Lemma 5.3. *Let G be a graph such that $\chi(G) = p+1$, $\mathrm{cl}(G) = p$ and let G be p-saturated. Then:*
(a) $\mathrm{cl}(G-v) = \mathrm{cl}(G), \ \forall v \in V(G)$;
(b) $\pi(\overline{G}) \geq p$.

Proof. Let $V_1 \cup \ldots \cup V_{p+1}$ be a $(p+1)$-chromatic partition of $V(G)$. Since this partition is p-saturated, $\mathrm{cl}(G - V_i) = p$, $i = 1, \ldots, p+1$. From these equalities it follows that $\mathrm{cl}(G-v) = \mathrm{cl}(G) = p$ for all $v \in V(G)$. Lemma 5.2(b) implies the inequality $\pi(\overline{G}) \geq p$. □

Lemma 5.4. *Let G be a graph such that $|V(G)| = 2p+1$, $\chi(G) = p+1$, $\mathrm{cl}(G) = p$, and let G be p-saturated. Then the graph $\overline{G}$ is connected.*

Proof. According to Lemma 5.3(b), $\pi(\overline{G}) \geq p$. Let $V(G) = \{v_1, \ldots, v_{2p+1}\}$ and let $\{v_1, v_2\}, \ldots, \{v_{2p-1}, v_{2p}\}$ be a matching of $\overline{G}$. Then

$$\{v_1, v_2\} \cup \ldots \cup \{v_{2p-1}, v_{2p}\} \cup \{v_{2p+1}\}$$

is a $(p+1)$-chromatic partition of G. The connected component of $\overline{G}$, which contains v_{2p+1}, will be denoted by M. By Lemma 5.3(a), $\overline{G}$ has no isolated vertices. Hence $|M| \geq 2$. Obviously, if one of the vertices v_{2k-1}, v_{2k} belongs to M, then $\{v_{2k-1}, v_{2k}\} \subseteq M$. Hence we may assume that

$$M = \{v_1, v_2, \ldots, v_{2s-1}, v_{2s}, v_{2p+1}\}, \quad 1 \leq s \leq p.$$

Suppose that $\overline{G}$ is not connected. Then $s < p$. Since G is p-saturated, M contains an $(s+1)$-clique Q of G. It is clear that $\Gamma_{\overline{G}}(Q) \subseteq M$. Thus, $|\Gamma_{\overline{G}}(Q)| \leq s$. Since $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$ (see Lemma 5.3(a)), this contradicts Lemma 5.2(a) and proves Lemma 5.4. □

Lemma 5.5. *Let G be a graph such that $\chi(G) = p+1$, $\mathrm{cl}(G) = p$, and let G be also p-saturated. Then $|V(G)| \geq 2p+1$ and $|V(G)| = 2p+1$ only if $G = \overline{C}_{2p+1}$.*

Proof. It follows from Lemma 5.3(a) that

$$\mathrm{cl}(G-v) = \mathrm{cl}(G), \quad \forall v \in V(G). \tag{5.1}$$

By Lemma 5.2(c), $|V(G)| \geq 2p+1$. Let $|V(G)| = 2p+1$. Consider the graph H such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(H)$, $E(H) \supseteq E(G)$ and $\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$. According to (5.1), Lemma 5.1 and Lemma 5.2(c), $\chi(H) \leq p+1$. Since $\chi(H) \geq \chi(G) = p+1$, we have $\chi(H) = p+1$. Obviously, each $(p+1)$-chromatic partition of H is also a $(p+1)$-chromatic partition of G. Hence H is also p-saturated. By Lemma 5.4, $\overline{H}$ is connected. It follows from $\mathrm{cl}(H+e) > \mathrm{cl}(H)$, $\forall e \in E(\overline{H})$, that $\alpha(\overline{H}-e) > \alpha(\overline{H})$, $\forall e \in E(\overline{H})$. So, $\overline{H}$ is an α-critical and connected graph. According to Theorem B, $\overline{H} = C_{2p+1}$. Thus G is a subgraph of $\overline{C}_{2p+1}$. By Proposition 3.1, $G = \overline{C}_{2p+1}$. □

6. A PROOF OF THEOREM 2.1

Let $V_1 \cup \ldots \cup V_r$ be an r-chromatic partition of G such that $V' = V_1 \cup \ldots \cup V_{p+1}$ contains no $(p+1)$-clique of G. Let $G' = G[V']$ and $V'' = V(G) \setminus V'$. By Proposition 2.2, the graph G' is p-saturated. Hence $\mathrm{cl}(G') = p$. Obviously, $\chi(G') = p+1$. According to Lemma 5.5, $|V'| \geq 2p+1$. Since $|V''| \geq r-p-1$, we have $|V(G)| \geq r+p$. Let $|V(G)| = r+p$. Then $|V(G')| = 2p+1$ and $|V''| = r-p-1$. By Lemma 5.5, $G' = \overline{C}_{2p+1}$. Thus G is a subgraph of $K_{r-p-1} + \overline{C}_{2p+1}$. It follows from Proposition 3.2 that $G = K_{r-p-1} + \overline{C}_{2p+1}$.

7. ON THE VERTEX FOLKMAN GRAPHS

Definition 7.1. Let G be a graph and let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers. The r-partition $V_1 \cup \ldots \cup V_r$ of $V(G)$ is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the set V_i contains no a_i-clique of G. The symbol $G \xrightarrow{v} (a_1, \ldots, a_r)$ means that every r-partition of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

Let $m = \sum_{i=1}^{r}(a_i - 1) + 1$. Consider an r-partition $V(K_{m-1}) = V_1 \cup \ldots \cup V_r$, where $|V_i| = a_i - 1$. Obviously, this r-partition is $(a_1, \ldots, a_r)$-free. Hence $K_{m-1} \overset{v}{\nrightarrow} (a_1, \ldots, a_r)$. It is clear that $K_m \xrightarrow{v} (a_1, \ldots, a_r)$. Thus, from $\mathrm{cl}(G) \geq m$ it follows that $G \xrightarrow{v} (a_1, \ldots, a_r)$. Clearly, $G \xrightarrow{v} (a_1, \ldots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman proves in [3] that for every $a_1, \ldots, a_r$ there exists a graph $G \xrightarrow{v} (a_1, \ldots, a_r)$ with $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$. The graph G, such that $G \xrightarrow{v} (a_1, \ldots, a_r)$, is called a vertex $(a_1, \ldots, a_r)$-Folkman graph.

It is clear that

Proposition 7.1. *For any permutation φ of the symmetric group S_r we have*

$$G \xrightarrow{v} (a_1, \ldots, a_r) \iff G \xrightarrow{v} (a_{\varphi(1)}, \ldots, a_{\varphi(r)}).$$

For the positive integers $a_1, \ldots, a_r$, $r \geq 2$, we put

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}. \tag{7.1}$$

Theorem 7.1. *Let positive integers $a_1, \ldots, a_r$, $r \geq 2$, m and p satisfy* (7.1) *and $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$. Then $\chi(G) \geq m$ and if $\chi(G) = m$, the graph G is p-saturated.*

Proof. Suppose $\chi(G) \leq m-1$ and $V(G) = V_1 \cup \ldots \cup V_{m-1}$ is an $(m-1)$-chromatic partition of G. Let $V(K_{m-1}) = \{z_1, \ldots, z_{m-1}\}$ and let $W_1 \cup \ldots \cup W_r$ be an r-partition of $V(K_{m-1})$ such that $|W_i| = a_i - 1$. Consider the map $V(G) \overset{\varphi}{\longrightarrow} V(K_{m-1})$, where $v \overset{\varphi}{\longrightarrow} z_i$ for all $v \in V_i$. We put $V'_k = \varphi^{-1}(W_k)$, $k = 1, \ldots, r$. Since V'_k is an union of $a_k - 1$ independent sets of G, V'_k contains no a_k-clique, $k = 1, \ldots, r$. So, $V'_1 \cup \ldots \cup V'_r$ is an $(a_1, \ldots, a_r)$-free partition of G, which is a contradiction.

Let $\chi(G) = m$. Suppose that G is not p-saturated and let $V_1 \cup \ldots \cup V_m$ be an m-chromatic partition of G such that $V' = V_1 \cup \ldots \cup V_p$ contains no p-clique of G. By Proposition 7.1, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_r = p$. We put $G' = G - V'$. Obviously, $\chi(G') = m - p = m - a_r = \sum_{i=1}^{r-1}(a_i - 1)$. From these equalities it follows that G' has an $(a_1, \ldots, a_{r-1})$-free $(r-1)$-partition $W_1 \cup \ldots \cup W_{r-1}$. But then $W_1 \cup \ldots \cup W_{r-1} \cup V'$ is an $(a_1, \ldots, a_r)$-free r-partition of G, which is a contradiction. This ends the proof of Theorem 7.1. □

Theorem 7.2. *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers and let m and p satisfy* (7.1). *Let the graph G be such that $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$ and $\mathrm{cl}(G) < m$. Then $\pi(\overline{G}) \geq p$.*

Proof. We prove the inequality $\pi(\overline{G}) \geq p$ by induction on m. It follows from $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$ that $\mathrm{cl}(G) \geq p$. Since $\mathrm{cl}(G) < m$, $m \geq p+1$. By this inequality, the minimal admissible value of m is $p+1$.

1. Let $m = p+1$. According to Proposition 7.1, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_r = p$. From $m = p+1$ it follows that $a_1 = \cdots = a_{r-2} = 1$, $a_{r-1} = 2$ and $\mathrm{cl}(G) = p$. Hence $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$ implies $G \overset{v}{\longrightarrow} (2, p)$. From $G \overset{v}{\longrightarrow} (2, p)$ it follows $\mathrm{cl}(G - v) \geq p$ for all $v \in V(G)$. So, $\mathrm{cl}(G - v) = \mathrm{cl}(G) = p$ for all $v \in V(G)$. According to Lemma 5.2(b), $\pi(\overline{G}) \geq p$.

2. Let $m \geq p+2$. If $\mathrm{cl}(G - v) = \mathrm{cl}(G)$, $\forall v \in V(G)$, from Lemma 5.2(b) it follows that $\pi(\overline{G}) \geq \mathrm{cl}(G)$. Hence $\pi(\overline{G}) \geq p$. Suppose $\mathrm{cl}(G - v_0) < \mathrm{cl}(G)$ for some $v_0 \in V(G)$. Since $\mathrm{cl}(G) < m$, $\mathrm{cl}(G - v_0) < m - 1$. We may assume that $a_1 \leq \cdots \leq a_r = p$. It follows from $m \geq p+2$ that $a_{r-1} \geq 2$. Obviously, $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$ implies $G - v_0 \overset{v}{\longrightarrow} (a_1, \ldots, a_{r-2}, a_{r-1} - 1, a_r)$. Applying the inductive hypothesis for $G - v_0$, we conclude that $\pi(\overline{G} - v_0) \geq p$.

Hence, $\pi(\overline{G}) \geq p$. □

Theorem 7.3 ([7]). *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers and m and p satisfy* (7.1). *If $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$ and $\mathrm{cl}(G) < m$, then $|V(G)| \geq m + p$.*

Another proof of Theorem 7.3. According to Theorem 7.1, $\chi(G) \geq m$, and accordingly to Theorem 7.2, $\pi(\overline{G}) \geq p$. It follows from Proposition 2.1 that $|V(G)| \geq m + p$. □

Theorem 7.4 ([8]). *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers, and let m and p satisfy* (7.1). *If $G \xrightarrow{v} (a_1, \ldots, a_r)$, $\mathrm{cl}(G) < m$ and $|V(G)| = m + p$, then $G = K_{m-p-1} + \overline{C}_{2p+1}$.*

Another proof of Theorem 7.4. It follows from Proposition 2.1 and Theorem 7.2 that $|V(G)| \geq \chi(G) + p$. Since $|V(G)| = m + p$, we conclude that $\chi(G) \leq m$. By Theorem 7.1, $\chi(G) = m$ and G is p-saturated. It follows from Theorem 2.1 and $|V(G)| = m + p$ that G is not $(p+1)$-saturated and $G = K_{m-p-1} + \overline{C}_{2p+1}$.

It is proved in [6] that $K_{m-p-1} + \overline{C}_{2p+1} \xrightarrow{v} (a_1, \ldots, a_r)$.

8. EDGE FOLKMAN GRAPHS

Definition 8.1. Let $a_1, \ldots, a_r$, $a_i \geq 2$, $r \geq 2$, be integers. Let G be a graph and let

$$E(G) = E_1 \cup \ldots \cup E_r$$

be an r-colouring of $E(G)$. This r-colouring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the graph G contains no monochromatic a_i-clique of colour i. The symbol $G \xrightarrow{e} (a_1, \ldots, a_r)$ means that every r-colouring of $E(G)$ is not $(a_1, \ldots, a_r)$-free.

Obviously, if $\mathrm{cl}(G) \geq R(a_1, \ldots, a_r)$, where $R(a_1, \ldots, a_r)$ is the Ramsey number, then $G \xrightarrow{e} (a_1, \ldots, a_r)$. It is clear that $G \xrightarrow{e} (a_1, \ldots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. The existence of a graph $G \xrightarrow{e} (a_1, \ldots, a_r)$ with $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$ was proved in the case $r = 2$ by Folkman in [3] and for arbitrary r by Nesetril and Rodl in [16].

Theorem 8.1. *Let $a_1, \ldots, a_r$, $a_i \geq 2$, $r \geq 2$, be integers and let $G \xrightarrow{e} (a_1, \ldots, a_r)$. Then*

(a) *$\chi(G) \geq R$, where $R = R(a_1, \ldots, a_r)$;*

(b) *suppose that $\chi(G) = R$, $\mathrm{cl}(G) < R$ and there exists an r-colouring*

$$E(K_R) = E_1 \cup \ldots \cup E_r \tag{8.1}$$

with the unique monochromatic a_i-clique P of colour i and without monochromatic a_j-clique of colour j, $j \neq i$. Then G is a_i-saturated and if $K_{R-a_i-1} + \overline{C}_{2a_i+1} \overset{e}{\not\rightarrow} (a_1, \ldots, a_r)$, then $|V(G)| > R + a_i$.

Proof. The proof of the inequality (a) is due to Lin in [5]. To prove the proposition (b) of Theorem 8.1, suppose to the contrary that $V_1 \cup \ldots \cup V_R$ is an R-chromatic partition of $V(G)$ such that $V_1 \cup \ldots \cup V_{a_i}$ contains no a_i-clique. Let $V(K_R) = \{z_1, \ldots, z_R\}$ and $P = \{z_1, \ldots, z_{a_i}\}$. Consider the map $V(G) \xrightarrow{\varphi} V(K_R)$, where $v \xrightarrow{\varphi} z_i$, $\forall v \in V_i$. Let $E_1' \cup \ldots \cup E_r'$ be the r-colouring of $E(G)$, where $[u, v] \in E_i' \iff [\varphi(u), \varphi(v)] \in E_i$ of (8.1). From $G \xrightarrow{e} (a_1, \ldots, a_r)$ it follows that in this r-colouring there exists a monochromatic a_k-clique Q of colour k. Obviously, $\varphi(Q)$ is a monochromatic a_k-clique of colour k in (8.1). By the properties of the

r-colouring (8.1) it follows that $i = k$ and $\varphi(Q) = P \doteq \{z_1, \ldots, z_{a_i}\}$. Hence $Q \subseteq V_1 \cup \ldots \cup V_{a_i}$. This contradicts the assumption that $V_1 \cup \ldots \cup V_{a_i}$ contains no a_i-clique and proves that G is a_i-saturated.

According to Theorem 2.1, $|V(G)| \geq \chi(G) + a_i = R + a_i$. Since $K_{R-a_i-1} + \overline{C}_{2a_i+1} \overset{e}{\not\longrightarrow} (a_1, \ldots, a_r)$, $G \neq K_{R-a_i-1} + \overline{C}_{2a_i+1}$. From Theorem 2.1, $|V(G)| > R + a_i$. The proof of Theorem 8.1 is completed. □

Theorem 8.1 generalizes the results from [12].

Consider the graphs G such that $G \overset{e}{\longrightarrow} (3,4)$ and $\mathrm{cl}(G) < 9$. We put $N(3,4;9) = \min\{|V(G)| : G \overset{e}{\longrightarrow} (3,4) \text{ and } \mathrm{cl}(G) < 9\}$.

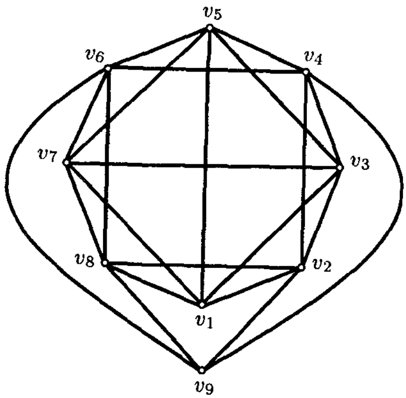

Fig. 1. The graph F

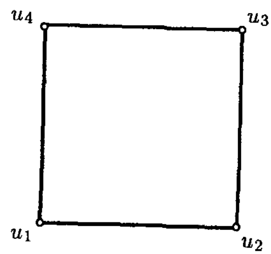

Fig. 2. The graph F_1

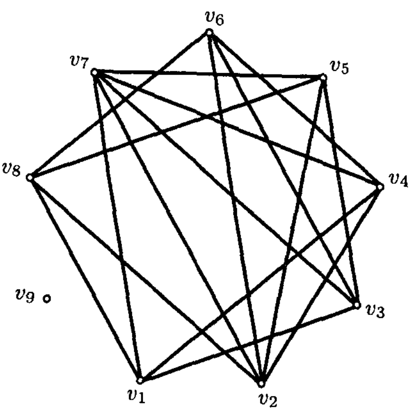

Fig. 3. The graph F_2

Corollary 8.1 ([10]). $N(3,4;9) = 14$.

Proof. It is proved in [11] and [15] that $K_4 + C_5 + C_5 \overset{e}{\longrightarrow} (3,4)$. Hence $N(3,4;9) \leq 14$. We prove the inequality $N(3,4;9) \geq 14$. Since $R(3,4) = 9$, from Theorem 8.1 follows $\chi(G) \geq 9$.

Case 1. $\chi(G) \geq 10$. Since $\text{cl}(G) \leq 8$, Theorem 1 in [13] implies $|V(G)| \geq 14$.

Case 2. $\chi(G) = 9$. By F, F_1 and F_2 we denote the graphs which are given in Fig. 1, Fig. 2 and Fig. 3, respectively. In Fig. 1 is given the unique 9-vertex graph F with $\alpha(F) = 2$ and containing an unique 4-clique ($[v_1, v_3, v_5, v_7]$), [14]. Hence the 2-colouring $E(K_9) = E_1 \cup E_2$, where $E_2 = E(F)$, contains an unique 4-clique of 2nd colour and contains no 3-cliques of 1st colour. Let

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Consider the 2-colouring $E(K_4 + \overline{C}_9) = E_1 \cup E_2$, where $E(K_4) \cap E_2 = E(F_1)$, $E(\overline{C}_9) \cap E_2 = E(F_2)$ and $[u_i, v_j] \in E_2 \iff a_{ij} = 2$. This 2-colouring is (3, 4)-free, [10]. By Theorem 8.1, $|V(G)| \geq 14$.

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Received on May 9, 2001

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

COMPUTATION OF THE VERTEX FOLKMAN NUMBERS $F(2,2,2,3;5)$ AND $F(2,3,3;5)$

NEDYALKO NENOV

In this note we show that the exact value of the vertex Folkman numbers $F(2,2,2,3;5)$ and $F(2,3,3;5)$ is 12.

Keywords: vertex Folkman graph, vertex Folkman number

2000 MSC: 05C55

1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$ (the clique number of G).

If $W \subseteq V(G)$, then: $G[W]$ is the subgraph of G induced by W and $G - W$ is the subgraph of G induced by $V(G) \setminus W$. We shall use also the following notations:

$\overline{G}$ — the complement of the graph G;

$\alpha(G)$ — the independence number of G;

$N_G(v)$, $v \in V(G)$ — the set of all vertices of G adjacent to v;

K_n — the complete graph of n vertices;

C_n — the simple cycle of n vertices;

$\chi(G)$ — the chromatic number of G.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y],\ x \in V(G_1),\ y \in V(G_2)\}$.

The Ramsey number $R(p,q)$ is the smallest natural n such that for an arbitrary n-vertex graph G either $\alpha(G) \geq p$ or $\mathrm{cl}(G) \geq q$. We need the equality $R(3,3) = 6$, [3].

2. VERTEX FOLKMAN NUMBERS AND THE MAIN RESULT

Definition 2.1. Let G be a graph, $a_1, \ldots, a_r$ be positive integers and let

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \emptyset,\ i \neq j,$$

be an r-coloring of the vertices of G. This coloring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the graph G does not contain a monochromatic a_i-clique of color i. The symbol $G \to (a_1, \ldots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

The graph G such that $G \to (a_1, \ldots, a_r)$ is called a vertex Folkman graph. We put

$$F(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \to (a_1, \ldots, a_r) \text{ and } \mathrm{cl}(G) < q\}.$$

It is clear that from $G \to (a_1, \ldots, a_r)$ it follows that $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman, [2], proves that there exists a graph G such that $G \to (a_1, \ldots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$. Therefore, if $q > \max\{a_1, \ldots, a_r\}$, then the numbers $F(a_1, \ldots, a_r; q)$ exist. Those numbers are called vertex Folkman numbers.

Let $a_1, \ldots, a_r$ be positive integers. We put

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}. \tag{1}$$

Obviously, $K_m \to (a_1, \ldots, a_r)$ and $K_{m-1} \nrightarrow (a_1, \ldots, a_r)$. Hence, if $q \geq m+1$, then $F(a_1, \ldots, a_r; q) = m$. The numbers $F(a_1, \ldots, a_r; m)$ exist only if $m \geq p+1$. For those numbers the following is known:

Theorem A ([4]). *Let $a_1, \ldots, a_r$ be positive integers and let m and p satisfy (1), where $m \geq p+1$. Then $F(a_1, \ldots, a_r; m) = m + p$. If $G \to (a_1, \ldots, a_r)$, $\mathrm{cl}(G) < m$ and $|V(G)| = m + p$, then $G = K_{m-p-1} + \overline{C}_{2p+1}$.*

Remark. The proof of Theorem A, given in [4], is based on [4, Lemma 1, p. 251]. But the proof of this lemma is not correct, because the sentence "If we delete both endpoints of any its edges adjacent to $\{x, y\}$, then $\alpha(G)$ decreases again." is not true (see p.252).

A correct proof of Theorem A is given in [13] (see also p.66, Theorem 7.4 in this volume).

The numbers $F(a_1, \ldots, a_r; m-1)$ exist only if $m \geq p+2$. For those numbers the following is known:

Theorem B ([13]). *Let $a_1, \ldots, a_r$ be positive integers. Let m and p satisfy (1), where $m \geq p+2$. Then $F(a_1, \ldots, a_r; m-1) \geq m+p+2$.*

Theorem C ([14]). *Let $a_1, \ldots, a_r$ be positive integers and let m and p satisfy* (1). *Let $m \geq p+2$, $G \to (a_1, \ldots, a_r)$ and $\mathrm{cl}(G) < m-1$. Then:*

(a) $|V(G)| \geq m + p + \alpha(G) - 1$;

(b) *if* $|V(G)| = m + p + \alpha(G) - 1$, *then* $|V(G)| \geq m + 3p$.

It is clear that for each permutation φ of the symmetric group S_r

$$G \to (a_1, \ldots, a_r) \iff G \to (a_{\varphi(1)}, \ldots, a_{\varphi(r)}).$$

Note that if $a_1 = 1$, then $F(a_1, \ldots, a_r; q) = F(a_2, \ldots, a_r; q)$. Therefore, we can assume that $2 \leq a_1 \leq \cdots \leq a_r$.

The next theorem implies that, in the special situation $a_1 = \cdots = a_r = 2$, $r \geq 5$, the inequality from Theorem B is exact.

Theorem D.

$$F(\underbrace{2, \ldots, 2}_{r}; r) = \begin{cases} 11, & r = 3 \text{ or } r = 4; \\ r+5, & r \geq 5. \end{cases}$$

It is clear that $G \to (\underbrace{2, \ldots, 2}_{r}) \iff \chi(G) \geq r+1$.

Mycielski in [5] presents an 11-vertex graph G such that $G \to (2,2,2)$ and $\mathrm{cl}(G) = 2$, proving that $F(2,2,2;3) \leq 11$. Chvátal, [1], proves that Mycielski graph is the smallest such graph and hence $F(2,2,2;3) = 11$. The inequality $F(2,2,2,2;4) \geq 11$ is proved in [8] and inequality $F(2,2,2,2;4) \leq 11$ is proved in [7] and [12] (see also [9]). The equality $F(\underbrace{2, \ldots, 2}_{r}; r) = r+5$, $r \geq 5$, is proved in [7], [12] and later in [4]. Only few other numbers of the type $F(a_1, \ldots, a_r; m-1)$ are known, namely: $F(3,3;4) = 14$ (the inequality $F(3,3;4) \leq 14$ is proved in [6] and the opposite inequality $F(3,3;4) \geq 14$ is verified by means of computers in [18]); $F(3,4;5) = 13$, [10]; $F(2,2,4;5) = 13$, [11]; $F(4,4;6) = 14$, [15]; $F(2,2,2,4;6) = F(2,3,4;6) = 14$, [16].

In this paper we will calculate another two numbers of this type.

Theorem 2.1. $F(2,2,2,3;5) = F(2,3,3;5) = 12$.

3. THE LEMMAS

We consider the graph P, whose complementary graph $\overline{P}$ is given in Fig. 1. For this graph we put

$$A = \{a_1, \ldots, a_8\}, \quad B = \{b_1, b_2, b_3, b_4\}.$$

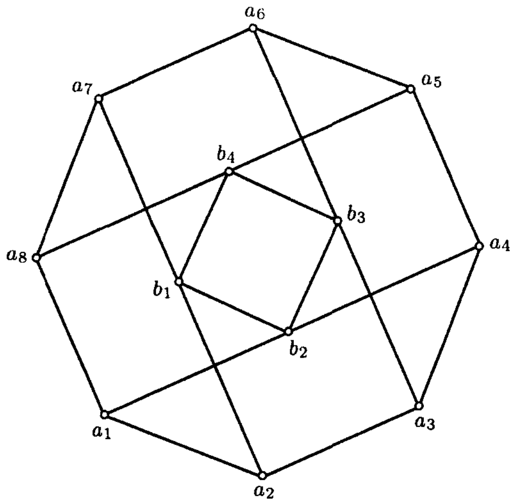

Fig. 1. Graph $\overline{P}$

Lemma 3.1 (Main Lemma). $P \not\to (2,3,3)$.

To prove the main Lemma, we make use of the next lemmas.

Lemma 3.2. *Let $W \subseteq V(\overline{P})$ and $\overline{P}[W] = C_5$.*
(a) *If $W \cap B = \{b_1\}$, then $W = \{b_1, a_1, a_2, a_7, a_8\}$.*
(b) *If $W \cap B = \{b_2\}$, then $W = \{b_2, a_1, a_2, a_3, a_4\}$.*
(c) *If $W \cap B = \{b_3\}$, then $W = \{b_3, a_3, a_4, a_5, a_6\}$.*
(d) *If $W \cap B = \{b_4\}$, then $W = \{b_4, a_5, a_6, a_7, a_8\}$.*

Proof. It is sufficient to prove the proposition (a).
Let $W \cap B = \{b_1\}$. From $b_2, b_4 \notin W$ and $\overline{P}[W] = C_5$ it follows that $a_2, a_7 \in W$. From $a_7 \in W$ it follows that $a_8 \in W$ or $a_6 \in W$. From $a_2 \in W$ it follows that $a_1 \in W$ or $a_3 \in W$. Since in $\{a_1, a_3, a_6, a_8\}$ only a_1 and a_8 are adjacent in $\overline{P}$, we have $W = \{b_1, a_1, a_2, a_7, a_8\}$. □

Lemma 3.3. *Let $W \subseteq V(\overline{P})$, $\overline{P}[W] = C_5$ and $|W \cap B| = 2$. Then the two vertices of $W \cap B$ are adjacent in $\overline{P}$.*

Proof. Assume the contrary and let for example $W = \{b_1, b_3\}$. From $\overline{P}[W] = C_5$ it follows that there exists $u \in W$ such that $u \in N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3)$. Since $N_{\overline{P}}(b_1) \cap N_{\overline{P}}(b_3) = \{b_2, b_4\}$, this contradicts equality $W = \{b_1, b_3\}$. □

Lemma 3.4. *Let $W \subseteq V(\overline{P})$ and $\overline{P}[W] = C_5$.*
(a) *If $W \cap B = \{b_1, b_2\}$, then $W = \{b_1, b_2, a_1, a_7, a_8\}$ or $W = \{b_1, b_2, a_2, a_3, a_4\}$.*
(b) *If $W \cap B = \{b_2, b_3\}$, then $W = \{b_2, b_3, a_1, a_2, a_3\}$ or $W = \{b_2, b_3, a_4, a_5, a_6\}$.*
(c) *If $W \cap B = \{b_3, b_4\}$, then $W = \{b_3, b_4, a_3, a_4, a_5\}$ or $W = \{b_3, b_4, a_6, a_7, a_8\}$.*

(d) *If* $W \cap B = \{b_1, b_4\}$, *then* $W = \{b_1, b_4, a_5, a_6, a_7\}$ *or* $W = \{b_1, b_4, a_1, a_2, a_8\}$.

Proof. It is sufficient to prove the proposition (a).

Let $W \cap B = \{b_1, b_2\}$. From $b_1 \in W$ and $b_4 \notin W$ it follows that $a_2 \in W$ or $a_7 \in W$. Let $a_2 \in W$. Since $\overline{P}[a_1, a_2, b_1, b_2] = C_4$, we have $a_1 \notin W$. Hence, $a_3 \in W$. Therefore, from $\overline{P}[W] = C_5$ it follows that $W = \{b_1, b_2, a_2, a_3, a_4\}$. Let $a_7 \in W$. From $\overline{P}[W] = C_5$ it follows that $a_6 \in W$ or $a_8 \in W$. Since $N_{\overline{P}}(a_6) \cap N_{\overline{P}}(b_2) = \{b_3\}$ and $b_3 \notin W$, we have $a_8 \in W$. From $N_{\overline{P}}(a_8) \cap N_{\overline{P}}(b_2) = \{a_1\}$ it follows that $W = \{b_1, b_2, a_1, a_7, a_8\}$. □

Lemma 3.5. *Let* $W \subseteq V(\overline{P})$ *and* $\overline{P}[W] = C_5$.

