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ON THE VERTEX FOLKMAN NUMBERS $F_V(\underbrace{2,\dots,2}_{R};R-1)$ AND $F_V(\underbrace{2,\dots,2}_{R};R-2)$

NEDYALKO NENOV

For a graph G the symbol $G \xrightarrow{v} (a_1,\dots,a_r)$ means that in every r-coloring of the vertices of G, for some $i \in \{1,2,\dots,r\}$ there exists a monochromatic a_i-clique of color i. The vertex Folkman numbers

$$F_v(a_1,\dots,a_r;q) = \min\{|V(G)| : G \xrightarrow{v} (a_1,\dots,a_r) \text{ and } K_q \nsubseteq G\}$$

are considered. We prove that

$$F_v(\underbrace{2,\dots,2}_{r};r-1) = r+7\,, \quad r \geq 6 \quad \text{and} \quad F_v(\underbrace{2,\dots,2}_{r};r-2) = r+9\,, \quad r \geq 8\,.$$

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1. INTRODUCTION

We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$. In this paper we shall also use the following notation:

- $V(G)$ is the vertex set of the graph G;
- $E(G)$ is the edge set of the graph G;

- $\overline{G}$ is the complement of G;
- $G[V]$, $V \subseteq V(G)$ is the subgraph of G induced by V;
- $G - V$, $V \subseteq V(G)$ is the subgraph of G induced by $V(G) \setminus V$;
- $\alpha(G)$ is the vertex independence number of G;
- $\chi(G)$ is the chromatic number of G;
- $f(G) = \chi(G) - \mathrm{cl}(G)$;
- K_n is the complete graph on n vertices;
- C_n is the simple cycle on n vertices.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

The Ramsey number $R(p, q)$ is the smallest natural n such that for every n-vertex graph G either $\mathrm{cl}(G) \geq p$ or $\alpha(G) \geq q$. An exposition of the results on the Ramsey numbers is given in [25]. In Table 1.1 we list the known Ramsey numbers $R(p, 3)$ (see [25]).

p	3	4	5	6	7	8	9	10
$R(p,3)$	6	9	14	18	23	28	36	40–43

Table 1.1: The known Ramsey numbers

Definition. *Let $a_1, \ldots, a_r$ be positive integers. We say that the r-coloring*

$$V(G) = V_1 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset,\ i \neq j$$

of the vertices of the graph G is $(a_1, \ldots, a_r)$-free, if V_i does not contain an a_i-clique for each $i \in \{1, \ldots, r\}$. The symbol $G \overset{v}{\rightarrow} (a_1, \ldots, a_r)$ means that there is no $(a_1, \ldots, a_r)$-free coloring of the vertices of G.

Let $a_1, \ldots, a_r$ and q be natural numbers. Define

$$H_v(a_1, \ldots, a_r; q) = \{G : G \overset{v}{\rightarrow} (a_1, \ldots, a_r) \text{ and } \mathrm{cl}(G) < q\},$$
$$F_v(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \in H_v(a_1, \ldots, a_r; q)\}.$$

The graph $G \in H_v(a_1, \ldots, a_r; q)$ is said to be an extremal graph in $H_v(a_1, \ldots, a_r; q)$, if $|V(G)| = F_v(a_1, \ldots, a_r; q)$.

It is clear that $G \overset{v}{\rightarrow} (a_1, \ldots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman [3] proved that there exists a graph G such that $G \overset{v}{\rightarrow} (a_1, \ldots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$. Therefore

$$F_v(a_1, \ldots, a_r; q) \text{ exists} \iff q > \max\{a_1, \ldots, a_r\}. \tag{1.1}$$

The numbers $F_v(a_1, \dots, a_r; q)$ are called vertex Folkman numbers.

If $a_1, \dots, a_r$ are positive integers, $r \geq 2$ and $a_i = 1$ then it is easily seen that

$$G \xrightarrow{v} (a_1, \dots, a_i, \dots, a_r) \iff G \xrightarrow{v} (a_1, \dots, a_{i-1}, a_{i+1}, a_r).$$

Thus it suffices to consider only such numbers $F_v(a_1, \dots, a_r; q)$ for which $a_i \geq 2$, $i = 1, \dots, r$. In this paper we consider the vertex Folkman numbers $F_v(2, \dots, 2; q)$. Set

$$(\underbrace{2, \dots, 2}_{r}) = (2_r) \quad \text{and} \quad F_v(\underbrace{2, \dots, 2}_{r}; q) = F_v(2_r; q).$$

By (1.1),

$$F_v(2_r; q) \text{ exists} \iff q \geq 3. \tag{1.2}$$

It is clear that

$$G \xrightarrow{v} (2_r) \iff \chi(G) \geq r + 1. \tag{1.3}$$

Since $K_{r+1} \xrightarrow{v} (2_r)$ and $K_r \overset{v}{\nrightarrow} (2_r)$, we have

$$F_v(2_r; q) = r + 1 \quad \text{if} \quad q \geq r + 2.$$

In [2] Dirac proved the following result.

Theorem 1.1. ([2]) *Let G be a graph such that $\chi(G) \geq r + 1$ and $\mathrm{cl}(G) \leq r$. Then*

(a) $|V(G)| \geq r + 3$*;*

(b) *If* $|V(G)| = r + 3$*, then* $G = K_{r-3} + C_5$*.*

According to (1.3), Theorem 1.1 admits the following equivalent form:

Theorem 1.2. *Let $r \geq 2$ be a positive integer. Then*

(a) $F_v(2_r; r + 1) = r + 3$*;*

(b) $K_{r-2} + C_5$ *is the only extremal graph in* $H_v(2_r; r + 1)$*.*

In [14] Łuczak, Ruciński and Urbański defined for arbitrary positive integers $a_1, \dots, a_r$ the numbers

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \dots, a_r\}. \tag{1.4}$$

They proved the following extension of Theorem 1.2.

Theorem 1.3. ([14]) *Let $a_1, \dots, a_r$ be positive integers and m and p be defined by* (1.4)*. Let $m \geq p + 1$. Then*

(a) $F_v(a_1, \ldots, a_r; m) = m + p$;

(b) $K_{m-p-1} + \overline{C}_{2p+1}$ *is the only extremal graph in* $H_v(a_1, \ldots, a_r; m)$.

For another extension of Theorem 1.1 see [21].

From (1.1) it follows that the numbers $F_v(a_1, \ldots, a_r; m-1)$ exist if and only if $m \geq p+2$. The exact values of all numbers $F_v(a_1, \ldots, a_r; m-1)$ for which $p = \max\{a_1, \ldots, a_r\} \leq 4$ are known. A detailed exposition of these results was given in [13] and [23]. We do not know any exact values of $F_v(a_1, \ldots, a_r; m-1)$ in the case when $\max\{a_1, \ldots, a_r\} \geq 5$. Here we shall note only the values $F_v(a_1, \ldots, a_r; m-1)$ when $a_1 = a_2 = \cdots = a_r = 2$, i.e. of the numbers $F_v(2_r; r)$. From (1.2) these numbers exist if and only if $r \geq 3$. If $r = 3$ and $r = 4$ we have that

$$F_v(2_3; 3) = 11; \tag{1.5}$$

$$F_v(2_4; 4) = 11. \tag{1.6}$$

The inequality $F_v(2_3; 3) \leq 11$ was proved in [15] and the opposite inequality $F_v(2_3; 3) \geq 11$ was proved in [1]. The equality (1.6) was proved in [18] (see also [19]). If $r \geq 5$ we have the following result.

Theorem 1.4. ([17], see also 24]) *Let* $r \geq 5$. *Then:*

(a) $F_v(2_r; r) = r + 5$;

(b) $K_{r-5} + C_5 + C_5$ *is the only extremal graph in* $H_v(2_r; r)$.

Theorem 1.4(a) was proved also in [8] and [14].

According to (1.2), the number $F_v(2_r; r-1)$ exists if and only if $r \geq 4$. In [17] we proved that

$$F_v(2_r; r-1) = r + 7 \quad \text{if} \quad r \geq 8. \tag{1.7}$$

In this paper we improve (1.7) by proving the following result:

Theorem 1.5. *Let* $r \geq 4$ *be an integer. Then:*

(a) $F_v(2_r; r-1) \geq r + 7$;

(b) $F_v(2_r; r-1) = r + 7\,,\quad$ *if* $r \geq 6$;

(c) $F_v(2_5; 4) \leq 16$.

In [9] Jensen and Royle showed that

$$F_v(2_4; 3) = 22. \tag{1.8}$$

We see from Theorem 1.5 and (1.8) that $F_v(2_5; 4)$ is the only unknown number of the kind $F(2_r; r-1)$[1].

From (1.2) it follows that the Folkman number $F(2_r; r-2)$ exists if and only if $r \geq 5$. In [16] we proved that $F_v(2_r; r-2) = r+9$ if $r \geq 11$. In this paper we improve this result as follows:

Theorem 1.6. *Let $r \geq 5$ be an integer. Then:*

(a) $F_v(2_r; r-2) \geq r+9$;

(b) $F_v(2_r; r-2) = r+9\,, \quad \textit{if} \quad r \geq 8.$

The numbers $F_v(2_r; r-2)$, $5 \leq r \leq 7$, are unknown.

2. AUXILIARY RESULTS

Let G be an arbitrary graph. Define

$$f(G) = \chi(G) - \mathrm{cl}(G).$$

Lemma 2.1. *Let G be a graph such that $f(G) \leq 2$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. Since $\chi(G) \geq \mathrm{cl}(G)$, we have $f(G) \geq 0$. For $f(G) = 0$ the inequality is trivial. Let $f(G) = 1$ and $\chi(G) = r+1$. Then $\mathrm{cl}(G) = r$. Note that $r \geq 2$ because of $\chi(G) \neq \mathrm{cl}(G)$. By (1.3) we have $G \in H_v(2_r; r+1)$. Thus, from Theorem 1.2(a) it follows that $|V(G)| \geq r+3 = 2f(G) + \chi(G)$. Let $f(G) = 2$ and $\chi(G) = r+1$. Then $\mathrm{cl}(G) = r-1$. Since $\chi(G) \neq \mathrm{cl}(G)$, $\mathrm{cl}(G) = r-1 \geq 2$, i.e. $r \geq 3$. From Theorem 1.4(a), (1.5) and (1.6) we obtain that $|V(G)| \geq r+5 = \chi(G) + 2f(G)$. This completes the proof of Lemma 2.1. □

Let $G = G_1 + G_2$. Obviously,

$$\chi(G) = \chi(G_1) + \chi(G_2); \tag{2.1}$$

$$\mathrm{cl}(G) = \mathrm{cl}(G_1) + \mathrm{cl}(G_2). \tag{2.2}$$

Hence,

$$f(G) = f(G_1) + f(G_2). \tag{2.3}$$

[1]Meanwhile, it has been proved that $F_v(2_5; 4) = 16$, see J. Lathrop, S. Radziszowski, Computing the Folkman Number $F_v(2, 2, 2, 2, 2; 4)$, Journal of Combinatorial Mathematics and Combinatorial Computing, 78 (2011), 213–222.

A graph G is said to be vertex-critical chromatic if $\chi(G-v) < \chi(G)$ for all $v \in V(G)$. We shall use the following result in the proof of Theorem 1.6.

Theorem 2.1. ([4], see also [5]) *Let G be a vertex-critical chromatic graph and $\chi(G) \geq 2$. If $|V(G)| < 2\chi(G) - 1$, then $G = G_1 + G_2$, where $V(G_i) \neq \emptyset$, $i = 1, 2$.*

Remark. In the original statement of Theorem 2.1 the graph G is supposed to be edge-critical chromatic (and not vertex-critical chromatic). Since each vertex-critical chromatic graph G contains an edge-critical chromatic subgraph H such that $\chi(G) = \chi(H)$ and $V(G) = V(H)$, the above statement is equivalent to the original one. It is also more convenient for the proof of Theorem 1.6.

Let G be a graph and $A \subseteq V(G)$ be an independent set of vertices of the graph G. It is easy to see that

$$G \overset{v}{\rightarrow} (2_r),\ r \geq 2 \ \Rightarrow\ G - A \overset{v}{\rightarrow} (2_{r-1}). \tag{2.4}$$

Lemma 2.2. *Let $G \in H_v(2_r; q)$, $q \geq 3$ and $|V(G)| = F_v(2_r; q)$. Then*

(a) *G is a vertex-critical $(r+1)$-chromatic graph;*

(b) *If $q < r+3$, then $\mathrm{cl}(G) = q - 1$.*

Proof. By (1.3), $\chi(G) \geq r+1$. Assume that (a) is false. Then there would exist $v \in V(G)$ such that $\chi(G - v) \geq r + 1$. According to (1.3), $G - v \in H_v(2_r; q)$. This contradicts the equality $|V(G)| = F_v(2_r; q)$.

Assume that (b) is false, i.e. $\mathrm{cl}(G) \leq q-2$. Then from $q < r+3$ it follows that $\mathrm{cl}(G) < r+1$. Since $\chi(G) \geq r+1$ there are $a, b \in V(G)$ such that $[a, b] \notin E(G)$. Consider the subgraph $G_1 = G - \{a, b\}$. We have $r \geq 2$, because $\chi(G) \neq \mathrm{cl}(G)$. Thus, from (2.4) and $\mathrm{cl}(G) \leq q-2$ it follows that $G_1 \in H_v(2_{r-1}; q-1)$. Obviously, $G_1 \in H_v(2_{r-1}; q-1)$ leads to $K_1 + G_1 \in H_v(2_r; q)$. This contradicts the equality $|V(G)| = F_v(2_r; q)$, because $|V(K_1 + G_1)| = |V(G)| - 1$. Lemma 2.2 is proved. □

Lemma 2.3. *Let $G \in H_v(2_r; q)$, $r \geq 2$. Then*

$$|V(G)| \geq F_v(2_{r-1}; q) + \alpha(G).$$

Proof. Let $A \subseteq V(G)$ be an independent set such that $|A| = \alpha(G)$. Consider the subgraph $G_1 = G - A$. According to (2.4), $G_1 \in H_v(2_{r-1}; q)$. Hence $|V(G_1)| \geq F_v(2_{r-1}; q)$. Since $|V(G)| = |V(G_1)| + \alpha(G)$, Lemma 2.3 is proved. □

We shall use also the following three results:

$$F_v(2, 2, p; p+1) \geq 2p + 4, \qquad \text{see [20] ;} \tag{2.5}$$

$$F_v(2, 2, 4; 5) = 13, \qquad \text{see [22].} \tag{2.6}$$

Theorem 2.2. ([12]) *Let G be a graph,* $\mathrm{cl}(G) \leq p$ *and* $|V(G)| \geq p+2$, $p \geq 2$. *Let G also possess the following two properties:*

(i) $G \overset{v}{\not\to} (2,2,p)$*;*

(ii) *If* $V(G) = V_1 \cup V_2 \cup V_3$ *is a* $(2,2,p)$*-free 3-coloring, then* $|V_1| + |V_2| \leq 3$.

Then $G = K_1 + G_1$.

3. AN UPPER BOUND FOR THE NUMBERS $F_v(2_r;q)$

Consider the graph P whose complementary graph $\overline{P}$ is depicted in Figure 1. This graph is a well-known construction of Greenwood and Gleason [6], which

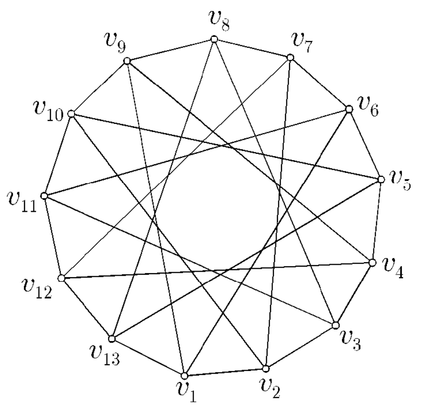

Figure 1: Graph $\overline{P}$

shows that $R(5,3) \geq 14$, since $|V(P)| = 13$ and

$$\alpha(P) = 2; \tag{3.1}$$

$$\mathrm{cl}(P) = 4 \qquad \text{(see [6])}. \tag{3.2}$$

From $|V(P)| = 13$ and (3.1) it follows that $\chi(P) \geq 7$. Since $\{v_1\} \cup \{v_2, v_3\} \cup \cdots \cup \{v_{12}, v_{13}\}$ is a 7-chromatic partition of $V(P)$, we have

$$\chi(P) = 7. \tag{3.3}$$

Let r and s be non-negative integers and $r \geq 3s+6$. Define

$$\tilde{P} = K_{r-3s-6} + P + \underbrace{C_5 + \cdots + C_5}_{s}.$$

From (2.1), (2.2), (3.2) and (3.3) we obtain that $\chi(\tilde{P}) = r+1$ and $\mathrm{cl}(\tilde{P}) = r-s-2$. By (1.3), it follows that $\tilde{P} \in H_v(2_r; r-s-1)$ and thus

$$F_v(2_r; r-s-1) \leq |V(\tilde{P})|.$$

Since $|V(\tilde{P})| = r + 2s + 7$, we proved the following

Theorem 3.1. *Let r and s be non-negative integers and $r \geq 3s + 6$. Then*

$$F_v(2_r; r-s-1) \leq r + 2s + 7.$$

Remark. Since $r \geq 3s + 6$ we have $r - s - 1 > 2$. Thus, according to (1.2), the numbers $F_v(2_r; r-s-1)$ exist.