(a) *If* $W \cap B = \{b_1, b_2, b_3\}$, *then*

$$W = \{b_1, b_2, b_3, a_2, a_3\} \quad or \quad W = \{b_1, b_2, b_3, a_6, a_7\}.$$

(b) *If* $W \cap B = \{b_2, b_3, b_4\}$, *then*

$$W = \{b_2, b_3, b_4, a_4, a_5\} \quad or \quad W = \{b_2, b_3, b_4, a_1, a_8\}.$$

(c) *If* $W \cap B = \{b_1, b_3, b_4\}$, *then*

$$W = \{b_1, b_3, b_4, a_6, a_7\} \quad or \quad W = \{b_1, b_3, b_4, a_2, a_3\}.$$

(d) *If* $W \cap B = \{b_1, b_2, b_4\}$, *then*

$$W = \{b_1, b_2, b_4, a_1, a_8\} \quad or \quad W = \{b_1, b_2, b_4, a_4, a_5\}.$$

Proof. It is sufficient to prove the proposition (a). Let $W \cap B = \{b_1, b_2, b_3\}$. From $b_1 \in W$ and $\overline{P}[W] = C_5$ it follows that $a_2 \in W$ or $a_7 \in W$. Let $a_2 \in W$. Since $N_{\overline{P}}(a_2) \cap N_{\overline{P}}(b_3) = \{a_3\}$, we have $W = \{b_1, b_2, b_3, a_2, a_3\}$. If $a_7 \in W$, then from $N_{\overline{P}}(a_7) \cap N_{\overline{P}}(b_3) = \{a_6\}$ it follows that $W = \{b_1, b_2, b_3, a_6, a_7\}$. □

4. A PROOF OF THE MAIN LEMMA

Assume that $P \nrightarrow (2,3,3)$ and let $V_1 \cup V_2 \cup V_3$ be a $(2,3,3)$-free 3-coloring of $V(P)$. From $\alpha(P) = 2$ it follows that

$$|V_1| \leq 2. \tag{2}$$

Since V_i, $i = 2,3$, contains no 3-clique, from $\alpha(P) = 2$ and $R(3,3) = 6$ it follows that

$$|V_i| \leq 5, \quad i = 2,3. \tag{3}$$

The equality $|V(P)| = 12$ together with (2) and (3) imply that $|V_1| = 2$, $|V_2| = |V_3| = 5$. We put $G_i = \overline{P}[V_i]$, $i = 2,3$. Since $\alpha(G_i) = \mathrm{cl}(G_i) = 2$, from $|V_i| = 5$, $i = 2,3$, it follows that $G_2 = G_3 = C_5$. Obviously, $\overline{P}[A] = C_8$. Hence $V_i \cap B \neq \emptyset$, $i = 2,3$. Assume that $|V_2 \cap B| \leq |V_3 \cap B|$. From $|B| = 4$ it follows that $1 \leq |V_2 \cap B| \leq 2$.

Case 1. $|V_2 \cap B| = 1$. Without a loss of generality we can assume that $V_2 \cap B = \{b_1\}$. According to Lemma 3.2(a), $V_2 = \{b_1, a_1, a_2, a_7, a_8\}$.

Subcase 1a. $|V_3 \cap B| = 1$. Suppose that $V_3 \cap B = \{b_2\}$ or $V_3 \cap B = \{b_4\}$. Then, according to Lemma 3.2, $V_2 \cap V_3 \neq \emptyset$, which is a contradiction. Let $V_3 \cap B = \{b_3\}$.

Then $V_3 = \{b_3, a_3, a_4, a_5, a_6\}$ (see Lemma 3.2(c)). Hence $V_1 = \{b_2, b_4\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 1b. $|V_3 \cap B| = 2$. According to Lemma 3.3, $V_3 \cap B = \{b_2, b_3\}$ or $V_3 \cap B = \{b_3, b_4\}$. Without a loss of generality we can assume that $V_3 \cap B = \{b_2, b_3\}$. From $V_2 \cap V_3 = \emptyset$ and Lemma 3.4(b) it follows that $V_3 = \{b_2, b_3, a_4, a_5, a_6\}$. Hence $V_1 = \{a_3, b_4\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 1c. $|V_3 \cap B| = 3$. It is clear that $V_3 \cap B = \{b_2, b_3, b_4\}$. From $V_2 \cap V_3 = \emptyset$ and Lemma 3.5(b) it follows that $V_3 = \{b_2, b_3, b_4, a_4, a_5\}$. Hence $V_1 = \{a_3, a_6\}$. This contradicts the assumption that V_1 is an independent set in P.

Case 2. $|V_2 \cap B| = 2$. It is clear that $|V_3 \cap B| = 2$. According to Lemma 3.3, we can assume that $V_2 \cap B = \{b_1, b_2\}$ and $V_3 \cap B = \{b_3, b_4\}$. Because of the Lemma 3.4(a) we have the following two subcases:

Subcase 2a. $V_2 = \{b_1, b_2, a_2, a_3, a_4\}$. From Lemma 3.4(c) and $V_2 \cap V_3 = \emptyset$ it follows that $V_3 = \{b_3, b_4, a_6, a_7, a_8\}$. Hence $V_1 = \{a_1, a_5\}$. This contradicts the assumption that V_1 is independent in P.

Subcase 2b. $V_2 = \{b_1, b_2, a_1, a_7, a_8\}$. From Lemma 3.4(c) and $V_2 \cap V_3 = \emptyset$ it follows that $V_3 = \{b_3, b_4, a_3, a_4, a_5\}$. Hence $V_2 = \{a_2, a_6\}$. This contradicts the assumption that V_1 is independent in P.

5. A PROOF OF THEOREM 2.1

It is obvious that from $G \to (2,3,3)$ it follows that $G \to (2,2,2,3)$. Therefore

$$F(2,2,2,3;5) \leq F(2,3,3;5).$$

From the above inequality it becomes clear that it is sufficient to prove that $F(2,3,3;5) \leq 12$ and $F(2,2,2,3;5) \geq 12$.

1. Proof of the inequality $F(2,3,3;5) \leq 12$. According to the main Lemma, $P \to (2,3,3)$. Since $\text{cl}(P) = 4$ and $|V(P)| = 12$, we have $F(2,3,3;5) \leq 12$.

2. Proof of the inequality $F(2,2,2,3;5) \geq 12$. According to Theorem B, $F(2,2,2,3;5) \geq 11$. Assume that $F(2,2,2,3;5) = 11$ and let G be a graph such that $|V(G)| = 11$, $\text{cl}(G) < 5$ and $G \to (2,2,2,3)$. From Theorem C(a) it follows that $\alpha(G) \leq 3$. According to Theorem C(b), $\alpha(G) \neq 3$. Hence

$$\alpha(G) = 2. \tag{4}$$

Assume that there exist $u, v \in V(G)$ such that $N_G(u) \supseteq N_G(v)$. It is clear that $\{u, v\} \notin E(G)$. From $F(2,2,2,3;5) \geq 11$ it follows that $G - v \nrightarrow (2,2,2,3)$. Consider an arbitrary $(2,2,2,3)$-free 4-coloring of $G - v$. If we color the vertex v with the same color as the vertex u, we will obtain $(2,2,2,3)$-free 4-coloring of G, which is a contradiction. Therefore:

$$N_G(v) \not\subseteq N_G(u) \quad \text{for all } u, v \in V(G). \tag{5}$$

If $|N_G(v)| = |V(G)| - 1$ for some $v \in V(G)$, then $\mathrm{cl}(G - v) < 4$ and $G - v \rightarrow (2, 2, 2, 2)$. This contradicts Theorem D. Hence, $|N_G(v)| \neq |V(G)| - 1$, $\forall v \in V(G)$. This, together with (5) imply that

$$|N_G(v)| \leq |V(G)| - 3 \quad \text{for all } v \in V(G). \tag{6}$$

From $F(2, 2, 4; 5) = 13$, [11], it follows that $G \nrightarrow (2, 2, 4)$. Let $V_1 \cup V_2 \cup V_3$ be $(2, 2, 4)$-free 3-coloring of $V(G)$. It follows from (4) that $|V_1| \leq 2$, $|V_2| \leq 2$. According to (6) and (4), we may assume that $|V_1| = |V_2| = 2$. We put $G_1 = G[V_3]$. It is clear that from $G \rightarrow (2, 2, 2, 3)$ it follows that $G_1 \rightarrow (2, 3)$. According to Theorem A, $G_1 = \overline{C}_7$ (Fig. 2). Let $V_1 = \{a, b\}$, $V_2 = \{c, d\}$ and $G_2 = G[a, b, c, d]$. From (4) it follows that $E(G_2)$ contains two independent edges. Without a loss of generality we can assume that $[a, c]$, $[b, d] \in E(G_2)$. It is sufficient to consider the next two cases.

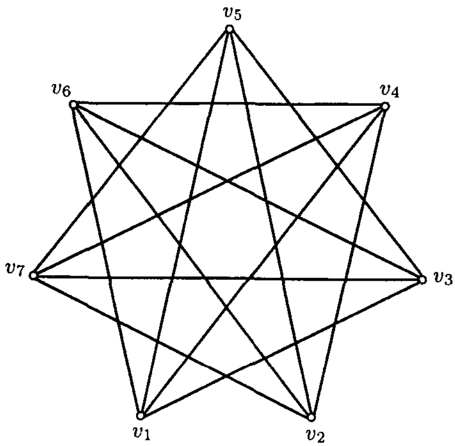

Fig. 2. Graph $\overline{C}_7$

Case 1. $E(G_2) = \{[a, c], [b, d]\}$. From $\mathrm{cl}(G) < 5$ it follows that one of the vertices a, c is not adjacent to some of the vertices $v_1, \ldots, v_7$ (see Fig. 2). Without a loss of generality we may assume that v_1 and a are not adjacent. Consider the 4-coloring

$$\{v_4, v_5\} \cup \{v_6, v_7\} \cup \{c, d\} \cup \{v_1, v_2, v_3, a, b\}.$$

Since $G \rightarrow (2, 2, 2, 3)$, we have that $\{v_1, v_2, v_3, a, b\}$ contains a 3-clique. Hence $v_1, v_3 \in N_G(b)$. Similarly, $v_1, v_6 \in N_G(b)$. So, $v_1, v_3, v_6 \in N_G(b)$. Similarly, $v_1, v_3, v_6 \in N_G(d)$. Hence $\{v_1, v_3, v_6, b, d\}$ is a 5-clique, which is a contradiction.

Case 2. $E(G_2) \supseteq \{[a, c], [b, d], [a, d]\}$. As in case 1, we may assume that a and v_1 are not adjacent. Then from (4) it follows that $v_2, v_7 \in N_G(a)$. From (4) it follows also that a is adjacent to some of the vertices v_4, v_5. Without a loss of generality we may assume that v_4 and a are adjacent. So,

$$v_2, v_4, v_7 \in N_G(a). \tag{7}$$

From (7) and $\mathrm{cl}(G) < 5$ it follows that d is not adjacent to any of the vertices v_2, v_4, v_7. Hence, it is sufficient to consider the next three subcases.

Subcase 2a. The vertex d is not adjacent to v_2. Consider the 4-coloring

$$\{v_5, v_6\} \cup \{v_1, v_7\} \cup \{a, b\} \cup \{v_2, v_3, v_4, c, d\} \tag{8}$$

of $V(G)$. From $G \to (2,2,2,3)$ it follows that $\{v_2, v_3, v_4, c, d\}$ contains a 3-clique. Hence, $v_2, v_4 \in N_G(c)$. Similarly, $v_2, v_7 \in N_G(c)$. From (7) it follows that $\{v_2, v_4, v_7, a, c\}$ is a 5-clique, which contradicts $\mathrm{cl}(G) < 5$.

Subcase 2b. The vertex d is not adjacent to v_4. Consider the 4-coloring (8). As in the subcase 2a it follows that $v_2, v_4 \in N_G(c)$. Similarly, from the 4-coloring

$$\{v_1, v_7\} \cup \{v_2, v_3\} \cup \{a, b\} \cup \{v_4, v_5, v_6, c, d\}$$

it follows that $v_4, v_6 \in N_G(c)$. So,

$$v_2, v_4, v_6 \in N_G(c). \tag{9}$$

According to (7), (9) and $\mathrm{cl}(G) < 5$, the vertex c is not adjacent to v_7. Consider the 4-coloring

$$\{v_3, v_4\} \cup \{v_5, v_6\} \cup \{a, b\} \cup \{v_1, v_2, v_7, c, d\}.$$

Since $G \to (2,2,2,3)$, then $\{v_1, v_2, v_7, c, d\}$ contains a 3-clique. Hence, $v_2, v_7 \in N_G(d)$. Similarly, from $G \to (2,2,2,3)$ and the 4-coloring

$$\{v_1, v_2\} \cup \{v_3, v_4\} \cup \{a, b\} \cup \{v_5, v_6, v_7, c, d\}$$

it follows that $v_5, v_7 \in N_G(d)$. Then

$$v_2, v_5, v_7 \in N_G(d). \tag{10}$$

From (7), (9) and $\mathrm{cl}(G) < 5$ it follows that a and v_6 are not adjacent. From (7), (10) and $\mathrm{cl}(G) < 5$ it follows that a and v_5 are not adjacent. So, the vertex a is not adjacent to v_5 and v_6, which contradicts (4).

Subcase 2c. The vertex d is not adjacent to v_7. This subcase is analogous with subcase 2b.

6. THE EXTREMAL GRAPHS

By $G - e$, $e \in E(G)$, we denote the subgraph of G such that $V(G-e) = V(G)$ and $E(G-e) = E(G) \setminus \{e\}$.

Consider the graph $\overline{P}$ from Fig. 1. For this graph we set: $P_0 = P$, $P_1 = P - [a_1, a_6]$, $P_2 = P - [a_1, a_5]$, $P_3 = P_1 - [a_2, a_5]$, $P_4 = P_1 - [a_4, a_7]$, $P_5 = P_1 - [a_3, a_7]$, $P_6 = P_2 - [a_4, a_8]$, $P_7 = P_2 - [a_3, a_7]$, $P_8 = P_3 - [a_4, a_7]$, $P_9 = P_7 - [a_2, a_6]$, $P_{10} = P_8 - [a_3, a_8]$, $P_{11} = P_9 - [a_4, a_8]$.

We need the next theorem.

Theorem E, [17]. *Let the graph G be such that $|V(G)| = 12$, $\mathrm{cl}(G) = 4$ and $\alpha(G) = 2$. Then G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$.*

Definition 6.1. We say that the graph G is extremal if $|V(G)| = 12$, $\mathrm{cl}(G) < 5$, $G \to (2,3,3)$ or $G \to (2,2,2,3)$.

According to Theorem C(a), if G is extremal, then $\alpha(G) \leq 4$. From Theorem C(b) it follows that $\alpha(G) \neq 4$. Hence $\alpha(G) = 2$ or $\alpha(G) = 3$. In this section we describe all critical graphs G with $\alpha(G) = 2$.

Theorem 6.1. *Let G be extremal graph such that $G \to (2,3,3)$ and $\alpha(G) = 2$. Then G is isomorphic to the graph P.*

Proof. According to Theorem E, the graph G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$. The 3-coloring

$$\{a_7, a_8\} \cup \{b_2, a_1, a_4, a_5, a_6\} \cup \{b_1, b_3, b_4, a_2, a_3\}$$

of P_1 is $(2,3,3)$-free and the 3-coloring

$$\{a_1, a_5\} \cup \{b_1, b_2, a_2, a_3, a_4\} \cup \{b_3, b_4, a_6, a_7, a_8\}$$

of P_2 is $(2,3,3)$-free. Hence G is not a subgraph of P_1 and P_2. Thus $G = P$. □

Theorem 6.2. $P_i \to (2,2,2,3)$, $i = 0, \dots, 11$. *If an extremal graph G is such that $G \to (2,2,2,3)$ and $\alpha(G) = 2$, then G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$.*

Proof. Let $V_1 \cup V_2 \cup V_3 \cup V_4$ be a 4-coloring of $V(P_i)$ and V_i, $i = 1,2,3$, be independent. From $\alpha(P_i) = 2$ it follows that $|V_i| \leq 2$, $i = 1,2,3$. Hence $|V_4| \geq 6$. From $\alpha(P_i) = 2$ and $R(3,3) = 6$ it follows that V_4 contains a 3-clique. Thus P_i does not have a $(2,2,2,3)$-free 4-coloring and hence $P_i \to (2,2,2,3)$. According to Theorem E, the graph G is isomorphic to one of the graphs P_i, $i = 0, \dots, 11$. □

7. THE VERTEX FOLKMAN NUMBERS $F(2, \dots, 2, p; q)$ AND THE RAMSEY NUMBERS $R(3, q)$

Theorem 7.1. *Let $p \geq 2$, r and q be positive integers such that*

$$R(3,p) + 2r < R(3,q). \tag{11}$$

Then $F(\underbrace{2, \dots, 2}_{r}, p; q) \leq R(3,p) + 2r$.

Proof. Let G be a graph such that $|V(G)| = R(3,p) + 2r$, $\mathrm{cl}(G) < q$ and

$$\alpha(G) = 2. \tag{12}$$

According to (11), the graph G exists. Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-coloring of $V(G)$. Suppose that V_i, $i = 1, \ldots, r$, are independent. From (12) it follows that $|V_i| \leq 2$, $i = 1, \ldots, r$. Hence $|V_{r+1}| \geq R(3,p)$. According to the definition of $R(3,p)$ and (12), V_{r+1} contains a p-clique. Thus G does not have a $(\underbrace{2,\ldots,2}_{r},p)$-free coloring and hence $G \to (\underbrace{2,\ldots,2}_{r},p)$. From $\text{cl}(G) < q$ and $|V(G)| = R(3,p) + 2r$ it follows that $F(\underbrace{2,\ldots,2}_{r},p;q) \leq R(3,p) + 2r$. □

Consider the table of the known Ramsey numbers $R(3,p)$, [19]:

p	3	4	5	6	7	8	9	10
$R(3,p)$	6	9	14	18	23	28	36	40–43

From this table and Theorem 7.1 it follows:

$F(2,2,4;5) \leq 13$ (in [11] it is proved $F(2,2,4;5) = 13$);
$F(2,2,6;7) \leq 22$ (in [11] it is proved $F(2,2,6;7) \leq 26$);
$F(2,2,7;8) \leq 27$ (in [11] it is proved $F(2,2,7;8) \leq 30$);
$F(2,2,8;9) \leq 32$ (in [11] it is proved $F(2,2,8;9) \leq 34$);
$F(2,2,9;10) \leq 40$ if $R(3,10) \neq 40$ (in [11] it is proved $F(2,2,9;10) \leq 38$);
$F(2,2,2,3;5) \leq 12$ (according to Theorem 2.1, $F(2,2,2,3;5) = 12$);
$F(2,2,2,5;7) \leq 20$ (in [11] it is proved $F(2,2,2,5;7) \leq 23$).

8. ON THE NUMBERS $F(\underbrace{2,\ldots,2}_{r},p;p+r-1)$

We put $F(\underbrace{2,\ldots,2}_{r},p;p+r-1) = F_r(2,p)$.

The proof of Theorem 5 from [13] establishes the following statement:

Theorem F. *Let $G \to (\underbrace{2,\ldots,2}_{s},p)$. Then $K_r + G \to (\underbrace{2,\ldots,2}_{r+s},p)$ for any r.*

From Theorem 2.1, Theorem F and Theorem B it follows that

$$r + 8 \leq F_r(2,3) \leq r + 9, \quad r \geq 3.$$

The exact value of $F_2(2,3) = F(2,2,3;4)$ is unknown.

From Theorem B, Theorem F and the inequalities $F_2(2,6) \leq 22$, $F_2(2,7) \leq 27$, $F_2(2,8) \leq 32$ and $F(2,2,2,5;7) \leq 20$ it follows that

$$r + 14 \leq F_r(2,6) \leq r + 20, \quad r \geq 2;$$
$$r + 16 \leq F_r(2,7) \leq r + 25, \quad r \geq 2;$$

$$r + 18 \leq F_r(2,8) \leq r + 30, \quad r \geq 2;$$
$$r + 12 \leq F_r(2,5) \leq r + 17, \quad r \geq 3.$$

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Received on October 17, 2001

Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

ON A CONJECTURE FOR THE IDENTITIES IN MATRIX ALGEBRAS WITH INVOLUTION [1]

TSETSKA GRIGOROVA RASHKOVA

In a previous paper the author made a conjecture on the minimal degree $4n$ of the polynomials, which are identities for the matrix algebra of order $2n$ with symplectic involution considered as polynomials both in symmetric and skew-symmetric due to the involution variables.
In the present paper we establish that the conjecture is not true at least for the case of the matrix algebra of fourth order by giving an example of such an identity of degree seven, which is a Bergman type identity.
For the matrix algebra of sixth order with symplectic involution we describe the class of all Bergman type identities both in symmetric and skew-symmetric variables of minimal degree (which appeared to be 14). For arbitrary polynomials being identities of the considered type the question of their minimal degree is still open.

Keywords: Bergman type polynomials, symplectic involution, *-identities in symmetric or skew-symmetric variables

2000 MSC: 16R50, 16R10

Let K be a field of characteristics zero with elements α, β, etc. We call Bergman type polynomials the following class of homogeneous polynomials introduced by Formanek [2] and Bergman [1] for investigating matrix identities by means of commutative algebra.

To a homogeneous polynomial in commuting variables

$$g(t_1, \dots, t_{n+1}) = \sum \alpha_p t_1^{p_1} \dots t_{n+1}^{p_{n+1}} \in K[t_1, \dots, t_{n+1}] \qquad (1)$$

[1]Partially supported by the Bulgarian Foundation for Scientific Research under Grant MM1106/2001.

we relate a polynomial $v(g)$ from the free associative algebra $K\langle x, y_1, \ldots, y_n\rangle$,

$$v(g) = v(g)(x, y_1, \ldots, y_n) = \sum \alpha_p x^{p_1} y_1 \ldots x^{p_n} y_n x^{p_{n+1}}. \tag{2}$$

Any homogeneous and multilinear in $y_1, \ldots, y_n$ polynomial $f(x, y_1, \ldots, y_n)$ can be written as

$$f(x, y_1, \ldots, y_n) = \sum_{i=(i_1,\ldots,i_n)\in Sym(n)} v(g_i)(x, y_{i_1}, \ldots, y_{i_n}), \tag{3}$$

where $g_i \in K[t_1, \ldots, t_{n+1}]$.

We consider Bergman type polynomials on subalgebras of the matrix algebra $M_{2n}(K, *)$ with symplectic involution defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}, \tag{4}$$

where A, B, C, D are $(n \times n)$-matrices and t is the usual transpose.

Details on polynomial and $*$-polynomial identities one may find in [6, 4].

For an algebra R with involution $*$ we have $(R, *) = R^+ \oplus R^-$, where $R^+ = \{r \in R \mid r^* = r\}$ and $R^- = \{r \in R \mid r^* = -r\}$. Let $K\langle X\rangle$ be the free associative algebra. We call $f(x_1, \ldots, x_m) \in K\langle X\rangle$ a $*$-polynomial identity for the algebra $(R, *)$ both in symmetric and skew-symmetric variables if $f(r_1^+, \ldots, r_m^+) = 0$ for all $r_1^+, \ldots, r_m^+ \in R^+$ and $f(r_1^-, \ldots, r_m^-) = 0$ for all $r_1^-, \ldots, r_m^- \in R^-$.

One of the reasons to study this kind of identities is the following. The algebra R^- is a Lie algebra with respect to the new multiplication $[r_1^-, r_2^-] = r_1^- r_2^- - r_2^- r_1^-$, $r_1^-, r_2^- \in R^-$, and the identities in skew-symmetric variables for $(R, *)$ are weak polynomial identities for the pair (R, R^-), i.e. the identities of the related representation of R^-. Similarly, R^+ is a Jordan algebra with respect to the multiplication $r_1^+ \circ r_2^+ = r_1^+ r_2^+ + r_2^+ r_1^+$, $r_1^+, r_2^+ \in R^+$, and the identities in symmetric variables are weak polynomial identities for the pair (R, R^+). In this way, the identities both in symmetric and skew-symmetric variables are the weak identities which hold for both pairs (R, R^-) and (R, R^+).

In [5] we have discussed the minimal degree of such identities and have made the following conjecture:

Conjecture [5, Conjecture 3.1]. The minimal degree of a $*$-identity both in symmetric and skew-symmetric variables in $M_{2n}(K, *)$ for $n \geq 2$ is equal to $4n$.

In the present paper we show that *this conjecture is not true* (at least for $n = 2$). Considering the matrix algebra $M_4(K, *)$, we give an example of a Bergman type polynomial, which is a $*$-identity both in symmetric and skew-symmetric variables of degree 7.

For the matrix algebra of sixth order with symplectic involution we describe the class of all Bergman type identities both in symmetric and skew-symmetric variables of degree 14, which is their minimal degree.

These considerations are consequences of the following main theorem:

Theorem 1. *Any Bergman type identity in skew-symmetric variables for the algebra $M_{2n}(K,*)$ with symplectic involution is a $*$-identity in symmetric variables as well.*

For proving the theorem we need some preliminary results.

Proposition 1. [4, Theorem 1] *Let a polynomial $f(x, y_1, \ldots, y_n)$ of type* (3) *be a $*$-identity in skew-symmetric variables for $M_{2n}(K,*)$. Then the polynomial*

$$\prod_{\substack{1 \le p < q \le n+1 \\ (p,q) \ne (1,n+1)}} (t_p^2 - t_q^2)(t_1 - t_{n+1})$$

divides the polynomials g_i from (1) *for all $i = (i_1, \ldots, i_n)$.*

Lemma 1. *Any generic symmetric matrix (with respect to the symplectic involution) is diagonalizable.*

Proof. It is a well-known fact [6, Theorem 2.5.10] that a $(2n \times 2n)$-matrix x, symmetric with respect to the symplectic involution, satisfies an equation $p(x) = 0$ of degree n and its characteristic polynomial is $p^2(x)$. Thus any generic symmetric matrix has at most n different characteristic values. A generic symmetric matrix a of order $2n$ has the following presentation in the form (4): A is a generic $(n \times n)$-matrix, D is its transpose, and B and C are generic skew-symmetric $(n \times n)$-matrices. If it has less than n different characteristic values, then the same will hold for any generic symmetric matrix and also for

$$x = \sum_{i=1}^{n} \rho_i (e_{ii} + e_{n+i,n+i}), \tag{5}$$

where ρ_i are algebraically independent variables. This is a contradiction, because x has n different characteristic values. Thus any generic symmetric matrix is diagonalizable and we may consider x in the form (5).

Lemma 2. *For any polynomial $g(t_1, \ldots, t_{n+1})$ of type* (1) *divisible by the product $\prod_{i<j}(t_i - t_j)$, the associated polynomial $f(x, y_1, \ldots, y_n)$ of type* (3) *vanishes on $M_{2n}(K,*)$ for x being symmetric due to the involution and for arbitrary $y_1, \ldots, y_n \in M_{2n}(K,*)$.*

Proof. According to Lemma 1, the matrix x can be replaced by $\bar{x} = \sum_{i=1}^{n} \rho_i (e_{ii} + e_{n+i,n+i})$. The linearity of f in $y_1, \ldots, y_n$ allows to consider the variables $y_i, i = 1, \ldots, n$, as matrix units from $M_{2n}(K,*)$ with nonzero product, namely $\bar{y_i} = e_{p_i p_{i+1}}$ for $p_i = 1, \ldots, 2n$. Thus we get

$$f(\bar{x}, \bar{y_1}, \ldots, \bar{y_n}) = v(g)(\bar{x}, \bar{y_1}, \ldots, \bar{y_n}) = g(\bar{\rho}_{p_1}, \ldots, \bar{\rho}_{p_n}, \bar{\rho}_{p_{n+1}}) e_{p_1 p_{n+1}},$$

where $\bar{\rho}_p = \rho_p$ if $p \leq n$, and $\bar{\rho}_p = \rho_{p-n}$ if $n+1 \leq p \leq 2n$. Since $\bar{\rho}_{p_i} \in \{\rho_1, \dots, \rho_n\}$, $i = 1, \dots, n+1$, we have $\bar{\rho}_{p_i} = \bar{\rho}_{p_j}$ for some $i \neq j$. This means that $g(\bar{\rho}_{p_1}, \dots, \bar{\rho}_{p_n}, \bar{\rho}_{p_{n+1}}) = 0$ as $t_i - t_j$ divides the polynomial $g(t_1, \dots, t_{n+1})$.

Proof of Theorem 1. Let f be a Bergman type identity in skew-symmetric variables. According to Proposition 1, the considered polynomial in commuting variables is divisible by the product $\prod_{i<j}(t_i - t_j)$, $1 \leq i < j \leq n+1$. Lemma 2 gives that f is a Bergman type identity in symmetric variables as well.

It is known [3, pp. 318–319] that $[[x_1^2, x_2]^2, x_1] = 0$ is a $*$-identity of minimal degree in skew-symmetric variables for $M_4(K,*)$.

The following proposition illustrates Theorem 1.

Proposition 2. *The linearization in x_2 of the polynomial $[[x_1^2, x_2]^2, x_1]$ is a Bergman type $*$-identity for $M_4(K,*)$ both in symmetric and skew-symmetric variables.*

Proof. The proposition follows immediately from Theorem 1, because $[[x_1^2, x_2]^2, x_1]$ is an identity in skew-symmetric variables for $M_4(K,*)$ and its linearization in x_2 is of Bergman type. We shall give an alternative proof and shall show that it is a consequence of an identity of special form.

First, for a Bergman type identity in symmetric variables of degree n we describe the general form of its consequence in symmetric variables of degree $n+1$.

Let

$$\begin{aligned} f(x, y_1, y_2) &= v(g_{(1,2)})(x, y_1, y_2) + v(g_{(2,1)})(x, y_2, y_1) \\ &= v(g_1)(x, y_1, y_2) + v(g_2)(x, y_2, y_1) \\ &= f_1(x, y_1, y_2) + f_2(x, y_2, y_1) \end{aligned} \tag{6}$$

be a Bergman type $*$-identity in symmetric variables. We consider its consequence (in symmetric variables)

$$\begin{aligned} A &= \alpha f(y_1 = xy_1 + y_1x) + \beta f(y_2 = xy_2 + y_2x) + \gamma(xf + fx) \\ &= \alpha[f(y_1 = xy_1) + f(y_1 = y_1x)] \\ &+ \beta[f(y_2 = xy_2) + f(y_2 = y_2x)] + \gamma(xf + fx) \\ &= \alpha x f_1 + \alpha f_2(y_1 = xy_1) + \alpha f_1(y_1 = y_1x) + \alpha f_2 x \\ &+ \beta f_1(y_1 = y_1x) + \beta x f_2 + \beta f_1 x + \beta f_2(y_1 = xy_1) \\ &+ \gamma x f_1 + \gamma x f_2 + \gamma f_1 x + \gamma f_2 x = A_1(x, y_1, y_2) + A_2(x, y_2, y_1). \end{aligned}$$

The commutative polynomials corresponding to the parts $A_1(x, y_1, y_2)$ and $A_2(x, y_2, y_1)$ are respectively

$$\begin{aligned} g_{11} = g_{(1,2)} &= [(\alpha+\gamma)t_1 + (\alpha+\beta)t_2 + (\beta+\gamma)t_3]g_1, \\ g_{21} = g_{(2,1)} &= [(\beta+\gamma)t_1 + (\alpha+\beta)t_2 + (\alpha+\gamma)t_3]g_2. \end{aligned} \tag{7}$$

1. For $\alpha = 1, \beta = \gamma = 0$ (7) gives

$$g_{11} = g_1' = (t_1 + t_2)g_1,$$
$$g_{21} = g_2' = (t_2 + t_3)g_2.$$

2. For $\alpha = \gamma = 0,\ \beta = 1$ we get

$$g_{11} = g_1'' = (t_2 + t_3)g_1,$$
$$g_{21} = g_2'' = (t_1 + t_2)g_2.$$

The linearization in y of the pointed in [5, Part 3] identity $[[x, y]^2, x] = 0$ in symmetric variables is a Bergman type identity of type (3) for which (following (6))

$$g_{(1,2)} = g_{(2,1)} = g_0 = (t_1 - t_2)(t_1 - t_3)(t_2 - t_3). \tag{8}$$

The linearization in y of the identity $[[x^2, y]^2, x] = 0$ (in skew-symmetric variables) is a Bergman type identity of type (3). In this case (7) gives

$$\begin{aligned} g_{(1,2)} = g_{(2,1)} = g &= (t_1^2 - t_2^2)(t_1 - t_3)(t_2^2 - t_3^2) \\ &= (t_1 + t_2)(t_2 + t_3)g_0. \end{aligned} \tag{9}$$

We want to show that the linearization of $[[x^2, y]^2, x] = 0$, which corresponds to (9), is a consequence in symmetric variables of the linearization of $[[x, y]^2, x] = 0$, corresponding to (8).

Applying the first case 1 to the identity in symmetric variables

$$f(x, y_1, y_2) = v(g_0)(x, y_1, y_2) + v(g_0)(x, y_2, y_1),$$

we obtain the identity $f'(x, y_1, y_2)$ for which $g_1' = (t_1 + t_2)g_0$ and $g_2' = (t_2 + t_3)g_0$.

Now we apply the second step to $f'(x, y_1, y_2)$ and get $f''(x, y_1, y_2)$ for which $g_1'' = (t_2 + t_3)(t_1 + t_2)g_0$ and $g_2'' = (t_1 + t_2)(t_2 + t_3)g_0$. Thus $f''(x, y_1, y_2)$ is a consequence of $f(x, y_1, y_2)$ in symmetric variables and holds for $M_4(K, *)$ since $g_1'' = g_2'' = g$ is as in (9).

We point that the identity in skew-symmetric variables of degree 7 [3, Theorem 5] is not a $*$-identity in symmetric variables. Hence the Bergman type of the considered polynomials is essential.

Theorem 2. *All Bergman type polynomials f of degree 7, which are $*$-identities both in symmetric and skew-symmetric variables, are of the form kf_0, $k \in K$, for $f_0(x, y_1, y_2) = f_1 + f_2 = v(g)(x, y_1, y_2) + v(g)(x, y_2, y_1)$, where $g = (t_1^2 - t_2^2)(t_1 - t_3)(t_2^2 - t_3^2)$.*

Proof. According to Theorem 1, Proposition 1 and the notations in Theorem 2, we write f as $f = \alpha f_1 + \beta f_2$ for $\alpha, \beta \in K$. As $f(y_1 \leftrightarrow y_2) = 0$ is an identity too, we get $\alpha f_2 + \beta f_1 = 0$. It means that $(\alpha - \beta)f_1 + (\beta - \alpha)f_2 = 0$. If $\alpha - \beta \neq 0$, then $f_1 - f_2 = 0$ is an identity for $M_4(K, *)$. The identity given before Proposition

2 leads to the identity $f_1 + f_2 = 0$. Thus $f_1 = 0$ and $f_2 = 0$ are identities in skew-symmetric variables. Calculating $f_i(\rho_1(e_{11} - e_{33}) + \rho_2(e_{22} - e_{44}), e_{12} - e_{43}, e_{14} + e_{23})$ for $i = 1, 2$, we get

$$\begin{aligned} f_1 &= 2\rho_1(\rho_1^2 - \rho_2^2)^2 e_{13} \neq 0, \\ f_2 &= -2\rho_1(\rho_1^2 - \rho_2^2)^2 e_{13} \neq 0, \end{aligned}$$

a contradiction. Thus $\alpha - \beta = 0$ and $f = kf_0$.

Now we continue the investigations in $M_6(K, *)$.

For $n = 3$ the commutative polynomial in Proposition 1 is denoted by g_0, and $v(g_0)(x, y_{i_1}, y_{i_2}, y_{i_3})$ – by $f_0(x, y_{i_1}, y_{i_2}, y_{i_3})$.

Proposition 3. [4, Theorem 3] *The polynomial*

$$P(x, y_1, y_2, y_3) = f_0(x, y_{i_1}, y_{i_2}, y_{i_3}) + f_0(x, y_{i_3}, y_{i_2}, y_{i_1})$$

for all (i_1, i_2, i_3) *is a* $*$*-identity of degree* 14 *in skew-symmetric variables for* $M_6(K, *)$.

Using the notations
$f_1 = v(g_0)(x, y_1, y_2, y_3)$, $f_2 = v(g_0)(x, y_1, y_3, y_2)$,
$f_3 = v(g_0)(x, y_2, y_1, y_3)$, $f_4 = v(g_0)(x, y_3, y_2, y_1)$,
$f_5 = v(g_0)(x, y_2, y_3, y_1)$ and $f_6 = v(g_0)(x, y_3, y_1, y_2)$,
we get that

$$\alpha(f_1 + f_4) + \beta(f_2 + f_5) + \gamma(f_3 + f_6) = 0, \quad \alpha, \beta, \gamma \in K, \tag{10}$$

is a $*$-identity in skew-symmetric variables.

We describe the class of all Bergman type identities of degree 14 in skew-symmetric variables for $M_6(K, *)$.

Theorem 3. *All Bergman type* $*$*-identities of degree* 14 *in skew-symmetric variables for* $M_6(K, *)$ *have the form* (10).

Proof. According to Proposition 1, any identity of the considered type has the form $\sum_{i=1}^{6} \alpha_i f_i$. As $\alpha_1(f_1 + f_4) = 0$, $\alpha_2(f_2 + f_5) = 0$ and $\alpha_3(f_3 + f_6) = 0$ are identities in skew-symmetric variables, we get the identity

$$f = (\alpha_4 - \alpha_1)f_4 + (\alpha_5 - \alpha_2)f_5 + (\alpha_6 - \alpha_3)f_6 = 0.$$

Considering $f(y_1 \leftrightarrow y_2) = 0$, $f(y_2 \leftrightarrow y_3) = 0$ and $f(y_3 \leftrightarrow y_1) = 0$, we get the following system:

$$\begin{aligned} (\alpha_4 - \alpha_1)f_4 + (\alpha_5 - \alpha_2)f_5 + (\alpha_6 - \alpha_3)f_6 = 0, \\ (\alpha_4 - \alpha_1)f_6 + (\alpha_5 - \alpha_2)f_2 + (\alpha_6 - \alpha_3)f_4 = 0, \\ (\alpha_4 - \alpha_1)f_5 + (\alpha_5 - \alpha_2)f_4 + (\alpha_6 - \alpha_3)f_3 = 0, \\ (\alpha_4 - \alpha_1)f_1 + (\alpha_5 - \alpha_2)f_3 + (\alpha_6 - \alpha_3)f_2 = 0. \end{aligned}$$

Using the notations $a = (\alpha_4 - \alpha_1)$, $b = (\alpha_5 - \alpha_2)$ and $c = (\alpha_6 - \alpha_3)$ and Proposition 3, the matrix A of the considered homogeneous system for the unknowns f_4, f_5 and f_6 has the form

$$\begin{pmatrix} a & b & c \\ c & -b & a \\ b & a & -c \\ -a & -c & -b \end{pmatrix}.$$

Elementary transformations on the matrix and *Mathematica* calculations prove that special polynomials are not $*$-identities in skew-symmetric variables and they lead to the only trivial solution for a, b, c. Thus $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_5$ and $\alpha_3 = \alpha_6$, and we get (10).

The main Theorem 1 applied for $M_6(K, *)$ gives

Proposition 4. *All Bergman type $*$-identities of degree* 14 *both in symmetric and skew-symmetric variables for $M_6(K, *)$ have the form* (10).

Acknowledgments. The author is grateful to V. Drensky for his essential comments on the text improving the final version of the paper.

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Received on November 20, 2001

Centre of Applied Mathematics and Informatics
"A.Kanchev" University of Rousse
8, Studentska str., 7017 Rousse
BULGARIA
E-mail: tcetcka@ami.ru.acad.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

A GENERALIZATION OF REDFIELD'S MASTER THEOREM [1]

VALENTIN VANKOV ILIEV

Generalizations of Redfield's master theorem and the superposition theorem are proved by using decomposition of the tensor product of several induced monomial representations of the symmetric group S_d into transitive constituents. As direct consequences, several corollaries concerning superpositions of graphs are obtained.

Keywords: monomial representations of the symmetric group, automorphism groups of superpositions of graphs

2000 MSC: 20C30, 05A15, 05C30

INTRODUCTION

In the present paper we prove a generalization of Redfield's master theorem as a direct consequence of the decomposition of the tensor product of several induced monomial representations of the symmetric group into its transitive summands. The underlying permutation representations give rise to the original Redfield's group-reduced distributions, or, equivalently, to Read's equivalence relation of "T-similarity" and superpositions. The most important examples of superpositions are the superpositions of several graphs $\Gamma_1, \dots, \Gamma_k$, each on the same number of vertices. A superposition of $\Gamma_1, \dots, \Gamma_k$ is a graph that is obtained by superposing Γ_m on the same set of vertices and by keeping their edges apart. The superposition theorem counts the number of superpositions of the graphs Γ_m in terms of their

[1]Research partially supported by the Bulgarian Foundation of Scientific Research under Grant MM-1106/2001.

automorphism groups $W_m \leq S_d$. The corresponding generalization enables us to count those superpositions whose automorphism groups have certain properties in case one of the automorphism groups W_m, $m = 1, \dots, k$, has an one-dimensional character of special type.

The paper is stratified as follows. In Section 1 we note that the tensor product of several induced monomial representations of the symmetric group S_d is a monomial one. Then we discuss the corresponding permutation representation of S_d and, in particular, show that there is a canonical bijection between the S_d-orbit space thus obtained and the set of Read's equivalence classes from [4, Sec. 3]. Section 2 contains two equivalent statements that generalize Redfield's master theorem and a generalization of the superposition theorem. In Section 3 we find the number of all superpositions with certain properties of several graphs.

1. TENSOR PRODUCT OF INDUCED MONOMIAL REPRESENTATIONS OF S_d

Throughout the paper we assume that K is an algebraically closed field of characteristic zero and that all group characters are K-valued.

Let R^d be the Abelian group consisting of all generalized characters of the symmetric group S_d, and let Λ^d be the Abelian group of homogeneous degree d symmetric functions with integer coefficients in a countable set of variables $x_0, x_1, x_2, \dots$ If u, $v \in R^d$, we denote by $\langle u, v \rangle$ their (integer-valued) scalar product. According to [3, Ch. I, Sec. 4], we can define an integer-valued scalar product $\langle\ ,\ \rangle$ on the group Λ^d, such that the characteristic map $ch\colon R^d \to \Lambda^d$ (see [3, Ch. I, Sec. 7]) is an isometric isomorphism of Abelian groups.

Let $W \leq S_d$ be a permutation group and $\chi\colon W \to K$ be an one-dimensional character. We set

$$Z(\chi; p_1, \dots, p_d) = \frac{1}{|W|} \sum_{\sigma \in W} \chi(\sigma) p_1^{c_1(\sigma)} \dots p_d^{c_d(\sigma)},$$

where $p_s = \sum_{i=0}^{\infty} x_i^s$ are the power sums, and $c_s(\sigma)$ is the number of cycles of length s in the cyclic decomposition of the permutation σ. The symmetric function $Z(\chi) = Z(\chi; p_1, \dots, p_d)$ is said to be generalized cyclic index of the group W. For $\zeta \in S_d$, we denote by $\varrho(\zeta)$ the corresponding partition $(1^{c_1(\zeta)}, \dots, d^{c_d(\zeta)})$ of the natural number d.

The tensor product of two finite-dimensional K-linear representations of S_d with characters u and v has character uv. If $f = ch(u)$ and $g = ch(v)$, where u and v are generalized characters of S_d, one defines internal product $f * g$ of two symmetric functions $f, g \in \Lambda^d$ by $f * g = ch(uv)$. With respect to the internal product, the Abelian group Λ^d becomes a commutative and associative ring such that the complete symmetric function $h_d = ch(1_{S_d})$ is an identity element (see [3, Ch. I, Sec. 7]).

Let W be a subgroup of the symmetric group S_d and let $\chi: W \to K$ be an one-dimensional character of W. The field K has a natural structure of left KW-module given by $\sigma c = \chi(\sigma)c$, where $\sigma \in W$, $c \in K$. We denote by K_χ the corresponding one-dimensional K-linear representation of W. Let I be a left transversal of W in S_d. The induced monomial representation $Ind_W^{S_d}(\chi) = KS_d \otimes_{KW} K_\chi$ has a natural basis $(e_i)_{i\in I}$, $e_i = i \otimes 1$, as a K-linear space. Since for any $\zeta \in S_d$ and $i \in I$ there exist unique $j \in I$ and $\sigma \in W$ such that $\zeta i = j\sigma$, we obtain a group homomorphism $s: S_d \to S(I)$ defined by the formula

$$(s(\zeta)(i))^{-1}\zeta i \in W.$$

Moreover, the permutation group $s(S_d)$ is transitive on the set I. We have $\zeta e_i = \zeta(i \otimes 1) = (\zeta i) \otimes 1 = (j\sigma) \otimes 1 = j \otimes (\sigma 1)$. Therefore the action of S_d on $Ind_W^{S_d}(\chi)$ is given by

$$\zeta e_i = \beta_i(\zeta) e_{s(\zeta)(i)},$$

where $\beta_i(\zeta) = \chi(\sigma) = \chi((s(\zeta)(i))^{-1}\zeta i)$.

For the rest of the paper we introduce the following notation:

$(W_m)_{m=1}^k$ is a finite family of subgroups of the symmetric group S_d;

$(\chi_m)_{m=1}^k$, $\chi_m: W_m \to K$, is a family of one-dimensional characters;

I_m, $(e_i)_{i\in I_m}$, $s_m: S_d \to S(I_m)$ and $(\beta_i^{(m)})_{i\in I_m}$, are the above ingredients for the induced monomial representation $Ind_{W_m}^{S_d}(\chi_m)$, where $m = 1, \ldots, k$.

The rule

$$\zeta(i_1, \ldots, i_k) = (s_1(\zeta)(i_1), \ldots, s_k(\zeta)(i_k)), \tag{1.1}$$

where $(i_1, \ldots, i_k) \in I_1 \times \cdots \times I_k$ and $\zeta \in S_d$, defines an action of the group S_d on the set $I = I_1 \times \cdots \times I_k$.

We denote by W° the group W with the opposite group structure. The Cartesian product of groups $S_d \times W_1^\circ \times \cdots \times W_k^\circ$ acts on the set $S_d \times \cdots \times S_d$ by virtue of the rules

$$(\zeta, w_1, \ldots, w_k)(a_1, \ldots, a_k) = (\zeta a_1 w_1, \ldots, \zeta a_k w_k) \tag{1.2}$$

and

$$(\zeta, w_1, \ldots, w_k).(a_1, \ldots, a_k) = (w_1^{-1} a_1 \zeta^{-1}, \ldots, w_k^{-1} a_k \zeta^{-1}). \tag{1.3}$$

The next obvious lemma follows from the definitions of the actions of the corresponding groups and paves the way for some combinatorial applications.

Lemma 1.1. *The following four statements hold:*

(i) *two k-tuples $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ are in the same S_d-orbit in I if and only if there exist $\zeta \in S_d$ and $w_m \in W_m$ such that $j_m = \zeta i_m w_m$ for $m = 1, \ldots, k$;*

(ii) *the stabilizer of the k-tuple $(i_1, \ldots, i_k) \in I$ in the symmetric group S_d is the intersection $i_1 W_1 i_1^{-1} \cap \ldots \cap i_k W_k i_k^{-1}$;*

(iii) *the inclusion $I_1 \times \cdots \times I_k \subset S_d \times \cdots \times S_d$ induces a bijection between the orbit space $S_d \backslash I$ and the orbit space $S_d \times W_1^\circ \times \cdots \times W_k^\circ \backslash S_d \times \cdots \times S_d$ with respect to the action (1.2);*

(iv) *the inversion*

$$S_d \times \cdots \times S_d \to S_d \times \cdots \times S_d,\ (a_1, \ldots, a_k) \mapsto (a_1^{-1}, \ldots, a_k^{-1}),$$

is an isomorphism of the actions (1.2) *and* (1.3) *of the group* $S_d \times W_1^\circ \times \cdots \times W_k^\circ$.

Remark 1. (i) The set of the orbits of the action (1.2) coincides with the factor-set of $S_d \times \cdots \times S_d$ with respect to the equivalence relation "T-similarity", defined in [4, Sec. 3]. Therefore, according to Lemma 1.1, (iii), there is a bijection between the orbit space $S_d \backslash I$ and the set of all distinct superpositions of k graphs with d vertices each (multiple edges and loops allowed), see [4, Sec. 4]. Moreover, the stabilizer $i_1 W_1 i_1^{-1} \cap \ldots \cap i_k W_k i_k^{-1}$ is the automorphism group of the superposition $(i_1, \ldots, i_k)$.

(ii) The orbits of the action (1.3) are Redfield's group-reduced distributions, determined in [5, p. 434]. Lemma 1.1, (iii), (iv), yields that there is a bijection between the orbit space $S_d \backslash I$ and the set of all distinct group-reduced distributions.

Proposition 1.1. *The tensor product*

$$Ind_{W_1}^{S_d}(\chi_1) \otimes_K \cdots \otimes_K Ind_{W_k}^{S_d}(\chi_k) \tag{1.4}$$

is a monomial K-linear representation of S_d with basis $(e_i = e_{i_1} \otimes \cdots \otimes e_{i_k})_{i \in I}$, the action of S_d being given by the rule

$$\zeta e_i = \beta_i(\zeta) e_{s(\zeta)(i)},$$

where $\beta_i(\zeta) = \beta_{i_1}^{(1)}(\zeta) \ldots \beta_{i_k}^{(k)}(\zeta)$.

Proof. It is clear that the family $(e_i)_{i \in I}$ is a basis for the K-linear space (1.4). We have $\zeta e_i = \zeta e_{i_1} \otimes \cdots \otimes \zeta e_{i_k} = \beta_{i_1}^{(1)}(\zeta) \ldots \beta_{i_k}^{(k)}(\zeta) e_{s_1(\zeta)(i_1)} \otimes \cdots \otimes e_{s_k(\zeta)(i_k)} = \beta_i(\zeta) e_{s(\zeta)(i)}$. In particular, (1.4) is a monomial representation of S_d. □

Due to [1, Lemma 1], the characteristic of the tensor product (1.4) is the internal product $Z(\chi_1) * \cdots * Z(\chi_k)$. We set

$$C(W_1, \ldots, W_k) =$$

$$\{(\sigma_1, \ldots, \sigma_k) \in W_1 \times \cdots \times W_k \mid c_s(\sigma_1) = \cdots = c_s(\sigma_k),\ s = 1, \ldots, d\}.$$

Obviously, $((1), \ldots, (1)) \in C(W_1, \ldots, W_k)$.

For any $\sigma = (\sigma_1, \ldots, \sigma_k) \in C(W_1, \ldots, W_k)$ we define $c_s(\sigma) = c_s(\sigma_1) = \cdots = c_s(\sigma_k)$ for $s = 1, \ldots, d$. Moreover, we set $z_\sigma = \prod_{s=1}^d s^{c_s(\sigma)} c_s(\sigma)!$.

The next proposition links the present definition of internal product to Read's one from [4, Subsec. 3.3].

Proposition 1.2. *It holds*

$$Z(\chi_1) * \cdots * Z(\chi_k)$$

$$= \frac{1}{|W_1| \ldots |W_k|} \sum_{\sigma \in C(W_1, \ldots, W_k)} z_\sigma^{k-1} \chi_1(\sigma_1) \ldots \chi_k(\sigma_k) p_1^{c_1(\sigma)} \ldots p_d^{c_d(\sigma)}.$$

Proof. The proof is an immediate consequence of [3, Ch. I, Sec. 7, (7.12)]. □

2. REDFIELD'S ANSATZ

In this section we generalize the Redfield's master theorem and the superposition theorem.

Theorem 2.1. *It holds*

$$Ind_{W_1}^{S_d}(\chi_1) \otimes_K \cdots \otimes_K Ind_{W_k}^{S_d}(\chi_k)$$

$$\simeq \oplus_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} Ind^{S_d}_{\omega_1 W_1 \omega_1^{-1}\cap\ldots\cap\omega_k W_k \omega_k^{-1}}(\psi_{(\omega_1,\ldots,\omega_k)}),$$

where $T(W_1,\ldots,W_k)$ is a system of distinct representatives of the S_d-orbits in the Cartesian product $I = I_1 \times \cdots \times I_k$ with respect to the action (1.1) *of S_d, and $\psi_{(\omega_1,\ldots,\omega_k)}$ is the one-dimensional character of the group $\omega_1 W_1 \omega_1^{-1} \cap \ldots \cap \omega_k W_k \omega_k^{-1}$, which is the restriction of the expression*

$$\beta_\omega(\zeta) = \chi_1((s_1(\zeta)(\omega_1))^{-1}\zeta\omega_1)\ldots\chi_k((s_k(\zeta)(\omega_k))^{-1}\zeta\omega_k)$$

from Proposition 1.1.

Proof. Due to Proposition 1.1, the tensor product (1.4) is a monomial representation of S_d, so it gives an induced monomial representation on each S_d-orbit in the set I and (1.4) is the direct sum of these transitive constituents. Now, Lemma 1.1, (ii), finishes the proof. □

Transferring this result by virtue of the characteristic map *ch* on the Abelian group Λ^d, we obtain a direct generalization of the Redfield's master theorem.

Theorem 2.2. *It holds*

$$Z(\chi_1) * \cdots * Z(\chi_k) = \sum_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} Z(\psi_{(\omega_1,\ldots,\omega_k)}).$$

Following R. C. Read, if A is a polynomial in several variables $p_1,\ldots,p_d$, we denote by $N(A)$ the sum of its coefficients.

Theorem 2.3. *The number of the elements $\omega \in T(W_1,\ldots,W_k)$ such that $\psi_{(\omega_1,\ldots,\omega_k)} = 1$ on the stabilizer $\omega_1 W_1 \omega_1^{-1} \cap \ldots \cap \omega_k W_k \omega_k^{-1}$ is*

$$N(Z(\chi_1) * \cdots * Z(\chi_k)).$$

Proof. Applying the operation N on the two sides of the equality from Theorem 2.2, we obtain

$$N(Z(\chi_1) * \cdots * Z(\chi_k)) = \sum_{(\omega_1,\ldots,\omega_k)\in T(W_1,\ldots,W_k)} N(Z(\psi_{(\omega_1,\ldots,\omega_k)})).$$

Given a group $G \le S_d$ and an one-dimensional character $\psi: G \to K$, we have $N(Z(\psi)) = \langle \psi, 1_G \rangle_G$, where $\langle \ , \ \rangle$ is the standard scalar product of functions on the group G (see [3, Ch. I, Sec. 7]). Since $\langle \psi, 1_G \rangle_G = 0$ when $\psi \neq 1_G$, and $\langle \psi, 1_G \rangle_G = 1$ when $\psi = 1_G$, the proof is done. □

Remark 2. When $\chi_m = 1_{W_m}$ for $m = 1, \dots, k$, Theorem 2.2 (respectively, Theorem 2.3) turns into the Redfield's master theorem (respectively, turns into the superposition theorem).

3. GRAPHICAL COROLLARIES

Here is how the above machinery applies to the graph theory. Combining Theorem 2.3 and Remark 1, we establish Theorem 3.1 and several graphical corollaries of it. In accordance with Remark 1, these statements can also be formulated in the language of Redfield's ranges and the associated range-groups, see [5, p. 434].

Let $\Gamma_1, \dots, \Gamma_k$ be graphs with d vertices (loops and multiple edges allowed) and let $W_1 \le S_d, \dots, W_k \le S_d$ be their automorphism groups, respectively. Let $\chi_m: W_m \to K$ be an one-dimensional character of W_m, $m = 1, \dots, k$. Suppose that $\chi_2 = 1_{W_2}, \dots, \chi_k = 1_{W_k}$, and set $W = W_1$, $\chi = \chi_1$.

Theorem 3.1. *Let $\mathcal{G}$ be a set of subgroups of the symmetric group S_d, which is closed with respect to conjugations. Let $H \le W$ be the kernel of the character χ. Let us assume that the set of all subgroups of W, which belong to $\mathcal{G}$, coincides with the set of all subgroups of H. Then the number of those superpositions of the graphs $\Gamma_1, \dots, \Gamma_k$, whose automorphism groups belong to $\mathcal{G}$, is $N(Z(\chi) * Z(1_{W_2}) * \dots * Z(1_{W_k}))$.*

Proof. For any subgroup $H' \le W$ we have $H' \in \mathcal{G}$ if and only if $\chi_{|H'} = 1_{H'}$. The automorphism group $A_\omega = \omega_1 W_1 \omega_1^{-1} \cap \dots \cap \omega_k W_k \omega_k^{-1}$ of any superposition $\omega = (\omega_1, \dots, \omega_k)$ is a subgroup of $\omega_1 W \omega_1^{-1}$. Obviously, the subgroups of $\omega_1 W \omega_1^{-1}$ from $\mathcal{G}$ are exactly the subgroups of the kernel $\omega_1 H \omega_1^{-1}$ of the one-dimensional character $\chi(\omega_1^{-1}\zeta\omega_1)$ of the group $\omega_1 W \omega_1^{-1}$. On the other hand, the one-dimensional character $\psi_\omega(\zeta)$ of A_ω from Theorem 2.2 is the restriction of $\chi(\omega_1^{-1}\zeta\omega_1)$. Thus, $A_\omega \in \mathcal{G}$ if and only if $\psi_\omega(\zeta)$ is identically 1 on A_ω. Therefore Theorem 2.3 implies the result. □

Given a cyclic group of order b and a divisor a of b, let $\varrho^{(a)}$ be an one-dimensional character of this cyclic group, whose kernel has order a. If $\mathcal{G}$ is the set of all cyclic subgroups of S_d of order that divides a, Theorem 3.1 yields

Corollary 3.1. *If the group $W = W_1$ is cyclic of order b and if a is a divisor of b, then the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, which have a cyclic automorphism group of order dividing a, is*

$$N(Z(\varrho^{(a)}) * Z(1_{W_2}) * \dots * Z(1_{W_k})).$$

In the particular case $a = 1$, we obtain

Corollary 3.2. *If the permutation group $W = W_1$ is cyclic, then the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, having a trivial automorphism group, is*

$$N(Z(\varrho) * Z(1_{W_2}) * \cdots * Z(1_{W_k})),$$

where $\varrho\colon W \to K$ is an injective one-dimensional character of W.

Now, let $\mathcal{G}$ be the set of all subgroups of S_d, consisting of even permutations. Then Theorem 3.1 implies

Corollary 3.3. *The number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism group consist of even permutations, is*

$$N(Z(\varepsilon) * Z(1_{W_2}) * \cdots * Z(1_{W_k})),$$

where $\varepsilon\colon W_1 \to K$ is the restriction of the alternating character of S_d on W_1.

Let r be a natural number. We suppose that:

(a) $W = W_1$ has a normal solvable subgroup R of order r such that the factor-group W/R is cyclic of order relatively prime to r.

Then the group W itself is solvable. According to the generalized Sylow theorems (cf [2, Ch. 9, Theorem 9.3.1]), R is the only subgroup of W of order r. Moreover, any subgroup of W of order that divides r is contained in R.

Denote by π an one-dimensional character of W with kernel R. If $\mathcal{G}$ is the set of all subgroups of S_d of order dividing r, we obtain

Corollary 3.4. *If the group $W = W_1$ satisfies condition* (a), *then*

$$N(Z(\pi) * Z(1_{W_2}) * \cdots * Z(1_{W_k}))$$

is the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism groups are of order dividing r.