4. PROOF OF THEOREM 1.5

Proof of Theorem 1.5(a) Let $G \in H_v(2_r; r-1)$. We need to show that $|V(G)| \geq r + 7$. From Lemma 2.3 we have

$$|V(G)| \geq F_v(2_{r-1}; r-1) + \alpha(G).$$

By (1.5), (1.6) and Theorem 1.4(a) we deduce $F_v(2_{r-1}; r-1) \geq r + 4$. Hence

$$|V(G)| \geq r + 4 + \alpha(G). \tag{4.1}$$

We prove the inequality $|V(G)| \geq r + 7$ by induction with respect to r. From Table 1.1 we see that

$$R(r-1, 3) < r + 6 \ \text{ if } r = 4 \quad \text{or} \quad r = 5. \tag{4.2}$$

Obviously, from $G \in H_v(2_r; r-1)$ it follows that $\chi(G) \neq \mathrm{cl}(G)$. Thus, $\alpha(G) \geq 2$. From (4.1) we obtain $|V(G)| \geq r + 6$. From this inequality and (4.2) we see that $|V(G)| > R(r-1, 3)$ if $r = 4$ or $r = 5$. Since $\mathrm{cl}(G) < r-1$, it follows that $\alpha(G) \geq 3$. Now from (4.1) we obtain that $|V(G)| \geq r + 7$ if $r = 4$ or $r = 5$.

Let $r \geq 6$. We shall consider separately two cases:

Case 1. $G \overset{v}{\nrightarrow} (2, 2, r-2)$. From Theorem 2.2 we see that only following two subcases are possible:

Subcase 1a. $G = K_1 + G_1$. From $G \in H_v(2_r, r-1)$ it follows that $G_1 \in H_v(2_{r-1}; r-2)$. By the induction hypothesis, $|V(G_1)| \geq r+6$. Therefore, $|V(G)| \geq r + 7$.

Subcase 1b. There is a $(2, 2, r-2)$-free 3-coloring $V(G) = V_1 \cup V_2 \cup V_3$ such that $|V_1| + |V_2| \geq 4$. Let us consider the subgraph $\tilde{G} = G[V_3]$. By assumption $\tilde{G}$ does not contain an $(r-2)$-clique, i.e. $\mathrm{cl}(\tilde{G}) < r-2$. Since V_1 and V_2 are independent sets and $G \overset{v}{\rightarrow} (2_r)$, it follows from (2.4) that $\tilde{G} \overset{v}{\rightarrow} (2_{r-2})$. Thus, $\tilde{G} \in H_v(2_{r-2}; r-2)$. By (1.6) and Theorem 1.4(a), $|V(\tilde{G})| \geq r+3$. As $|V_1|+|V_2| \geq 4$, we have $|V(G)| \geq r+7$.

Case 2. $G \overset{v}{\rightarrow} (2, 2, r-2)$. Since $\mathrm{cl}(G) < r-1$, $G \in H_v(2, 2, r-2; r-1)$. From (2.5) it follows that $|V(G)| \geq 2(r-2)+4 = 2r$. Hence, if $2r \geq r+7$, i.e. $r \geq 7$, then $|V(G)| \geq r+7$. Let $r = 6$. Then $G \in H_v(2, 2, 4; 5)$. By (2.6) we conclude that $|V(G)| \geq 13$.

Proof of Theorem 1.5(b) Let $r \geq 6$. According to Theorem 1.5(a) we have $F_v(2_r; r-1) \geq r+7$. From Theorem 3.1 ($s = 0$) we obtain the opposite inequality $F_v(2_r; r-1) \leq r+7$.

Proof of Theorem 1.5(c) There is a 16-vertex graph G such that $\alpha(G) = 3$ and $\mathrm{cl}(G) = 3$, because $R(4, 4) = 18$ (see [6]). From $|V(G)| = 16$ and $\alpha(G) = 3$ obviously it follows that $\chi(G) \geq 6$. By (1.3), $G \overset{v}{\rightarrow} (2_5)$. So, $G \in H_v(2_5; 4)$. Hence $F_v(2_5; 4) \leq |V(G)| = 16$.

Theorem 1.5 is proved. □

Corollary 4.1 *Let G be a graph such that $f(G) \leq 3$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. If $f(G) \leq 2$, then Corollary 4.1 follows from Lemma 2.1. Let $f(G) = 3$ and $\chi(G) = r+1$, then $\mathrm{cl}(G) = r-2$. Since $\chi(G) \neq \mathrm{cl}(G)$, it follows that $\mathrm{cl}(G) \geq 2$. Thus, $r \geq 4$. By (1.3) we get $G \in H_v(2_r; r-1)$. From Theorem 1.5(a) we obtain $|V(G)| \geq r+7 = \chi(G) + 2f(G)$. □

Remark. In $H_v(2_r; r-1)$, $r \geq 8$, there are more than one extremal graph. For instance, in $H_v(2_8; 7)$ besides $K_2 + P$ (see Theorem 3.1), the graph $C_5 + C_5 + C_5$ is extremal, too.

5. PROOF OF THEOREM 1.6

Proof of Theorem 1.6(a) Let $G \in H_v(2_r; r-2)$. We need to show that $|V(G)| \geq r+9$. From Lemma 2.3 we have

$$|V(G)| \geq F_v(2_{r-1}; r-2) + \alpha(G).$$

By Theorem 1.5(a), $F_v(2_{r-1}; r-2) \geq r+6$. Thus,

$$|V(G)| \geq r+6+\alpha(G). \tag{5.1}$$

We prove the inequality $|V(G)| \geq r+9$ by induction with respect to r. From Table 1.1 we see that

$$R(r-2,3) < r+8, \quad 5 \leq r \leq 7. \tag{5.2}$$

Obviously, from $G \in H_v(2_r; r-2)$ it follows that $\chi(G) \neq \mathrm{cl}(G)$. Thus, $\alpha(G) \geq 2$. From (5.1) we obtain $|V(G)| \geq r+8$. This, together with (5.2), implies $|V(G)| > R(r-2,3)$ if $5 \leq r \leq 7$. Since $\mathrm{cl}(G) < r-2$, $\alpha(G) \geq 3$. By the inequality (5.1), $|V(G)| \geq r+9$, $5 \leq r \leq 7$.

Let $r \geq 8$. Obviously, it suffices to consider only the situation when

$$|V(G)| = F_v(2_r; r-2). \tag{5.3}$$

By (5.3) and Lemma 2.2 we have that

$$G \textit{ is a vertex-critical } (r+1)\textit{-chromatic graph;} \tag{5.4}$$

and

$$\mathrm{cl}(G) = r-3. \tag{5.5}$$

From (5.4) and (5.5) it follows that

$$f(G) = 4. \tag{5.6}$$

We shall consider separately two cases.

Case 1. $|V(G)| < 2r+1$. By (5.4) and Theorem 2.1 we obtain that

$$G = G_1 + G_2. \tag{5.7}$$

From (5.7), (2.1) and (5.4) obviously it follows that

$$G_i,\ i=1,2 \quad \textit{is a vertex-critical chromatic graph.} \tag{5.8}$$

Let $f(G_1) = 0$. Then, according to (5.8) G_1 is a complete graph. Thus, it follows from (5.7) that $G = K_1 + G'$. It is clear that

$$G \in H_v(2_r; r-2) \Rightarrow G' \in H_v(2_{r-1}; r-3).$$

By the induction hypothesis, $|V(G')| \geq r+8$. Hence, $|V(G)| \geq r+9$. Let $f(G_i) \neq 0$, $i = 1,2$. We see from (5.7), (2.3) and (5.6) that $f(G_i) \leq 3$, $i = 1,2$. By Corollary 4.1 we conclude that

$$|V(G_i)| \geq \chi(G_i) + 2f(G_i), \quad i = 1,2.$$

Summing these inequalities and using (2.1) and (2.3) we obtain

$$|V(G)| \geq \chi(G) + 2f(G). \tag{5.9}$$

According to (5.4), $\chi(G) = r+1$. Finally, from (5.9) and (5.6) it follows that $|V(G)| \geq r+9$.

Case 2. $|V(G)| \geq 2r+1$. Since $r \geq 8$, then $2r+1 \geq r+9$. Hence $|V(G)| \geq r+9$.

Proof of Theorem 1.6(b) By Theorem 1.6(a), $F_v(2_r; r-2) \geq r+9$. Therefore, we need to prove the opposite inequality $F_v(2_r; r-2) \leq r+9$ if $r \geq 8$. If $r \geq 9$, this inequality follows from Theorem 3.1 ($s = 1$). Let $r = 8$. By $R(6,3) = 18$ [11] (see also [7]), there is a graph Q such that $|V(Q)| = 17$, $\alpha(Q) = 2$ and $\mathrm{cl}(Q) = 5$. From $|V(Q)| = 17$ and $\alpha(Q) = 2$ obviously it follows that $\chi(Q) \geq 9$. Thus, by (1.3), $Q \overset{v}{\to} (2_8)$. Hence $Q \in H_v(2_8; 6)$ and $F_v(2_8; 6) \leq |V(Q)| = 17$. Theorem 1.6 is proved. □

Corollary 5.1. *Let G be a graph such that $f(G) \leq 4$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. If $f(G) \leq 3$, then Corollary 5.1 follows from Corollary 4.1. Let $f(G) = 4$ and $\chi(G) = r+1$, then $\mathrm{cl}(G) = r-3$. Since $\chi(G) \neq \mathrm{cl}(G)$, we have $\mathrm{cl}(G) \geq 2$, and consequently, $r \geq 5$. By (1.3), $G \in H_v(2_r; r-2)$. Using Theorem 1.6(a), we get $|V(G)| \geq r+9 = \chi(G) + 2f(G)$. □

Let $r \geq 3s+8$. Define

$$\tilde{Q} = K_{r-3s-8} + Q + \underbrace{C_5 + \cdots + C_5}_{s},$$

where graph Q is given in the proof of Theorem 1.6(b). Since $\mathrm{cl}(Q) = 5$ and $\chi(Q) \geq 9$, we have by (2.1) and (2.2) that $\mathrm{cl}(\tilde{Q}) = r-s-3$ and $\chi(\tilde{Q}) \geq r+1$. According to (1.3), $\tilde{Q} \in H_v(2_r; r-s-2)$. Thus, $F_v(2_r; r-s-2) \leq |V(\tilde{Q})|$. Since $|V(\tilde{Q})| = r+2s+9$, we obtain the following

Theorem 5.1. *Let r and s be non-negative integers and $r \geq 3s+8$. Then*

$$F_v(2_r; r-s-2) \leq r+2s+9.$$

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NORMALLY GENERATED SUBSPACES OF LOGARITHMIC CANONICAL SECTIONS

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The logarithmic-canonical bundle $\Omega^2_{A'}(T')$ of a smooth toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a ball quotient $\mathbb{B}/\Gamma$ is known to be sufficiently ample over the Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$. The present work develops criteria for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated over $\widehat{A}$, i.e., to determine a regular immersive projective morphism of $\widehat{A}$ with normal image. These are applied to a specific example $A'_1 = (\mathbb{B}/\Gamma_1)'$ over the Gauss numbers. The first section organizes some preliminaries. The second one provides two sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$.

Keywords: Modular forms, holomorphic sections of line bundles, abelian functions, sufficiently ample and normally generated line bundles.

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1. PRELIMINARIES

Throughout, let $\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\} = SU_{2,1}/S(U_2 \times U_1)$ be the complex two dimensional ball and $\Gamma \subset SU_{2,1}$ be a lattice, acting freely on $\mathbb{B}$. The compact $\mathbb{B}/\Gamma$ are of general type. The non-compact $\mathbb{B}/\Gamma$ admit smooth toroidal compactification $(\mathbb{B}/\Gamma)'$ by a disjoint union $T' = \cup_{i=1}^h T'_i$ of smooth irreducible elliptic curves T'_i. From now on, we concentrate on $A' = (\mathbb{B}/\Gamma)'$ with abelian minimal model A. In such a case, the lattice Γ, the ball quotient $\mathbb{B}/\Gamma$ and its compactifications are said to be co-abelian.

The contraction $\xi : A' \to A$ of the rational (-1)-curves on A' restricts to a biregular morphism $\xi : T_i' \to \xi(T_i') = T_i$, as far as an abelian surface A does not support rational curves. In such a way, ξ produces the multi-elliptic divisor $T = \xi(T') = \sum_{i=1}^{h} T_i \subset A$, i.e., a divisor with smooth elliptic irreducible components T_i. According to Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$, any irreducible component of the exceptional divisor of ξ intersects T' in at least two points. Therefore $\xi : A' \to A$ is the blow-up of A at the singular locus $T^{\text{sing}} = \sum_{1 \leq i < j \leq h} T_i \cap T_j$ of T. Holzapfel has shown in [5] that the blow-up A' of an abelian surface A at the singular locus $T^{\text{sing}} = \sum_{1 \leq i < j \leq h} T_i \cap T_j$ of a multi-elliptic divisor $T = \sum_{i=1}^{h} T_i$ is the toroidal compactification $A' = (\mathbb{B}/\Gamma)'$ of a smooth ball quotient $\mathbb{B}/\Gamma$ if and only if $A = E \times E$ is the Cartesian square of an elliptic curve E and

$$\sum_{i=1}^{h} \text{card}(T_i \cap T^{\text{sing}}) = 4\text{card}(T^{\text{sing}}). \tag{1.1}$$

In order to describe the smooth irreducible elliptic curves T_i on A and their intersections, let us note that the inclusions $T_i \subset A = E \times E$ are morphisms of abelian varieties. Consequently, they lift to affine linear maps of the corresponding universal covers and

$$T_i = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, a_i u + b_i v + c_i \in \pi_1(E)\}$$

for some $a_i, b_i, c_i \in \mathbb{C}$. The fundamental group

$$\pi_1(T_i) = \{t \in \mathbb{C} \,|\, b_i t + \pi_1(E) = -a_i t + \pi_1(E) = \pi_1(E)\} = a_i^{-1}\pi_1(E) \cap b_i^{-1}\pi_1(E).$$

If Γ is an arithmetic lattice then the elliptic curve E has complex multiplication by an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$. As a result, Γ is commensurable with the full Picard modular group $SU_{2,1}(\mathcal{O}_{-d})$ over the integers ring $\mathcal{O}_{-d}$ of $\mathbb{Q}(\sqrt{-d})$. Such Γ are called Picard modular groups. Moreover, all T_i are defined over K. For simplicity, we assume that $\pi_1(E) = \mathcal{O}_{-d}$, in order to have maximal endomorphism ring $End(E) = \mathcal{O}_{-d}$. Since $K = \mathbb{Q}(\sqrt{-d})$ is the fraction field of $\mathcal{O}_{-d}$, one can choose $a_i, b_i \in \mathcal{O}_{-d}$. Thus, $\pi_1(T_i) \supseteq \mathcal{O}_{-d}$, $a_i\pi_1(E) + b_i\pi_1(E) \subseteq \mathcal{O}_{-d}$ and T_i has minimal fundamental group $\pi_1(T_i) = \mathcal{O}_{-d}$ exactly when $a_i\pi_1(E) + b_i\pi_1(E) = \pi_1(E) = \mathcal{O}_{-d}$. In particular, if K is of class number 1, then all the smooth elliptic curves $T_i \subset A = \mathbb{C}^2/\left(\mathcal{O}_{-d} \times \mathcal{O}_{-d}\right)$, defined over $K = \mathbb{Q}(\sqrt{-d})$, have minimal fundamental groups $\pi_1(T_i) = \mathcal{O}_{-d}$. From now on, we do not restrict the class number of $K = \mathbb{Q}(\sqrt{-d})$, but confine only to smooth irreducible elliptic curves T_i with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$. If $b_i \neq 0$, then

$$T_i^{(1)} = \{(b_i t + \pi_1(E), -a_i t - b_i^{-1} c_i + \pi_1(E)) \,|\, t \in \mathbb{C}\} \subseteq T_i.$$

Moreover, the complete pre-image of $T_i^{(1)}$ in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A is $\pi_1(T_i)$-invariant family of complex lines. Therefore, $T_i^{(1)}$ is an elliptic curve and coincides with T_i.