Now, we formulate an important version of Corollary 3.4. Let q be a prime number. Suppose that:

(b) the group $W = W_1$ has a normal q-subgroup R such that the factor-group W/R is cyclic of order relatively prime to q.

In accordance with Sylow theorems (see [2, Ch. 4, Theorems 4.2.1 - 4.2.3]), R is the only Sylow q-subgroup of W. Moreover, any q-subgroup of W is contained in R.

Denote by ι an one-dimensional character of W with kernel R. If $\mathcal{G}$ is the set of all q-subgroups of S_d, then we get

Corollary 3.5. *If the group $W = W_1$ satisfies condition* (b), *then*

$$N(Z(\iota) * Z(1_{W_2}) * \cdots * Z(1_{W_k}))$$

is the number of all superpositions of $\Gamma_1, \dots, \Gamma_k$, whose automorphism group is a q-group.

Remark 3. Examples of abstract groups $W = W_1$ which satisfy the hypothesis (a) (respectively, the hypothesis (b)) can be obtained by constructing a semi-direct product of a solvable group R of order r (respectively, a q-group R) with a cyclic group C of order relatively prime to r (respectively, relatively prime to q). The Schur-Zassenhaus' theorem (see [6, Ch. IV, Sec. 8, IV.7.c]) asserts that there are no other examples. In the symmetric group S_d, it is enough to choose $R \leq S_d$ and $C \leq S_d$ with the above-mentioned properties so that $RC = CR$, $R \cap C = \{(1)\}$ and R is a normal subgroup of the group $W_1 = RC$.

Acknowledgments. The author would like to thank prof. R. Robinson who noted that Corollary 3.1 could be obtained by using Case 2 of Theorem from [5], p. 449, where Redfield has counted the superpositions by automorphism group when one of the range groups is cyclic. The author would like to thank also prof. F. Harary for his moral support and the referee for the helpful remarks which improved the readability of the paper.

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Received on November 13, 2001

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev str., bl. 8, 1113 Sofia
BULGARIA
E-mail: viliev@math.bas.bg
viliev@aubg.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

SHARP ESTIMATES FOR THE FIFTH COEFFICIENT OF THE INVERSE FUNCTIONS OF THE TOTALLY MONOTONIC FUNCTIONS

PAVEL G. TODOROV

We find the minimum and the maximum of the fifth coefficient for the inverse functions of the totally monotonic functions.

Keywords: minimum and maximum, inverse functions, totally monotonic functions

2000 MSC: primary 30C45, 30C50; secondary 12D10, 26C10, 26C15, 30B10, 30C10, 30C15

1. INTRODUCTION

Let T denote the class of totally monotonic functions

$$w = \varphi(z) = \int_0^1 \frac{z\, d\mu(t)}{1 - zt} = \sum_{n=1}^{\infty} a_n z^n, \quad z \notin [1, +\infty], \tag{1.1}$$

where $\mu(t)$ is a probability measure on $[0,1]$ and

$$a_n = \int_0^1 t^{n-1}\, d\mu(t), \quad n = 1, 2, \ldots, \quad a_1 = 1. \tag{1.2}$$

In [1] it it noted that the largest common region of convergence of all Taylor series at the point $w = 0$ of the inverse functions $z = \psi(w)$ of the functions (1.1) is the disk $|w| < 1/2$. Let

$$z = \psi(w) = \sum_{n=1}^{\infty} b_n w^n, \quad |w| < \frac{1}{2}, \quad b_1 = 1, \tag{1.3}$$

be such series, where in [1] the coefficients b_n are determined explicitly by the coefficients a_n in (1.2). In [1] we found the minimum and the maximum of the coefficients b_2, b_3 and b_4 and conjectured that the extrema of all coefficiens b_n, $n = 2, 3, 4, \ldots$, in (1.3) are attained only for the rational functions of the form

$$\varphi(z) = cz + \frac{(1-c)z}{1-z} \in T, \quad 0 \le c \le 1, \tag{1.4}$$

for suitable values of c, and, in addition,

$$b_{2m} \ge -1, \quad m = 1, 2, \ldots, \tag{1.5}$$

and

$$b_{2m+1} \le 1, \quad m = 1, 2, \ldots, \tag{1.6}$$

where the equalities in (1.5) and (1.6) hold only for the function

$$\psi(w) = \frac{w}{1+w} = \sum_{n=1}^{\infty} (-1)^{n-1} w^n, \tag{1.7}$$

inverse of the function (1.4) for $c = 0$, respectively.

Now we verify these conjectures for the fifth coefficient b_5 in (1.3) as well.

2. SHARP ESTIMATES FOR b_5

In [1, p. 41, Theorem 4] we have proved that the minimum (the maximum) of the coefficients b_n, $n \ge 2$, in (1.3) in the class T is attained only either in the subclass of functions (1.4) or in the subclass of functions

$$\varphi(z) = \sum_{k=1}^{p} \frac{c_k z}{1 - t_k z} \in T, \tag{2.1}$$

where

$$1 \le p \le m, \quad n = 2m, \quad m = 1, 2, \ldots, \tag{2.2}$$

$$1 \le p \le m+1, \quad n = 2m+1, \quad m = 1, 2, \ldots, \tag{2.3}$$

$$0 \le t_1 \le t_2 \le \cdots \le t_p \le 1, \quad 0 \le c_k \le 1, \quad \sum_{k=1}^{p} c_k = 1, \tag{2.4}$$

and $t_1, t_2, \ldots, t_p$ are among the numbers 0 and 1 and the roots in the interval $0 \le t \le 1$ of the equation

$$P(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s}(s-1)t^{s-2} = 0, \quad n \ge 3. \tag{2.5}$$

The function

$$Q(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} t^{s-1}, \quad Q'(t) = P(t), \quad n \ge 3, \tag{2.6}$$

has equal values at any two adjacent points of the sequence $t_1, t_2, \ldots, t_p$ for $p \ge 2$, i.e.

$$Q(t_1) = Q(t_2) = \cdots = Q(t_p), \quad p \ge 2. \tag{2.7}$$

The equations (2.7) are necessary conditions for the extrema of b_n with respect to $c_1, c_2, \ldots, c_p$. In fact, b_n depends on $a_2, a_3, \ldots, a_n$, which by (2.1) are equal to

$$a_s = \sum_{k=1}^{p} c_k t_k^{s-1}, \quad 2 \le s \le n, \quad n \ge 3, \quad p \ge 2. \tag{2.8}$$

From (2.8) and the last equation in (2.4) we have

$$\frac{\partial a_s}{\partial c_k} = t_k^{s-1} - t_{k+1}^{s-1}, \quad 1 \le k \le p, \quad p \ge 2, \quad t_{p+1} = t_1. \tag{2.9}$$

Having in mind (2.9) and (2.6), we obtain the formula

$$\frac{\partial b_n}{\partial c_k} = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} \cdot \frac{\partial a_s}{\partial c_k} = Q(t_k) - Q(t_{k+1}), \tag{2.10}$$
$$n \ge 3, \quad 1 \le k \le p, \quad p \ge 2, \quad Q(t_{p+1}) = Q(t_1).$$

Since $\partial b_n / \partial c_k = 0$ at the extrema of b_n, formula (2.10) yields (2.7).

Theorem 2.1. *The coefficient b_5 in (1.3) satisfies the sharp inequalities*

$$-0.1317545\ldots = 14c^4 - 35c^3 + 30c^2 - 10c + 1 \le b_5 \le 1, \tag{2.11}$$

where

$$c = 0.294997\ldots, \quad 0.294997 < c < 0.294998, \tag{2.12}$$

is the smallest positive root of the equation

$$56c^3 - 105c^2 + 60c - 10 = 0, \tag{2.13}$$

and the equalities hold only for the following extremal functions: on the left-hand side of (2.11) — *for the inverse function of the function* (1.4) *for* c *determined by* (2.11)–(2.13), *and on the right-hand side of* (2.11) — *for the function* (1.7).

Proof. In terms of the coefficients a_2, a_3, a_4, a_5 in (1.2), the coefficient b_5 in (1.3) has the following explicit form (see Theorem 3 and its proof in [1]):

$$b_5 = -a_5 + 6a_2a_4 + 3a_3^2 - 21a_2^2a_3 + 14a_2^4. \tag{2.14}$$

According to our general theorem, expressed by means of (2.1)–(2.7) applied to $n = 5$ and (2.14), the only possible extremal functions for b_5 are the functions of the form (1.4), and

$$\varphi(z) = \frac{z}{1-tz} \in T, \quad 0 \le t \le 1, \tag{2.15}$$

$$\varphi(z) = cz + \frac{(1-c)z}{1-tz} \in T, \quad 0 < c < 1, \quad 0 < t < 1, \tag{2.16}$$

$$\varphi(z) = \frac{cz}{1-tz} + \frac{(1-c)z}{1-z} \in T, \quad 0 < c < 1, \quad 0 < t < 1, \tag{2.17}$$

$$\varphi(z) = \frac{cz}{1-t_1z} + \frac{(1-c)z}{1-t_2z} \in T, \quad 0 < c < 1, \quad 0 < t_1 < t_2 < 1, \tag{2.18}$$

$$\varphi(z) = c_1z + \frac{c_2z}{1-tz} + \frac{c_3z}{1-z} \in T, \tag{2.19}$$
$$0 < c_{1,2,3} < 1, \quad c_1 + c_2 + c_3 = 1, \quad 0 < t < 1$$

(in general t is different for each function), with the corresponding equations

$$P(t) = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3 - 21a_2^2)2t + 18a_2t^2 - 4t^3 = 0 \tag{2.20}$$

in t and functions

$$Q(t) = \left(6a_4 - 42a_2a_3 + 56a_2^3\right)t + \left(6a_3 - 21a_2^2\right)t^2 + 6a_2t^3 - t^4 \tag{2.21}$$

with $Q'(t) = P(t)$. For the latter and for the corresponding functions (1.4), (2.16)–(2.19) we have the equations

$$Q(0) = Q(1), \tag{2.22}$$
$$P(t) = 0, \quad Q(0) = Q(t), \tag{2.23}$$
$$P(t) = 0, \quad Q(t) = Q(1), \tag{2.24}$$
$$P(t_1) = 0, \quad P(t_2) = 0, \quad Q(t_1) = Q(t_2), \tag{2.25}$$
$$P(t) = 0, \quad Q(0) = Q(t), \quad Q(t) = Q(1), \quad Q(1) = Q(0). \tag{2.26}$$

(i) First, we examine the function (1.4). From it we find the Taylor coefficients

$$a_2 = 1-c, \quad a_3 = 1-c, \quad a_4 = 1-c, \quad a_5 = 1-c. \tag{2.27}$$

From (2.27) and (2.14) we obtain that

$$b_5 = 14c^4 - 35c^3 + 30c^2 - 10c + 1 \tag{2.28}$$
$$= 14(c-1)\left(c - \frac{1}{2}\right)\left(c - \frac{7+\sqrt{21}}{14}\right)\left(c - \frac{7-\sqrt{21}}{14}\right) := b_5(c), \quad 0 \le c \le 1.$$

It follows from (2.28) that the derivative equation

$$b_5'(c) = 56c^3 - 105c^2 + 60c - 10 = 0 \tag{2.29}$$

has three real roots

$$\begin{aligned} c' &= 0.294997\ldots, \quad 0.294997 < c' < 0.294998, \\ c'' &= 0.652\ldots, \quad 0.652 < c'' < 0.653, \\ c''' &= 0.9270\ldots, \quad 0.9270 < c''' < 0.9271 \end{aligned} \tag{2.30}$$

for which

$$\begin{aligned} \min b_5(c') &= -0.1317545\ldots, \\ \max b_5(c'') &= 0.062235\ldots, \\ \min b_5(c''') &= -0.03281\ldots \end{aligned} \tag{2.31}$$

In addition,

$$b_5(0) = 1, \quad b_5(1) = 0. \tag{2.32}$$

The derivative equation (2.29) follows from formula (2.10) and equation (2.22) as well. In fact, we have

$$b_5'(c) = Q(0) - Q(1) = 56c^3 - 105c^2 + 60c - 10 = 0 \tag{2.33}$$

by (2.10) for $n = 5$, $k = 1$, $p = 2$, $c_1 = c$, $t_1 = 0$, $t_2 = 1$, and (2.21) for the values (2.27) and (2.22).

(ii) Second, we examine the function (2.15). Converting (2.15) or by means of the coefficients of (2.15) and (2.14), we obtain

$$b_5 = t^4, \quad 0 \le t \le 1, \quad \min b_5 = 0, \quad \max b_5 = 1. \tag{2.34}$$

(iii) Third, we examine the function (2.16). From (2.16) we find the coefficients

$$a_2 = (1-c)t, \quad a_3 = (1-c)t^2, \quad a_4 = (1-c)t^3, \quad a_5 = (1-c)t^4. \tag{2.35}$$

From (2.20), (2.21) and (2.23) we obtain

$$\frac{1}{3t}\left[P(t) - \frac{1}{t}Q(t)\right] = 2a_3 - 7a_2^2 + 4a_2 t - t^2 = 0. \tag{2.36}$$

It follows from (2.35)–(2.36) that

$$7(1-c)^2 - 6(1-c) + 1 = 0. \tag{2.37}$$

From (2.37) and (2.35) we find

$$a_2^{\pm} = \frac{3 \pm \sqrt{2}}{7} t, \quad a_3^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^2, \quad a_4^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^3, \quad a_5^{\pm} = \frac{3 \pm \sqrt{2}}{7} t^4, \tag{2.38}$$

respectively. Now (2.38) and (2.14) yield

$$b_5^{\pm} = t^4 \frac{-13 \pm 16\sqrt{2}}{343}, \quad 0 < t < 1, \tag{2.39}$$

respectively. Equations (2.39) lead to the corresponding boundaries

$$\inf b_5^+ = 0, \quad \sup b_5^+ = 0.0280682\ldots, \quad 0 < t < 1, \tag{2.40}$$
$$\inf b_5^- = -0.10387\ldots, \quad \sup b_5^- = 0, \quad 0 < t < 1. \tag{2.41}$$

(iv) Fourth, we examine the function (2.17). From (2.17) we find the coefficients

$$a_2 = c(t-1)+1, \ a_3 = c(t^2-1)+1, \ a_4 = c(t^3-1)+1, \ a_5 = c(t^4-1)+1. \tag{2.42}$$

From (2.20), (2.21) and (2.24) we obtain

$$\begin{aligned} &\frac{1}{t-1}\left\{P(t) - \frac{1}{t-1}[Q(t) - Q(1)]\right\} \\ &= 6a_3 - 21a_2^2 + 6a_2(2t+1) - 3t^2 - 2t - 1 = 0. \end{aligned} \tag{2.43}$$

It follows from (2.42)–(2.43) that

$$21(1-t)^2c^2 - 6(1-t)(5-3t)c + (3t^2 + 2t + 16) = 0. \tag{2.44}$$

The discriminant of the equation (2.44) in c is

$$3(1-t)^2[2t(3t-52) - 37] < 0, \quad 0 < t < 1. \tag{2.45}$$

From (2.45) we conclude that the equation (2.44) has no real roots for c, and hence, the function (2.17) is not extremal for b_5.

(v) Fifth, we examine the function (2.18). From it we find the coefficients

$$\begin{aligned} a_2 &= c(t_1 - t_2) + t_2, \quad a_3 = c(t_1^2 - t_2^2) + t_2^2, \\ a_4 &= c(t_1^3 - t_2^3) + t_2^3, \quad a_5 = c(t_1^4 - t_2^4) + t_2^4. \end{aligned} \tag{2.46}$$

On the other hand, from (2.20), (2.21) and (2.25) we obtain

$$\frac{1}{2(t_1 - t_2)}\left\{P(t_1) + P(t_2) - \frac{2}{t_1 - t_2}[Q(t_1) - Q(t_2)]\right\} = 3a_2 - t_1 - t_2 = 0, \tag{2.47}$$

$$a_2 = \frac{t_1 + t_2}{3}.$$

Further, from (2.21) and (2.25) we get

$$\frac{1}{2(t_1-t_2)}[P(t_1)-P(t_2)] = 6a_3 - 21a_2^2 + 9a_2(t_1+t_2) - 2(t_1^2+t_1t_2+t_2^2) = 0. \tag{2.48}$$

It follows from (2.47) and (2.48) that

$$a_3 = \frac{2t_1^2+t_1t_2+2t_2^2}{9}. \tag{2.49}$$

Finally, from (2.20), the first equation in (2.25), (2.47) and (2.49) we obtain

$$a_4 = \frac{2(t_1+t_2)(7t_1^2-4t_1t_2+7t_2^2)}{81}. \tag{2.50}$$

Now, identifying the both expressions of a_2 in (2.46) and (2.47), we find

$$c = \frac{t_1-2t_2}{3(t_1-t_2)}. \tag{2.51}$$

Having in mind (2.51), the identification of the corresponding expressions of a_3 and a_4 in (2.46), (2.49) and (2.50) leads to the system of equations

$$t_1^2 - 4t_1t_2 + t_2^2 = 0, \quad 13t_1^2 - 46t_1t_2 + 13t_2^2 = 0. \tag{2.52}$$

Setting $t_2 = kt_1$ in (2.52), we obtain the equations

$$k^2 - 4k + 1 = 0, \quad 13k^2 - 46k + 13 = 0. \tag{2.53}$$

But equations (2.53) have no common root, whence it follows that the function (2.18) is not extremal for b_5.

(vi) Sixth, we examine the function (2.19). From (2.19) we find the coefficients

$$a_2 = c_2t + c_3, \quad a_3 = c_2t^2 + c_3, \quad a_4 = c_2t^3 + c_3, \quad a_5 = c_2t^4 + c_3. \tag{2.54}$$

On the other hand, from (2.21) and (2.26) we have $Q(0) = 0$.

$$\frac{1}{t}Q(t) = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3 - 21a_2^2)t + 6a_2t^2 - t^3 = 0, \tag{2.55}$$

$$\frac{1}{t-1}[Q(t)-Q(1)] = 6a_4 - 42a_2a_3 + 56a_2^3 + (6a_3-21a_2^2)(t+1) \tag{2.56}$$

$$+ 6a_2(t^2+t+1) - (t+1)(t^2+1) = 0,$$

$$Q(1) = (6a_4 - 42a_2a_3 + 56a_2^3) + (6a_3 - 21a_2^2) + 6a_2 - 1 = 0. \tag{2.57}$$

Subtracting (2.57) from (2.56), we obtain

$$(6a_3 - 21a_2^2)t + 6a_2(t^2+t) - t^3 - t^2 - t = 0. \tag{2.58}$$

If we add (2.58) to (2.56) and subtracting this sum from (2.20), then we find

$$a_2 = \frac{2t+1}{6}. \tag{2.59}$$

It follows from (2.58)–(2.59) that

$$a_3 = \frac{16t^2+4t+7}{72}. \tag{2.60}$$

From (2.57) and (2.60) we find

$$a_4 = \frac{224t^3+48t^2+42t+91}{1296}. \tag{2.61}$$

Now the identification of the corresponding expressions of a_2 and a_3 in (2.54) and (2.59)–(2.60) leads to the values

$$c_2 = \frac{16t^2-20t-5}{72t(t-1)}, \quad c_3 = \frac{8t^2+8t-7}{72(t-1)}. \tag{2.62}$$

From (2.54) and (2.62) we find

$$a_4 = \frac{16t^3-4t^2-t+7}{72} \tag{2.63}$$

and

$$a_5 = \frac{16t^4-4t^3-9t^2-t+7}{72}. \tag{2.64}$$

The identification of (2.61) and (2.63) yields the equation

$$64t^3-120t^2-60t+35=0. \tag{2.65}$$

The equation (2.65) has three real roots lying in the intervals $(-\infty,0)$, $(0,1)$ and $(1,+\infty)$, respectively, where the root in the interval $(0,1)$ is

$$t' = 0.3668\ldots, \quad 0.3668 < t' < 0.3669. \tag{2.66}$$

The value (2.14) for (2.59)–(2.60) and (2.63)–(2.64) is

$$b_5 = \frac{128t^4-320t^3-240t^2+280t-91}{5184} := b_5(t). \tag{2.67}$$

From (2.66)–(2.67) we obtain

$$b_5(t') = -0.0065704\ldots \tag{2.68}$$

Now comparison of (2.30)–(2.32), (2.34), (2.40)–(2.41) and (2.68) leads to (2.11)–(2.13), which completes the proof of Theorem 2.1.

For the next coefficients $b_6, b_7, \ldots$ we can proceed in the same way. In accordance with our conjecture, for the functions (1.4) we can expect that the functions (2.1)–(2.4) different from the functions (1.4) are not extremal for b_n, $n \geq 6$.

3. AN EXPLICIT FORM OF THE COEFFICIENTS OF THE INVERSE FUNCTIONS OF THE FUNCTIONS (1.4)

The function (1.4) can be rewritten in the form

$$w = \varphi_c(z) := \frac{z(1-cz)}{1-z} = z + (1-c)\sum_{n=2}^{\infty} z^n \in T, \quad |z| < 1, \quad 0 \le c \le 1 \tag{3.1}$$

The branch of the two-valued inverse function of (3.1) determined by the values $z = 0$ for $w = 0$ is the function

$$\begin{aligned} z = \psi_c(w) &= \frac{1 + w - \sqrt{1 - 2(2c-1)w + w^2}}{2c} \\ &\equiv \frac{2w}{1 + w + \sqrt{1 - 2(2c-1)w + w^2}} \end{aligned} \tag{3.2}$$

with $\sqrt{1} = 1$, analytic and univalent in the w-plane cut along the two two-times-describable rays

$$w = \varphi_c(1 + iy) = 2c - 1 + i\left(cy + \frac{1-c}{y}\right), \quad -\infty \le y \le +\infty,$$

connecting the branch points

$$w_{1,2} = \varphi_c(z_{1,2}) = 2c - 1 \pm 2i\sqrt{c(1-c)}, \quad z_{1,2} = 1 \pm i\frac{\sqrt{c(1-c)}}{c},$$

through the point at infinity, which correspond to the equations

$$\frac{\partial \varphi_c(z_{1,2})}{\partial z} = 0.$$

According to our earlier results for the univalence of the class T of functions (1.1), their derivatives $\varphi'(z)$ vanish on the straight line $\operatorname{Re} z = 1$ only for the functions (1.4) with $0 < c < 1$ (see [2, pp. 417–418, Theorem 1; Eq. (4) contains a misprint where an inequality sign is reversed], [3, p. 120, Theorem 1], [4] and [5]). Hence the image of the half-plane $\operatorname{Re} z \le 1$ by each function (1.1) of the class T except the functions (1.4) for $0 < c < 1$ has exterior points.

Theorem 3.1. *The inverse function* (3.2) *has the Taylor expansion*

$$z = \psi_c(w) = \sum_{n=1}^{\infty} b_n(c) w^n, \quad b_1(c) = 1, \quad |w| < 1, \tag{3.3}$$

where

$$b_n(c) = \sum_{\nu=0}^{n-1} \frac{(-1)^{n-1-\nu}}{\nu+1}\binom{2\nu}{\nu}\binom{n-1+\nu}{n-1-\nu} c^\nu, \quad n = 1, 2, \ldots \tag{3.4}$$

Proof. By the first representation in (3.2) we obtain

$$z = \psi_c(w) = \frac{1}{2c}(1+w)\left(1 - \sqrt{1 - \frac{4cw}{(1+w)^2}}\right). \tag{3.5}$$

It follows for sufficiently small values of $|w|$ that

$$\begin{aligned} 1 - \sqrt{1 - \frac{4cw}{(1+w)^2}} &= 2\sum_{\nu=1}^{\infty} \frac{1}{\nu}\binom{2\nu-2}{\nu-1} c^\nu w^\nu (1+w)^{-2\nu} \\ &= 2\sum_{\nu=1}^{\infty} \frac{1}{\nu}\binom{2\nu-2}{\nu-1} c^\nu \sum_{n=\nu}^{\infty} (-1)^{n-\nu}\binom{\nu+n-1}{n-\nu} w^n \\ &= 2\sum_{n=1}^{\infty} w^n \sum_{\nu=1}^{n} \frac{(-1)^{n-\nu}}{\nu}\binom{2\nu-2}{\nu-1}\binom{\nu+n-1}{n-\nu} c^\nu. \end{aligned} \tag{3.6}$$

Now (3.5)–(3.6) lead to (3.3)–(3.4), which completes the proof of Theorem 3.1.

In particular, for $c = 1$ and $c = 1/2$, the coefficients of (3.2) are $b_n(1) = 0$, $n \geq 2$, and

$$b_{2n}\left(\frac{1}{2}\right) = \frac{(-1)^n}{n2^{2n-1}}\binom{2n-2}{n-1}, \quad b_{2n+1}\left(\frac{1}{2}\right) = 0, \quad n \geq 1,$$

respectively, which compared with (3.4) yield the corresponding identities.

Formula (2.10) for $n \geq 3$, $k = 1$, $p = 2$, $c_1 = c$, $t_1 = 0$, $t_2 = 1$ is reduced to the formula

$$b_n'(c) = Q(0) - Q(1),$$

where $b_n'(c)$, $Q(0) = 0$ and $Q(1)$ are determined by (3.4) and (2.6), respectively (for $n = 5$ this formula is noted in (2.33)).

Let

$$m_n = \min_T b_n, \quad M_n = \max_T b_n, \quad n = 2, 3, \ldots, \tag{3.7}$$

where b_n, $n \geq 2$, are those in (1.3).

If the conjecture for the function (1.4) is true, then

$$m_n = \min_{0 \leq c \leq 1} b_n(c), \quad M_n = \max_{0 \leq c \leq 1} b_n(c), \quad n = 2, 3, \ldots, \tag{3.8}$$

where $b_n(c)$, $n \geq 2$, are those in (3.4).

For $n = 2, 3, 4, 5$ it follows from (3.4) that

$$\begin{aligned} b_2(c) &= -1 + c, \quad b_3(c) = 1 - 3c + 2c^2, \\ b_4(c) &= -1 + 6c - 10c^2 + 5c^3, \\ b_5(c) &= 1 - 10c + 30c^2 - 35c^3 + 14c^4. \end{aligned} \tag{3.9}$$

By formulas (3.7)–(3.8) applied to the polynomials (3.9) we obtain the explicit values of m_n and M_n for $n = 2, 3, 4, 5$ as follows:

$$m_2 = -1 \ (c = 0), \quad M_2 \ (c = 1); \tag{3.10}$$

$$m_3 = -\frac{1}{8} \ \left(c = \frac{3}{4}\right), \quad M_3 = 1 \ (c = 0); \tag{3.11}$$

$$m_4 = -1 \ (c = 0), \tag{3.12}$$

$$M_4 = \frac{5 + 4\sqrt{10}}{135} = 0.13073415\ldots \ \left(c = \frac{10 - \sqrt{10}}{15} = 0.45584816\ldots\right);$$

$$m_5 = -0.1317545\ldots \ (c = 0.294997\ldots), \quad M_5 = 1 \ (c = 0); \tag{3.13}$$

where m_n and M_n for $n = 2, 3, 4, 5$ are realized only by the functions (1.4) (or (3.1)–(3.2)) for the values of c indicated in the parentheses and (2.12)–(2.13), respectively.

The equations (3.10)–(3.12) for $n = 2, 3, 4$ and the equations (3.13) for $n = 5$ are proved in [1] and Theorem 2.1 above, respectively.

For $n = 6, 7, \ldots$ the values of $m_6, m_7, \ldots, M_6, M_7, \ldots$ can be obtained by the conjectural formulas (3.8) applied to the polynomials (3.4) for $n = 6, 7, \ldots$

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Received on December 3, 2001

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

ON THE ANALOGY BETWEEN THE MAXWELL ELECTROMAGNETIC FIELD AND THE ELASTIC CONTINUUM

CHRISTO I. CHRISTOV

We attribute the action at a distance in electromagnetic phenomena to the effect of the internal stresses in an absolute continuous medium called *metacontinuum*. We show that Maxwell equations are straightforward corollaries from the governing equations of an incompressible elastic medium with the shear waves corresponding to the electromagnetic waves. The main advantage of the new description is that it enables one to incorporate the nonlinearity, whose manifestations turn out to be the presence of the so-called Lorentz force, and a Galilean invariance of the model. Another generalization of the model consists in acknowledging a high-grade elasticity which introduces dispersion into the governing system. A self-similar solution is found for the plane dispersive shear waves and shown to result in red-shift even when the source of light is not moving. In order to detect the existence of an absolute continuum, a revision of the classical Michelson-Morley experiment is proposed in which the effect is of first order and is not canceled by the FitzGerald-Lorentz contraction.

Keywords: Maxwell equations, scalar and vector potentials, incompressible elastic continuum, dispersive effects in electromagnetic field, Doppler effect

2000 MSC: 35Q60, 70S20, 74A05, 74A10, 74B05

INTRODUCTION

Electromagnetic phenomenon is an epitome of an action at a distance. The latter is hard to imagine without some kind of a material carrier and the nineteen centuries tradition has been always connected with some mechanical construct. In

Cauchy's and Hamilton's vision it was a lattice whose continuum approximation yielded the elastic-body model. In Maxwell's imagination the medium had internal degrees of freedom and the action at a distance was mediated by infinitesimal "gears" transmitting the momentum from point to point. McCullagh and Sommerfeld quantified this idea as elastic body with special rotational elasticity (see [14] for an exhaustive review of these theories). After Lord Kelvin came up with the model of fluid aether and its vortex theory of matter, the coinage "aether" assumed almost exclusively fluid meaning. Paying tribute to the tradition, all of the models for the presumably absolute continuous media underlying the physical world were called "aether", although some of them (the different elastic models, for instance) were quite different from an "aetherial fluid" (where the coinage comes from).

In the 19th century, the question of entrainment of the aether (aether-drift) was perceived as pivotal and the nil result of Michelson and Morley experiment [10, 11] was understood as a blow to the whole edifice of aether theories. However, the notion of a material carrier of the long-distance interactions could not be dismissed altogether and the conceptual vacuum was filled by the concept of "physical vacuum" or the "field" which possessed all the properties of the disgraced aether, e.g., action at a distance, but it was deliberately exempted from the obligation to be checked for aether-drift effect. It was then advertised as a thing in itself not connected to any "primitive mechanistism".

In our point of view, the *field* (*physical vacuum, aether*) can only be understood from the point of view of an absolute material continuum in which the *internal stresses* are the transmitter of the long-range interactions. In order to distinguish it from the mechanical continuous media (bodies, liquids, gases, etc.), we call the continuum-mechanics model of the unified field *metacontinuum* in the sense that it is beyond (*meta*) the observable phenomena and is their progenitor.

A valid candidate for the luminiferous field is the elastic medium, because, as shown from Cauchy himself (see [14]), it gives a good quantitative prediction for the shear-wave phenomena (light). Building upon our previous work [1, 2], we show here that the Maxwell equations follow from the linearized governing equations of the metacontinuum provided that the electric and magnetic fields are properly understood as manifestation of the *meta* internal stresses. The main difference from the Cauchy's volatile elastic aether (zero dilational elastic modulus) is that we consider the opposite limiting case: an elastic continuum with infinitely large dilational modulus (virtually incompressible elastic medium).

The model proposed here should not be confused with McCullagh's model of pseudo-elastic continuum with restoring couples, which tried to explain the unusual form of Maxwell equations apparently not fitting into the picture of continuum mechanics (see [14, 7] for reference and further developments).

Our model naturally incorporates the Galilean invariance, while the Maxwell equations are not Galilean invariant and there is no feasible way to make them such.

In the end, we address the problems connected with experimental identification of the presence of an absolute continuum and propose a certain revisiting of the Michelson and Morley experiment.

1. CAUCHY VERSUS MAXWELL

We begin with the linearized equations of the elastic continuum. For small velocities the Lagrangian and Eulerian descriptions coincide and for the displacements $\boldsymbol{u}$ of a Hookean elastic medium one gets the linear vector wave equation

$$\begin{aligned}\mu_0 \frac{\partial \boldsymbol{v}}{\partial t} \equiv \mu_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} &= \eta \Delta \boldsymbol{u} + (\lambda + \eta)\nabla(\nabla \cdot \boldsymbol{u}) \\ &\equiv -\eta \nabla \times \nabla \times \boldsymbol{u} + (\lambda + 2\eta)\nabla(\nabla \cdot \boldsymbol{u}),\end{aligned} \tag{1.1}$$

where $\boldsymbol{u}, \boldsymbol{v}$ are the displacement and velocity vectors, respectively; η, λ are the Lamé elasticity coefficients and μ_0 is the density of metacontinuum in material (Lagrangian) coordinates. In this paper we concern ourselves with a continuous medium of constant elastic coefficients η, λ and constant density μ_0.

Equations (1.1) govern both the shear and the compression/dilation motions. The former are controlled by the shear Lamé coefficient η, while the latter — by the dilational (second) Lamé coefficient λ, and more specifically, by the sum $(\lambda + 2\eta)$. The phase speeds of propagation of the respective small disturbances are

$$\begin{aligned} c &= \left(\frac{\eta}{\mu_0}\right)^{\frac{1}{2}}, \qquad c_s = \left(\frac{2\eta + \lambda}{\mu_0}\right)^{\frac{1}{2}}, \\ \frac{c^2}{c_s^2} &= \delta, \qquad \delta = \frac{\eta}{2\eta + \lambda}. \end{aligned} \tag{1.2}$$

Here c, c_s are the speeds of shear and compression waves, respectively. In what follows, we call them *speed of light* and *speed of sound*, respectively. To reduce the number of governing parameters, one needs to make an assumption about the second Lamé coefficient or, which is the same, about the *speed of sound*, c_s. There are two options: to consider a so-called "volatile" continuum with $c_s = 0$ or an incompressible continuum with $c_s \to \infty$. The latter is a continuum whose speed of sound is much greater than the speed of light, i.e., $\delta \ll 1$.

Here we examine the limiting case of a virtually incompressible continuum when $\lambda \gg \eta$ ($\delta \ll 1$). In this case (1.1) can be recast as follows:

$$\delta \left(c^{-2} \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + \nabla \times \nabla \times \boldsymbol{u} \right) = \nabla(\nabla \cdot \boldsymbol{u}), \tag{1.3}$$

and the displacement $\boldsymbol{u}$ can be developed into asymptotic power series with respect to δ:

$$\boldsymbol{u} = \boldsymbol{u}_0 + \delta \boldsymbol{u}_1 + \cdots \tag{1.4}$$

Introducing (1.4) into (1.3) and combining the terms with like powers, for the first two terms we obtain

$$\nabla(\nabla \boldsymbol{u}_0) = 0, \tag{1.5}$$

$$c^{-2}\frac{\partial^2 \boldsymbol{u}_0}{\partial t^2} + \nabla\times\nabla\times \boldsymbol{u}_0 = \nabla(\nabla\cdot \boldsymbol{u}_1). \tag{1.6}$$

From (1.5) one can deduce

$$\nabla\cdot \boldsymbol{u}_0 = \text{const}, \quad \text{or} \quad \nabla\cdot \boldsymbol{v}_0 = 0, \tag{1.7}$$

which is also a linear approximation to the incompressibility condition for a continuum. In the general model of nonlinear elasticity with finite deformations, the incompressibility condition is imposed on the Jacobian of transformation from material to geometrical variables, but in the first-order approximation in δ the equation (1.7) holds true.

Henceforth we omit the index '0' for the variable $\boldsymbol{u}$ without fear of confusion. We denote formally

$$\varphi \stackrel{\text{def}}{=} -(\lambda+2\eta)\nabla\cdot \boldsymbol{u}_1, \qquad \boldsymbol{A} \stackrel{\text{def}}{=} \boldsymbol{v}_0, \tag{1.8}$$

and recast the linearized Cauchy balance (1.6) in dimensional form as follows:

$$\mu_0\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\varphi + \nabla\cdot\boldsymbol{\tau}, \tag{1.9}$$

where $\boldsymbol{\tau}$ is the deviator stress tensor for which the following relation is obtained from the constitutive relation (the Hooke law) for elastic body, namely:

$$\boldsymbol{\tau} = \eta\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T) - 2\eta(\nabla\cdot\boldsymbol{u})\boldsymbol{I}, \tag{1.10}$$

where $\boldsymbol{I}$ stands for the unit tensor. For the divergence of $\boldsymbol{\tau}$ one has

$$\nabla\cdot\boldsymbol{\tau} = -\eta\,\nabla\times(\nabla\times\boldsymbol{u}) \stackrel{\text{def}}{=} \boldsymbol{E}, \tag{1.11}$$

where $\boldsymbol{E}$ stands for the vector to which the action of the purely shear part of internal stresses is actually reduced. It has the meaning of a point-wise distributed body force and we shall call it "electric force." In terms of $\boldsymbol{E}$, Eq. (1.9) recasts to

$$\boldsymbol{E} = -\mu_0\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\varphi, \tag{1.12}$$

which involves $\boldsymbol{A}$ and φ. In the same vein we define a "magnetic induction", $\boldsymbol{B}$, and "magnetic field", $\boldsymbol{H}$, as follows:

$$\boldsymbol{B} = \mu_0\nabla\times\boldsymbol{A} = \mu_0\boldsymbol{H}, \quad \boldsymbol{H} \stackrel{\text{def}}{=} \nabla\times\boldsymbol{A}. \tag{1.13}$$

The system of equations (1.12), (1.13) is nothing else but the equations of electrodynamics in terms of $\boldsymbol{A}$ and φ, which play respectively the role of the well-known vector and scalar potentials of the electromagnetic field. In the framework of the present approach, however, these potentials are not certain non-physical quantities introduced merely for convenience. Rather they appear to be the most natural variables: velocity and pressure of elastic luminiferous continuum. Note that the density, μ_0, of the elastic *meta*continuum appears as the magnetic permittivity of the Maxwell field.

Now one can derive the original Maxwell equations. Taking the operation *curl* of (1.12) and acknowledging (1.13), one obtains

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.14}$$

which is nothing else but the first of Maxwell equations (the Faraday law). Respectively, from eqs.(1.11), (1.13) and (1.8) one obtains

$$\frac{1}{\eta}\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times (\nabla \times \frac{\partial \boldsymbol{u}}{\partial t}) \equiv \nabla \times \boldsymbol{H}. \tag{1.15}$$

The last equation is precisely the "second Maxwell equation" provided that the shear elastic modulus of metacontinuum is interpreted as the inverse of electric permittivity $\eta = \varepsilon_0^{-1}$. This equation has been postulated by Maxwell [9] as an improvement over the Ampere law incorporating the so-called displacement current $\partial \boldsymbol{E}/\partial t$ in the Biot-Savart form. For the case of a void space, however, when no charges or currents are present, the second Maxwell equation lives a life of its own and the Ampere law plays merely heuristic role in its derivation. It is broadly accepted now that the second Maxwell equation is verified by a number of experiments. Here we have shown that it is also a corollary of the elastic constitutive relation for the metacontinuum and is responsible for the propagation of the shear stresses (action at a distance) in *metacontinuum.*

Thus the two main (time dependent) equations of the Maxwell form have already been derived. The condition $\operatorname{div} \boldsymbol{H} = 0$ (third Maxwell equation) follows directly from the very definition of magnetic field. Similarly, taking the divergence of both sides of (1.11), one immediately obtains the fourth Maxwell equation $\operatorname{div} \boldsymbol{E} = 0$. Thus we have shown that the Maxwell equations follow from the linearized governing equations of the Hookean elastic medium whose dilational modulus is much larger than the shear one.

Thus the first objective of the present work has been achieved. We have shown that the linearized equations of elastic continuum admit what can be called *Maxwell form.* In the framework of such a paradigm, each point of the elastic continuum experiences a body force $\boldsymbol{E}$ to which the action of the internal elastic stresses is reduced. We call it "electric force."

Let us note that the governing equations of the metacontinuum are *Galilean* invariant, while the linearized version (1.1) (and hence, the Maxwell form) has lost this important property. The lack of Galilean invariance of Maxwell equations is their strangest property.

A far reaching consequence of the previous section is that it gives a clue of how to seek for a Galilean invariance of the equations of the luminiferous field (the *metacontinuum*). In the classical continua the Galilean invariance is connected to the convective (advective) nonlinearity of the governing equations. The question of what kind of effects are to be expected due to the presence of the advective nonlinear terms stands. In other words, which interactions in the metacontinuum are perceived as additional forces if the observer is limited by the concept of linear Maxwell field.

Looking for confirmation of the possible nonlinearity of the model, one stumbles into a very strange (from Maxwell's point of view) entity called "Lorentz force". Consider the governing equations of an elastic continuum in the so-called Lamb form (see, e.g., [12]):

$$\mu_0 \left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \boldsymbol{v} \times \operatorname{rot} \boldsymbol{v} \right) + \nabla\varphi = -\boldsymbol{E}, \tag{2.1}$$

where we have already substituted the notations for the above defined scalar potential and electric field.

This form allows one to assess the forces acting at a given material point of the metacontinuum due to the convective accelerations of the latter. The gradient part of the convective acceleration can not be observed independently from the pressure gradient φ in the metacontinuum. In fact, one can measure only the quantity $\varphi_1 \equiv \varphi + \frac{1}{2}\boldsymbol{v}^2$. Thus the only observable effect of the nonlinearity is connected with the last term of the acceleration. By virtue of our definition of magnetic induction (1.13), the term under consideration adopts the form

$$F_l = \mu_0 \boldsymbol{v} \times \mathrm{B}. \tag{2.2}$$

Equation (2.2) expresses the force acting in each material point of the metacontinuum. This force is a part of the inertial force in the metacontinuum. In order to find its relation to the Lorentz force, one has to insert a test charge propagating through the continuum, to integrate (2.2) over the support of the charge and to get something proportional to the Lorentz force experienced by a moving charge. The exact coefficient of proportionality can be checked only after the notion of charge is incorporated into the model (see, [2]). Yet the expression (2.2) has an important bearing, because it involves the same quantities as the Lorentz force. It points out the direction in which the governing equations (1.1) of the electromagnetic dynamics can be generalized so that to become *Galilean* invariant.

The paradigm based on incompressible metacontinuum has already been instrument in restoring the Galilean invariance in the electromagnetic phenomena. However, the significance of the new concept is important in much wider field. It opens a more natural way of treating the mechanical properties of the unified field.

Let us consider now the (1+1)D-motions (plane waves). This is a good approximation for the spherical waves far from the emitting source. The governing equations for nonlinear elastic waves reduce to

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left(1 + \frac{\partial w}{\partial x}\right) \frac{\partial^2 w}{\partial x^2} - \kappa \frac{\partial^4 w}{\partial x^4}, \tag{3.1}$$

where $w = u_y(x,t)$ is one of the transversal components of the vector of elastic displacements $\boldsymbol{u}$ and c is the phase speed of the shear waves (speed of light). Respectively, κ is the dispersion coefficient. It is completely natural to have some higher grade elasticity in the metacontinuum, which results in the fourth-order terms.

Consider now the evolution of the wave profile $w(x,t)$ in the frame moving with velocity c in the positive x-direction and introduce new independent coordinates and sought function:

$$t_1 = \frac{1}{2} t, \quad x_1 = x - ct, \qquad w(t,x) = w_1(t_1, x_1).$$

The different derivatives are expressed as follows:

$$\begin{aligned} &\frac{\partial w}{\partial x} = \frac{\partial w_1}{\partial x_1}, \quad \frac{\partial w}{\partial t} = \frac{1}{2}\frac{\partial w_1}{\partial t_1} - c\frac{\partial w_1}{\partial x_1}, \\ &\frac{\partial^2 w}{\partial t^2} = \frac{1}{4}\frac{\partial^2 w_1}{\partial t_1^2} - c\frac{\partial^2 w_1}{\partial t_1 \partial x_1} + c^2 \frac{\partial^2 w_1}{\partial x_1^2}. \end{aligned} \tag{3.2}$$

If we consider only motions for which the evolution in the moving frame is very slow, then we can disregard the local time derivatives with respect to the local spatial derivatives in the sense that

$$\left| \frac{\partial^m w_1}{\partial t_1 \partial x_1^{m-1}} \right| \ll c \left| \frac{\partial^m w_1}{\partial x_1^m} \right|, \quad \left| \frac{\partial^2 w_1}{\partial t_1^2} \right| \ll c \left| \frac{\partial^2 w_1}{\partial t_1 \partial x_1} \right|.$$

Upon introducing (3.2) into eq. (3.1) and neglecting the terms according to the above scheme, we arrive at the following approximate equation:

$$-c\frac{\partial^2 w_1}{\partial x_1 \partial t_1} = \frac{\partial}{\partial x_1} \left[\frac{c^2}{2} \left(\frac{\partial w_1}{\partial x_1}\right)^2 - \kappa \frac{\partial^4 w_1}{\partial x_1^4} \right]. \tag{3.3}$$

Since the exact spatial position of the source is immaterial, one can chose it in a manner so that after one integration with respect to x_1 the integration constant to be set equal to zero. Thus eq.(3.3) recasts

$$\frac{\partial w_1}{\partial t_1} = -\frac{c}{2}\left(\frac{\partial w_1}{\partial x_1}\right)^2 + \frac{\kappa}{c}\frac{\partial^3 w_1}{\partial x_1^3}. \tag{3.4}$$

The dimension for κ is L^4T^{-2}, where L is a length scale, T is a time scale. Inverting these relations, we get $T = c^2\kappa^{-\frac{1}{2}}$ and $L = \kappa^{\frac{1}{2}}c^{-1}$. Then one can introduce dimensionless variables according to the scheme

$$w_1 = \frac{\kappa^{\frac{1}{2}}}{c}\hat{w}, \qquad t_1 = \frac{\kappa^{\frac{1}{2}}}{c^2}\hat{t}, \quad x_1 = \frac{\kappa^{\frac{1}{2}}}{c}\hat{x},$$

and to render (3.4) to the following dimensionless equation:

$$\frac{\partial \hat{w}}{\partial \hat{t}} = -\frac{1}{2}\left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^2 + \frac{\partial^3 \hat{w}}{\partial \hat{x}^3}. \tag{3.5}$$

In what follows, we omit the "hats" without fear of confusion. The above equation admits a self-similar solution of the following type:

$$w = t^{-b}F(\eta), \qquad \eta = \frac{x}{t^a}. \tag{3.6}$$

To demonstrate this, we find the expression for the different terms:

$$\frac{\partial w}{\partial t} = t^{-1-b}[F(\eta) - a\eta F'(\eta)], \quad \frac{\partial w}{\partial x} = t^{-b-a}F'(\eta),$$
$$\frac{\partial^3 w}{\partial x^3} = t^{-b-3a}F'''(\eta),$$

and introduce them in (3.5) to obtain

$$t^{-1-b}[F(\eta) - a\eta F'(\eta)] = -\frac{1}{2}t^{-2b-2a}\left(F'(\eta)\right)^2 + t^{-b-3a}F'''(\eta).$$

A self-similar solution is possible only if

$$-1 - b = -2b - 2a, \quad -1 - b = -b - 3a,$$

i.e.

$$a = b = \frac{1}{3}. \tag{3.7}$$

This kind of solution has been found in [3] for the Boussinesq equation. The numerical simulation [3] has confirmed the self-similar behavior of the solution: the support increases with $t^{1/3}$, while the amplitude of the pulse decreases proportionally to $t^{-\frac{1}{3}}$.

What are the implications of the existence of a self-similar solution? The second of equations (3.6) yields that the characteristic spatial scale of the solution increases with $t^{\frac{1}{3}}$. This means that for harmonic waves the wave length will increase accordingly. Since we consider a moving frame, then the dimensionless time t is measured by the distance $r = ct$ travelled by the solution from the source. Then in our model the Hubble coefficient of proportionality between the relative red shift and the distance from the source will not be constant, but will decrease as $r^{-2/3}$. It is well-known that as the observations push to the far limits of the Universe, the Hubble constant decreases. And the controversy is still raging about the value of the Hubble "constant".

The data for the Hubble constant from a single investigation is not enough to identify the coefficient κ. It will take a more sophisticated approach compiling all the available data for all different distances. This warrants a special investigation, which goes beyond the framework of the present paper. What we can claim here is that far from the source one can not distinguish between the red-shifting due to the presence of dispersion and the one that results from a Doppler effect (if present). This means that when a dispersion is present, then the "red shift" can be alternatively explained without the help of "Big-Bang" hypothesis.

4. ABSOLUTE MEDIUM AND THE FIRST-ORDER DOPPLER EFFECT

The foremost implication of the above described relationship between the Maxwell equations and the equations of elastic continuum is that there is a place for absolute continuum. It is clear that one cannot propose anything so radical without proposing an experimental scheme to detect the absolute continuum or at least to comment on the known ones. We do not mean some more radical revision like [13], where new hypothesis about the isotropy of the speed of light has been tested. Rather we propose revisiting the classical Michelson experiment.

To use interferometry for the verification of Doppler effect has been proposed by Maxwell [8]. It was widely believed that discovering a Doppler effect would prove the existence of an absolute medium at rest. Way before the actual experiment was performed by Michelson, Maxwell pointed out that any interferometry experiment, involving splitting and reflecting the same ray, inevitably renders the sought effect of *second order* with respect to the small parameter $d = v_e c^{-1}$ (v_e stands for the velocity of Earth with respect to the quiescent medium). In the worst case $d \approx 10^{-4}$, which corresponds to the orbital velocity of Earth. If the whole solar system is moving faster through the void, then d could be larger and the effect would be more appreciable.

It was Michelson [10] who implemented such a precise experiment. And after a nil effect was observed, a prolonged discussion began about the accuracy of Michelson's experimental observations. The set-up was later on refined by Michelson and Morley [11] (MM, for brevity) and the absence of the expected type of interference was confirmed more decisively.

In our opinion, the nil effect of MM experiment cannot disprove the existence of absolute medium, because the only conclusion that can be drown strictly from the nil effect is not that the absolute medium does not exist, but rather that in the medium where the light is being propagated there occurs an apparent contraction of the spatial scales in the direction of motion of the source (FitzGerald-Lorenz contraction) proportional to the factor

$$1 - \frac{v_e^2}{2c^2} \approx \sqrt{1 - \frac{v_e^2}{c^2}},$$

which exactly compensates for the expected second-order effect.

All this means that the real proof of the existence or non-existence of the *meta-continuum* can be furnished only through measuring the first-order effect. We deliberately exclude from consideration any kind of non-optical experiment and leave beyond our scope the optical experiments in dense matter (water filled columns, etc). A situation in which the first-order effect is not canceled can be created if *two different* sources of light are employed with sufficiently well synchronized frequencies. Lasers with the required level of stabilization of the frequency are available nowadays, but it goes beyond the frame of the present work to deal with the specific problems of the hardware. It suffices only to mention that if the two sources are synchronized up to 10^{-6}, the accuracy would be of order of 1% compared to the magnitude of the sought effect which is at least of order of 10^{-4}.

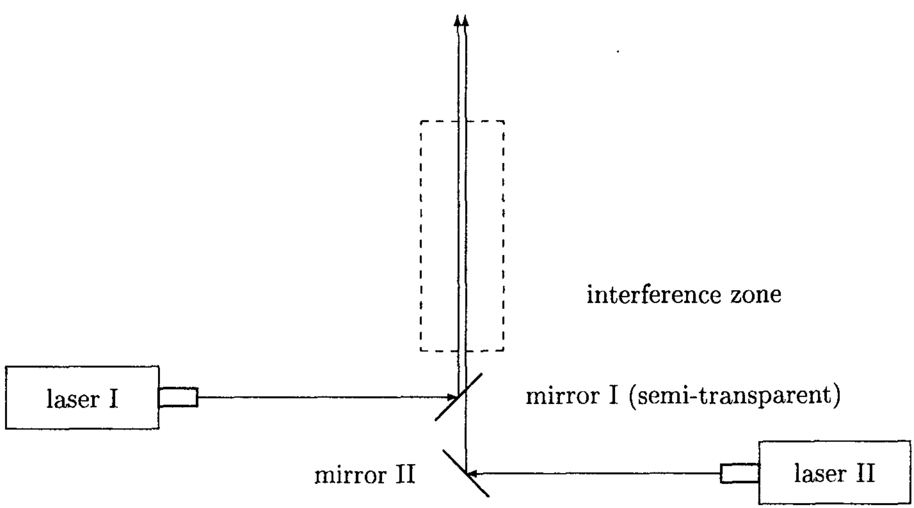

Fig. 1. Principal scheme of the interferometry experiment

In Fig. 1 the scheme of a possible first-order experiment is presented. Consider two "identical" (synchronized within 10^{-6}) sources of monochromatic light, which move together in the same direction with the same velocity. The first of them emits a plane wave propagating in the direction of motion and the second one — in the opposite direction.

By means of one mirror and one semi-transparent mirror, the two plane waves are made co-linear. The role of the mirrors is to change the direction of propagation of each wave without destroying its plane nature. The beam of the second laser is reflected by the mirror II changing its direction on 90° and making it to pass through a semi-transparent mirror whose reflecting surface serves to change the direction of the beam of the first laser on 90°. Beyond the semi-transparent mirror the two beams are parallel and can produce an easily detectable interference pattern. A snapshot of the region of interference would reveal strips of different intensity, gradually transforming into each other, so that the modulation frequency can be estimated.

It is interesting to note that Jaseda et al. [5, 6] already have used two lasers in an interferometry experiment in order to verify quantitatively the FG-L contraction, but in their experiments the lasers beams are parallel, while in the proposed here experiment they are anti-parallel since now it is not the contraction that needs verification, but the very existence of the first-order of Doppler effect.

Following [4], we outline here the derivation of the Doppler effect. The plane harmonic waves propagating in a presumably quiescent medium are given by the following formula:

$$F_{\pm}(x,t) \equiv e^{i(k_r x \mp \omega_r t)}, \quad k_{\pm} = \frac{\omega_r}{c}, \quad \kappa_{\pm} = \frac{c}{\omega_r}, \tag{4.1}$$

where $\omega_{\pm}$ are the frequencies. The upper sign in the notations refers to the wave propagating in the positive direction, while the lower sign — to the wave propagating in the negative direction. These waves have to satisfy the boundary condition on the moving boundaries (the sources):

$$F_{\pm}(\pm v_e t, t) \equiv e^{i\omega_0 t}, \tag{4.2}$$

where v_e is the velocity of the moving frame relatively to the metacontinuum. If the sources were at rest, then they would have produced waves with wave number $k_0 = \omega_0/c$ and wave length $\kappa_0 = k_0^{-1}$. The boundary condition (4.2) yields the following relation for the parameters of the propagating wave:

$$\omega_{\pm} = \omega_0 \left(1 \mp \frac{v_e}{c}\right)^{-1}, \quad k_{\pm} = \frac{\omega_0}{c} \left(1 \mp \frac{v_e}{c}\right)^{-1}, \quad \kappa_{\pm} = \kappa_0 \left(1 \mp \frac{v_e}{c}\right). \tag{4.3}$$

After the reflection, the two waves are propagating as plane waves in the positive direction of z-axis (vertical in Fig. 1: $F_{\pm}(z,t) \equiv e^{i(k_{\pm} z - \omega_{\pm} t)}$). Then in the interference region one has a wave which is the superposition of two of them for a given moment of time (say, $t = 0$):

$$\mathrm{Re}\, |F_+(z,t) + F_-(z,t)| = 2\cos\left(\frac{k_+ + k_-}{2} z\right) \cos\left(\frac{k_+ - k_-}{2} z\right), \tag{4.4}$$

which is a modulated wave with a wave number of the carrier $\frac{1}{2}(k_+ + k_-) = k_0 + O(d^2)$ and $\frac{1}{2}(k_+ - k_-) = dk_0 + O(d^3)$ for the modulation. Respectively, the expressions for wave lengths valid to the second order are κ_0 and $\kappa_m = \kappa_0 d^{-1}$. For $d = 10^{-4}$ and red-light lasers, the length of the wave is $\kappa_0 \approx 6.3 \cdot 10^{-5}$ cm, and then for the length of modulation wave one has $\kappa_m = 0.63$ cm and the strips produced must be easily detectable on an optical table of standard dimensions.

5. CONCLUDING REMARKS

In the present paper we have shown how the Maxwell equations can be derived from the linearized equations of an incompressible elastic continuum. Thus for the electromagnetic phenomena, we have arrived at the notion of underlying continuous medium, which we call metacontinuum. The approach of the present work admits feasible generalization to include inertial effects in the metacontinuum. This may explain the so-called Lorentz force as an effect of the advective nonlinearity of the governing equations. Moreover, incorporating the advective part of the time derivative makes the proposed model Galilean invariant, i.e. a Galilean Electrodynamics is developed.

The new concept provides a sound foundation for incorporating additional information about the electromagnetic phenomena, opening the way for pertinent generalizations. As a featuring example of such a generalization, the incorporation of dispersion (high-grade elasticity) in the governing equations is considered. A self-similar solution is found for the latter case in which the dispersion acts as to increase the spatial scale (wave-length) of the solution with the cubic root of the distance from the source of wave.

In the end, we describe the changes which are to be made in Michelson-Morley experiment in order to be able to detect a first-order Doppler effect in an absolute medium. We show that the first-order Doppler effect is not affected by the FitzGerald-Lorentz contraction.

Acknowledgments. This work is supported by Grant LEQSF(1999-02)-RD-A-49 from the Louisiana Board of Regents.

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Received December 30, 2001

Department of Mathematics
University of Louisiana at Lafayette
Lafayette, LA, 70504-1010
USA
E-mail: christov@louisiana.edu

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA “ST. KLIMENT OHRIDSKI”

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

ON THE GRAY-HERVELLA CLASSES OF AH-STRUCTURES ON SIX-DIMENSIONAL SUBMANIFOLDS OF CAYLEY ALGEBRA

M. BANARU

It is proved that only eight Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional general-type submanifolds of the octave algebra.