The notations from the next lemma will be used throughout:

Lemma 1. *Let* $T_s = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s \in \mathcal{O}_d\}$ *and* $D_s = \{(u+\mathcal{O}_{-d}, v+\mathcal{O}_{-d}) \,|\, a_s u + b_s v + c_s + \mu_s \in \mathcal{O}_{-d}\}$ *for* $1 \leq s \leq 3$ *be elliptic curves with minimal fundamental groups* $\pi_1(T_s) = \pi_1(D_s) = \mathcal{O}_{-d}$ *on* $A = (\mathbb{C}/\mathcal{O}_{-d}) \times (\mathbb{C}/\mathcal{O}_{-d})$ *and*

$$\Delta_{ij} := \det \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}, \quad \Delta := \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Then for any even permutation $\{i, j, l\}$ *of* $\{1, 2, 3\}$ *there hold the following:*

(i) *the intersection number is* $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$, *where* $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})} : \mathbb{Q}(\sqrt{-d}) \to \mathbb{Q}$ *stands for the norm;*

(ii) $T_i \cap T_j \subset D_l$ *if and only if* $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ *and both* $\Delta_{ij}^{-1}\Delta_{jl}$ *and* $\Delta_{ij}^{-1}\Delta_{li}$ *belong to* $End(E) = \mathcal{O}_{-d}$*;*

(iii) $T_1 \cap T_2 \cap T_3 = \emptyset$ *if and only if* $\Delta \notin \Delta_{12}\mathcal{O}_{-d} + \Delta_{23}\mathcal{O}_{-d} + \Delta_{31}\mathcal{O}_{-d}$.

Proof. (i) If $T_i \cap T_j = \emptyset$, then the liftings of T_i, T_j to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are discrete families of mutually parallel lines. In such a case, we say briefly that T_i and T_j are parallel. That allows to choose $a_j = a_i$, $b_j = b_i$ and to calculate $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(0) = 0 = T_i.T_j$. When $T_i \cap T_j \neq \emptyset$, one can move the origin $\check{o}_A = (\check{o}_E, \check{o}_E) \in A$ in $T_i \cap T_j$ and represent

$$T_i = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, t \in \mathbb{C}\}, \quad T_j = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_j u + b_j v \in \mathcal{O}_{-d}\}.$$

Then the intersection is

$$T_i \cap T_j = \{(b_i t + \mathcal{O}_{-d}, -a_i t + \mathcal{O}_{-d}) \,|\, \Delta_{ij} t \in \mathcal{O}_{-d} \subset \mathbb{C}\} \simeq$$

$$(\Delta_{ij}^{-1}\mathcal{O}_{-d})/(b_i^{-1}\mathcal{O}_{-d} \cap a_i^{-1}\mathcal{O}_{-d}) = \left(\Delta_{ij}^{-1}\mathcal{O}_{-d}\right)/\mathcal{O}_{-d} \simeq \mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}.$$

For an arbitrary lattice $\Lambda \subset \mathbb{C}$, let us denote by $\mathcal{F}(\Lambda)$ a Λ-fundamental domain on $\mathbb{C}$. As far as $\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})$ is the $\mathcal{O}_{-d}/\Delta_{ij}\mathcal{O}_{-d}$-orbit of $\mathcal{F}(\mathcal{O}_{-d})$, the index equals

$$[\mathcal{O}_{-d} : \Delta_{ij}\mathcal{O}_{-d}] = \frac{\mathrm{vol}\mathcal{F}(\Delta_{ij}\mathcal{O}_{-d})}{\mathrm{vol}\mathcal{F}(\mathcal{O}_{-d})} = \frac{\mathrm{vol}\mathcal{F}(|\Delta_{ij}|\mathcal{O}_{-d})}{\mathrm{vol}\mathcal{F}(\mathcal{O}_{-d})} = |\Delta_{ij}|^2 = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}).$$

(ii) The intersection $T_i \cap T_j$ consists of the $\pi_1(A)$-equivalence classes of the solutions $(u, v) \in \mathbb{C}^2$ of

$$\left| \begin{array}{l} a_i u + b_i v = \lambda_1 - c_i \\ a_j u + b_j v = \lambda_2 - c_j \end{array} \right.$$

for arbitrary $\lambda_1, \lambda_2 \in \pi_1(E) = \mathcal{O}_{-d}$. A point

$$(\Delta_{ij}^{-1}(b_i c_j - b_j c_i) + \Delta_{ij}^{-1}(b_j \lambda_1 - b_i \lambda_2),\ \Delta_{ij}^{-1}(a_j c_i - a_i c_j) + \Delta_{ij}^{-1}(a_i \lambda_2 - a_j \lambda_1))$$

belongs to the lifting of D_l if and only if

$$\begin{aligned} &-\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}(c_i\Delta_{jl} + c_j\Delta_{li}) + c_l + \mu_l \\ &= -\Delta_{ij}^{-1}\Delta_{jl}\lambda_1 - \Delta_{ij}^{-1}\Delta_{li}\lambda_2 + \Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d} \end{aligned}$$

for $\forall \lambda_1, \lambda_2 \in \pi_1(E)$. That, in turn, is equivalent to $\Delta_{ij}^{-1}\Delta + \mu_l \in \pi_1(E) = \mathcal{O}_{-d}$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in End(E) = \mathcal{O}_{-d}$.

(iii) For arbitrary $\lambda_1, \lambda_2, \lambda_3 \in \pi_1(E) = \mathcal{O}_{-d}$, the linear system

$$\left| \begin{array}{l} a_1 u + b_1 v = \lambda_1 - c_1 \\ a_2 u + b_2 v = \lambda_2 - c_2 \\ a_3 u + b_3 v = \lambda_3 - c_3 \end{array} \right.$$

has no solutions exactly when

$$\det \begin{pmatrix} a_1 & b_1 & \lambda_1 - c_1 \\ a_2 & b_2 & \lambda_2 - c_2 \\ a_3 & b_3 & \lambda_3 - c_3 \end{pmatrix} = \Delta_{23}\lambda_1 + \Delta_{31}\lambda_2 + \Delta_{12}\lambda_3 - \Delta \neq 0.$$

Lemma 1 is proved. □

The non-arithmetic lattices $\Gamma \subset SU_{2,1}$ correspond to abelian surfaces $A = E \times E$, whose elliptic factors E have minimal endomorphism rings $End(E) = \mathbb{Z}$. Then the liftings of the elliptic curves $T_i \subset A$ with $\pi_1(T_i) = \pi_1(E)$ to the universal cover $\widetilde{A} = \mathbb{C}^2$ of A are given by $a_i u + b_i v + c_i \in \pi_1(E)$ with $a_i, b_i \in \mathbb{Z}$. As a result, the intersection numbers $T_i.T_j = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij})$ are comparatively large and there are very few chances for construction of a multi-elliptic divisor $T = \sum\limits_{i=1}^{h} \subset A$, subject to (1.1). This is a sort of a motivation for restricting our attention to the arithmetic case.

The smooth irreducible elliptic curves $T_i' \subset A'$ contract to the Γ-orbits $\kappa_i = \Gamma(p) \in \partial_\Gamma \mathbb{B}/\Gamma$ of the Γ-rational boundary points $p \in \partial_\Gamma \mathbb{B}$. These κ_i are called cusps. The resulting Baily-Borel compactification $\widehat{A} = \widehat{\mathbb{B}/\Gamma} = (\mathbb{B}/\Gamma) \cup (\partial_\Gamma \mathbb{B}/\Gamma)$ is a normal projective surface.

Definition 2. *Let Γ be a Picard modular group, $\gamma \in \Gamma$ and $Jac(\gamma) = \frac{\partial(\gamma_1,\gamma_2)}{\partial(z_1,z_2)}$ be the Jacobian matrix of $\gamma = (\gamma_1, \gamma_2) : \mathbb{B} \to \mathbb{B} \subset \mathbb{C}^2$. The global holomorphic functions $\delta : \mathbb{B} \to \mathbb{C}$ with transformation law*

$$\gamma^*(\delta)(z) = \delta\gamma(z) = [\det Jac(\gamma)]^{-n}\,\delta(z) \quad \textit{for } \forall \gamma \in \Gamma, \forall z \in \mathbb{B}$$

are called Γ*-modular forms of weight* n*.*

The Γ*-modular forms of weight* n *constitute a* $\mathbb{C}$*-linear space, which is denoted by* $[\Gamma, n]$*.*

Definition 3. *A* Γ*-modular form* $\delta \in [\Gamma, n]$ *is cuspidal if* $\delta(\kappa_i) = 0$ *at all the cusps* $\kappa_i \in \partial_\Gamma \mathbb{B}/\Gamma$*.*

The cuspidal Γ*-modular forms of weight* n *form the subspace* $[\Gamma, n]_{\text{cusp}}$ *of* $[\Gamma, n]$*.*

For any natural number n there is a $\mathbb{C}$-linear embedding

$$j_n : H^0(\mathbb{B}, \mathcal{O}_\mathbb{B}) \longrightarrow H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)$$

$$j_n(\delta)(z) = \delta(z)(dz_1 \wedge dz_2)^{\otimes n}$$

of the global holomorphic functions on the ball in the global holomorphic sections of the n-th pluri-canonical bundle $\left(\Omega^2_\mathbb{B}\right)^{\otimes n}$. It restricts to an isomorphism

$$j_n : [\Gamma, n] \longrightarrow H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma$$

of the Γ-modular forms of weight n with the Γ-invariant holomorphic sections of $\left(\Omega^2_\mathbb{B}\right)^{\otimes n}$. Note that the subspace $H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma$ of $H^0\left(\mathbb{B}/\Gamma, \left(\Omega^2_{\mathbb{B}/\Gamma}\right)^{\otimes n}\right)$ acts on $\widehat{A} = \widehat{\mathbb{B}/\Gamma}$, extending over the cusps $\partial_\Gamma \mathbb{B}/\Gamma$ of codimension 2 in $\widehat{A}$.

The tensor product $\Omega^2_{A'}(T') = \Omega^2_{A'} \otimes_\mathbb{C} \mathcal{O}_{A'}(T')$ is called logarithmic canonical bundle of A', while $\Omega^2_{A'}(T')^{\otimes n}$ are referred to as logarithmic pluri-canonical bundles. Hemperly has observed in [3] that

$$j_n[\Gamma, n] = H^0\left(\mathbb{B}, \left(\Omega^2_\mathbb{B}\right)^{\otimes n}\right)^\Gamma = H^0\left(A', \Omega^2_{A'}(T')^{\otimes n}\right)$$

as long as the holomorphic sections from these spaces have one and the same coordinate transformation law. A classical result of Baily-Borel establishes that $\Omega^2_{A'}(T')$ is sufficiently ample on $\widehat{A}$. The present article provides sufficient conditions for the ampleness of $\Omega^2_{A'}(T')$ on $\widehat{A}$.

Note that the canonical bundle

$$K_{A'} = \xi^* K_A + \mathcal{O}_{A'}(L) = \xi^* \mathcal{O}_A + \mathcal{O}_{A'}(L) = \mathcal{O}_{A'}(L)$$

is associated with the exceptional divisor $L = \xi^{-1}(T^{\text{sing}})$ of $\xi : A' \to A$. If s is a global meromorphic section of $\Omega^2_{A'}$ and t is a global meromorphic section of $\mathcal{O}_{A'}(T')$, then the tensoring

$$(s \otimes_\mathbb{C} t)^{\otimes(-n)} : H^0\left(A', \Omega^2_{A'}(T')^{\otimes n}\right) \longrightarrow \mathcal{L}_{A'}(n(L + T'))$$

is a $\mathbb{C}$-linear isomorphism with

$$\mathcal{L}_{A'}(n(L + T')) = \{f \in \mathfrak{Mer}(A') \mid (f) + n(L + T') \geq 0\}.$$

The isomorphism $\xi^* : \mathfrak{Mer}(A) \to \mathfrak{Mer}(A')$ of the meromorphic function fields induces a linear isomorphism

$$(\xi^*)^{-1} : \mathcal{L}_{A'}(n(L+T')) \longrightarrow \mathcal{L}_A(nT, nT^{\text{sing}}),$$

where $m_p : \text{Div}(A) \to \mathbb{Z}$ stands for the multiplicity at a point $p \in A$ and

$$\mathcal{L}_A(nT, nT^{\text{sing}}) = \{f \in \mathfrak{Mer}(A) \mid (f) + nT \geq 0,\ m_p(f) + n \geq 0 \text{ for } \forall p \in T^{\text{sing}}\}.$$

The linear isomorphisms

$$\tau_n := (\xi^*)^{-1}(s \otimes_{\mathbb{C}} t)^{\otimes(-n)} : j_n[\Gamma, n] \longrightarrow \mathcal{L}_A(nT, nT^{\text{sing}})$$

are called transfers of modular forms of weight n to abelian functions.

For any $\delta \in [\Gamma, 1]$, note that $\delta(\kappa_i) \neq 0$ if and only if $T_i \subset (\tau_1 j_1(\delta))_\infty$. Observe also that $\tau_1 j_1[\Gamma, 1]_{\text{cusp}} = \{f \in \mathcal{L}_A(T, T^{\text{sing}}) \mid (f)_\infty = \emptyset\} = \mathbb{C}$ and fix the cuspidal form $\eta_o = (\tau_1 j_1)^{-1}(1)$ of weight 1.

Towards the construction of abelian functions $f \in \mathcal{L}_A(T, T^{\text{sing}})$, let us recall from [7] that any elliptic function $g : E \to \mathbb{P}^1$ can be represented as

$$g(z) = C_o \prod_{i=1}^{k} \frac{\sigma(z - \alpha_i)}{\sigma(z - \beta_i)}, \tag{1.2}$$

where

$$\sigma(z) = z \prod_{\lambda \in \pi_1(E) \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right)^{\frac{z}{\lambda} + \frac{1}{2}\left(\frac{z}{\lambda}\right)^2}$$

is the Weierstrass σ-function, $\alpha_i, \beta_i, C_o \in \mathbb{C}$ and $\sum_{i=1}^{k} \alpha_i \equiv \sum_{i=1}^{k} \beta_i (\text{mod}\, \pi_1(E))$. The points of $E = \mathbb{C}/\pi_1(E)$ are of the form $\overline{a} = a + \pi_1(E)$ for some $a \in \mathbb{C}$. The elliptic function (1.2) takes all the values from $\mathbb{P}^1$ with one and a same multiplicity k. Moreover, if $g^{-1}(x) = \{\overline{p_i(x)} \in E \mid 1 \leq i \leq k\}$ for some $x \in \mathbb{C} \subset \mathbb{P}^1$, then $\sum_{i=1}^{k} \overline{p_i(x)} = \sum_{i=1}^{k} \overline{\beta_i}$. Observe that $\sigma : \mathbb{C} \to \mathbb{C}$ is a non-periodic entire function, but its divisor $(\sigma)_{\mathbb{C}} = \pi_1(E)$ on $\mathbb{C}$ is $\pi_1(E)$-invariant. That enables to define the divisor $(\sigma)_E = \check{o}_E$ of σ on E. In global holomorphic coordinates $(u, v) \in \mathbb{C}^2$, the divisor

$$(\sigma(a_i u + b_i v + c_i))_{\mathbb{C}^2} = \{(u, v) \in \mathbb{C}^2 \mid a_i u + b_i v + c_i \in \pi_1(E) = \mathcal{O}_{-d}\}$$

is the complete pre-image of T_i in the universal cover $\widetilde{A} = \mathbb{C}^2$ of A. That allows to define the divisor

$$(\sigma(a_i u + b_i v + c_i)) = (\sigma(a_i u + b_i v + c_i))_A = T_i.$$

Let $f \in \mathcal{L}_A(T)$ be an abelian function with pole divisor $(f)_\infty = \sum\limits_{i=1}^{k} T_i$, after an eventual permutation of the irreducible components of T. Then

$$f_\infty := \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i) \text{ and } f_0 := f f_\infty \tag{1.3}$$

are (non-periodic) entire functions on $\mathbb{C}^2$. Let $\zeta = \frac{\sigma'}{\sigma}$ be Weierstrass' ζ-function, $\eta : \pi_1(E) \to \mathbb{C}$ be the $\mathbb{Z}$-linear homomorphism, satisfying $\zeta(z + \lambda) = \zeta(z) + \eta(\lambda)$ for all $z \in \mathbb{C}$, $\lambda \in \pi_1(E)$ and

$$\varepsilon(\lambda) = \begin{cases} 1 & \text{for } \lambda \in 2\pi_1(E), \\ -1 & \text{for } \lambda \in \pi_1(E) \setminus 2\pi_1(E). \end{cases}$$

Recall from [6] the $\pi_1(E)$-transformation law

$$\frac{\sigma(z+\lambda)}{\sigma(z)} = \varepsilon(\lambda) e^{\eta(\lambda)\left(z + \frac{\lambda}{2}\right)} \quad \text{for} \quad \forall \lambda \in \pi_1(E), \forall z \in \mathbb{C}.$$

Under the assumption (1.3), the $\pi_1(A)$-periodicity of f is equivalent to

$$\frac{f_0(u+\lambda, v)}{f_0(u,v)} = \frac{f_\infty(u+\lambda, v)}{f_\infty(u,v)} = \prod_{i=1}^{k} \varepsilon(a_i\lambda) e^{\eta(a_i\lambda)\left(a_i u + b_i v + c_i + \frac{a_i\lambda}{2}\right)}$$

and

$$\frac{f_0(u, v+\lambda)}{f_0(u,v)} = \frac{f_\infty(u, v+\lambda)}{f_\infty(u,v)} = \prod_{i=1}^{k} \varepsilon(b_i\lambda) e^{\eta(b_i\lambda)\left(a_i u + b_i v + c_i + \frac{b_i\lambda}{2}\right)}$$

for $\forall \lambda \in \pi_1(E) = \mathcal{O}_{-d}$, $\forall (u,v) \in \mathbb{C}^2$. We choose

$$f_0(u,v) = \prod_{i=1}^{k} \sigma(a_i u + b_i v + c_i + \mu_i)$$

and reduce the $\pi_1(A)$-periodicity of f to

$$1 = \frac{f(u+\lambda, v)}{f(u,v)} = e^{\sum_{i=1}^{k} \eta(a_i\lambda)\mu_i}, \; 1 = \frac{f(u, v+\lambda)}{f(u,v)} = e^{\sum_{i=1}^{k} \eta(b_i\lambda)\mu_i} \quad \forall \lambda \in \mathcal{O}_{-d}, \forall (u,v) \in \mathbb{C}^2.$$

Let us mention that Holzapfel has studied $f \in \mathcal{L}_A(T)$ of the above form with at most three non-parallel irreducible components of $(f)_\infty$, intersecting pairwise in single points. The next lemma provides a sufficient (but not necessary) condition for $\pi_1(A)$-periodicity of a σ-quotient, whose pole divisor has an arbitrary number of irreducible components with arbitrary intersection numbers.