Keywords: almost Hermitian manifold, Gray-Hervella classification, semi-Kählerian structure
2000 MSC: 53C55

1. PRELIMINARIES

In this paper we consider the Gray-Hervella classes of almost Hermitian (AH) structures [11]. As it is well-known [1, 11], a $2n$-dimensional manifold M^{2n} with a Riemannian metric $g = \langle \cdot, \cdot \rangle$ and an almost complex structure J is called almost Hermitian if the following condition holds:

$$\langle JX, JY \rangle = \langle X, Y \rangle, \quad \forall X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of smooth vector fields on M^{2n}. All considered manifolds, tensor fields and similar objects are assumed to be of the class C^∞. We recall that the fundamental form of an almost Hermitian manifold is determined by

$$F(X, Y) = \langle X, JY \rangle, \quad X, Y \in \aleph(M^{2n}).$$

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a G-structure, where G is the unitary group $U(n)$ [1]. Its elements are the frames adapted to the structure (A-frames). They look as follows:

$$(p, \varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{\widehat{1}}, \ldots, \varepsilon_{\widehat{n}}),$$

where $p \in M^{2n}$, ε_a are the eigenvectors corresponding to the eigenvalue $i = \sqrt{-1}$, and $\varepsilon_{\widehat{a}}$ are the eigenvectors corresponding to the eigenvalue $-i$. Here $a = 1, \ldots, n$; $\widehat{a} = a + n$. Therefore, the matrix of the almost complex structure in an A-frame at the point p looks as follows:

$$(J_j^k) = \left(\begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array}\right), \qquad (1.1)$$

where I_n is the identity matrix; $k, j = 1, \ldots, 2n$. By direct computing, it is easy to obtain that in A-frame the matrices of the Riemannian metric g and of the fundamental form F look as follows, respectively:

$$(g_{kj}) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right), \quad (F_{kj}) = \left(\begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array}\right). \qquad (1.2)$$

2. CARTAN-KIRICHENKO STRUCTURAL EQUATIONS OF AN AH-STRUCTURE

The form of the Levi-Civita connection ∇ is determined by the forms system $\{\omega_j^k\}$ on the space of the complex frames stratification over an almost Hermitian manifold. Similarly, the deplacement form ω is determined by the forms system $\{\omega^k\}$. The Cartan structural equations of the stratification space over almost Hermitian manifold look as follows:

$$\begin{aligned} &1)\ d\omega^k = \omega_j^k \wedge \omega^j; \\ &2)\ d\omega_j^k = \omega_l^k \wedge \omega_j^l + \frac{1}{2} R_{jml}^k \omega^m \wedge \omega^l, \end{aligned} \qquad (2.1)$$

where $\{R_{jml}^k\}$ are the components of the Riemannian curvature tensor (or of the Riemann-Christoffel tensor [13]). Here and further $k, j, m, l = 1, \ldots, 2n$.
As J and g are the tensors of (1,1)- and (2,0)-type, respectively, and as $\nabla g = 0$, then the components of these tensors must satisfy the following system of differential equations:

$$\begin{aligned} &1)\ dJ_j^k + J_l^k \omega_j^l - J_j^l \omega_l^k = J_{j,l}^k \omega^l; \\ &2)\ dg_{kj} + g_{lj} \omega_k^l + g_{kl} \omega_j^l = 0, \end{aligned} \qquad (2.2)$$

where $\{J^k_{j,l}\}$ are the components of ∇J. Taking into account (1.1) and (1.2), we can rewrite $(2.2)_1$ as follows:

$$1)\ \omega^a_{\hat{b}} = -\frac{i}{2}J^a_{\hat{b},k}\omega^k;\quad 2)\ \omega^{\hat{a}}_b = -\frac{i}{2}J^{\hat{a}}_{b,k}\omega^k;\quad 3)\ J^a_{b,k}=0;\quad 4)\ J^{\hat{a}}_{\hat{b},k}=0. \tag{2.3}$$

Similarly, from $(2.2)_2$ we obtain:

$$1)\ \omega^a_{\hat{b}} + \omega^b_{\hat{a}} = 0;\quad 2)\ \omega^a_b + \omega^{\hat{b}}_{\hat{a}} = 0;\quad 3)\ \omega^{\hat{a}}_b + \omega^{\hat{b}}_a = 0. \tag{2.4}$$

Here and further $a,b,c,d = 1,\dots,n$; $\hat{a} = a+n$. Substituting (2.3) and (2.4) in Cartan structural equations (2.1), we get

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + \omega^a_{\hat{b}} \wedge \omega^{\hat{b}} = \omega^a_b \wedge \omega^b - \frac{i}{2}J^a_{\hat{b},c}\omega^c \wedge \omega^{\hat{b}} + \frac{i}{2}J^a_{[\hat{b},\hat{c}]}\omega^{\hat{c}} \wedge \omega^{\hat{b}};\\ d\omega^{\hat{a}} &= \omega^{\hat{a}}_b \wedge \omega^b + \omega^{\hat{a}}_{\hat{b}} \wedge \omega^{\hat{b}} = \omega^{\hat{a}}_{\hat{b}} \wedge \omega^{\hat{b}} + \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}\omega^{\hat{c}} \wedge \omega^b - \frac{i}{2}J^{\hat{a}}_{[b,c]}\omega^c \wedge \omega^b. \end{aligned} \tag{2.5}$$

We denote $\omega_k = g_{kj}\omega^j$. In particular, $\omega_a = \omega^{\hat{a}} = \overline{\omega^a}$. Taking into account this fact as well as (2.4), we can rewrite (2.5) as follows:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c;\\ d\omega_a &= -\,\omega^b_a \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c, \end{aligned} \tag{2.6}$$

where

$$B^{ab}{}_c = -\frac{i}{2}J^a_{\hat{b},c};\quad B_{ab}{}^c = \frac{i}{2}J^{\hat{a}}_{b,\hat{c}};\quad B^{abc} = \frac{i}{2}J^a_{[\hat{b},\hat{c}]};\quad B_{abc} = -\frac{i}{2}J^{\hat{a}}_{[b,c]}. \tag{2.7}$$

The functions $\{B^{ab}{}_c\}$, $\{B_{ab}{}^c\}$, $\{B^{abc}\}$, $\{B_{abc}\}$ serve as components of complex tensors of an almost Hermitian manifold (M^{2n}, J, g) [2], because, considering the differential continuations of (2.7), it is not difficult to see that

$$\begin{aligned} &dB^{ab}{}_c + B^{ab}{}_d\omega^d_c - B^{db}{}_c\omega^a_d - B^{ad}{}_c\omega^b_d = B^{ab}{}_{c,k}\omega^k;\\ &dB_{ab}{}^c - B_{ab}{}^d\omega^c_d + B_{db}{}^c\omega^d_a + B_{ad}{}^c\omega^d_b = B_{ab}{}^c{}_{,k}\omega^k;\\ &dB^{abc} - B^{dbc}\omega^a_d - B^{adc}\omega^b_d - B^{abd}\omega^c_d = B^{abc}{}_{,k}\omega^k;\\ &dB_{abc} + B_{dbc}\omega^d_a + B_{adc}\omega^d_b + B_{abd}\omega^d_c = B_{abc,k}\omega^k. \end{aligned}$$

Definition 2.1 ([4]). The tensors with the components $\{B^{ab}{}_c\}$ and $\{B_{ab}{}^c\}$ are called virtual Kirichenko tensors of first and second order, respectively.

Definition 2.2 ([4]). The tensors with the components $\{B^{abc}\}$ and $\{B_{abc}\}$ are called structural Kirichenko tensors of first and second order, respectively.

Definition 2.3. The equations (2.6) are called Cartan-Kirichenko structural equations of an almost Hermitian structure on the manifold M^{2n}.

We remark that according to (2.4)

$$J^{a}_{\hat{b},k} + J^{b}_{\hat{a},k} = 0.$$

So, we have

$$B^{ab}{}_{c} + B^{ba}{}_{c} = 0.$$

Similarly,

$$B_{ab}{}^{c} + B_{ba}{}^{c} = 0.$$

Thus, we have proved

Proposition 2.1. *The virtual Kirichenko tensors of an almost Hermitian manifold are skew-symmetric relative to the first pair of indices.*

From (2.7) we obtain the following result:

Proposition 2.2. *The structural Kirichenko tensors of an almost Hermitian manifold are skew-symmetric relative to the second pair of indices.*

Owing to the reality of ∇J, from the given definitions we have

Proposition 2.3. $B^{ab}{}_{c} = \overline{B_{ab}{}^{c}}$, $B^{abc} = \overline{B_{abc}}$.

3. THE MAIN RESULT

Let $\mathbf{O} \equiv \mathbf{R}^8$ be the Cayley algebra. As it is well-known [10], two non-isomorphic three-fold vector cross products are defined on it by means of the relations

$$P_1(X,Y,Z) = -X(\overline{Y}Z) + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X,Y,Z \in \mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$, and $X \to \overline{X}$ is the conjugation operator. Moreover, any other three-fold vector cross product in the octave algebra is isomorphic to one of the two above-mentioned.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1,2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [10]. The point $p \in M^6$ is called general [12] if

$$e_0 \notin T_p(M^6) \not\subset L(e_0)^{\perp},$$

where e_0 is the unit of Cayley algebra and $L(e_0)^{\perp}$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [12, 13].

Naturally, we can consider the following question:

Which of the sixteen Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional submanifolds of Cayley algebra?

A partial answer is known just for a six-dimensional special $M^6 \subset \mathbf{O}$, or the so-called Calabi submanifolds [13]. We recall that an almost Hermitian submanifold $M^6 \subset \mathbf{O}$ is called special [13] if

$$e_0 \notin T_p(M^6) \subset L(e_0)^{\perp}.$$

Such six-dimensional almost Hermitian submanifolds of the octave algebra were studied by A. Gray [8 – 10], E. Calabi [6], K. Yano and T. Sumitomo [17].

We answer this question in the case of general-type $M^6 \subset \mathbf{O}$, i.e. we shall prove the following

Main Theorem. *Just the eight Gray-Hervella classes of almost Hermitian structures can be represented on six-dimensional general-type submanifolds of Cayley algebra, namely:*

$$K,\ AK,\ NK,\ SH,\ QK,\ W_1 \oplus W_3,\ W_2 \oplus W_3,\ SK.$$

Proof. Let us use the characterization of Gray-Hervella classes of AH-structures in terms of Kirichenko tensors [2], as listed in Table 1.

Table 1

Characterization of Gray-Hervella classes of AH-structures in terms of Kirichenko tensors [2]

Class	Condition
K	$B^{abc} = 0, \quad B^{ab}{}_c = 0$
$NK = W_1$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = 0$
$AK = W_2$	$B^{(abc)} = 0, \quad B^{ab}{}_c = 0$
$SH = W_3$	$B^{abc} = 0, \quad B^{ab}{}_b = 0$
W_4	$B^{abc} = 0, \quad B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$
$QK = W_1 \oplus W_2$	$B^{ab}{}_c = 0$
$H = W_3 \oplus W_4$	$B^{abc} = 0$
$VG = W_1 \oplus W_4$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$
$W_1 \oplus W_3$	$B^{abc} = -B^{bac}, \quad B^{ab}{}_b = 0$
$W_2 \oplus W_3$	$B^{(abc)} = 0, \quad B^{ab}{}_b = 0$
$W_2 \oplus W_4$	$B^{(abc)} = 0, \quad B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$
$SK = W_1 \oplus W_2 \oplus W_3$	$B^{ab}{}_b = 0$
$G1 = W_1 \oplus W_3 \oplus W_4$	$B^{abc} = -B^{bac}$
$G2 = W_2 \oplus W_3 \oplus W_4$	$B^{(abc)} = 0$
$W_1 \oplus W_2 \oplus W_4$	$B^{ab}{}_c = \alpha^{[a}\delta^{b]}_c$
W	no condition

Now, we write out the Cartan-Kirichenko structural equations for six-dimensional general-type almost Hermitian submanifolds of Cayley algebra [13]:

$$\begin{aligned} d\omega^a &= \omega^a_b \wedge \omega^b + \frac{1}{\sqrt{2}}\varepsilon^{ah[b}D_h{}^{c]}\omega_b \wedge \omega_c + \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}\omega^c \wedge \omega_b; \\ d\omega_a &= -\,\omega^b_a \wedge \omega_b + \frac{1}{\sqrt{2}}\varepsilon_{ah[b}D^h{}_{c]}\omega_b \wedge \omega_c + \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}\omega_c \wedge \omega^b. \end{aligned} \tag{3.1}$$

Here $\varepsilon^{abc} = \varepsilon^{abc}_{123}$, $\varepsilon_{abc} = \varepsilon^{123}_{abc}$ are the components of Kronecker tensor of third order [16]:

$$\begin{aligned} &D_{cj} = \mp T^8_{cj} + iT^7_{cj}, \quad D_{\widehat{c}j} = \mp T^8_{\widehat{c}j} - iT^7_{\widehat{c}j}; \\ &D_h{}^c = D_{h\widehat{c}}, \quad D^h{}_c = D_{\widehat{h}c}, \quad D^{hc} = D_{\widehat{h}\widehat{c}}, \end{aligned} \tag{3.2}$$

where $\{T^{\varphi}_{kj}\}$, $\varphi = 7, 8$, are components of the configuration tensor (using Gray's terminology [8] or of the Euler curvature tensor [7]). Now, we assume that the indices a, b, c, d, h range from 1 to 3; the indices k, j range from 1 to 6, and we set $\widehat{a} = a + 3$.

Comparing (2.6) and (3.1), we get the following relations for virtual Kirichenko tensors of an almost Hermitian $M^6 \subset \mathbf{O}$ [5]:

$$B_{ab}{}^c = \frac{1}{\sqrt{2}}\varepsilon_{abh}D^{hc}, \quad B^{ab}{}_c = \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hc}.$$

From (3.2) it follows that the tensor D_{hb} is symmetric relative to the indices h and b; ε^{abh} is skew-symmetric relative to these indices. Hence, $\varepsilon^{abh}D_{hb} = 0$, and therefore

$$B^{ab}{}_b = \frac{1}{\sqrt{2}}\varepsilon^{abh}D_{hb} = 0.$$

So, for an arbitrary almost Hermitian structure induced by means of a three fold vector cross product on six-dimensional submanifolds of Cayley algebra, the following identity is fulfilled:

$$B^{ab}{}_b \equiv 0.$$

Consequently, as it is clear from the given table, all these almost Hermitian structures must be semi-Kählerian (SK). The class of SK-structures containes only eight classes of AH-structures, namely:

$$K,\ NK,\ AK,\ SH,\ QK,\ W_1 \oplus W_3,\ W_2 \oplus W_3,\ SK. \quad \square$$

We remark that this result is similar to A. Gray's conclusion that a six-dimensional special submanifold $M^6 \subset \mathbf{O}$ is semi-Kählerian [9, 11]. So, all six-dimesional almost Hermitian submanifolds of Cayley algebra are semi-Kählerian, i.e. they belong to one of the eight above-mentioned Gray-Hervella classes of AH-manifolds.

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Received on October 25, 2001
Revised December 19, 2001

Smolensk University of Humanities
2, Gertsen str., 214014 Smolensk
RUSSIA
E-mail: banaru@keytown.com

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Tome 95

BIVARIATE INTERPOLATION BY (m,n)–SPLINES

IRINA GEORGIEVA

We study the problem of interpolation and approximation of bivariate functions by (m, n)-splines, that is, by functions $\phi(x, y)$ for which $\phi^{(m,n)}(x, y)$ is a piece-wise constant.

Keywords: reconstruction, multivariate polynomials, interpolation, splines

2000 MSC: 41A63, 41A15

1. INTRODUCTION

The interpolation methods are a basic tool for approximation of functions. While, in the univariate case, most of the interpolation problems admit a nice treatment, often yielding a closed form expression for the interpolating function, the study of the corresponding multivariate problems encounters serious difficulties. For example, the interpolation by multivariate algebraic polynomials is not always regular. One of the central directions of investigation in this field is the construction of appropriate configurations of nodes for which the problem is regular. Similar difficulties occur in interpolation by other multivariate classes and, in particular, by splines. In this paper, we consider a standard problem of interpolation of bivariate functions on a rectangular grid by a special class of splines, which we call *(m, n)-splines.* Let us give the precise definition.

Suppose $G := [a, b] \times [c, d]$ is a given rectangular domain on the plane. Let us introduce a grid on G defined by the lines $x_i = a + i\dfrac{b-a}{M}, y_j = c + j\dfrac{d-c}{N}$, $i = 1, \dots, M$, $j = 1, \dots, N$. In this way we get a partition of G into a sum of small rectangles $\square_{ij} = [x_{i-1}, x_i) \times [y_{j-1}, y_j)$. We say that a function $s(x, y)$ is an (m, n)-*spline* on G if

$$\frac{\partial^{m+n} s}{\partial x^m \partial y^n}(x, y) = c_{ij} \text{ for } (x, y) \in \square_{ij},$$

where c_{ij} are constants.

In what follows, for the sake of definiteness, we assume that G is the unit square, i.e., $G = [0, 1]^2$. Let us denote by $\square_{ij}^c$ the closure of $\square_{ij}$.

We consider the following interpolation problem:

For a given sufficiently smooth function f and a set of MN distinct points $\{t_{ij}\}$ in G, find an (m, n)-spline $s(x, y)$ such that

$$s(t_{ij}) = f(t_{ij}), \quad i = 1, \ldots, M, \quad j = 1, \ldots, N, \tag{1.1}$$

and satisfying the boundary conditions

$$\frac{\partial^i s}{\partial y^i}(0, y) = \frac{\partial^i f}{\partial y^i}(0, y), \quad i = 0, \ldots, m-1, \quad y \in [0, 1],$$

$$\frac{\partial^j s}{\partial x^j}(x, 0) = \frac{\partial^j f}{\partial x^j}(x, 0), \quad j = 0, \ldots, n-1, \quad x \in [0, 1].$$

We show that the interpolation problem (1.1) has a unique solution for any choice of the nodes $t_{ij} = (\xi_{ij}, \eta_{ij})$ such that

$$\left\{ \begin{array}{ll} x_{i-1} < \xi_{ij} \leq x_i, & i = 1, \ldots, M \\ y_{j-1} < \eta_{ij} \leq y_j, & j = 1, \ldots, N \end{array} \right\}.$$

The solution is given explicitly for some small (m, n). We study also the question of approximation of the functions f by the corresponding interpolating spline s and give an error estimate for $(m, n) = (1, 1)$, $(1, 2)$ and $(2, 2)$.

2. PRELIMINARIES

The notion of a blending function is frequently used in this paper. Let us recall the definition (cf. [1]).

Functions from the space

$$C^{m,n}_{[0,1]^2} = \left\{ f : \frac{\partial^{k+l} f}{\partial x^k \partial y^l} \in C_{[0,1]^2}, \ k = 1, \ldots, m, \ l = 1, \ldots, n \right\},$$

satisfying the conditions

$$\frac{\partial^{k+l} f}{\partial x^k \partial y^l} = 0,$$

are said to be *blending functions* of order (m, n).

We shall denote the space of all blending functions of order (m, n) by $B^{m,n}_{[0,1]^2}$.

The next representation (given in [2]) of any sufficiently smooth function $f(x,y)$ in terms of blending functions of order (m,n) will be used in the sequel. It is based on the Taylor-type operators T_x^m and T_y^n, defined as the Taylor expansion of $f(x,y)$ at $(0,y)$, $(x,0)$, respectively, of order m, respectively n. In other words,

$$T_x^m f := f(0,y) + \frac{1}{1!}\frac{\partial}{\partial x} f(0,y)x + \cdots + \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial x^{m-1}} f(0,y)x^{m-1}.$$

Lemma 2.1. *For any function $f \in C^{m,n}_{[0,1]^2}$, the following representation holds:*

$$\begin{aligned} f(x,y) &= T_x^m f(x,y) + T_y^n f(x,y) - T_x^m T_y^n f(x,y) \\ &+ \frac{1}{(m-1)!(n-1)!}\int_0^1\int_0^1 (x-t)_+^{m-1}(y-\tau)_+^{n-1}\frac{\partial^{m+n} f}{\partial t^m \partial \tau^n} f(t,\tau)\,dt\,d\tau. \end{aligned}$$

Proof. According to the Taylor's formula with integral representation of the reminder, we have

$$f(x,y) = T_x^m f(x,y) + \int_0^1 \frac{(x-t)_+^{m-1}}{(m-1)!}\frac{\partial^m}{\partial t^m} f(t,y)\,dt.$$

Applying again Taylor's formula to $f^{(m,0)}(t,y)$, this time with respect to y at $y=0$, we get

$$\frac{\partial^m}{\partial t^m} f(t,y) = T_y^n \frac{\partial^m}{\partial t^m} f(t,y) + \int_0^1 \frac{(y-v)_+^{n-1}}{(n-1)!}\frac{\partial^{m+n}}{\partial t^m \partial v^n} f(t,v)\,dv.$$

Inserting the last expression of $\frac{\partial^m}{\partial t^m} f(t,y)$ in the first equality and taking into account that, by the commutativity of the differentiation operator and T_n^y,

$$\int_0^1 \frac{(x-t)_+^{m-1}}{(m-1)!} T_y^n \frac{\partial^m}{\partial t^m} f(t,y)\,dt = T_y^n\Big[f(x,y) - T_x^m f(x,y)\Big],$$

we obtain the wanted equality.

Let us mention that $B_f(x,y) := T_x^m f(x,y) + T_y^n f(x,y) - T_x^m T_y^n f(x,y)$ is a blending function of order (m,n). Moreover, the restriction of B_f and its partial derivatives $B_f^{(i,j)}$ on the lines $x=0$ and $y=0$ coincide with the corresponding values of f and its derivatives there for $i=0,\dots,m-1$, $j=0,\dots,n-1$.

Therefore, in view of Lemma 2.1, any (m,n)-spline f can be represented as a sum of an appropriate blending function B_f of order (m,n) and a convolution of the kernel

$$K(x,y,t,\tau) := \frac{(x-t)_+^{m-1}}{(m-1)!}\frac{(y-\tau)_+^{n-1}}{(n-1)!}$$

with a piecewise constant function $c(t,\tau)$,

$$c(t,\tau) := c_{ij} \quad \text{for} \quad (t,\tau) \in \square_{ij}.$$

Next we introduce a class of (m,n)-splines with a final support, the so-called B-splines, which will be used as a basis in the space of (m,n)-splines. In order to do this, we consider an infinite rectangular net in the plane:

$$\{x_i = i/M,\ y_j = j/N,\ -\infty < i,j < \infty,\ i,j - integers\}.$$

As in the introduction, we denote

$$\square_{ij} = \{(x,y) : x_{i-1} \le x < x_i, y_{j-1} \le y < y_j\}.$$

With any pair (i,j) of indices we associate the B-splines $B_{ij}^{(m,n)}$ of two variables of order (m,n) defined by

$$B_{ij}^{(m,n)}(x,y) := (\cdot - x)_+^n[x_i,\dots,x_{i+m+1}](\cdot - y)_+^n[y_j,\dots,y_{j+n+1}] = B_i^m(x)B_j^n(y).$$

For simplicity of the notations we shall often omit the upper indices m and n, when it is possible. Let us denote by D_{ij} the support of $B_{ij}(x,y)$. It is the Cartesian product of the supports of the univariate B-splines $B_i^{(m)}(x)$ and $B_j^{(n)}(y)$, namely, $D_{ij} = (x_i, x_{i+m+1}) \times (y_j, y_{j+n+1})$. Notice that in our notations the lower and left most rectangle, included in D_{ij}, is $\square_{i+1,j+1}$.

Lemma 2.2. *For any finite set $I := (I_1, I_2) \subset Z \times Z$ of indices, the B-splines $B_{ij}^{(m,n)}(x,y)$, $(i,j) \in I$, are linearly independent in $\Re^2$.*

Proof. Assume the contrary. Then there exists a linear combination

$$g(x,y) := \sum_{(i,j)\in I} \alpha_{ij} B_{ij}(x,y)$$

with at least one non-zero coefficient α_{ij}, which vanishes identically on the plane $\Re^2$. We introduce the lexicografic order in I. Let (i_0, j_0) be the first member of I. If $t_{i_0 j_0}$ belongs to the interior of $\square_{i_0 j_0} \subset D_{i_0 j_0}$, we have $B_{i_0 j_0}(t_{i_0 j_0}) \neq 0$ and hence $\alpha_{i_0 j_0} = 0$. Let $(\tilde{\imath}, \tilde{\jmath})$ be the next member of I. Quite analogously, we get $\alpha_{\tilde{\imath},\tilde{\jmath}} = 0$. Hence $\alpha_{ij} = 0$ for all $(i,j) \in I$. □

Lemma 2.3. *The functions $\{B_{ij}\}_{i=0}^{M-1}{}_{j=0}^{N-1}$ are linearly independent in $[0,1]^2$.*

The proof is similar to that of Lemma 2.2 and we omit it here.

Let us consider the subspace of (m,n)-splines

$$S_{m,n}^0 = \left\{ s \in S_{m,n} : \frac{\partial^i s}{\partial y^i}(0,y) = 0,\quad i = 0,\dots,m-1, \right.$$

$$\left. \frac{\partial^j s}{\partial x^j}(x,0) = 0,\quad j = 0,\dots,n-1 \right\}.$$

Corollary 2.1. *The B-splines $\{B_{ij}\}_{i=0}^{M-1}{}_{j=0}^{N-1}$ form a basis of $S_{m,n}^0$.*

Proof. We have proved that $\{B_{ij}\}_{i=0}^{M-1}{}_{j=0}^{N-1}$ are linearly independent. It is obvious that $B_{ij} \in S^0_{m,n}$ (since $B_{ij}(x,y) = B_i^{(m)}(x)B_j^{(n)}(y)$). Besides, $dim S^0_{m,n} = MN$=number of $B_{ij}(x,y)$.

3. THE INTERPOLATION THEOREM

The regularity of the interpolation problem by univariate splines is completely characterized by the *interlacing condition* of Schoenberg and Whitney [3], [4]. There is not yet such a characterization result in the multivariate case. In the next theorem we prove the regularity of the bivariate interpolation by (m,n)-splines for a quite general class of node configurations.

Theorem 3.1. *If*

$$\left\{ \begin{array}{l} x_{i-1} < \xi_{ij} \leq x_i, \ i = 1, \dots, M \\ y_{j-1} < \eta_{ij} \leq y_j, \ j = 1, \dots, N \end{array} \right\},$$

then the interpolation problem (1.1) *has a unique solution.*

Proof. There exists a unique blending function $b(x,y) \in B^{m,n}_{[0,1]^2}$ such that

$$\frac{\partial^i b}{\partial y^i}(0,y) = \frac{\partial f^i}{\partial y^i}(0,y), \quad i = 0, \dots, m-1, \quad y \in [0,1],$$

$$\frac{\partial^j b}{\partial x^j}(x,0) = \frac{\partial^j f}{\partial x^j}(x,0), \quad j = 0, \dots, n-1, \quad x \in [0,1].$$

Let us consider the values $\tilde{f}_{ij} = f(t_{ij}) - b(t_{ij})$. We claim that there exists a unique spline $s^0_{m,n} \in S^0_{m,n}$, which satisfies the interpolation conditions

$$s^0_{m,n}(t_{ij}) = \tilde{f}_{ij}.$$

Indeed, let us consider the corresponding homogeneous problem $s^0_{m,n}(t_{ij}) = 0$, $i = 1, \dots, M$, $j = 1, \dots, N$.

Lemma 2.3 gives a representation of $s^0_{m,n}$ in the form

$$s^0_{m,n} = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}.$$

Then $0 = s^0_{m,n}(t_{11}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}(t_{11}) = \alpha_{00} B_{00}(t_{11})$. Since $t_{11} \in \Box_{11} \subset D_{00}$, we have $B_{00}(t_{11}) \neq 0$ and $\alpha_{00} = 0$.

Further, $0 = s^0_{m,n}(t_{12}) = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \alpha_{ij} B_{ij}(t_{12}) = \alpha_{00} B_{00}(t_{12}) + \alpha_{01} B_{01}(t_{12}) = 0 + \alpha_{01} B_{01}(t_{12})$, hence $\alpha_{01} = 0$. We continue the process and finally get a diagonal matrix with $B_{i-1,j-1}(t_{ij})$ sitting in the diagonal. These numbers are different from zero since $t_{ij} \in \Box_{ij} \subset D_{i-1,j-1}$. Hence the homogeneous problem has only the trivial zero solution. This means that the non-homogenous problem has a unique solution. We assert that $s(x,y) = s^0_{m,n}(x,y) + b(x,y)$ is a solution of the original interpolation problem. Indeed,

$$\begin{aligned} s^{(m,n)}(x,y) &= \left(s^0_{(m,n)}\right)^{(m,n)}(x,y) + b^{(m,n)}(x,y) \\ &= \left(s^0_{(m,n)}\right)^{(m,n)}(x,y) + 0 = c_{i,j} \end{aligned}$$

when $(x,y) \in \square_{ij}$, $i = 1,\ldots,M$, $j = 1,\ldots,N$. Hence $s(x,y) \in S_{m,n}$. Besides,

$$s(t_{ij}) = s^0_{m,n}(t_{ij}) + b(t_{ij}) = f(t_{ij}) - b(t_{ij}) + b(t_{ij}) = f(t_{ij})$$

for $i = 1,\ldots,M$, $j = 1,\ldots,N$. Obviously, $s(x,y)$ satisfies also the matching conditions along the segments $[0,y]$, $0 \leq y \leq 1$, and $[x,0]$, $0 \leq x \leq 1$, since $s^0_{(m,n)}(x,y)$ was chosen to satisfy the zero conditions.

Let us suppose that there are two solutions of the interpolation problem: $s_1(x,y)$ and $s_2(x,y)$. If $s(x,y) = s_1(x,y) - s_1(x,y)$, then s satisfies the zero boundary conditions (on the segments). Hence $s \in S^0_{m,n}$ and from the condition $s(t_{ij}) = 0$, $i = 1,\ldots,M$, $j = 1,\ldots,N$, we get that $s \equiv 0$. The uniqueness is proved.

4. PARTICULAR CASES

4.1. AN ESTIMATE OF THE ERROR IN THE CASE $(m,n) = (1,1)$ AND $t_{ij} = (i/M, j/N)$

We analyse further the interpolating spline in case of low orders m, n. Consider the rectangular net of points

$$x_i = i/M,\ y_j = j/N,\ i = 1,\ldots,M,\ j = 1,\ldots,N.$$

Let us denote

$$\begin{aligned} \Delta_{xy} &= f(x,y) + f(0,0) - f(x,0) - f(0,y), \\ \Delta_{ij} &= f(x_i,y_j) + f(x_{i-1},y_{j-1}) - f(x_i,y_{j-1}) - f(x_{i-1},y_j), \\ \Delta_1 &= f(x_{i-1},y) + f(0,y_{j-1}) - f(x_{i-1},y_{j-1}) - f(0,y), \\ \Delta_2 &= f(x,y_{j-1}) + f(x_{i-1},0) - f(x,0) - f(x_{i-1},y_{j-1}), \\ \Delta_3 &= f(x,y) + f(x_{i-1},y_{j-1}) - f(x,y_{j-1}) - f(x_{i-1},y). \end{aligned}$$

We shall approximate the function $f(x,y)$ by interpolating $(1,1)$-splines, that is, by functions of the form

$$s(f,x,y) = f(x,0) + f(0,y) - f(0,0) + \int_0^x \int_0^y c(u,v)\,du\,dv,$$

where

$$c(u,v) = \{c_{ij} \ for \ (u,v) \in \Box_{ij}, \ i = 1,\dots,M, \ j = 1,\dots,N\}$$

and the constants $\{c_{ij}\}_{i=1}^{M} {}_{j=1}^{N}$ are chosen to satisfy the interpolating conditions

$$s(f,x_i,y_j) = f(x_i,y_j), \quad i = 1,\dots,M, \ j = 1,\dots,N.$$

We shall derive an expression for the error of approximation in terms of modulus of continuity $\omega(f,\delta_1,\delta_2)$. Recall that

$$\omega(f,\delta_1,\delta_2) = \sup_{|h_1|\le\delta_1} \sup_{|h_2|\le\delta_2} |f(x+h_1,y+h_2) - f(x,y)|.$$

In order to estimate the error, we need the values of $\{c_{ij}\}$, which we calculate below using the above-mentioned interpolatory conditions. By the first interpolatory condition we have

$$s(f,x_1,y_1) = f(x_1,0) + f(0,y_1) - f(0,0) + \int_0^{x_1}\int_0^{y_1} c_{11}\,du\,dv \ = f(x_1,y_1),$$

which is easily reduced to $\Delta_{11} = hc_{11}$, where $h = \frac{1}{MN}$. Hence $c_{11} = \frac{\Delta_{11}}{h}$. From the interpolatory conditions at the point (x_1,y_2) we get

$$f(x_1,y_2) = f(x_1,0) + f(0,y_2) - f(0,0) + \int_0^{x_1}\int_0^{y_1} c_{11}\,du\,dv + \int_0^{x_1}\int_{y_1}^{y_2} c_{12}\,du\,dv.$$

To find c_{12}, we use the above formula and the value of c_{11}, just found. We obtain $c_{12} = \frac{\Delta_{12}}{h}$. Similarly, we get that $c_{1j} = \frac{\Delta_{1j}}{h}$ for $j = 1,\dots,N$. We continue with the calculations of c_{21} up to c_{2N} and so on, till c_{MN}. In this way , we get that $c_{ij} = \frac{\Delta_{ij}}{h}$, $i = 1,\dots,M$, $j = 1,\dots,N$.