Lemma 4. *If*

$$\sum_{i=1}^{k} a_i\mu_i = \sum_{i=1}^{k} \overline{a_i}\mu_i = \sum_{i=1}^{k} b_i\mu_i = \sum_{i=1}^{k} \overline{b_i}\mu_i = 0\,, \tag{1.4}$$

then the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_iu + b_iv + c_i + \mu_i)}{\sigma(a_iu + b_iv + c_i)} \tag{1.5}$$

is $\mathcal{O}_{-d} \times \mathcal{O}_{-d}$ -periodic.

Proof. Let us recall from [1] that the integers ring of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ is of the form $\mathcal{O}_{-d} = \mathbb{Z} + 2\omega\mathbb{Z}$ for

$$2\omega = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1 (\operatorname{mod} 4), \\ \frac{-1+\sqrt{-d}}{2} & \text{for } -d \equiv 1 (\operatorname{mod} 4). \end{cases}$$

Any $\nu \in \mathcal{O}_{-d}$ has unique representation $\nu = x + 2\omega y$ with

$$x = \frac{2\omega\overline{\nu} - 2\overline{\omega}\nu}{2\omega - 2\overline{\omega}} \in \mathbb{Z}, \quad y = \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} \in \mathbb{Z}.$$

Legendre's equality

$$\eta(2\omega) - 2\omega\eta(1) = 2\pi\sqrt{-1},$$

(cf.[6]) implies that

$$\eta(\nu) = \nu\eta(1) + \frac{\nu - \overline{\nu}}{2\omega - 2\overline{\omega}} 2\pi\sqrt{-1} \quad \text{ for } \forall \nu \in \mathcal{O}_{-d}.$$

As a result,

$$\sum_{i=1}^{k} \eta(a_i\lambda)\mu_i = \left(\sum_{i=1}^{k} a_i\mu_i\right)\lambda\eta(1) + \left(\sum_{i=1}^{k} a_i\mu_i\right)\frac{2\pi\sqrt{-1}\lambda}{2\omega - 2\overline{\omega}} - \left(\sum_{i=1}^{k} \overline{a_i}\mu_i\right)\frac{\overline{\lambda}2\pi\sqrt{-1}}{2\omega - 2\overline{\omega}}.$$

Lemma 4 is proved □

Mutually parallel smooth elliptic curves $T_1, \dots, T_k$ admit liftings

$$T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_1u + b_1v + c_i \in \mathcal{O}_{-d}\}.$$

For arbitrary $\mu_j \in \mathbb{C}$ with $\sum\limits_{i=1}^{k} \mu_i = 0$, the σ-quotient

$$f(u,v) = \prod_{i=1}^{k} \frac{\sigma(a_1u + b_1v + c_i + \mu_i)}{\sigma(a_1u + b_1v + c_i)} \tag{1.6}$$

belongs to $\mathcal{L}_A(T, T^{\text{sing}})$ and has smooth pole divisor $(f)_\infty = \sum_{i=1}^{k} T_i$. Following [4], we say that (1.6) is a k-fold parallel σ-quotient. A σ-quotient (1.5) has smooth pole divisor if and only if it is k-fold parallel.

Definition 5. *A special σ-quotient of order k is a function of the form (1.5), which is subject to (1.4), has singular pole divisor $(f)_\infty$ and $\mu_i \notin \mathcal{O}_{-d}$ for all $1 \leq i \leq k$.*

Lemma 6. *If $f \in \mathcal{L}_A(T, T^{\text{sing}})$ is a special σ-quotient of order $k \geq 2$, then at any point $p \in (f)_\infty^{\text{sing}}$ the multiplicity $m_p(f)_\infty$ satisfies*

$$2 \leq m_p(f)_\infty \leq \left[\frac{k+1}{2} \right],$$

where $\left[\frac{k+1}{2}\right]$ is the greatest natural number, non-exceeding $\frac{k+1}{2}$.

In particular, $\mathcal{L}_A(T, T^{\text{isng}})$ does not contain a special σ-quotient of order 2.

Proof. The smoothness of the irreducible components $T_1, \dots, T_k$ of $(f)_\infty$ results in $(f)_\infty^{\text{sing}} \subset \sum_{1 \leq i < j \leq k} (T_i \cap T_j)$ and implies that $m_p(f)_\infty \geq 2$ for all $p \in (f)_\infty^{\text{sing}}$. Suppose that $m_p(f)_\infty = m$ for some $2 \leq m \leq k$. After an eventual permutation of $T_1, \dots, T_k$, one can assume that $p \in T_1 \cap \dots \cap T_m$ and $p \notin T_{m+1} + \dots + T_k$. Then

$$m_p(f) + 1 = m_p(f)_0 - m_p(f)_\infty + 1 = m_p(f)_0 - m + 1 \geq 0$$

requires the existence of $D_{m+1}, \dots, D_{2m-1} \subset (f)_0 = \sum_{i=1}^{k} D_i$ with $p \in D_{m+1} \cap \dots \cap D_{2m-1}$, after a further permutation of $D_{m+1}, \dots, D_k$. Now $2m - 1 \leq k$ implies that $m_p(f)_\infty = m \leq \left[\frac{k+1}{2}\right]$.

In particular, for $k = 2$ the inequality $2 \leq m_p(f)_\infty \leq \left[\frac{3}{2}\right]$ cannot be satisfied. □

Proposition 7. *If*

$$f(u, v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i + \mu_i)}{\sigma(a_i u + b_i v + c_i)} \tag{1.7}$$

is a special σ-quotient from $\mathcal{L}_A(T, T^{\text{sing}})$, then $T_1 \cap T_2 \cap T_3 = \emptyset$ and the intersection numbers $T_1.T_2 = T_2.T_3 = T_3.T_1 \in \mathbb{N}$ are equal.

Proof. By Lemma 6 there follows $m_p(f)_\infty = 2$ for $\forall p \in (f)_\infty^{\text{sing}}$. In particular, $(f)_\infty = T_1 + T_2 + T_3$ has no triple point and $T_1 \cap T_2 \cap T_3 = \emptyset$. Further, for any $p \in T_i \cap T_j$ the condition $m_p(f) + 1 \geq 0$ requires that $p \in D_l$, therefore $\mu_l \in \mathcal{O}_{-d} - \Delta_{ij}^{-1}\Delta$ and $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}$, according to Lemma 1 (ii). A cyclic change of the even permutation $\{i, j, l\}$ by $\{j, l, i\}$ and $\{l, i, j\}$ results in $\Delta_{jl}^{-1}\Delta_{li}, \Delta_{jl}^{-1}\Delta_{ij} \in \mathcal{O}_{-d}$ and, respectively, $\Delta_{li}^{-1}\Delta_{ij}, \Delta_{li}^{-1}\Delta_{jl} \in \mathcal{O}_{-d}$. Consequently, $\Delta_{ij}^{-1}\Delta_{jl}, \Delta_{ij}^{-1}\Delta_{li} \in \mathcal{O}_{-d}^*$, whereas $N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{ij}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{jl}) = N_{\mathbb{Q}}^{\mathbb{Q}(\sqrt{-d})}(\Delta_{li})$. Now, by Lemma 1 (i) it follows that $T_i.T_j = T_j.T_l = T_l.T_i$. □

Definition 8. *The divisor $T_1 + T_2 + T_3$ with three smooth elliptic irreducible components is called a triangle if $T_1 \cap T_2 \cap T_3 = \emptyset$ and $T_1.T_2 = T_2.T_3 = T_3.T_1 = 1$.*

Examples of special σ-quotients with triangular pole divisors are constructed by Holzapfel in [4]. We show that any triangular divisor can be realized as a pole divisor of a special σ-quotient $f \in \mathcal{L}_A(T, T^{\text{sing}})$ and provide a general formula for such f.

Proposition 9. *Let $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a'_i u + b'_i v + c'_i \in \mathcal{O}_{-d}\}$ with $1 \leq i \leq 3$ be the smooth irreducible elliptic components of a triangle $T_1 + T_2 + T_3$ and $a_i = \Delta'_{jl} a'_i$, $b_i = \Delta'_{jl} b'_i$, $c_i = \Delta'_{jl} c'_i$. Then $a_1 + a_2 + a_3 = 0$, $b_1 + b_2 + b_3 = 0$, $\Delta_{12}^{-1}\Delta \not\in \mathcal{O}_{-d}$ and for any $\nu \in \mathcal{O}_{-d}$ the function*

$$f(u, v) = \prod_{i=1}^{3} \frac{\sigma(a_i u + b_i v + c_i - \Delta_{12}^{-1}\Delta + \nu)}{\sigma(a_i u + b_i v + c_i)} \tag{1.8}$$

is a special σ-quotient from $\mathcal{L}_A(T, T^{\text{sing}})$ with pole divisor $(f)_\infty = T_1 + T_2 + T_3$.

Proof. Let $v'_i = \begin{pmatrix} a'_i \\ b'_i \end{pmatrix}$ for $1 \leq i \leq 3$. Expanding along the third row, one obtains

$$0 = \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ a'_1 & a'_2 & a'_3 \end{vmatrix} = \Delta'_{23} a'_1 + \Delta'_{31} a'_2 + \Delta'_{12} a'_3 = 0,$$

$$0 = \begin{vmatrix} a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \\ b'_1 & b'_2 & b'_3 \end{vmatrix} = \Delta'_{23} b'_1 + \Delta'_{31} b'_2 + \Delta'_{12} b'_3 = 0,$$

and concludes that

$$v_1 + v_2 + v_3 = \Delta'_{23} v'_1 + \Delta'_{31} v'_2 + \Delta'_{12} v'_3 = 0_{2\times 1}, \quad \Delta_{12} = \Delta_{23} = \Delta_{31}. \tag{1.9}$$

Now, according to Lemma 1 (iii), $T_1 \cap T_2 \cap T_3 = \emptyset$ is equivalent to $\Delta \not\in \Delta_{12}\mathcal{O}_{-d}$. Then the condition $m_p(f)_0 \geq m_p(f)_\infty - 1$ for $\forall p \in (f)_\infty^{\text{sing}}$ reduces to $T_i \cap T_j \subset D_l$ for any even permutation $\{i, j, l\}$ of $\{1, 2, 3\}$. Making use of Lemma 1 (ii), one can choose $\mu_1 = \mu_2 = \mu_3 = \nu - \Delta_{12}^{-1}\Delta \not\in \mathcal{O}_{-d}$. Then (1.9) implies (1.4) from Lemma 4 and reveals that (1.8) is a special σ-quotient from $\mathcal{L}_A(T, T^{\text{sing}})$. □

Definition 10. *The special σ-quotients (1.8) from $\mathcal{L}_A(T, T^{\text{sing}})$ with triangular pole divisors $(f)_\infty = T_1 + T_2 + T_3$ are called triangular.*

For elliptic curves $T_i = \{(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) \,|\, a_i u + b_i v + c_i \in \mathcal{O}_{-d}\}$, $1 \leq i \leq 2$ with minimal fundamental groups $\pi_1(T_i) = \pi_1(E) = \mathcal{O}_{-d}$ and intersection number $T_1.T_2 = 1$, Lemma 1 (i) implies that

$$M = \begin{pmatrix} a_2 & b_2 \\ a_1 & b_1 \end{pmatrix} \in GL_2(\mathcal{O}_{-d}).$$

As a result, there arises an automorphism

$$\varphi : A \longrightarrow A,$$

$$\varphi(u + \mathcal{O}_{-d}, v + \mathcal{O}_{-d}) = \left[M \left(\begin{array}{c} \overline{u} \\ \overline{v} \end{array} \right) + \left(\begin{array}{c} \overline{c_2} \\ \overline{c_1} \end{array} \right) \right]^t$$

with $\varphi(T_1) = E \times \check{o}_E$, $\varphi(T_2) = \check{o}_E \times E$. Making use of $\sigma(\alpha z) = \alpha\sigma(z)$ for $\forall \alpha \in \mathcal{O}^*_{-d}$, $\forall z \in \mathbb{C}$, one observes that any triangular σ-quotient can be reduced by an automorphism of A to the form

$$f_{012}(u, v) = \frac{\sigma(u + a_0^{-1} c_0)\sigma(v + b_0^{-1} c_0)\sigma(a_0 u + b_0 v)}{\sigma(u)\sigma(v)\sigma(a_0 u + b_0 v + c_0)} \qquad (1.10)$$

with $a_0, b_0 \in \mathcal{O}^*_{-d}$, $c_0 \notin \mathcal{O}_{-d}$.

We are going to describe the complete divisor of a triangular σ-quotient.

Definition 11. *The divisor* $D = \sum_{i=0}^{2} D_i - \sum_{i=0}^{2} T_i$ *with smooth elliptic irreducible components* D_i, T_j *is called a tetrahedron (cf. Figure 1) if:*

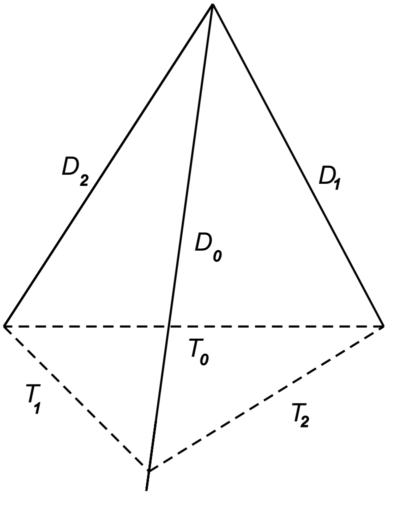

Figure 1: Tetrahedron

(i) $\sum_{i=0}^{2} T_i$ *is a triangle;*

(ii) D_i *are parallel to* T_i *for all* $0 \leq i \leq 2$*;*

(iii) $D_0 \cap D_1 \cap D_2 = D_0 \cap D_1 = D_1 \cap D_2 = D_2 \cap D_0 = \{p_0\}$ *for some point* $p_0 \in A$*;*

(iv) $\left(\sum_{i=0}^{2} D_i\right) \cap \left(\sum_{i=0}^{2} T_i\right) = \left(\sum_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum_{i=0}^{2} D_i\right)^{\text{smooth}}$.

Definition 12. *An inscribed (ordered) pair of triangles (cf. Figure 2) is a divisor* $D = \sum_{i=0}^{2} D_i - \sum_{i=0}^{2} T_i$*, such that:*

(i) $\sum_{i=0}^{2} D_i$ *and* $\sum_{i=0}^{2} T_i$ *are triangles;*

(ii) D_i *are parallel to* T_i *for all* $0 \leq i \leq 2$*;*

(iii) $\left(\sum_{i=0}^{2} D_i\right) \cap \left(\sum_{i=0}^{2} T_i\right) = \left(\sum_{i=0}^{2} T_i\right)^{\text{sing}} \subset \left(\sum_{i=0}^{2} D_i\right)^{\text{smooth}}$.

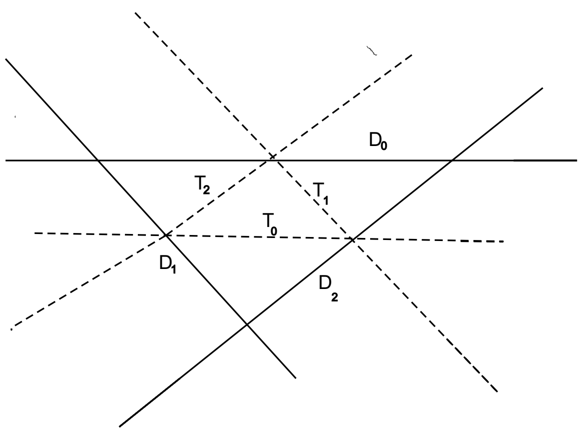

Figure 2: Inscribed (ordered) pair of triangles

An explicit calculation of the singular points of the complete divisor yields the following

Corollary 13. *Let (1.10) with* $a_0, b_0 \in \mathcal{O}^*_{-d}$*,* $c_0 \notin \mathcal{O}_{-d}$ *be a triangular* σ*-quotient with complete divisor* $(f_{012}) = \sum_{i=0}^{2} D_i - \sum_{i=0}^{2} T_i$*. Then:*

(i) $c_0 + \mathcal{O}_{-d} \in E_{2-\text{tor}}$ *is a* 2*-torsion point if and only if* (f_{012}) *is a tetrahedron;*

(ii) $c_0 + \mathcal{O}_{-d} \notin E_{2-tor}$ *exactly when* (f_{012}) *is an inscribed pair of triangles.*

In either case, the multiplicity $m_p(f_{012}) = -1$ *at all* $p \in (f_{012})_\infty \cap T^{\text{sing}}$.

In [4] Holzapfel introduces the idea for detecting the linear independence of co-abelian modular forms by the poles of the corresponding transfers to abelian functions. Instead of his strongly descending divisor condition, we use a natural complete decreasing flag on $[\Gamma, 1]$. That enables to supply a criterion for some modular forms to constitute a basis of $[\Gamma, 1]$ and to show that $[\Gamma, 1]$ has always a basis of the considered form.

Observe that the subspaces

$$V_i = j_1[\Gamma, 1]_i := \{\omega \in j_1[\Gamma, 1] \,|\, \omega(\kappa_1) = \ldots = \omega(\kappa_{i-1}) = 0\}$$

of $V_1 = j_1[\Gamma, 1]$ form a non-increasing flag

$$j_1[\Gamma, 1] = V_1 \supseteq V_2 \supseteq \ldots \supseteq V_{m-1} \supseteq V_m \supseteq \ldots \supseteq V_h \supseteq V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}.$$

For any $\omega, \omega' \in V_i$ one has $\omega'(\kappa_i)\omega - \omega(\kappa_i)\omega' \in V_{i+1}$, so that $0 \leq \dim_{\mathbb{C}} (V_i/V_{i+1}) \leq 1$ for all $1 \leq i \leq h$. We prove that there is a permutation of the cusps $\kappa_1, \ldots, \kappa_h$, so that $V_i/V_{i+1} \simeq \mathbb{C}$ for $1 \leq i \leq m$ and $V_{m+1} = V_{m+2} = \ldots = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}} \simeq \mathbb{C}$. If so, then $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1$.

Proposition 14. *If the pole divisors of $f_i \in \mathcal{L}_A(T, T^{\mathrm{sing}})$ are subject to*

$$T_i \subset (f_i)_\infty \subseteq T_i + T_{i+1} + \ldots + T_h \quad \text{for all} \quad 1 \leq i \leq m,$$

then $\omega_i = \tau_1^{-1}(f_i) \in j_1[\Gamma, 1]$ with $1 \leq i \leq m$ form a basis of a complement of $V_{m+1} = j_1[\Gamma, 1]_{m+1}$.

In particular, if $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\mathrm{cusp}}$, then $j_1(\eta_o), \omega_1, \ldots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$.

Proof. It suffices to show that for arbitrary $b_1, \ldots, b_m \in \mathbb{C}$ the linear system

$$\sum_{i=1}^{m} \omega_i(\kappa_j) t_i = b_j, \quad 1 \leq j \leq m \tag{1.11}$$

has a unique solution $(t_1, \ldots, t_m)$. On one hand, that implies the linear independence of $\omega_1, \ldots, \omega_m$ over $\mathbb{C}$. On the other hand, for any $\omega \in j_1[\Gamma, 1]$ there is uniquely determined $\sum_{i=1}^{m} c_i\omega_i$ with $\omega_0 = \omega - \sum_{i=1}^{m} c_i\omega_i \in j_1[\Gamma, 1]_{m+1} = V_{m+1}$. In other words, $j_1[\Gamma, 1] = \mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m) \oplus V_{m+1}$, so that $\omega_1, \ldots, \omega_m$ is a basis of the complement $\mathrm{Span}_{\mathbb{C}}(\omega_1, \ldots, \omega_m)$ of V_{m+1}.

Towards the existence of a unique solution of (1.11), note that the requirement $T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \ldots + T_h$ is equivalent to $\omega_i(\kappa_i) \neq 0$ and $\omega_i(\kappa_1) = \omega_i(\kappa_2) = \ldots = \omega_i(\kappa_{i-1}) = 0$. Thus, (1.11) is of the form

$$\begin{pmatrix} \omega_1(\kappa_1) & \ldots & 0 & \ldots & 0 \\ \omega_1(\kappa_i) & \ldots & \omega_i(\kappa_i) & \ldots & 0 \\ \omega_1(\kappa_m) & \ldots & \omega_i(\kappa_m) & \ldots & \omega_m(\kappa_m) \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_i \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix},$$

with non-degenerate, lower-triangular coefficient matrix and has unique solution for all $b_1, \dots, b_m \in \mathbb{C}$.

In the case of $V_{m+1} = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}$, note that $j_1[\Gamma, 1]_{\text{cusp}} = \mathbb{C} j_1(\eta_o)$ with $\tau_1 j_1(\eta_o) = 1 \in \mathcal{L}_A(T, T^{\text{sing}})$, so that $j_1(\eta_o), \omega_1, \dots, \omega_m$ is a $\mathbb{C}$-basis of $j_1[\Gamma, 1]$. □

The next proposition establishes that $j_1[\Gamma, 1]$ has always a $\mathbb{C}$-basis of the considered form.

Proposition 15. *Let $\Gamma \subset SU_{2,1}$ be a freely acting, co-abelian Picard modular group and* $\dim_{\mathbb{C}}[\Gamma, 1] = m+1$. *Then there is a permutation $\{\kappa_1, \dots, \kappa_m, \kappa_{m+1}, \dots, \kappa_h\}$ of the Γ-cusps, such that*

$$V_1/V_2 \simeq V_2/V_3 \simeq \cdots \simeq V_m/V_{m+1} \simeq \mathbb{C}, \quad V_{m+1} = V_{m+2} = \cdots = V_{h+1} = j_1[\Gamma, 1]_{\text{cusp}}.$$

Any $\omega_i \in V_i \setminus V_{i+1}$ transfers to $\tau_1(\omega_i) \in \mathcal{L}_A(T, T^{\text{sing}})$ with

$$T_i \subset (\tau_1(\omega_i))_\infty \subseteq T_i + T_{i+1} + \cdots + T_h \quad \textit{for} \quad 1 \leq i \leq m$$

and $j_1(\eta_o), \omega_1, \dots, \omega_m$ is a $\mathbb{C}$-basis of $V_1 = j_1[\Gamma, 1]$.

In particular, if $T_{h-1}.T_h = 1$ then $V_{h-1} = j_1[\Gamma, 1]_{\text{cusp}}$ and $\dim[\Gamma, 1] \leq h - 1$.

Proof. If $V_1 = V_{h+1}$, then there is nothing to be proved. From now on, we assume that $\dim V_1/V_{h+1} = m \in \mathbb{N}$. By induction on $1 \leq i \leq m$, we establish the existence of $\omega_j \in V_j \setminus V_{j+1}$ for all $1 \leq j \leq i$. First of all, for any $\omega_1 \in V_1 \setminus V_{h+1}$ there exists a cusp κ_1 with $\omega_1(\kappa_1) \neq 0$. Then for an arbitrary permutation of the remaining cusps, one has $\omega_1 \in V_1 \setminus V_2$. If we have chosen $\omega_j \in V_j \setminus V_{j+1}$ for $1 \leq j \leq i-1$ and $V_i \supsetneqq V_{h+1}$, then for an arbitrary $\omega_i \in V_i \setminus V_{h+1}$ there exists a permutation of $\{\kappa_i, \kappa_{i+1}, \dots, \kappa_h\}$, such that $\omega_i(\kappa_i) \neq 0$. Clearly, $\omega_i \in V_i \setminus V_{i+1}$ and we have obtained a basis $j_1(\eta_o), \omega_1, \dots, \omega_m$ of $V_1 = j_1[\Gamma, 1]$. The conditions $\omega_i \in V_i \setminus V_{i+1}$ amount to $T_i \subset (\tau_1(\omega_i))_\infty$ and $T_j \nsubseteqq (\tau_1(\omega_i))_\infty$ for all $1 \leq j \leq i-1$.

If $T_{h-1}.T_h = 1$, then up to an automorphism of A, one can assume that $T_{h-1} = E \times \check{o}_E$ and $T_h = \check{o}_E \times E$. We claim that $\mathcal{L}_A((E \times \check{o}_E) + (\check{o}_E \times E)) = \mathbb{C}$, so that $\dim_{\mathbb{C}}[\Gamma, 1] = m + 1 \leq h - 1$. Indeed, for an arbitrary $Q \in E \setminus \check{o}_E$ the restriction $f|_{E \times Q}$ is an elliptic function of order 1. Therefore $f|_{E \times Q} \equiv C(Q) \in \mathbb{C}$ is a constant. Similarly, $f|_{P \times E} \equiv C'(P) \in \mathbb{C}$ for any $P \in E \setminus \check{o}_E$. As a result, $C'(P) = f(P, Q) = C(Q)$ for all $Q \in E$ and $f|_A$ is constant. □

Proposition 16. (Holzapfel [5]) *Let us fix the half-periods $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{i}{2}$, $\omega_3 = \omega_1 + \omega_2$ of the lattice $\pi_1(E) = \mathcal{O}_{-1} = \mathbb{Z} + i\mathbb{Z}$, the 2-torsion points $Q_0 :=$* $0 (\operatorname{mod} \mathbb{Z} + i\mathbb{Z}) \in E$, *$Q_j := \omega_j (\operatorname{mod} \mathbb{Z} + i\mathbb{Z}) \in E$ for $1 \leq j \leq 3$ and $Q_{ij} := (Q_i, Q_j) \in A$. Consider the elliptic curves*

$$T_k = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - i^k v \in \pi_1(E)\} \quad \textit{for} \quad 1 \leq k \leq 4,$$

$$T_{4+k} = \{(u + \pi_1(E), v + \pi_1(E)) \mid u - \omega_k \in \pi_1(E)\} \quad \textit{for} \quad 1 \leq k \leq 2,$$

$$T_{6+k} = \{(u + \pi_1(E), v + \pi_1(E)) \,|\, v - \omega_k \in \pi_1(E)\} \quad \textit{for} \quad 1 \leq k \leq 2.$$

Then the blow-up of A at the singular points

$$S_1 = Q_{00}, \quad S_2 = Q_{33}, \quad S_3 = Q_{11}, \quad S_4 = Q_{12}, \quad S_5 = Q_{21}, \quad S_6 = Q_{22}$$

of $T^{(6,8)}_{\sqrt{-1}} = \sum_{k=1}^{8} T_k$ is the toroidal compactification $(\mathbb{B}/\Gamma_1)'$ of a ball quotient $\mathbb{B}/\Gamma_1$ by a freely acting Picard modular group Γ_1 over the Gaussian integers $\mathbb{Z}[i]$.

The self-intersection matrix $M(6,8) \in \mathbb{Z}_{6\times 8}$ of $T^{(6,8)}_{\sqrt{-1}}$ is defined to have entries $M(6,8)_{ij} = 1$ for $S_i \in T_j$ and $M(6,8)_{ij} = 0$ for $S_i \not\in T_j$. Straightforwardly,

$$M(6,8) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

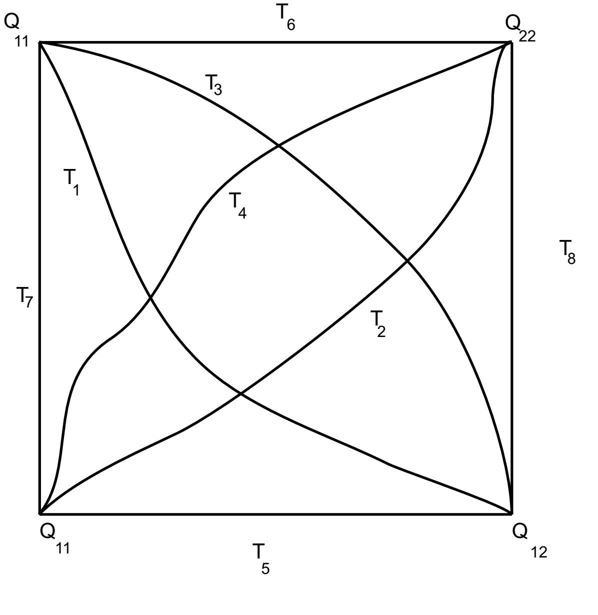

Figure 3: The incidence relations of $T^{(6,8)}_{\sqrt{-1}}$ and $\sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij} \subset \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\text{sing}}$.

According to $Q_{00}, Q_{33} \in T_k$ or $\forall 1 \leq k \leq 4$, there are no triangles $T_i + T_j + T_k \subset T^{(6,8)}_{\sqrt{-1}}$ with $1 \leq i < j \leq 4$, $1 \leq i < j < k \leq 8$. Bearing in mind that

$\left(T^{(6,8)}_{\sqrt{-1}}\right)^{\text{sing}} \cap \left(\sum_{k=5}^{8} T_k\right) = \sum_{i=1}^{2}\sum_{j=1}^{2} Q_{ij}$, one makes use of Figure 3 and recognizes the triangles $T_{2k-1} + T_{4+m} + T_{6+m}$, $T_{2k} + T_{4+m} + T_{9-m}$ with $1 \leq k, m \leq 2$. An immediate application of Proposition 9 with $\nu = 2\omega_m$ and, respectively, $\nu = \omega_3 + \omega_m + (-1)^{k+1}\omega_{3-m}$, yields the following

Corollary 17. *The space* $\mathcal{L}_A\left(T^{(6,8)}_{\sqrt{-1}}, \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\text{sing}}\right)$ *contains the binary parallel*

$$f_{56}(u,v) = \frac{\sigma(u-\omega_1-\mu_1)\sigma(u-\omega_2+\mu_1)}{\sigma(u-\omega_1)\sigma(u-\omega_2)},$$

$$f_{78}(u,v) = \frac{\sigma(v-\omega_1-\mu_2)\sigma(v-\omega_2+\mu_2)}{\sigma(v-\omega_1)\sigma(v-\omega_2)}$$

and the triangular σ*-quotients*

$$f_{2k-1,4+m,6+m}(u,v) = \frac{\sigma(u+(-1)^k iv+\omega_3)\sigma(-u+\omega_m+\omega_3)\sigma((-1)^{k+1}iv+(-1)^k i\omega_m+\omega_3)}{\sigma(u+(-1)^k iv)\sigma(-u+\omega_m)\sigma((-1)^{k+1}iv+(-1)^k i\omega_m)}$$

$$f_{2k,4+m,9-m}(u,v) = \frac{\sigma(u+(-1)^{k+1}v+\omega_3)\sigma(-u+\omega_m+\omega_3)\sigma((-1)^k v+(-1)^{k+1}\omega_{3-m}+\omega_3)}{\sigma(u+(-1)^{k+1}v)\sigma(-u+\omega_m)\sigma((-1)^k v+(-1)^{k+1}\omega_{3-m})}$$

with arbitrary $1 \leq k, m \leq 2$.