Now we are prepared to estimate the error. Let us suppose that the point (x,y) is in $\Box_{ij}$. Consider the identity

$$\begin{aligned} f(x,y) - s(f,x,y) &= f(x,y) + f(0,0) - f(x,0) - f(0,y) - \int_0^{x}\int_0^{y} c(u,v)\,du\,dv \\ &= \Delta_{xy} - \int_0^{x}\int_0^{y} c(u,v)\,du\,dv. \end{aligned}$$

Let us denote, respectively:

- by h_1 – the area of the rectangle with vertices (x_1,y), $(0,y_{j-1})$, (x_1,y_{j-1}) and $(0,y)$;
- by h_2 – the area of the rectangle with vertices (x,y_1), $(x_{i-1},0)$, $(x,0)$ and (x_{i-1},y_1);
- by h_3 – the area of the rectangle with vertices (x,y), $(x_{i-1}, \ y_{j-1})$, (x,y_{j-1}) and (x_{i-1},y).

Then

$$\begin{aligned}\int_0^x \int_0^y c(u,v)\,du\,dv &= h\sum_{k=1}^{i-1}\sum_{l=1}^{j-1} c_{kl} + h_1 \sum_{k=1}^{i-1} c_{kj} + h_2 \sum_{l=1}^{j-1} c_{il} + h_3 c_{ij} \\ &= \sum_{k=1}^{i-1}\sum_{l=1}^{j-1} \Delta_{kl} + \frac{h_1}{h}\sum_{k=1}^{i-1} \Delta_{kj} + \frac{h_2}{h}\sum_{l=1}^{j-1} \Delta_{il} + \frac{h_3}{h}\Delta_{ij}.\end{aligned}$$

Hence

$$\begin{aligned}&f(x,y) - s(f,x,y) = \Delta_1 + \Delta_2 + \Delta_3 - \frac{h_1}{h}\sum_{k=1}^{i-1} \Delta_{kj} - \frac{h_2}{h}\sum_{l=1}^{j-1} \Delta_{il} - \frac{h_3}{h}\Delta_{ij} \\ &= -\frac{h_1}{h}\left[f(x_{i-1},y_j) + f(0,y) - f(x_{i-1},y) - f(0,y_j)\right] + \left(1 - \frac{h_1}{h}\right)\Delta_1 \\ &-\frac{h_2}{h}\left[f(x_i,y_{j-1}) + f(x,0) - f(x_i,0) - f(x,y_{j-1})\right] + \left(1 - \frac{h_2}{h}\right)\Delta_2 + \Delta_3 - \frac{h_3}{h}\Delta_{ij}.\end{aligned}$$

Using the properties of the modulus of continuity, we obtain the estimate

$$\begin{aligned}&|f(x,y) - s(f,x,y)| \le 2\left[\frac{h_1}{h}\omega_{1,1}\left(f,1,\frac{1}{N}\right) + \left(1-\frac{h_1}{h}\right)\omega_{1,1}\left(f,1,\frac{1}{N}\right)\right. \\ &\left. + \frac{h_2}{h}\omega_{1,1}\left(f,\frac{1}{M},1\right) + \left(1-\frac{h_2}{h}\right)\omega_{1,1}\left(f,\frac{1}{M},1\right) + 2\omega_{1,1}\left(f,\frac{1}{M},\frac{1}{N}\right)\right].\end{aligned}$$

In this simple case we can give explicitely the Lagrangian basis for the interpolation problem

$$\begin{aligned}S^0_{1,1} = \{ \quad s: \quad &\frac{\partial^2 s}{\partial x^1 \partial y^1}(x,y) = c_{ij},\ (x,y) \in \Box_{ij}, \\ &i = 1,\dots,M,\ j = 1,\dots,N,\ s(0,y) = 0,\ s(x,0) = 0\}.\end{aligned}$$

More precisely , we construct functions $\delta_{pq}(x,y) \in S^0_{1,1}$ such that

$$\delta_{pq}\left(\frac{k}{M},\frac{l}{N}\right) = \delta_{pk}\delta_{ql}\ ,\ k = 1,\dots,M,\ l = 1,\dots,N.$$

(Obviously, $\{\delta_{pq}\}_{p=1}^{M}{}_{q=1}^{N}$ are linearly independent and since their number is $M.N$, they indeed form a basis of $S^0_{1,1}$.) We seek $\delta_{pq}(x,y)$ in the form

$$\delta_{pq}(x,y) = \int_0^1 \int_0^1 (x-t)^0_+ (y-v)^0_+ c(t,v)\,dt\,dv,$$

where $c(t,v) = c_{ij}$ for $(t,v) \in \Box_{ij}$, $\quad i = 1,\dots,M,\ j = 1,\dots,N.$

We derive that

$$\delta_{pq}(x,y) = \begin{cases} MN\left(x-\frac{p-1}{M}\right)\left(y-\frac{q-1}{N}\right), & (x,y)\in\Box_{pq}, \\ MN\left(\frac{p+1}{M}-x\right)\left(y-\frac{q-1}{N}\right), & (x,y)\in\Box_{p+1,q}, \\ MN\left(x-\frac{p-1}{M}\right)\left(\frac{q+1}{N}-y\right), & (x,y)\in\Box_{p,q+1}, \\ MN\left(\frac{p+1}{M}-x\right)\left(\frac{q+1}{N}-y\right), & (x,y)\in\Box_{p+1,q+1}, \\ 0, \text{ elsewhere.} \end{cases}$$

The calculation of δ_{pq} is straightforward and we omit it here. Instead, we show how to compute the basic functions in the more complicated case $(m,n)=(1,2)$.

Having computed δ_{pq}, one can give explicitely the solution of the interpolation problem as

$$s(f,x,y) = f(0,y) + f(x,0) - f(0,0) + \sum_{p=1}^{M}\sum_{q=1}^{N} f(t_{pq})\delta_{pq}(x,y).$$

4.2. THE CASE $(m,n)=(1,2)$ AND $t_{ij}=(i/M, j/N)$

Here we construct the Lagrangian basic functions $\delta_{pq}(x,y) \in S^0_{1,2}$. By definition, they satisfy the conditions

$$\delta_{pq}\left(\frac{k}{M},\frac{l}{N}\right) = \delta_{pk}\delta_{ql}\ ,\ k=1,\ldots,M,\ l=1,\ldots,N. \tag{4.1}$$

We seek $\delta_{pq}(x,y)$ in the form

$$\delta_{pq}(x,y) = \int_0^1\int_0^1 (x-t)^0_+(y-v)^1_+ c(t,v)\,dt\,dv,$$

where $c(t,v)=c_{ij}$ for $(t,v)\in\Box_{ij}$, $\quad i=1,\ldots,M,\ j=1,\ldots,N$.

Our next purpose is to calculate the constants c_{ij}. We will determine them using the interpolatory conditions (4.1).

Assume first that $p>1$, $q>1$. Using the condition $0=\delta_{pq}\left(\frac{1}{M},\frac{1}{N}\right)$, we get

$$\begin{aligned} 0 &= \int_0^1\int_0^1 \left(\frac{1}{M}-t\right)^0_+\left(\frac{1}{N}-v\right)^1_+ c(t,v)\,dt\,dv \\ &= c_{11}\int_0^{1/M}\left(\frac{1}{M}-t\right)^0 dt\int_0^{1/N}\left(\frac{1}{N}-v\right)dv = c_{11}\frac{1}{2MN^2}, \end{aligned}$$

i.e., $c_{11} = 0$. Moreover, the condition $\delta_{pq}\left(\frac{1}{M}, \frac{l}{N}\right) = 0$ gives

$$\begin{aligned} 0 &= \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_0^{1/N} \left(\frac{l}{N} - v\right) dv c_{11} \\ &+ \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_{1/N}^{2/N} \left(\frac{l}{N} - v\right) dv c_{12} + \cdots \\ &+ \int_0^{1/M} \left(\frac{1}{M} - t\right)^0 dt \int_{(l-1)/N}^{l/N} \left(\frac{l}{N} - v\right) dv c_{1l}. \end{aligned}$$

For $l = 2$ we get $0 = \alpha_1.0 + \alpha_2.c_{12}$ and hence $c_{12} = 0$. Analogously, we find that $c_{13} = 0, \ldots, c_{1l} = 0$ for $l = 1, \ldots, N$. If $p > 2$, in the same manner we show that $c_{2l} = 0$ for $l = 1, \ldots, N$. Moreover , for all $k < p$, we have $c_{kl} = 0$ for $l = 1, \ldots, N$. Quite analogously, we get the same for all $l < q$ and $k = 1, \ldots, M$. In the case $p = 1$ (or $q = 1$) these null columns (rows) are missing.

Assume that $p = 1, \ldots, M, \quad q = 1, \ldots, N$. From the equality $1 = \delta_{pq}\left(\frac{p}{M}, \frac{q}{N}\right)$ we get

$$1 = c_{pq} \int_{(p-1)/M}^{p/M} \left(\frac{p}{M} - t\right)^0 dt \int_{(q-1)/N}^{q/N} \left(\frac{q}{N} - v\right)^1 dv = c_{pq} \frac{1}{M} \frac{1}{2N^2},$$

i.e., $c_{pq} = 2MN^2$. Let now $k \geq p$ and $l \geq q$, $(k-p)^2 + (l-q)^2 \neq 0$. Then

$$\begin{aligned} 0 &= \delta_{pq}\left(\frac{k}{M}, \frac{l}{N}\right) \\ &= \sum_{j=q}^{l} \sum_{i=p}^{k} c_{ij} \int_{(i-1)/M}^{i/M} \left(\frac{k}{M} - t\right)^0 dt \int_{(j-1)/N}^{j/N} \left(\frac{l}{N} - v\right)^1 dv \\ &= \sum_{j=q}^{l} \sum_{i=p}^{k} c_{ij} \frac{1}{M} \frac{(2l - 2j + 1)}{2N^2}. \end{aligned}$$

We get

$$c_{kl} = -\left[\sum_{j=q}^{l-1} \sum_{i=p}^{k} c_{ij}(2l - 2j + 1) + \sum_{i=p}^{k-1} c_{il}\right]. \tag{4.2}$$

If the upper index is less than the lower one in any of the sums, we interprete this sum as equal to zero. For $l = q,\ k = p + 1, \ldots, M$, we have

$$c_{kq} = -\sum_{i=p}^{k-1} c_{iq}. \tag{4.3}$$

For $k = p,\ l = q + 1, \ldots, N$,

$$c_{pl} = -\sum_{j=q}^{l-1} c_{pj}(2l - 2j + 1). \tag{4.4}$$

We put in (4.3) $k = p+1$ and get $c_{p+1,q} = -c_{pq} = -2MN^2$. For $k = p+2$: $c_{p+2,q} = -(c_{pq} + c_{p+1,q}) = 0$, for $k = p+3$: $c_{p+3,q} = -(c_{pq} + c_{p+1,q} + c_{p+2,q}) = 0$, and so on, $c_{kp} = 0$ for $k = p+2, \ldots, M$.

From (4.4) for $l = q+1$ we have that $c_{p,q+1} = -3c_{pq} = -6MN^2$. It can be shown by induction that $c_{pl} = (-1)^{l-q} 4c_{pq} = (-1)^{l-q} 8MN^2$ for $l = q+2, \ldots, N$. Indeed, from (4.4) for $l = q+2$,

$$c_{p,q+2} = -[5c_{pq} + 3(-3c_{pq})] = -(1)^{q+2-q} 4c_{pq}.$$

Let us suppose that the assertion holds for every natural number from $(q+2)$ till $(l-1)$. Then we shall prove that it is true also for $l = q+2n$. The case $l = q+2n+1$ holds analogously. To this purpose, we use (4.4):

$$\begin{aligned} c_{pl} = c_{p,q+2n} &= -[(4n+1)c_{pq} + (4n-1)c_{p,q+1} + (4n-3)c_{p,q+2} + \cdots + 3c_{p,l-1}] \\ &= -c_{pq}[-3.4 + 5.4 - 7.4 + \cdots + (4n-3).4 + (4n-1).3 + (4n+1)] \\ &= 4c_{pq} = -(1)^{l-q} 4c_{pq} = -(1)^{l-q} 8MN^2. \end{aligned}$$

By induction (on l) we shall show that $c_{pl} = -c_{p+1,l}$, $c_{kl} = 0$ for $k = p+2, \ldots, M$ and $l = q, \ldots, N$. The assertion holds for $l = q$. Let it hold for every natural number from q till $(l-1)$. Then, by (4.2),

$$c_{kl} = \left[\sum_{j=q}^{l-1} (c_{pj} + c_{p+1,j})(2l-2j+1) \right.$$

$$\left. + \sum_{j=q}^{l-1} \sum_{i=p+2}^{k-1} c_{ij}(2l-2j+1) + \sum_{i=p}^{k-1} c_{il} \right] = -\sum_{i=p}^{k-1} c_{il},$$

i.e., we get a relation similar to (4.3). Putting in it $k = p+1$, we get $c_{p+1,l} = -c_{pl}$. The substitution $k = p+2$ gives $c_{p+2,l} = 0$ and so on , we get $c_{Ml} = 0$. The assertion is proved.

So we have calculated $\{c_{ij}\}_{i=1}^{M}{}_{j=1}^{N}$ for $\delta_{pq}(x,y)$.

Now we are ready to give the explicite form of $\delta_{pq}(x,y)$. Let $(x,y) \in \Box_{kl}$. Having in mind that $c(t,v) = c_{ij}$ for $(x,y) \in \Box_{ij}$ and $c_{ij} = 0$ for some (i,j), we get

$$\begin{aligned} \delta_{pq}(x,y) &= \sum_{j=q}^{l-1} \sum_{i=p}^{k-1} c_{ij} \int_{(i-1)/M}^{i/M} (x-t)^0 \, dt \int_{(j-1)/N}^{j/N} (y-v)^1 \, dv \\ &+ \sum_{j=q}^{l-1} c_{kj} \int_{(k-1)/M}^{x} (x-t)^0 \, dt \int_{(j-1)/N}^{j/N} (y-v)^1 \, dv \\ &+ \sum_{i=p}^{k-1} c_{il} \int_{(i-1)/M}^{i/M} (x-t)^0 \, dt \int_{(l-1)/N}^{y} (y-v)^1 \, dv \end{aligned}$$

$$
\begin{aligned}
&+ \quad c_{kl} \int_{(k-1)/M}^{x} (x-t)^0 \, dt \int_{(l-1)/N}^{y} (y-v)^1 \, dv \\
&= \quad \sum_{j=q}^{l-1} \sum_{i=p}^{k-1} c_{ij} + \frac{1}{MN} \left(y - \frac{2j-1}{2N} \right) \\
&+ \quad \sum_{j=q}^{l-1} c_{kj} \frac{1}{N} \left(x - \frac{k-1}{M} \right) \left(y - \frac{2j-1}{2N} \right) \\
&+ \quad \sum_{i=p}^{k-1} c_{il} \frac{1}{2M} \left(y - \frac{l-1}{N} \right)^2 + c_{kl} \left(x - \frac{k-1}{M} \right) \frac{1}{2} \left(y - \frac{l-1}{N} \right)^2 .
\end{aligned}
$$

We will consider various cases for k and l :

1. $(x, y) \in \square_{pq}$, i.e., $k = p, \ l = q$:

$$
\delta_{pq}(x,y) = c_{pq} \left(x - \frac{p-1}{M} \right) \frac{1}{2} \left(y - \frac{q-1}{N} \right)^2 = MN^2 \left(x - \frac{p-1}{M} \right) \left(y - \frac{q-1}{N} \right)^2 .
$$

2. $(x, y) \in \square_{p,q+1}$, i.e., $k = p, \ l = q+1$:

$$
\delta_{pq}(x,y) = c_{pq} \left(x - \frac{p-1}{M} \right) \frac{1}{N} \left(y - \frac{2q-1}{2N} \right) + c_{p,q+1} \left(x - \frac{p-1}{M} \right) \frac{1}{2} \left(y - \frac{q}{N} \right)^2
$$

$$
= 3MN^2 \left(x - \frac{p-1}{M} \right) \left(\frac{q+1}{N} - y \right) \left(y - \frac{q}{N} + \frac{1}{3N} \right) .
$$

3. $(x, y) \in \square_{p+1,q}$:

$$
\delta_{pq}(x,y) = c_{pq} \frac{1}{2M} \left(y - \frac{q-1}{N} \right)^2 + c_{p+1,q} \left(x - \frac{p}{M} \right) \frac{1}{2} \left(y - \frac{q-1}{N} \right)^2
$$

$$
= MN^2 \left(y - \frac{q-1}{N} \right)^2 \left(\frac{p+1}{M} - x \right) .
$$

4. $(x, y) \in \square_{p+1,q+1}$:

$$
\delta_{pq}(x,y) = c_{pq} \frac{1}{MN} \left(y - \frac{2q-1}{2N} \right) + c_{p+1,q} \left(x - \frac{p}{M} \right) \frac{1}{N} \left(y - \frac{2q-1}{2N} \right)
$$

$$
+ c_{p,q+1} \frac{1}{2M} \left(y - \frac{q}{N} \right)^2 + c_{p+1,q+1} \left(x - \frac{p}{M} \right) \frac{1}{2} \left(y - \frac{q}{N} \right)^2
$$

$$
= 3MN^2 \left(\frac{p+1}{M} - x \right) \left(\frac{q+1}{N} - y \right) \left(y - \frac{q}{N} + \frac{1}{3N} \right) .
$$

5. $(x, y) \in \square_{k,l}$, for $k \le p-1$ or $l \le q-1$: Let, for instance, $k \le p-1$. Then

$$
\delta_{pq}(x,y) \quad = \quad \sum_{j=q}^{l-1} c_{kj} \left(x - \frac{k-1}{M} \right) \frac{1}{N} \left(y - \frac{2j-1}{2N} \right)
$$

$$+ \quad c_{kl}\left(x-\frac{k-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2=0,$$

since $c_{kj}=0$ for $j=q,\ldots,l$.

6. $(x,y)\in\Box_{p,l}$, for $l\geq q+2$:

$$\delta_{pq}(x,y)=\sum_{j=q}^{l-1}c_{pj}\left(x-\frac{p-1}{M}\right)\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)+c_{pl}\left(x-\frac{p-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2$$

$$=\left(x-\frac{p-1}{M}\right)\left[\sum_{j=q}^{l-1}c_{pj}\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)c_{pl}\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2\right],$$

since $c_{pq}=2MN^2$, $c_{p,q+1}=-6MN^2$, $c_{p,q+t}=(-1)^t 8MN^2$, the second multiplier A is equal to

$$2MN\left[y-\frac{2q-1}{2N}-3\left(y-\frac{2q+1}{2N}\right)+4B\right]+4MN^2(-1)^{l-q}\left(y-\frac{l-1}{N}\right)^2,$$

where

$$B=y-\frac{2(q+2)-1}{2N}-y+\frac{2(q+3)-1}{2N}+\cdots+(-1)^{l-q-1}\left(y-\frac{2(l-1)-1}{2N}\right),$$

i.e.,

$$A=4MN\left[\left(\frac{q+1}{N}-y\right)+2B\right].$$

We will calculate B first for the case of even summands, i.e., when $l-1-(q+2)+1=l-q-2$ is even. Then $(l-q)$ is even and

$$B=\frac{2}{2N}[-(q+2)+(q+3)-(q+4)+\cdots-(l-2)+(l-1)]=\frac{l-q-2}{2N}.$$

If $(l-q)$ is odd, then

$$B=\left[\frac{l-1-q-2}{2N}+y-\frac{2(l-1)-1}{2N}\right]=\left[y-\frac{l+q}{2N}\right],$$

and thus

$$\delta_{pq}=(-1)^{l-q+1}\left(x-\frac{p-1}{M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)4MN^2.$$

7. $(x,y)\in\Box_{p+1,l}$, for $l\geq q+2$:

$$\delta_{pq}(x,y)=\sum_{j=q}^{l-1}c_{pj}\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)+\sum_{j=q}^{l-1}c_{p+1,j}\frac{1}{N}\left(x-\frac{p}{M}\right)\left(y-\frac{2j-1}{2N}\right)$$

$$+c_{pl}\frac{1}{2M}\left(y-\frac{l-1}{N}\right)^2+c_{p+1,l}\left(x-\frac{p}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2.$$

Since $c_{p+1,j}=-c_{pj}$ for $j=1,\ldots,N$, then

$$\delta_{pq}(x,y)=\left(\frac{p+1}{M}-x\right)\left[\sum_{j=q}^{l-1}c_{pj}\frac{1}{N}\left(y-\frac{2j-1}{2N}\right)+c_{pl}\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2\right]$$

and as in item **6.**,

$$\delta_{pq}(x,y)=(-1)^{l-q+1}4MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right).$$

8. $(x,y)\in\square_{k,l}$, for $k\geq p+2$ and $l\geq q$: We represent δ_{pq} in the form

$$\sum_{j=q}^{l-1}(c_{pj}+c_{p+1,j})\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)+\sum_{j=q}^{l-1}\sum_{i=p+2}^{k-1}c_{ij}\frac{1}{MN}\left(y-\frac{2j-1}{2N}\right)$$

$$+\left[(c_{pj}+c_{p+1,j})+\sum_{i=p+2}^{k-1}c_{il}\right]\frac{1}{2M}\left(y-\frac{l-1}{N}\right)^2+c_{kl}\left(x-\frac{k-1}{M}\right)\frac{1}{2}\left(y-\frac{l-1}{N}\right)^2$$

and conclude that $\delta_{pq}(x,y)=0$ in this case. Therefore, we arrive at the following expressions for $\delta_{pq}(x,y)$:

$$MN^2\left(x-\frac{p-1}{M}\right)\left(y-\frac{q-1}{N}\right)^2 \quad \text{for } (x,y)\in\square_{pq};$$

$$MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{q-1}{N}\right)^2 \quad \text{for } (x,y)\in\square_{p+1,q};$$

$$3MN^2\left(x-\frac{p-1}{M}\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \quad \text{for } (x,y)\in\square_{p,q+1};$$

$$3MN^2\left(\frac{p+1}{M}-x\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \quad \text{for } (x,y)\in\square_{p+1,q+1};$$

$$(-1)^{l-q+1}4MN^2\left(x-\frac{p-1}{M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

for $(x,y)\in\square_{pl},\quad N\geq l\geq q+2;$

$$(-1)^{l-q+1}4MN^2\left(\frac{p+1}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

for $(x,y)\in\square_{p+1,q},\quad N\geq l\geq q+2;$

0, elsewhere.

This is also true for $p = M$ and $q = N$, but in these cases we consider only such indices that are less than or equal to M or N, respectively.

Consider the operator $I_{mn}[f]$, which puts in correspondence to a function f its interpolating (m,n)-spline at a fixed set of nodes $\{t_{pq}\}$. In the $(1,2)$-case we have constructed the Lagrangian basis and thus the interpolating spline $I_{1,2}[f]$ can be represented in the form

$$I_{1,2}[f](x,y) = \sum_{p=1}^{M}\sum_{q=1}^{N} \delta_{pq} f(t_{pq}).$$

Thus, for the norm $||I_{mn}||$ of this operator in the space $C(G)^\circ$ of continuous functions bounded by 1 in the unit square G, we get

$$||I_{mn}|| = \sup_{f\in C(G)^\circ} ||\sum_{p=1}^{M}\sum_{q=1}^{N} \delta_{pq} f(t_{pq})|| \le \sum_{p=1}^{M}\sum_{q=1}^{N} ||\delta_{pq}(x,y)||.$$

Bounds of the norm $||I_{mn}||$ are useful for estimating the error of approximation. That is why we give below such estimates in the case of the most frequently used norms. For simplicity of notation, we will omit the indices mn of I_{mn} and also we will write $\sum$ instead of $\sum_{p=1}^{M}\sum_{q=1}^{N}$.

For every spline s we have

$$\begin{aligned} ||f - If||_X &= ||f - s + Is - If||_X \le ||f - s||_X + ||I||_{L_\infty \to X} ||f - s||_{L_\infty} \\ &\le ||f - s||_{L_\infty}(1 + ||I||_{L_\infty \to X}). \end{aligned}$$

In the case $X = L_1$ we get the following bounds:

$$||\delta_{pq}||_{L_1} = \frac{2}{3MN}(N - q + 1) \quad \text{for } 1 \le p < M,\ 1 \le q < N,$$

$$||\delta_{Mq}||_{L_1} = \frac{N - q + 1}{3MN} \quad \text{for } 1 \le p < M,$$

$$||\delta_{pN}||_{L_1} = \frac{1}{3MN} \quad \text{for } 1 \le p < M,$$

$$||\delta_{MN}||_{L_1} = \frac{1}{6MN}.$$

Therefore,

$$\sum_{p=1}^{M}\sum_{q=1}^{N} ||\delta_{pq}||_{L_1} = \frac{N}{3}\left(1 - \frac{1}{2M}\right)\left(1 + \frac{1}{N} - \frac{1}{N^2}\right) \sim \frac{N}{3},$$

when M and N tend to infinity.

Hence

$$\|f - If\|_{L_1} \le \left[\frac{N}{3}\left(1 - \frac{1}{2M}\right)\left(1 + \frac{1}{N} - \frac{1}{N^2}\right) + 1\right] E_f^\infty,$$

where E_f^∞ is the best L_∞-approximation of f by $(1,2)$-splines.

In the case $X = L_2$ we calculate $\|\delta_{pq}\|_{L_2}$. For $1 \le p < M,\ 1 \le q < N$ we get

$$\|\delta_{pq}\|_{L_2}^2 = \frac{16}{45}\left(N - q + \frac{3}{2}\right).$$

Besides, for $1 \le q < N$,

$$\|\delta_{Mq}\|_{L_2}^2 = \frac{8}{45MN}\left(N - q + \frac{3}{2}\right),$$

for $1 \le p < M$,

$$\|\delta_{pN}\|_{L_2}^2 = \frac{2}{45MN} \qquad \text{and} \qquad \|\delta_{MN}\|_{L_2}^2 = \frac{1}{15MN}.$$

Then

$$\sum \|\delta_{pq}\|_{L_2}^2 = \frac{(N+2)(2 - 1/M)}{45} \sim \frac{2N}{45}.$$

Using the inequality $x_1 + \cdots + x_n \le \sqrt{n(x_1^2 + \cdots + x_n^2)}$, we find the estimate

$$\begin{aligned}\sum \|\delta_{pq}\|_{L_2} &\le \sqrt{MN \sum \|\delta_{pq}\|_{L_2}^2} = \sqrt{(M - 1/2)(N+2)N}\sqrt{\frac{2}{45}} \\ &\sim \sqrt{M}N\sqrt{\frac{2}{45}} \quad (\text{as } M, N \to \infty).\end{aligned}$$

Hence

$$\|I\|_{L_2} \le \sum \|\delta_{pq}\|_{L_2} \le \sqrt{(M - 1/2)(N+2)N}\sqrt{\frac{2}{45}} \sim \sqrt{M}N\sqrt{\frac{2}{45}},$$

$$\|f - If\|_{L_2} \le \left(\sqrt{(M - 1/2)(N+2)N}\sqrt{\frac{2}{45}} + 1\right) E_f^\infty.$$

Let $X = L_\infty$. Using that $\|\delta_{pq}\|_{L_\infty} = 1$ we get $\|If\|_{L_\infty} \le \sum 1.1 = MN$. Hence

$$\|I\|_{L_\infty \to L_\infty} \le MN$$

and

$$\|f - If\|_{L_\infty} \le (MN + 1)\, E_f^\infty.$$

We are going to use the estimate for $\sum \|\delta_{pq}\|_{L_2}^2$ to get a better estimate for $\|f - If\|_{L_1}$. Let $\chi_{pq}(x,y)$ be the characteristic function of the support of $\delta_{pq}(x,y)$. Then

$$\begin{aligned} \|If\|_{L_1} &= \|\sum \delta_{pq}\chi_{pq} f(t_{pq})\|_{L_1} \le \sum \|\delta_{pq}\chi_{pq} f(t_{pq})\|_{L_1} \\ &\le \sum \|\delta_{pq}\|_{L_2}\|\chi_{pq}\|_{L_2} \le \left(\sum \|\delta_{pq}\|_{L_2}^2\right)^{1/2} \left(\sum \|\chi_{pq}\|_{L_2}^2\right)^{1/2}. \end{aligned}$$

Since

$$\|\chi_{pq}\|_{L_2}^2 = \frac{2}{M}\left(1 - \frac{q-1}{N}\right)$$

for $1 \le p < M,\ 1 \le q \le N$ and

$$\|\chi_{Mq}\|_{L_2}^2 = \frac{1}{M}\left(1 - \frac{q-1}{N}\right),$$

we get

$$\left(\sum \|\chi_{pq}\|_{L_2}^2\right)^{1/2} = \sqrt{\left(2 - \frac{1}{M}\right)\frac{N+1}{2}}.$$

Then

$$\|I\|_{L_\infty \to L_1} = \sup_{\|f\|_{L_\infty} \le 1} \|If\|_{L_1} \le \frac{(2 - 1/M)\sqrt{(N+1)(N+2)}}{3\sqrt{10}} \sim \frac{2}{3\sqrt{10}} N$$

when M and N tend to infinity. Hence

$$\|f - If\|_{L_1} \le \left(\frac{(2 - 1/M)\sqrt{(N+1)(N+2)}}{3\sqrt{10}} + 1\right) E_f^\infty.$$

4.3. THE CASE $(m,n) = (2,2)$ AND $t_{ij} = (i/M, j/N)$

We seek $\delta_{pq}(x,y)$ of the form

$$\delta_{pq}(x,y) = \int_0^1 \int_0^1 (x-t)_+^1 (y-v)_+^1 c(t,v)\, dt\, dv,$$

where $c(t,v) = c_{ij}$ for $(t,v) \in \square_{ij},\ i = 1,\dots,M,\ j = 1,\dots,N$. The constants c_{ij} are determined by the interpolatory conditions

$$\delta_{pq}(t_{ij}) = \delta_{pi}\delta_{qj}, \quad i = 1,\dots,M,\ j = 1,\dots,N.$$

As in the previous section 4.2, we get

$$c_{pq} = 4M^2N^2,\ c_{p+1,q} = -3c_{pq},\ c_{kq} = (-1)^{k-p} 4c_{pq} \quad \text{for } k \ge p+2.$$

Moreover,

$$c_{k,q+1} = -3c_{kq} \text{ and } c_{kl} = (-1)^{l-q}4c_{kq} \text{ for } k \geq p+2.$$

Using the above expressions for c_{ij}, one can obtain that $\delta_{pq}(x,y)$ is equal to

$$M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(y-\frac{q-1}{N}\right)^2 \text{ if } (x,y)\in\square_{pq};$$

$$3M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(y-\frac{q-1}{N}\right)^2 \text{ if } (x,y)\in\square_{p+1,q};$$

$$3M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right) \text{ if } (x,y)\in\square_{p,q+1};$$

$$9M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right)$$

if $(x,y)\in\square_{p+1,q+1}$;

$$(-1)^{l-q+1}4M^2N^2\left(x-\frac{p-1}{M}\right)^2\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{pl},\;\; q+2\leq l\leq N$;

$$(-1)^{l-q+1}12M^2N^2\left(\frac{p+1}{M}-x\right)\left(x-\frac{p}{M}+\frac{1}{3M}\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{p+1,l},\; q+2\leq l\leq N$;

$$(-1)^{k-p+1}4M^2N^2\left(\frac{k}{M}-x\right)\left(x-\frac{k-1}{M}\right)\left(y-\frac{q-1}{N}\right)^2$$

if $(x,y)\in\square_{kq},\; p+2\leq k\leq M$;

$$(-1)^{k-p+1}12M^2N^2\left(x-\frac{k-1}{M}\right)\left(\frac{k}{M}-x\right)\left(\frac{q+1}{N}-y\right)\left(y-\frac{q}{N}+\frac{1}{3N}\right)$$

if $(x,y)\in\square_{k,q+1},\; p+2\leq k\leq M$;

$$(-1)^{l-q+k-p}16M^2N^2\left(x-\frac{k-1}{M}\right)\left(\frac{k}{M}-x\right)\left(y-\frac{l-1}{N}\right)\left(\frac{l}{N}-y\right)$$

if $(x,y)\in\square_{k,l},\; p+2\leq k\leq M,\; q+2\leq l\leq N$.