Proposition 14 provides the following

Corollary 18. *If* f_{pq} *and* f_{ijk} *are the binary parallel and triangular* σ*-quotients from the space* $\mathcal{L}_A\left(T^{(6,8)}_{\sqrt{-1}}, \left(T^{(6,8)}_{\sqrt{-1}}\right)^{\text{sing}}\right)$ *and* $\omega_{pq} = \tau_1^{-1}(f_{pq})$, $\omega_{ijk} = \tau_1^{-1}(f_{ijk})$, *then*

$$\omega_{157}, \quad \omega_{258}, \quad \omega_{368}, \quad \omega_{467}, \quad \omega_{56}, \quad \omega_{78}, \quad j_1(\eta_o)$$

is a $\mathbb{C}$*-basis of* $j_1[\Gamma_1, 1]$.

In particular, $\dim_{\mathbb{C}}[\Gamma, 1] = 7$.

2. SUFFICIENT CONDITIONS FOR THE NORMAL GENERATION OF A SPACE OF LOGARITHMIC CANONICAL SECTIONS

Definition 19. *A holomorphic line bundle* $\mathcal{E}$ *on an algebraic variety* X *is sufficiently ample if the holomorphic sections of a sufficiently large tensor power* $\mathcal{E}^{\otimes m}$ *provide a projective embedding of* X.

Definition 20. *A holomorphic line bundle $\mathcal{E}$ over an algebraic variety X is globally generated if the global holomorphic sections of $\mathcal{E}$ determine a regular projective morphism.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is globally generated if some (and therefore any) basis of V provides a regular projective morphism $X \to \mathbb{P}(V)$.

Definition 21. *A holomorphic line bundle $\mathcal{E}$ over an algebraic manifold X is normally generated if $\mathcal{E}$ is globally generated and $H^0(X, \mathcal{E})$ defines a projective immersion of X with normal image.*

A subspace $V \subseteq H^0(X, \mathcal{E})$ is normally generated if it is globally generated and the morphism $X \to \mathbb{P}(V)$ is a projective immersion with normal image.

The normal generation of a sufficiently ample line bundle is a classical topic under study. Various works provide normally generated and non-normally generated line bundles over curves and abelian varieties. According to [2], if $\mathcal{E}$ is a sufficiently ample line bundle on an abelian variety of dimension n, then $\mathcal{E}^{\otimes(n-1)}$ is normally generated. In particular, any sufficiently ample line bundle on an abelian surface is normally generated.

Our aim is to provide sufficient conditions for the normal generation of a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ over the Baily-Borel compactification $\widehat{A}$. That cannot be derived from the normal generation of a subspace $W \subseteq H^0(A, \mathcal{E})$ of holomorphic sections of a line bundle $\mathcal{E} \to A$. Namely, $\xi^* W$ cannot be a normally generated space of global holomorphic sections of $\xi^* \mathcal{E}$, as far as the morphism, associated with $\xi^* W$ is not immersive on the exceptional divisor $L = \xi^{-1}(T^{\text{sing}})$ of $\xi : A' \to A$.

Corollary 22. *Let X be an irreducible normal projective variety X and $f : X \to Y$ be a finite, regular, generically injective morphism onto Y. Then $f : X \to Y$ is a regular immersion with normal image Y.*

Proof. If $f : X \to Y$ is a regular morphism of degree $d \in \mathbb{N}$, then the generic fiber of f consists of d points, while the exceptional ones are constituted by $\leq d$ points. In particular, for $d = 1$, any regular, generically injective morphism is bijective onto its image. As a result, $f : X \to Y$ is a regular immersion with normal image. □

Our specific considerations will be based on the following immediate consequence of Corollary 22

Corollary 23. *Let X be an irreducible normal projective variety, $\mathcal{E} \to X$ be e holomorphic line bundle over X and $V \subseteq H^0(X, \mathcal{E})$ be a space of global holomorphic sections of $\mathcal{E}$. If $f : X \to \mathbb{P}(V)$ is a finite, regular, generically injective morphism then V is normally generated.*

Lemma 24. *A subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, is globally generated over $\widehat{A}$ if and only if it satisfies simultaneously the following two geometric conditions:*

(i) *for any irreducible component* T_i *of* T *there is* $\omega_i \in V$ *with* $(\tau_1(\omega_i))_\infty \supset T_i$*;*

(ii) *for any* $p \in T^{\rm sing}$ *there exists* $\omega_p \in V$ *with* $m_p(\tau_1(\omega_p)) = -1$*.*

Proof. The space V is globally generated over $\widehat{A}$ exactly when for any point $q \in \widehat{A}$ there is $v_q \in V$ with $v_q(q) \neq 0$. If $q \in \left(\widehat{\mathbb{B}/\Gamma}\right) \setminus \left(L \cup \sum_{i=1}^{h} \kappa_i\right)$, then $j_1(\eta_o)(q) \neq 0$. A modular form $\omega_i \in V$ does not vanish on the cusp κ_i if and only if $T_i \subset (\tau_1(\omega_i))_\infty$. A modular form $\omega_p \in V$ takes non-zero values on the rational (-1)-curve $\xi^{-1}(p)$ exactly when the multiplicity $m_p(\tau_1(\omega_p)) = -1$. □

From now on, we say briefly that a modular form $\omega \in H^0(A', \Omega^2_{A'}(T'))$ is binary parallel or triangular if its transfer $\tau_1(\omega) \in \mathcal{L}_A(T, T^{\rm sing})$ is binary parallel or, respectively, triangular.

Proposition 25. *Let us suppose that the subspace* $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ *contains the cuspidal form* $j_1(\eta_o)$*, two binary parallel forms* ω_{13}*,* ω_{24}*, a triangular* ω_{012} *with* $T_0 \cap T_3 \cap T_4 = \emptyset$ *and satisfies the following three conditions:*

(i) *for any* $i \not\in \{0, 1, \ldots, 4\}$ *there exists* $\omega_i \in V$ *with* $(\tau_1(\omega_i))_\infty \supset T_i$*;*

(ii) *for any* $p \in T^{\rm sing} \setminus \left(\sum_{j=0}^{4} T_j\right)$ *there exists* $\omega_p \in V$ *with* $m_p(\tau_1(\omega_p)) = -1$*;*

(iii) *for any* $1 \leq i < j \leq h$ *there is* $\omega_{ij} \in V$*, such that* $(\tau_1(\omega_{ij}))_\infty$ *contains exactly one of* T_i *or* T_j*.*

Then V *is normally generated.*

Proof. In the presence of Corollary 23, it suffices to establish that the projective morphism $f : \widehat{A} \to \mathbb{P}(V)$, associated with V is regular, finite and generically injective. Assumption (i) from the present proposition and $(\tau_1(\omega_{ij}))_\infty = T_i + T_j$, $(\tau_1(\omega_{012}))_\infty = T_0 + T_1 + T_2$ imply assumption (i) from Lemma 24. Further, no-one $p \in T^{\rm sing} \cap (T_1 + T_3)$ belongs to $(\tau_1(\omega_{13}))_0 = D_1 + D_3$, as far as T_1, T_3, D_1 and D_3 are mutually parallel and distinct. Therefore, $m_p(\tau_1(\omega_{13})) = -1$. Similarly, $m_p(\tau_1(\omega_{24})) = -1$ for $p \in T^{\rm sing} \cap (T_2 + T_4)$. By Corollary 13, $m_p(\tau_1(\omega_{012})) = -1$ for all $p \in T^{\rm sing} \cap \left(\sum_{i=0}^{2} T_i\right)$. Combining with assumption (ii) from the present proposition, one obtains (ii) from Lemma 24. That allows to conclude that $f : \widehat{A} \to \mathbb{P}(V)$ is regular.

The assumption (iii) guarantees that $f : \widehat{A} \to f(\widehat{A}) \subset \mathbb{P}(V)$ is finite. First of all, on $\widehat{A} \setminus [L + (\partial_\Gamma \mathbb{B}/\Gamma)] = (\mathbb{B}/\Gamma) \setminus L = A \setminus T$, the morphism

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13} \circ \xi = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24} \circ \xi = f_{24}\right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^2$$

is of degree 4. More precisely, if

$$f_{13}(u,v) = \frac{\sigma(u-\mu_1)\sigma(u-c_3+\mu_1)}{\sigma(u)\sigma(u-c_3)}, \; f_{24}(u,v) = \frac{\sigma(v-\mu_2)\sigma(v-c_4+\mu_2)}{\sigma(v)\sigma(v-c_4)}, \quad (2.1)$$

then for any $x, y \in \mathbb{P}^1$ the fiber is

$$(f_{13}, f_{24})^{-1}(x,y) = \{(P_i(x), Q_j(y)) \,|\, 1 \leq i, j \leq 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_1(y) + Q_2(y) = \overline{c_4}.$$

The condition (iii) provides the injectiveness of $f : \partial_\Gamma \mathbb{B}/\Gamma \to f(\partial_\Gamma \mathbb{B}/\Gamma)$, which suffices for $f : L \to f(L)$ to be discrete and, therefore, finite. Otherwise, f contracts some irreducible component $\xi^{-1}(p)$, $p \in T^{\text{sing}}$ of L. If $p \in T_i \cap T_j$ then $\kappa_i, \kappa_j \in \xi^{-1}(p)$, whereas $f(\kappa_i) = f(\kappa_j)$. Thus, $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ and, therefore, $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

The generic injectiveness of the projective morphism $f : \widehat{A} \to f(\widehat{A})$ follows from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{24}}{j_1(\eta_o)} = f_{24}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

This, in turn, is equivalent to the generic injectiveness of the rational surjective morphism

$$F = (f_{13}, f_{24}, f_{012}) : A \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Let us consider also the rational surjection $F_1 = (f_{13}, f_{24}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ and its factorization

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle F_1}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_{12}} & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

through F and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ onto the first two factors. The irreducible components T_1 and T_2 of the triangle $T_0 + T_1 + T_2$ have intersection number $T_1.T_2 = 1$. That allows to assume that $T_1 = \check{o}_E \times E$, $T_2 = E \times \check{o}_E$ and (1.10).

Suppose that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is not generically injective. By $F_1 = \mathrm{pr}_{12} \circ F$ and $\deg F_1 = 4$, the generic fiber of F on $F_1^{-1}(x,y)$ consists of 2 or 4 points. In either case, for any $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds at least one of the following pairs of relations:

Case (i): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y))$,
$f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_2(y))$;

Case (ii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_2(x), Q_2(y)),$
$f_{012}(P_1(x), Q_1(y)) = f_{012}(P_2(x), Q_1(y));$

Case (iii): $f_{012}(P_1(x), Q_2(y)) = f_{012}(P_1(x), Q_1(y)),$
$f_{012}(P_2(x), Q_2(y)) = f_{012}(P_2(x), Q_1(y)).$

We claim that the relations from at least one case are satisfied identically on $\mathbb{P}^1 \times \mathbb{P}^1$. Otherwise, the locus of either case is a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$ and their union is also a proper analytic subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$. The contradiction implies that for any $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$ there holds identically at least one of the Cases (i), (ii) or (iii). Note that (ii) and (iii) are equivalent under the transposition of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and, respectively, of $A = E \times E$.

Without loss of generality, one can suppose that $P_1(\infty) = \check{o}_E$ and $P_2(\infty) = \overline{c_3}$. In Case (i), up to a relabeling of $Q_1(y)$, $Q_2(y)$, one has $Q_1(\infty) = \check{o}_E$, $Q_2(\infty) = \overline{c_4}$. Then

$$\infty = f_{012}(\check{o}_E, \check{o}_E) = f_{012}(P_1(\infty), Q_1(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

However, $\overline{c_3} \neq \check{o}_E$, $\overline{c_4} \neq \check{o}_E$ and $T_3 \cap T_4 = \{(\overline{c_3}, \overline{c_4})\} \not\subseteq T_0$ reveal that $f_{012}(\overline{c_3}, \overline{c_4}) \neq \infty$, so that Case (i) does not hold identically on A. Similarly, in Case (ii), there follows

$$\infty = f_{012}(\check{o}_E, \overline{c_4}) = f_{012}(P_1(\infty), Q_2(\infty)) = f_{012}(P_2(\infty), Q_2(\infty)) = f_{012}(\overline{c_3}, \overline{c_4}).$$

The contradiction implies that $F : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. □

Here is another sufficient condition for a subspace $V \subseteq H^0(A', \Omega^2_{A'}(T'))$ to be normally generated.

Proposition 26. *Let V be a subspace of $H^0(A', \Omega^2_{A'}(T'))$, containing the cuspidal form $j_1(\eta_o)$, a binary parallel ω_{13}, triangular ω_{012}, ω_{234} with $T_0 \cap T_1 \cap T_4 = \emptyset$ and satisfying the following three conditions:*

(i) *for any $i \notin \{0, 1, \dots 4\}$ there exists $\omega_i \in V$ with $(\tau_1(\omega_i))_\infty \supset T_i$;*

(ii) *for any $p \in T^{\mathrm{sing}} \setminus \left(\sum_{j=0}^{4} T_j \right)$ there exists $\omega_p \in V$ with $m_p(\tau_1(\omega_p)) = -1$;*

(iii) *for any $1 \leq i < j \leq h$ there is $\omega_{ij} \in V$, such that $(\tau_1(\omega_{ij}))_\infty$ contains exactly one of T_i or T_j.*

Then V is normally generated.

Proof. As in Proposition 25, first we establish the regularity of the projective morphism $f : \widehat{A} \to f(\widehat{A})$.

Further, $f : \widehat{A} \to f(\widehat{A})$ is finite, as far as the fibers of its restriction on $(\mathbb{B}/\Gamma) \setminus L = A \setminus T$ are contained in the fibers of

$$\left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012} \right) : A \setminus T \longrightarrow \mathbb{C}^2.$$

Let $f_{012}(u, v)$ be of the form (1.10) and f_{13} be as in (2.1). Then for any $x, y \in \mathbb{P}^1$ the fiber

$$(f_{13}, f_{012})^{-1}(x, y) = \{(P_i(x), Q_{ij}(x, y)) \mid 1 \leq i, j \leq 2\}$$

with

$$P_1(x) + P_2(x) = \overline{c_3}, \quad Q_{i1}(x, y) + Q_{i2}(x, y) = -a_0 b_0^{-1} P_i(x) - b_0^{-1} \overline{c_0}$$

consists of at most four points. The reason is that for any fixed $P_i(x) \in E$ the elliptic function $f_{012}(P_i(x), \ )$ is of order 2. Thus, $(f_{13}, f_{012}) : A \setminus T \to \mathbb{C}^2$ is finite. The assumption (iii) implies that $f : L \cup (\partial_\Gamma \mathbb{B}/\Gamma) \to f\,(L \cup (\partial_\Gamma \mathbb{B}/\Gamma))$ is finite, so that $f : \widehat{A} \to f(\widehat{A})$ is a finite regular morphism.

We derive the generic injectiveness of $f : \widehat{A} \to f(\widehat{A})$ from the generic injectiveness of the affine morphism

$$F = \left(\frac{\omega_{13}}{j_1(\eta_o)} = f_{13}, \frac{\omega_{012}}{j_1(\eta_o)} = f_{012}, \frac{\omega_{234}}{j_1(\eta_o)} = f_{234} \right) : (\mathbb{B}/\Gamma) \setminus L = A \setminus T \longrightarrow \mathbb{C}^3.$$

To this end, let us factor the rational surjection $F_1 = (f_{13}, f_{012}) : A \to \mathbb{P}^1 \times \mathbb{P}^1$ through the rational surjection $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the projection $\mathrm{pr}_{12} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, along the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle F_1}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_{12}} & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array} .$$

If F is not generically injective, then at least one of the following three cases holds identically on $\mathbb{P}^1 \times \mathbb{P}^1$:

Case (i): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y))$,
$f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y))$;

Case (ii): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_2(x), Q_{22}(x, y))$,
$f_{234}(P_1(x), Q_{11}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y))$;

Case (iii): $f_{234}(P_1(x), Q_{12}(x, y)) = f_{234}(P_1(x), Q_{11}(x, y))$,
$f_{234}(P_2(x), Q_{22}(x, y)) = f_{234}(P_2(x), Q_{21}(x, y))$.

In either case, denote by $P_1(\infty) = \check{o}_E$ and $P_2(\infty) = \overline{c_3}$ the poles of the elliptic function f_{13} and note that $T_1 = P_1(\infty) \times E$, $T_3 = P_2(\infty) \times E$. Further, let

$Q_{i1}(\infty,\infty) = \check{o}_E$, so that $T_2 = E \times Q_{11}(\infty,\infty) = E \times Q_{21}(\infty,\infty)$. Finally, let $Q_{i2}(\infty,\infty) = -a_0 b_0^{-1} P_i(\infty) - b_0^{-1}\overline{c_0}$, in order to have

$$\{q_{10}\} = T_1 \cap T_0 = \{(P_1(\infty), Q_{12}(\infty,\infty))\},$$

$$\{q_{30}\} = T_3 \cap T_0 = \{(P_2(\infty), Q_{22}(\infty,\infty))\}.$$

Denote also

$$\{q_{12}\} = T_1 \cap T_2 = \{(P_1(\infty), Q_{11}(\infty,\infty))\},$$

$$\{q_{32}\} = T_3 \cap T_2 = \{(P_2(\infty), Q_{21}(\infty,\infty))\}.$$

Bearing in mind that $(f_{234})_\infty = T_2 + T_3 + T_4$, note that $f_{234}(q_{ij}) = \infty$ whenever $\{i,j\} \cap \{2,3,4\} \neq \emptyset$. In the Case (i) one has $f_{234}(q_{10}) = f_{234}(q_{32}) = \infty$. If $q_{10} \in T_2$, then $q_{10} \in T_0 \cap T_1 \cap T_2$, contrary to the assumption that $T_0 + T_1 + T_2$ is a triangle. On the other hand, $T_3 \cap T_1 = \emptyset$ guarantees that $q_{10} \notin T_3$. Therefore $q_{10} \in T_4$ and $q_{10} \in T_0 \cap T_1 \cap T_4 = \emptyset$. The contradiction rejects the Case (i). If the first relation of Case (ii) is identical on $\mathbb{P}^1 \times \mathbb{P}^1$, then $f_{234}(q_{10}) = f_{234}(q_{30}) = \infty$. As in the Case (i), that leads to an absurd. Finally, $f_{234}(q_{10}) = f_{234}(q_{12}) = \infty$ contradicts the hypotheses and establishes that $F = (f_{13}, f_{012}, f_{234}) : A \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is generically injective. □

An immediate application of Proposition 26 to the example from Proposition 16 yields the following

Corollary 27. *In the terms of Proposition 16, the subspace*

$$V_1 = \mathrm{Span}_{\mathbb{C}}(j_1(\eta_o), \omega_{56}, \omega_{157}, \omega_{267}, \omega_{368}, \omega_{458}) \subset H^0(A_1', \Omega^2_{A_1'}(T'))$$

is normally generated, i.e., determines a regular projective immersion

$$f : \widehat{\mathbb{B}/\Gamma_1} \to \mathbb{P}(V_1) = \mathbb{P}^5$$

with normal image.

If one applies Proposition 25 to the cuspidal form $j_1(\eta_o)$, the binary parallel ω_{56}, ω_{78} and triangular ω_{157}, then one needs to adjoin the triangular $\omega_{2,4+k,9-k}$, $\omega_{3,4+l,6+l}$, $\omega_{4,4+m,9-m}$ for some $k,l,m \in \{1,2\}$. The span of these modular forms is 7-dimensional and depletes the entire $[\Gamma_1, 1]$. It is clear that the normal generation of V_1 implies the normal generation of $H^0(A_1', \Omega^2_{A_1'}(T')) = j_1[\Gamma_1, 1]$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ASYMPTOTICALLY FASTEST SORTING ALGORITHM FOR ALMOST SORTED ARRAYS

STEFAN GERDJIKOV

The *patience sorting algorithm* was introduced by Mellows. If a given array has n elements and can be considered as a shuffle of m already sorted arrays, then the patience algorithm sorts the original array in $O(n \log m)$ time. In the current paper we show that this upper bound is worst-case optimal even if the minimum value of the parameter m is known in advance.

Keywords: Patience sorting algorithm, worst-case optimality, increasing subsequences

2000 Math. Subject Classification: 68W40

1. INTRODUCTION

We consider the problem of sorting a sequence of n distinct numbers. Although this problem is well studied and optimal $O(n \log n)$ worst-case and average-case algorithms have been developed [7], there is no exact estimate of the complexity of these algorithms with respect to the disorder in the sequence.

In the current paper we consider the *patience sorting algorithm* introduced by Mellows, [8, 9, 1, 2]. Essentially, this approach of sorting real numbers first splits the given array into a minimal number of increasingly sorted subarrays and afterwards merges the resulting arrays. Using a result of Fredman, [5], it can be easily shown that this algorithm runs in $O(n \log m)$-time for every sequence of size n that contains no decreasing subsequence of size $m + 1$. Note that m is not previously known to the algorithm. However, even if an upper bound for m is known in advance no better worst-case algorithm exists as we prove in Section 3.

The rest of this paper is organized as follows. In Section 2 we outline the patience sorting algorithm in details, prove its correctness and its time-complexity. In Section 3 we argue that the algorithm is worst-case optimal and in Section 4 we conclude.

2. PATIENCE ALGORITHM DESCRIPTION

In this section we assume that $a_1, \dots, a_n$ is a sequence of distinct numbers that is to be sorted in *increasing* order. To this end we describe an $O(n \log m)$-time algorithm where m is the size of the longest *decreasing* subsequence, i.e. m is maximal natural number with the property :

$$\text{there are } \; i(1) < i(2) < \cdots < i(m), \; \text{such that } \; a_{i(1)} > a_{i(2)} > \cdots > a_{i(m)}.$$

The patience algorithm consists of two steps, [8, 9]:

1. Split $a_1, \dots a_n$ into minimum number of increasing subsequences:

$$\{a_{1,1} < \cdots < a_{1,k_1}\}, \dots, \{a_{M,1} < \cdots < a_{M,k_M}\}.$$

2. Merge the resulting subsequences into an increasing array:

$$a_{\pi(1)} < a_{\pi(2)} < \cdots < a_{\pi(n)}.$$

Both these steps can be performed in time $O(n \log M)$ and using Dilworth's Theorem [4, 6] it is not difficult to see that $M = m$, which implies the result.

In the sequel we first prove that $M = m$ and then we briefly explain how to efficiently perform each of the two steps of the algorithm.

Given a sequence $a_1, \dots, a_n$, we introduce a partial ordering $\prec$ on the set $\{1, 2 \dots, n\}$ in the following way:

$$i \prec j \iff i < j \;\text{ and }\; a_i < a_j.$$

With this notation it is obvious that the following are equivalent:

- $i(1) \prec i(2) \cdots \prec i(k)$;
- $(a_{i(1)}, a_{i(2)}, \dots, a_{i(k)})$ is an increasing subsequence of $\{a_j\}_{j=1}^n$.

Thus each chain in $(\{1, \dots n\}, \prec)$ corresponds to an increasing subsequence in $a_1, \dots a_n$ and vice versa.

On the other hand, there is a similar relationship between the antichains in $(\{1, \dots, n\}, \prec)$ and the decreasing subsequences of $a_1, \dots a_n$. Specifically, we consider an antichain $\{i(1), \dots i(k)\}$, i.e. $i(j)$, $i(l)$ are incomparable with respect to $\prec$. We can assume that $i(1) < i(2) < \cdots < i(k)$. Now consider a pair $i(j) < i(l)$: since $i(j) \not\prec i(l)$, we deduce that $a_{i(j)} \not< a_{i(l)}$. Furthermore, $i(j) \neq i(l)$

and the members of the sequence a are all distinct numbers, which implies that $a_{i(j)} > a_{i(l)}$. Thus we have established that for every antichain $\{i(1), \ldots i(k)\}$ such that $i(1) < i(2) < \cdots < i(k)$,

$$a_{i(1)} > a_{i(2)} > \cdots > a_{i(k)} \text{ is a decreasing subsequence of } \{a_j\}_{j=1}^{n}.$$

Conversely, if $a_{i(1)} > a_{i(2)} > \cdots > a_{i(k)}$ is a decreasing subsequence, then $i(j) < i(l)$ implies $a_{i(j)} > a_{i(l)}$, i.e. $i(j)$ and $i(l)$ are incomparable with respect to $\prec$ and consequently determine an antichain.

Now the Dilworth's Theorem [4, 6] implies the following lemma:

Lemma 1. *Let m be the maximal length of a decreasing subsequence of $a_1, \ldots, a_n$ and let M be the minimal number of increasing subsequences of $a_1, \ldots, a_n$ in which $a_1, \ldots, a_n$ can be partitioned. Then $M = m$.*

Proof. By the discussion above, m is the size of a maximal antichain in $(\{1, 2, \ldots, n\}, \prec)$ and M is the minimum covering of $(\{1, 2, \ldots, n\}, \prec)$ with $\prec$-chains. Therefore, since $\prec$ is a partial ordering, Dilworth's Theorem [4, 6] implies $m = M$. □

Next we briefly describe the first part of the algorithm – determining the least number of increasing subsequences that cover $a_1, \ldots, a_n$. We basically follow the ideas presented in [5, 3]. The algorithm processes the elements a_i in increasing order of i. At each step i we keep a set of lists $L_1, \ldots, L_{m_i}$, such that $L_1, \ldots L_{m_i}$ form a minimum $\prec$-chain covering of the set $\{1, 2, \ldots, i\}$ and additionally for each $k \in L_{j+1}$ we keep a witness $w(k) \in L_j$ such that

$$w(k) < k \text{ and } w(k) \not\prec k,$$

which is equivalent to $w(k) < k$ and $a_{w(k)} > a_k$. Moreover, we maintain an array of the last elements $l[s] \in L_s$. Note that $a_{l[s+1]} < a_{w(l[s+1])} \leq a_{l(s)}$. The first inequality follows by the definition of the witnesses and the second follows by the fact that $w(l[s+1]) \preceq l[s]$ according to the definition of $l[s]$.

Now we describe how to maintain these invariants from step i to step $i + 1$.

1. Find the least s, such that $a_{l[s]} < a_{i+1}$.
2. If such an s does not exist, set $s = m_i + 1$, create a new list L_{m_i+1} and set $m_{i+1} = m_i + 1$, otherwise set $m_{i+1} = m_i$.
3. Insert $i + 1$ into L_s and set $l[s] = i + 1$.
4. If $s > 1$ set $w(i + 1) = l[s - 1]$.

Note that $i+1 > j$ for each $j \in \cup_{k=1}^{m_i} L_k$. Therefore $i+1 > l[s]$, and since $a_{l[s]} < a_{i+1}$, we obtain that $l[s] \prec i+1$. However, $l[s]$ is the maximal element of the list L_s, which implies that $L_s \cup \{i + 1\}$ is again a chain with maximal element $i + 1$. Next note that if $s > 1$, the choice of s implies that $a_{l[s-1]} > a_{i+1}$. Since $i+1 > l[s-1]$, we can safely define the witness of $i+1$ as $w(i+1) = l[s-1]$, as it is done in step 4. Finally, we argue that L_j is again a minimum covering of $\{1, 2, \ldots, i+1\}$ with $\prec$-chains. This is clear in the case $m_{i+1} = m_i$, i.e. if $s \leq m_i$. Assume that $s = m_i + 1$, then we can consider the sequence $\{w^k(i+1) \,|\, 0 \leq k \leq m_i\}$. Since $i+1 \in L_{m_i+1}$, the definition of the witness implies that $w^k(i+1) \in L_{m_i+1-k}$. Moreover, we have that $w^k(i+1) > w^{k+1}(i+1)$ and $a_{w^k(i+1)} < a_{w^{k+1}(i+1)}$. Therefore, the set $\{w^k(i+1) \,|\, 0 \leq k \leq m_i\}$ is an anti-chain of size $m_i + 1$ in $(\{1, 2, \ldots, i+1\}, \prec)$. Now by Dilworth's Theorem [4, 6] each covering with chains of $\{1, \ldots i+1\}$ contains at least $m_i + 1$ elements, and therefore $m_{i+1} = m_i + 1$.

This shows that the above algorithm determines a minimum covering with increasing subsequences. Next we prove the main result of this section:

Theorem 1. *There is an $O(n \log m)$-time algorithm that sorts an arbitrary sequence of distinct numbers $a_1, \ldots, a_n$ which contains no decreasing subsequence of length more than m.*

Proof. From the discussion above we know that the above algorithm provides a minimum covering with increasing subsequences. Now we consider its efficiency. Each of the steps 2, 3 and 4 can be performed in $O(1)$ time and step 1 can be performed in $O(\log m_i)$-time by binary searching the array $l[s]$ (recall that $a_{l[s]} > a_{l[s+1]}$). Since $m_i \leq m$ and we have n iterations in total, we obtain $O(n \log m)$-time algorithm to compute an optimal covering of $a_1, \ldots a_n$ with increasing subsequences.

Now, since $L_1, \ldots L_m$ are sorted in increasing order, we can easily merge them in $O(n \log m)$-time. One way to achieve this is to group the lists in pairs and merge the lists in every single pair. Each such step needs $O(n)$ time and reduces the number of lists twice. Thus in $O(\log m)$ iterations we end up with a single sorted list. Since we spend $O(n)$ time per iteration, the time bound follows.

Another possibility is to maintain a binary heap with up to m elements, each element corresponding to the least element of a list L_s which is still not sorted. At each step we extract the minimal element e from the heap and add it to the sorted output list (at the back). Next, if $e \in L_s$, we insert in the heap the next element of L_s. Clearly, we have $O(n)$ operations insert and extract minimal element from a heap with $O(m)$ elements. Therefore, each such operation can be performed in $O(\log m)$-time and the total time complexity results in $O(n \log m)$. □

3. OPTIMALITY

In this section we show that each algorithm which sorts correctly in increasing order a sequence of distinct numbers $a_1, \ldots, a_n$ needs to perform $\Theta(n \log m)$ comparisons where m is the length of the longest decreasing subsequence of $a_1, \ldots, a_n$.

This would imply that the algorithm we described in the preceding section is worst-case optimal. The approach we use is similar to that in [5].

To this end we first show that there are $e^{\Theta(n \log m)}$ permutations $a_{\pi(1)}, \dots, a_{\pi(n)}$ which contain no decreasing subsequence of length more than m.

Lemma 2. *Let $a_1 < a_2 < \cdots < a_n$ be distinct numbers and let $\Pi(m)$ be the set of permutations $\pi \in S_n$ such that $a_{\pi(1)}, \dots, a_{\pi(n)}$ contains no decreasing subsequence of length greater than m. Then*

$$|\Pi(m)| \geq \frac{m^n}{m!}\,.$$

Proof. We count the permutations $\pi \in S_n$ with the property that there exist integers m' and $k_1, \dots k_{m'}$, $k_{m'+1}$ such that:

$$m' \leq m \text{ and } 1 = k_1 < k_2 < \dots k_{m'} < k_{m'+1} = n+1$$
$$\forall j (a_{\pi(k_j)} < a_{\pi(k_j+1)} < \cdots < a_{\pi(k_{j+1}-1)})$$
$$\forall i \leq m' (a_{\pi(k_i)} > a_{\pi(k_{i+1})}).$$

In fact, $\{k_j, k_j+1, \dots, k_{j+1}-1\}_{j=1}^{m'}$ define m' chains in $(\{1, 2, \dots n\}, \prec)$,where $\prec$ is defined with respect to the sequence $a_{\pi(1)}, \dots, a_{\pi(n)}$. Consequently, by the discussion in the previous section, there is no decreasing subsequence of length more than m' in $a_{\pi(1)}, \dots, a_{\pi(n)}$. On the other hand, the elements $a_{\pi(k_j)}$ witness for such a decreasing sequence. Therefore, each such permutation π belongs to the set $\Pi(m)$.

All such permutations π can be generated in the following way:

- assign each element $i \in \{1, 2, \dots, n\}$ to exactly one of m sets B_j for $j \leq m$.
- discard all empty sets B_j.
- sort each $B_j \neq \emptyset$ in increasing order. In this fashion for each set B_j we obtain an increasing sequence b_j.
- arrange the sequences b_j's in decreasing order of their first elements. In this way we obtain the sequence $\pi(1), \dots, \pi(n)$.

Clearly, each permutation obtained in this way can be uniquely decomposed into the increasing sequences b_j's which witness that $\pi \in \Pi(m)$. Next observe that different families of sets $\{B_1, \dots, B_m\}$ and $\{B'_1, \dots B'_m\}$ determine different permutations π and π'. Indeed, if it were the case that $\pi = \pi'$, then these permutations would determine the same sequence of increasing sequences $b_1 = b'_1$, $b_2 = b'_2, \dots$, $b_{m'} = b'_{m'}$. Since each sequence b_j uniquely determines the set B_j, we conclude that $B_j = B'_j$ and since $\{B_1, \dots, B_m\}$ and $\{B'_1, \dots, B'_m\}$ define a partition of $\{1, 2, \dots, n\}$, we obtain that $\{B_1, \dots, B_m\} = \{B'_1, \dots, B'_m\}$.