Some technical calculations show that

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_1} = \frac{1}{9MN}\left(M^2+M-1\right)\left(N^2+N-1\right) \sim \frac{MN}{9}$$

when M and N tend to infinity.

Besides,

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_2}^2 = \frac{64}{225MN}\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right).$$

Hence

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_2} \le \frac{8}{15}\sqrt{\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right)} \sim \frac{8}{15}MN.$$

It is easy to see that

$$\sum_{p=1}^{M}\sum_{q=1}^{N}\|\delta_{pq}\|_{L_\infty} = MN.$$

The same way as in Section 4.2, one can get

$$\|f-If\|_{L_1} \le \left[\frac{1}{9MN}\left(M^2+M-1\right)\left(N^2+N-1\right)+1\right]E_f^{\infty},$$

where E_f^{∞} is the best L_∞ approximation of f with $(2,2)$-splines,

$$\|f-If\|_{L_2} \le \left[\frac{8}{15}\sqrt{\left(M^2+2M-\frac{21}{8}\right)\left(N^2+2N-\frac{21}{8}\right)}+1\right]E_f^{\infty}$$

Acknowledgments. The author is grateful to Peter Petrov for his help in the proof of the error estimate in Section 4.1.

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Received on November 28, 2001

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bontchev str., bl. 8, 1113 Sofia
BULGARIA
E-mail: irina@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 95

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 95

A GENERALIZATION OF THE VOIGT-REUSS BOUNDS FOR A BINARY MEDIUM

KONSTANTIN MARKOV, KERANKA ILIEVA

In the present note a certain generalization of the well-known Voigt-Reuss bounds on the effective conductivity κ^* of a binary medium is proposed. For a fixed binary constitution the scalar function $f(\alpha)$ that gives the undimensional effective conductivity as a function of the ratio α of the constituents conductivities is considered. Certain inequalities for the derivative $f'(\alpha)$ of this function, which include α, $f(\alpha)$ and the volume fractions of the constituents, are derived. The inequalities are sharp if these fractions are solely known. More precisely, they turn into equalities for the familiar laminate media loaded along and across the layers. The Voigt and Reuss bounds on κ^* follow from the proposed inequalities, but the latter are stronger than the former bounds, since estimates are put here on the rate at which the effective conductivity changes when the constituents properties are varied at a fixed binary constitution of the medium. It is in this sense, namely, when it is claimed that our inequalities generalize the Voigt-Reuss bounds.

Keywords: random two-phase media, variances, effective properties, bounds on the effective conductivity

2000 MSC: 60G60, 60H15

1. INTRODUCTION

The aim of this note is to report some inequalities, concerning the rate of variation of the effective properties of a two-phase medium when, at a fixed random constitution, the properties of the constituents are varied. The heat conduction context is chosen for the sake of simplicity. The random constitution is assumed statistically homogeneous and isotropic.

Let κ_1 and κ_2 be the conductivities of the constituents, $\kappa_1, \kappa_2 > 0$. The random conductivity field $\kappa(\boldsymbol{x})$ of the medium takes then the values κ_1 or κ_2, depending on whether $\boldsymbol{x}$ lies in the phase '1' or '2', respectively. If $\boldsymbol{E}$ denotes the prescribed macroscopic temperature gradient imposed upon the medium, the governing equations of the problem, at the absence of body sources, read

$$\nabla \cdot \boldsymbol{q}(\boldsymbol{x}) = 0, \quad \boldsymbol{q}(\boldsymbol{x}) = \kappa(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}), \tag{1.1a}$$

where $\boldsymbol{E}(\boldsymbol{x}) = \nabla\theta(\boldsymbol{x})$. Eq. (1.1a) is supplied with the condition

$$\langle \nabla\theta(\boldsymbol{x}) \rangle = \boldsymbol{E}, \tag{1.1b}$$

which plays the role of a "boundary" one. In (1.1a) $\boldsymbol{q}(\boldsymbol{x})$ is the flux vector and $\theta(\boldsymbol{x})$ is the random temperature field. Hereafter $\langle\cdot\rangle$ denotes ensemble averaging.

Having solved somehow the random problem (1.1), one can evaluate all multipoint moments of the temperature field $\theta(\boldsymbol{x})$ and the joint moments of $\theta(\boldsymbol{x})$ and $\kappa(\boldsymbol{x})$ by means of the known moments of the conductivity field, see [1, 9]. In particular, among the joint moments, the simplest one-point moment of the flux $\boldsymbol{q}(\boldsymbol{x})$ defines the well-known effective conductivity κ^* of the medium through the relation

$$\boldsymbol{Q} = \langle \boldsymbol{q}(\boldsymbol{x}) \rangle = \langle \kappa(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x}) \rangle = \kappa^* \boldsymbol{E} \tag{1.2}$$

(assuming statistical homogeneity and isotropy).

Note that the definition (1.2) of the effective conductivity κ^* reflects the "homogenization" of the problem under study in the sense that from a macroscopic point of view, when only the macroscopic values of the flux and temperature gradient are of interest, the medium behaves as if it were homogeneous with a certain macroscopic conductivity κ^*. This interpretation explains why κ^* and its counterparts, say, the effective elastic moduli, have been extensively studied in the literature on homogenization, see, e.g. [8, 14] *et al.*, as well as the recent survey [10]. A well-known fact is to be only recalled, namely that κ^* can be defined also "energetically":

$$\kappa^* E^2 = \left\langle \kappa(\boldsymbol{x}) |\nabla\theta(\boldsymbol{x})|^2 \right\rangle . \tag{1.3}$$

Besides κ^*, other statistical characteristics of the fields $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x})$ deserve attention and, above all, the (undimensional) variances of these fields, defined as follows:

$$\sigma_e^2 = \frac{\left\langle \boldsymbol{E}'(\boldsymbol{x})^2 \right\rangle}{E^2}, \quad \sigma_q^2 = \frac{\left\langle |\boldsymbol{q}'(\boldsymbol{x})|^2 \right\rangle}{Q^2} . \tag{1.4}$$

The primes denote in what follows the fluctuating parts of the respective random fields, so that, in particular, $\boldsymbol{E}'(\boldsymbol{x})f = \boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{E}$, and hence $\left\langle \boldsymbol{E}'(\boldsymbol{x}) \right\rangle = 0$.

It is to be noted immediately that for any two-point medium the variances σ_e^2 and σ_q^2 are simply interconnected:

$$\sigma_q^2 = -\frac{\kappa_1\kappa_2}{\kappa^{*2}}\sigma_e^2 - \frac{(\kappa^* - \kappa_1)(\kappa^* - \kappa_2)}{\kappa^{*2}} . \tag{1.5}$$

This formula, derived in [11], is a straightforward consequence of the fact that the medium under study is binary and hence the field $\kappa(x)$ takes the values κ_1 or κ_2 solely.

The variances (1.4) provide us with useful information about the deviation of the random fields under studies from their mean values. Also, they are connected to the mean energy of the appropriate fields, accumulated within the phases. That is why they have attracted some attention in the literature on heterogeneous media.

To the best of the authors' knowledge, an investigation of the variances, in addition to the effective properties in the scalar conductivity context, has been initiated by Beran *et al.*, [2, 4, 3]. In particular, Beran [2] has obtained bounds on the variances through the effective properties. The Beran estimates are quite crude and this is inevitable since they are applicable to *any* statistically homogeneous and isotropic medium. More restrictive bounds are derived in [11], but only for dispersions of spheres, correct to the order "square of concentration."

Note that an application of such variances, concerned with the deviation from the Hooke law in heterogeneous materials, can be found in the recent authors' paper [12].

The above mentioned results of Beran indicate that there may exist more intimate connection between variances and effective properties. Indeed, as shown first by Bergman [5], see also [7, 13], the variance is simply connected to the derivatives of the effective conductivity $\kappa^* = \kappa^*(\kappa_1, \kappa_1)$, treated as a function of the material properties κ_1 and κ_2 of the constituents in a binary medium, at a fixed random constitution. This is an interesting and important result, but its practical application is limited by the fact that very rarely rigorous analytical formulae for $\kappa^*(\kappa_1, \kappa_1)$ are known for realistic random constitution. Rigorous bounds on $\kappa^*(\kappa_1, \kappa_1)$ are well-known, of course, but they obviously cannot supply any estimates for the above-mentioned derivatives.

It turns out that the variances (1.4) can be simply represented by means of κ^* and its derivatives $\partial\kappa^*/\partial\kappa_1$ and $\partial\kappa^*/\partial\kappa_2$ with respect to the constituents properties, having fixed the random constitution. The appropriate formulae are direct consequences of the Bergman formula [5], which will be rederived in Section 2. In turn, this formula will yield certain inequalities between the effective conductivities κ^* and its derivatives $\partial\kappa^*/\partial\kappa_1$ and $\partial\kappa^*/\partial\kappa_2$ (Section 3). These inequalities, when transformed into dimensionless form, have as a consequence the Voigt and Reuss bounds (Section 4). Both these bounds are, to say the least, well-known. The important point, however, is that the inequalities derived here bound not only the effective properties, but also the rate of their change when the constituents properties are varied. The proposed inequalities are closely connected as well to the convexity of the function $f(\alpha)$, discussed in Section 5. The latter easily follows from the spectral representation of $f(\alpha)$, due again to Bergman [5]. A certain appealing geometrical interpretation of the Voigt bound is proposed as a consequence of the convexity of the function $f(\alpha)$, namely, that this function should lie below each of its tangents and, in particular, below its tangent, drawn at the point $\alpha = 1$.

For the sake of completeness, we shall provide here a derivation of the Bergman formula. It is a bit more rigorous than the original one due to Bergman [5], since ensemble (instead of volume) averaging will be utilized. In the papers [7, 13], where the same formula has been rederived later on, volume averaging is used, similarly to the original Bergman reasoning.

The starting point is the energy definition (1.3) of the effective conductivity κ^*. Let us change the conductivity field of the medium, $\kappa(\boldsymbol{x})$, by the infinitesimal quantity

$$\delta(\boldsymbol{x}) = \chi_1(\boldsymbol{x})\delta\kappa_1 + \chi_2(\boldsymbol{x})\delta\kappa_2, \tag{2.1}$$

where $\chi_1(\boldsymbol{x})$ and $\chi_2(\boldsymbol{x})$ are the characteristic functions of the regions, occupied by the constituents '1' and '2', respectively. Then, at fixed $\boldsymbol{E} = \nabla\theta(\boldsymbol{x})$, the field $\theta(\boldsymbol{x})$ will change by $\delta\theta(\boldsymbol{x})$ and the effective conductivity – by $\delta\kappa^*$. According to (1.3), we have

$$\begin{aligned}(\kappa^* + \delta\kappa^*)\, E^2 &= \left\langle (\kappa(\boldsymbol{x}) + \delta\kappa(\boldsymbol{x}))\, |\nabla\theta(\boldsymbol{x}) + \nabla\delta\theta(\boldsymbol{x})|^2 \right\rangle \\ &= \left\langle \kappa(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle + \left\langle \delta\kappa(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle + 2\underline{\left\langle \kappa(\boldsymbol{x})\, \nabla\theta(\boldsymbol{x}) \cdot \nabla\delta\theta(\boldsymbol{x}) \right\rangle}\,,\end{aligned} \tag{2.2}$$

having neglected terms of order $(\delta\kappa)^2$. The first term in the right-hand side of (3.2) equals $\kappa^* E^2$, see (1.3), and the underlined term there vanishes, since

$$\begin{aligned}\nabla \cdot \langle \kappa(\boldsymbol{x})\delta\theta(\boldsymbol{x})\nabla\theta(\boldsymbol{x}) \rangle &= \langle \delta\theta(\boldsymbol{x})\nabla \cdot (\kappa(\boldsymbol{x})\nabla\theta(\boldsymbol{x})) \rangle \\ &\quad + \langle \kappa(\boldsymbol{x})\, \nabla\theta(\boldsymbol{x}) \cdot \nabla\delta\theta(\boldsymbol{x}) \rangle = 0,\end{aligned}$$

having taken (1.1a) into account. The reason is that the field $\kappa(\boldsymbol{x})\delta\theta(\boldsymbol{x})\nabla\theta(\boldsymbol{x})$ is statistically homogeneous and therefore its mean value is constant.

Hence

$$\begin{aligned}\delta\kappa^* E^2 &= \left\langle \delta\kappa(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle \\ &= \delta\kappa_1 \left\langle \chi_1(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle + \delta\kappa_2 \left\langle \chi_2(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle.\end{aligned}$$

The latter implies

$$\frac{\partial\kappa^*}{\partial\kappa_i} = \frac{1}{E^2} \left\langle \chi_i(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})|^2 \right\rangle, \quad i = 1, 2, \tag{2.3}$$

which is exactly the Bergman formula [5]. It obviously means that the mean value of the temperature gradient square within the constituent 'i' is proportional to the derivative $\partial\kappa^*/\partial\kappa_i$, $i = 1, 2$.

Note first that

$$\frac{1}{E^2}\left\langle|\boldsymbol{E}(\boldsymbol{x})|^2\right\rangle = \frac{1}{E^2}\left\langle\chi_1(\boldsymbol{x})|\boldsymbol{E}(\boldsymbol{x})|^2\right\rangle + \frac{1}{E^2}\left\langle\chi_2(\boldsymbol{x})|\boldsymbol{E}(\boldsymbol{x})|^2\right\rangle = \frac{\partial\kappa^*}{\partial\kappa_1} + \frac{\partial\kappa^*}{\partial\kappa_2}$$

and hence

$$\begin{aligned}\sigma_e^2 = \frac{\left\langle|\boldsymbol{E}'(\boldsymbol{x})|^2\right\rangle}{E^2} &= \frac{1}{E^2}\left[\left\langle|\boldsymbol{E}(\boldsymbol{x})|^2\right\rangle - 2\boldsymbol{E}\cdot\langle\boldsymbol{E}(\boldsymbol{x})\rangle + E^2\right] \\ &= \frac{1}{E^2}\left\langle|\boldsymbol{E}(\boldsymbol{x})|^2\right\rangle - 1 = \frac{\partial\kappa^*}{\partial\kappa_1} + \frac{\partial\kappa^*}{\partial\kappa_2} - 1.\end{aligned} \tag{3.1}$$

Formula (3.1) provides us with the interconnection between the variance of the temperature gradient and the partial derivatives of the effective conductivity considered, at a fixed two-phase geometry, as a function of the constituents conductivities.

To recast (3.1) into dimensionless form, recall the obvious fact that $\kappa^* = \kappa^*(\kappa_1, \kappa_2)$ is a homogeneous function of first order, i.e.

$$\kappa^*(\lambda\kappa_1, \lambda\kappa_2) = \lambda\kappa^*(\kappa_1, \kappa_2), \quad \forall\,\lambda > 0.$$

This fact allows us to apply the Euler formula

$$\kappa_1\frac{\partial\kappa^*}{\partial\kappa_1} + \kappa_2\frac{\partial\kappa^*}{\partial\kappa_2} = \kappa^*,$$

i.e.

$$\frac{\partial\kappa^*}{\partial\kappa_1} = \frac{\kappa^*}{\kappa_1} - \frac{\kappa_2}{\kappa_1}\frac{\partial\kappa^*}{\partial\kappa_2}. \tag{3.2}$$

Let us now fix the conductivity κ_1 of the first of the constituents and introduce the dimensionless variables

$$\begin{aligned}\alpha &= \frac{\kappa_2}{\kappa_1}, \quad \alpha\in(0,\infty), \\ f(\alpha) &= \frac{\kappa^*}{\kappa_1}, \quad f(\alpha)\ge 0.\end{aligned} \tag{3.3}$$

Here $f(\alpha)$, for the fixed two-phase geometry under discussion, depends on the dimensionless ratio α solely. Using (3.2) and (3.3) into (3.1) gives

$$\begin{aligned}\sigma_e^2 &= \frac{\partial\kappa^*}{\partial\kappa_1} + \frac{\partial\kappa^*}{\partial\kappa_2} - 1 \\ &= \frac{\kappa^*}{\kappa_1} - \left(\frac{\kappa_2}{\kappa_1} - 1\right)\frac{\partial(\kappa^*/\kappa_1)}{\partial(\kappa_2/\kappa_1)} - 1,\end{aligned} \tag{3.4}$$

i.e.

$$\sigma_e^2 = f(\alpha) - 1 - f'(\alpha)(\alpha - 1). \tag{3.5}$$

Hence any theory that predicts $f(\alpha)$, i.e. the effective conductivity as a function of α automatically predicts the variance σ_e^2, since (3.4), as a consequence of the Bergman formula (2.3), is an *exact* relation.

The situation with the variance σ_q^2 of the heat flux is fully similar. In this case we should combine (1.5) and (3.6). The final result reads

$$\sigma_q^2 = \frac{\alpha(\alpha-1)f'(\alpha) - f(\alpha)\left(f(\alpha)-1\right)}{f^2(\alpha)}. \tag{3.6}$$

It remains now to note that both variances σ_e^2 and σ_q^2 are nonnegative, as it follows from their definitions (1.3). Together with (3.5) and (3.6), this obvious fact yields the inequalities

$$\frac{f(\alpha)\left(f(\alpha)-1\right)}{\alpha} \leq f'(\alpha)(\alpha-1) \leq f(\alpha) - 1. \tag{3.7}$$

Formula (3.7) is our generalization of the Voigt-Reuss bounds for a two-phase heterogeneous medium. The reason to call it generalization will become clear in the next section, where two basic consequences of (3.7) will be derived, namely, both the Voigt and Reuss bounds on the effective conductivity.

4. SOME CONSEQUENCES OF INEQUALITY (3.7)

Recall first the well-known perturbation expansion of the effective conductivity

$$\frac{\kappa}{\langle\kappa\rangle} = 1 - a_2\left(\frac{\kappa}{\langle\kappa\rangle}\right)^2 + \cdots, \tag{4.1}$$

due to Brown [6]. Here $\langle\kappa\rangle = c_1\kappa_1 + c_2\kappa_2$, $[\kappa] = \kappa_2 - \kappa_1$ and c_i is the volume fraction of the constituent 'i', $i = 1, 2$, so that $c_1 + c_2 = 1$. In (4.1) $a_2 = \frac{1}{3}c_1c_2$, but this fact will not be needed here, since it affects only the $(\alpha-1)^2$-term in the Taylor expansion (4.2) below.

In the dimensionless variables (3.3), Eq. (4.1) is recast as

$$f(\alpha) = \frac{\kappa^*}{\kappa_1} = 1 + c_2(\alpha - 1) + o\left(|\alpha - 1|\right), \quad |\alpha - 1| \ll 1. \tag{4.2}$$

Assume now that $\alpha > 1$. Rewrite the right-hand side inequality in (3.7) in the form

$$\frac{\mathrm{d}f(\alpha)}{f(\alpha)-1} \leq \frac{\mathrm{d}\alpha}{\alpha-1}, \quad \alpha > 1,$$

and integrate the latter from $1+\varepsilon$ to α, $\varepsilon > 0$. This gives

$$\ln\frac{f(\alpha)-1}{f(1+\varepsilon)-1} \leq \ln\frac{\alpha-1}{\varepsilon}$$

and hence

$$f(\alpha) - 1 \le \Big[f(1+\varepsilon) - 1 \Big] \frac{\alpha}{\varepsilon}.$$

Choosing now $\varepsilon \to 1 + 0$ in the latter inequality yields

$$f(\alpha) \le 1 + c_2(\alpha - 1), \tag{4.3}$$

having taken into account (4.2) as well. Repeating the above reasoning for $\alpha < 1$ produces the same result (4.3). Hence (4.3) holds for arbitrary values of the constituents conductivities κ_1 and κ_2. Using the definition (3.3) of α and $f(\alpha)$ allows us to recast (4.3) as

$$\kappa^* \le \kappa^{\mathrm{v}}, \quad \kappa^{\mathrm{v}} = c_1\kappa_1 + c_2\kappa_2, \tag{4.4}$$

and this is the familiar Voigt estimate upon the effective conductivity of the medium.

The treatment of the left-hand side inequality in (3.7) is fully similar. In this case, at $\alpha > 1$, we have

$$\frac{\mathrm{d}\alpha}{\alpha(\alpha - 1)} \le \frac{\mathrm{d}f(\alpha)}{f(\alpha)(f(\alpha) - 1)}. \tag{4.5}$$

An elementary integration of both sides of (4.5) over the interval $(1 + \varepsilon, \alpha)$ gives

$$\ln\frac{\alpha - 1}{\alpha} - \ln\frac{\varepsilon}{1+\varepsilon} \le \ln\frac{f(\alpha) - 1}{f(\alpha)} - \ln\frac{f(1+\varepsilon) - 1}{f(1+\varepsilon)},$$

which simplifies as

$$\frac{c_2(\alpha - 1)}{\alpha} \le \frac{f(\alpha) - 1}{f(\alpha)},$$

having again taken (4.2) into account. Hence

$$\frac{\alpha}{\alpha - c_2(\alpha - 1)} \le f(\alpha).$$

Recalling the definition (3.3), the latter can be recast as

$$\kappa^{\mathrm{r}} \le \kappa^*, \quad \kappa^{\mathrm{r}} = \left(\frac{c_1}{\kappa_1} + \frac{c_2}{\kappa_2}\right)^{-1} \tag{4.6}$$

and this is just the familiar Reuss bound on the effective conductivity.

It is important to point out that the inequalities (3.7) are *sharp*, i.e. they *cannot* be improved provided only the volume fractions c_1, c_2 are known.

Indeed, if the temperature gradient is along the layers of a laminate medium, the Voigt approximation (4.4) provides the exact value of the effective conductivity and hence

$$f(\alpha) = 1 + c_2(\alpha - 1).$$

The latter function turns the right-hand side of (3.7) into equality.

Similarly, if the temperature gradient is across the layers of a laminate medium, the Reuss value (4.6) represents exactly the effective conductivity . Then

$$f(\alpha) = \frac{\alpha}{\alpha - c_2(\alpha - 1)}$$

and this function assures equality sign in the left-hand side of (3.7).

5. DISCUSSION

Let us point out first that, since $f(1) = 1$,

$$f(\alpha) - 1 = f'(\xi)(\alpha - 1), \tag{5.1}$$

where $\xi \in (1, \alpha)$ or $\xi \in (\alpha, 1)$, depending on whether $\alpha > 1$ or $\alpha < 1$, respectively. (This is the well-known Lagrange theorem from the elementary calculus.) Together with the right-hand side of (3.7), Eq. (5.1) implies that for each α there exists an "intermediate" $\xi \in (1, \alpha)$ or $\xi \in (\alpha, 1)$, depending again on whether $\alpha > 1$ or $\alpha < 1$, such that

$$f'(\alpha) \leq f'(\xi), \quad \xi < \alpha. \tag{5.2}$$

From (5.2) it follows that $f''(1) < 0$ and hence the function $f(\alpha)$ is convex in a certain vicinity of $\alpha = 1$. (This is indeed so, because the coefficient a_2, proportional to $f''(1)$, in the Taylor expansion of $f(\alpha)$ about $\alpha = 1$ is negative, see the beginning of Section 4.) We do not know, however, whether (5.2) suffices to claim that the function $f(\alpha)$ is convex globally, i.e. $f''(\alpha) \leq 0$ on the whole semiaxis $\alpha \in (0, \infty)$.

However, the convexity of $f(\alpha)$ easily follows from the well-known spectral (pole) representation[1]

$$\begin{aligned} f(\alpha) &= 1 - F(s), \quad F(s) = \sum_n \frac{B_n}{s - s_n}, \\ s &= \frac{1}{1-\alpha}, \quad B_n \geq 0, \end{aligned} \tag{5.3}$$

due again to Bergman [5] (see also [9]). Indeed, a straightforward differentiation of (5.3) shows that $f''(\alpha) \leq 0$, $\forall \alpha \in (0, \infty)$ (recall that $B_n \geq 0$).

The convexity of $f(\alpha)$ means geometrically that the function lies below each of its tangents, i.e.

$$f(\alpha) \leq f'(\alpha_0)\,(\alpha - \alpha_0) + f(\alpha_0), \quad \forall\, \alpha, \alpha_0 \in (0, \infty). \tag{5.4}$$

In particular, the Voigt bound can be interpreted geometrically as the obvious fact that the function $f(\alpha)$ falls below its tangent, drawn at the point $\alpha = 1$, see Fig. 1.

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[1]The authors thank D. Bergman for this observation (in a private communication).

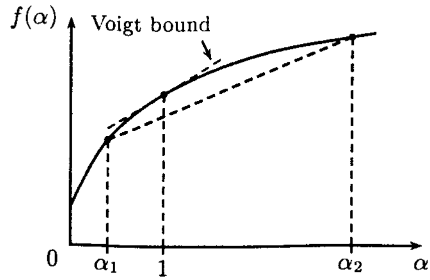

Fig. 1

Obviously, the inequality (5.4) reduces to the right-hand side of (3.7) if $\alpha_0 = 1$. It is clear, however, that (5.4) is of little practical use, because $f(\alpha_0)$ and $f'(\alpha_0)$ are unknown, in general, unless $\alpha_0 = 1$.

Recall also that the convexity of $f(\alpha)$ can be alternatively defined by the requirement that its graph in the interval (α_1, α_2) lies higher than the chord between the points $(\alpha_1, f(\alpha_1))$ and $(\alpha_2, f(\alpha_2))$, see Fig. 1. In other words, the following inequality holds:

$$\frac{f(\alpha_1) - f(\alpha_2)}{\alpha_1 - \alpha_2}\alpha + \frac{\alpha_1 f(\alpha_2) - \alpha_2 f(\alpha_1)}{\alpha_1 - \alpha_2} \leq f(\alpha), \tag{5.5}$$

$\forall\, \alpha \in (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in (0, \infty)$. Hence (5.5) provides a certain lower bound on the effective conductivity provided we have somehow measured the values of the latter for two given values α_1, α_2 of the ratio of the constituents conductivities. Observe, however, that (5.5) is a lower bound only in the interval $\alpha \in (\alpha_1, \alpha_2)$. Outside this interval (5.5) becomes an upper bound on $f(\alpha)$.

6. CONCLUDING REMARKS

We have derived certain inequalities, cf. (3.7), for the rate of change $f'(\alpha)$ of the dimensionless effective conductivity $f(\alpha)$ of a binary medium when the constituents properties are varied at fixed random constitution. The inequalities are of first order, in the sense that they include, besides $f(\alpha)$ and the dimensionless ratio α of the constituents conductivities, only the volume fractions of the constituents. They indicate that the above-mentioned rate of change $f'(\alpha)$ *cannot* be arbitrary for a realistic binary constitution. It is rather "guided" by the value $f(\alpha)$ of the effective conductivity at any given α. Presumably, higher-order counterparts of the inequalities (3.7) exist as well. They should provide tighter estimates for $f'(\alpha)$ at the expense of including the appropriate higher-order statistical information for the medium.

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Received December 10, 2001
Revised January 7, 2002

Faculty of Mathematics and Informatics
"K. Preslavski" University of Shumen
9700 Shumen
BULGARIA
E-mail: ilieva_k@abv.bg

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