Therefore, it suffices to bound from below the number of all different families $\{B_1, \dots, B_m\}$. It is easy to count that the assignment in the first part of the construction can be done in m^n different ways. Since each family $\{B_1, \dots, B_m\}$ can be generated by at most $m!$ permutations of the sets B_j, we obtain that the number of all different families $\{B_1, \dots, B_m\}$ is at least $\frac{m^n}{m!}$ and therefore there are at least $\frac{m^n}{m!}$ permutations such that $a_{\pi(1)} \dots, a_{\pi(n)}$ contains no decreasing subsequence of length more than m. Therefore $|\Pi(m)| \geq \frac{m^n}{m!}$. □

Corollary 1. *The number of permutations of a sequence $a_1, \dots, a_n$ of distinct numbers that contain no decreasing subsequence of length more than m is $e^{\Omega(n \log m)}$.*

Proof. We consider first the case $m \leq \frac{n}{2}$. According to Lemma 2, the number of permutations $\Pi(m)$ that contain no decreasing subsequence of length more than m is

$$|\Pi(m)| \geq \frac{m^n}{m!}.$$

By Stirling's formula, $m! = \sqrt{2\pi m}\, m^m\, e^{-m+o(1)}$. Hence, $m! = e^{m(\log m + O(1))}$. Therefore,

$$|\Pi(m)| \geq \frac{m^n}{e^{m(\log m + O(1))}} \geq e^{(n-m)\log m + O(m)} \geq e^{\frac{n}{2}\log m + O(m)} = e^{\Theta(n \log m)},$$

since $m \leq \frac{n}{2}$.

In the case $m > \frac{n}{2}$ we have $\log \frac{n}{2} \leq \log m \leq \log \frac{n}{2} + 1$. Now we use that

$$\Pi\left(\frac{n}{2}\right) \subseteq \Pi(m).$$

By the discussion above we obtain that

$$\left|\Pi\left(\frac{n}{2}\right)\right| = e^{\Omega(n \log \frac{n}{2})},$$

and since $\Omega(n \log \frac{n}{2}) = \Omega(n \log m)$ for $\frac{n}{2} \leq m \leq n$,it is easy to see that

$$|\Pi(m)| \geq \left|\Pi\left(\frac{n}{2}\right)\right| = e^{\Omega(n \log m)},$$

and the result follows in this case either. □

Corollary 2. *Each algorithm which correctly sorts in increasing order each sequence $a_1, \dots, a_n$ of distinct numbers which contains no decreasing subsequence of length more than m, has worst-case time-complexity $\Omega(n \log m)$.*

Proof. By Corollary 1, there are $e^{\Omega(n \log m)}$ different permutations of $a_1, \dots, a_n$ that the algorithm has to be capable to distinguish. Now, if the algorithm performs $o(n \log m)$ comparisons on each such instance, we can assign each such permutation to a leaf of a binary decision tree of height $o(n \log m)$. However, each such tree has $e^{o(n \log m)}$ leaves and therefore two different permutations will be assigned to the same leaf of the tree. Consequently, the algorithm will be unable to distinguish between them. □

As a corollary we obtain the following result:

Theorem 2. *The $O(n \log m)$-time sorting algorithm described in Section 2 is worst-case optimal.*

Proof. By Theorem 1 we have the correctness and the $O(n \log m)$ bound for the algorithm. On the other hand, Corollary 2 implies that any other algorithm that solves this problem is worst-case $\Omega(n \log m)$. □

4. CONCLUSION

We have studied the problem of sorting a sequence of n distinct numbers with respect to the size m of the longest decreasing subsequence that it contains. We described an $O(n \log m)$-time algorithm that solves this problem without any assumptions on m and we showed that this time-complexity is worst-case optimal even under the assumption that an upper bound for m is known in advance.

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LEAST ENUMERATIONS OF UNARY PARTIAL STRUCTURES

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In the present paper we consider structures with unary partial functions and partial predicates, called unary structures. Unary structures does not contain equality and inequality among the predicates of the structure. The main result obtained here is a characterization of the unary structures which have least enumerations, called degrees of the structures. As a corollary it is obtained a characterization of the unary structures which admit effective enumerations. There are some interesting results concerning the spectrum and the so-called quasi-degree of such structures.

Keywords: Enumeration, enumeration degree, enumeration operator, degree of a structure, type of a sequence of elements of a structure, Turing degree, universal set

2010 Math. Subject Classification: 03D45, 03D60, 03D75

1. INTRODUCTION

There are a lot of attempts to find a measure of the complexity of a given structure. Richter [8] has defined a degree of a structure as the least T-degree (if it exists) of all bijective total enumerations of the structure. Then it has been introduced a spectrum of a structure according to T-degrees, using only bijective total enumerations, too. There are a lot of investigations that show some sufficient conditions for a structure to have a least enumeration [1] and [11], and another with complicated structures without degree [8, 7, 2, 6]. They use the equality among the predicates of the structure. Soskov [12, 11] has generalized the notion of spectrum of a structure, using not only bijective enumeration, but all total ones. In that definition enumeration degrees are considered. This gives a possibility to consider not only totally defined structures, but partially defined, as well. Soskov [11] has generalized a spectrum of a partial structure, defining a partial spectrum, using partial enumerations.

Here we consider partial structures with unary functions and predicates, calling them unary. Since the equality and the inequality are not unary, they are not among the predicates of the structure. We consider such structures because they are simple enough, and as we will see, they are rich enough. For unary structures we find necessary and sufficient conditions for possessing least enumerations w.r.t. to e-degrees. As a corollary we obtain similar conditions w.r.t. to T-degrees.

In Section 2 we introduce the main definitions and preliminary results. In Section 3 we introduce a type and ∃-type of an element of a unary partial structure. Roughly speaking, a type (∃-type) of such an element is the set of all codes of open (existential) formulas, which are true on that element in that structure. A characterization of all unary structures, which admits least enumerations in the terms of a universal set of all types (∃-types) is given. We show that a unary partial structure admits a least enumeration if and only if there exist sequence of finite elements such that the ∃-type of that sequence is the least upper bound of all ∃-types of the structure and there exists a computable sequence of enumeration operators, such that the sequence of these enumeration operators applied to the upper bound "describes" all types of the elements of the structure. As a corollary we characterize the structures which admit effective enumerations. In this section we show that it is not possible to have a spectrum of a unary partial structure with denumerably many minimal elements.

In Section 4 we prove that a partial spectrum of a unary partial structure is upward closed for all partial enumerations. We show that for every set of r-degrees there is a structure with a set of types which "almost" coincides with the set of r-degrees, $r \in \{e, T\}$. Here we show several interesting examples, some of them concerning the so called quasi-degrees [11]. For example, we show that there are structures which don't have degrees, but they have quasi-degrees.

2. PRELIMINARIES

In this paper we use ω to denote the set of all natural numbers; $Dom(f)$, $Ran(f)$ and G_f to denote respectively the domain, the range and the graph of the function f; $\langle f\rangle$ or $\langle G_f\rangle$ to denote the set $\{\langle x_1,\dots,x_n,y\rangle | (x_1,\dots,x_n,y) \in G_f\}$, where $\langle .,\dots,.\rangle$ is some fixed coding function for all finite sequences of natural numbers.

We shall recall some definitions from [10, 3].

Let $\mathfrak{A} = \langle B; \theta_1,\dots,\theta_n; R_1,\dots,R_k\rangle$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1,\dots,\theta_n$ are partial unary functions in B and $R_1,\dots,R_k$ are unary partial predicates on B. We call such structures *unary*. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false.

Let $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k \rangle$ be a partial structure over the set ω. By $\langle \mathfrak{B} \rangle$ we denote the set $\langle \varphi_1 \rangle \oplus \cdots \oplus \langle \varphi_n \rangle \oplus \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_k \rangle$. Let W be a recursively enumerable set. For any set B let

$W(B) = \{x | \exists v(\langle v, x \rangle \in W \wedge E_v \subseteq B)\}$. In this case we say W is an enumeration operator. A sequence of enumeration operators $W_{z_0}, W_{z_1}, \ldots$ is said to be computable if there exists a recursive function h such that $h(n) = z_n$ for any natural n.

Definition 1. *Let $\mathcal{A}$ be a family of subsets of ω. A set $U \subseteq \omega^2$ is said to be universal for the family $\mathcal{A}$ if the following conditions hold:*

a) For every fixed $e \in \omega$, $\{x_1 | (e, x_1) \in U\} \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, then there exists e such that $A = \{x_1 | (e, x_1) \in U\}$.

Definition 2. *Let $\mathcal{F}$ be a family of unary partial functions. A binary partial function F is said to be universal for the family $\mathcal{F}$ if the following conditions hold:*

a) For every fixed $e \in \omega$, $\lambda x_1.F(e, x_1) \in \mathcal{F}$;

b) If $f \in \mathcal{F}$, then there exists e such that $f = \lambda x_1.F(e, x_1)$.

Definition 3. *An enumeration of a structure $\mathfrak{A}$ is any ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k \rangle$ is a partial unary structure on ω and α is a partial surjective mapping of ω onto B such that the following conditions hold:*

(i) $Dom(\alpha) \leq_e \langle \mathfrak{B} \rangle$;

(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ for every $x \in \omega$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ for every $x \in \omega$, $1 \leq j \leq k$.

An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *total* iff $Dom(\alpha) = \omega$.

An enumeration $\langle \alpha, \mathfrak{B} \rangle$ is said to be *effective* iff all functions and predicates in $\mathfrak{B}$ are computable.

Degree spectrum [12, 11] of the structure $\mathfrak{A}$ is the family

$$\mathbf{DS}(\mathfrak{A}) = \{d_e(\langle \mathfrak{B} \rangle) | \langle \alpha, \mathfrak{B} \rangle \text{ is a total enumeration of } \mathfrak{A}\}$$

Partial degree spectrum [11] of a structure $\mathfrak{A}$ is the family

$$\mathbf{PDS}(\mathfrak{A}) = \{d_e(\langle \mathfrak{B} \rangle) | \langle \alpha, \mathfrak{B} \rangle \text{ is an enumeration of } \mathfrak{A}\}.$$

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of the structure $\mathfrak{A}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of $\mathfrak{A}$ if and only if for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n unary functional symbols $\mathbf{f_1}, \ldots, \mathbf{f_n}$ and k unary predicate symbols $\mathbf{T_1}, \ldots, \mathbf{T_k}$. We add a new unary predicate symbol $\mathbf{T_0}$ which will represent the unary total predicate R_0, where $R_0(s) = 0$ for all $s \in B$.

Let us fix some denumerable set $X_1, X_2, \ldots$ of variables. We shall use capital letters X, Y, Z and the same letters by indexes to denote variables.

The definition of a term in the language $\mathcal{L}$ is the usual: every variable is a term; if τ is a term then $\mathbf{f_i}(\tau)$ is a term.

If τ is a term in the language $\mathcal{L}$, then we write $\tau(Y_1, \dots, Y_m)$ to denote that all variables which occur in the term τ are among $Y_1, \dots, Y_m$. If $a_1, \dots, a_m$ are elements of B and $\tau(Y_1, \dots, Y_m)$ is a term, then by $\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ we denote the value of the term τ in $\mathfrak{A}$ over the elements $a_1, \dots, a_m$, if it exists.

Termal predicate in the language $\mathcal{L}$ is defined by the following inductive clauses:

If $\mathbf{T} \in \{\mathbf{T_0}, \dots, \mathbf{T_k}\}$ and τ is a term, then $\mathbf{T}(\tau)$ and $\neg\mathbf{T}(\tau)$ are termal predicates.

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(Y_1, \dots, Y_m)$ be a termal predicate whose variables are among $Y_1, \dots, Y_m$ and let $a_1, \dots, a_m$ be elements of B. The value $\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ of Π over $a_1, \dots, a_r$ in $\mathfrak{A}$ is defined by the following inductive clauses:

If $\Pi = \mathbf{T}_j(\tau)$, $0 \leq j \leq k$, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong R_j(\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m))\,.$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong (\Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \supset 1, 0).$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong (\Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \supset \Pi^2_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m), 1).$$

Formulae of the kind $\exists Y_1' \dots \exists Y_l'(\Pi)$, where Π is a termal predicate are called *conditions*. Every variable which occurs in Π and is different from $Y_1', \dots, Y_l'$ is called free in the condition $\exists Y_1' \dots \exists Y_l'(\Pi)$.

Let $C = \exists Y_1' \dots \exists Y_l'(\Pi)$ be a condition, all free variables in C be among $Y_1, \dots, Y_m$, and $a_1, \dots, a_m$ be elements of B. The value $C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ is defined by the equivalence:

$$C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \iff \exists t_1 \dots \exists t_l (\Pi_{\mathfrak{A}}(Y_1'/t_1, \dots, Y_l'/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

We assume that some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$ is fixed. We shall use τ^v, Π^v, C^v to denote the corespondent one with code v.

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* in the structure $\mathfrak{A}$ if and only if there exists a recursive function γ of $r+1$ variables such that for all $n, x_1, \dots, x_r$, $C^{\gamma(n, x_1, \dots, x_r)}$ is a condition with free variables among $Z_1, \dots, Z_l$, $Y_1, \dots, Y_m$ and for some fixed elements $t_1, \dots, t_l$ of B the following equivalence is true:

$$(x_1, \dots, x_r, a_1, \dots, a_m) \in A \iff$$
$$\exists n \in \omega (C^{\gamma(n, x_1, \dots, x_r)}_{\mathfrak{A}}(Z_1/t_1, \dots, Z_l/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a partial structure. We shall give a generalized version of the normal enumerations [10] and call them normal *pseudo-enumerations.*

Define $f_i^*(p) = \langle i-1, p \rangle$, $i = 1, \dots, n$ and set $N_0 = \omega \backslash (Ran(f_1^*) \cup \dots \cup Ran(f_n^*))$. It is obvious that N_0 is a recursive set and let $\{\mathbf{p_0}, \mathbf{p_1}, \dots\} = N_0$, where $\mathbf{p_i} < \mathbf{p_j}$ if $i < j$.

Let $N_1 \subseteq N_0$. For every partial surjective mapping α^0 of N_1 onto B we define partial mapping α of ω onto B by the following inductive clauses:

(i) If $p \in N_1$, then $\alpha(p) \cong \alpha^0(p)$;

(ii) If $p = f_i^*(q)$, $\alpha(q) \cong a$ and $\theta_i(a) \cong b$, then $\alpha(p) \cong b$.

Let $D_1, \dots, D_n$ be partial predicates such that

$$D_i(x) \cong \begin{cases} 0, & \text{if } \theta_i(\alpha(x)) \text{ is defined,} \\ \text{undefined,} & \text{otherwise;} \end{cases}$$

and $f_1, \dots, f_n$ be partial functions such that

$$f_i(x) \cong \begin{cases} f_i^*(x), & \text{if } D_i(x) \cong 0, \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Let $\sigma_1, \dots, \sigma_k$ be the partial predicates defined by the equalities $\sigma_j(x) \cong R_j(\alpha(x))$, $j = 1, \dots, k$.

Let $\mathfrak{B}$ be the partial structure $\langle \omega; f_1, \dots, f_n; \sigma_1, \dots, \sigma_k \rangle$ and $\mathfrak{B}^*$ be the partial structure $\langle \omega; f_1^*, \dots, f_n^*; \sigma_1, \dots, \sigma_k \rangle$.

Every pair $\langle \alpha, \mathfrak{B} \rangle$ which is obtained by the method described above is called *normal pseudo-enumerations* of the structure $\mathfrak{A}$. The mapping α^0 again is called *basis* of the enumeration $\langle \alpha, \mathfrak{B} \rangle$. It is obvious that α^0 completely determines the normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$. Let us notice that in the general case a normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$ is not an enumeration at all. Nevertheless, we shall see that there are cases where they are enumerations and we shall use them.

In the case $N_1 = N_0$ normal pseudo-enumerations and normal enumerations coincide.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration. We shall reformulate all propositions for normal enumerations and shall formulate several new ones. The proofs are analogous to those for normal enumerations in [10] and we shall give proofs only of those which are different. Let us note that if $\theta_1, \dots, \theta_n$ are total, then the normal enumeration will be total.

Proposition 1. *$Dom(\alpha) \leq_e N_1 \oplus D_1 \oplus \dots \oplus D_n$.*

Proof. The result follows from

$$x \in Dom(\alpha) \iff x \in N_1 \vee \exists x_0 \exists x_1 \dots \exists x_l \exists i_1 \dots \exists i_l (1 \le i_1, \dots, i_l \le n \\ \& x_0 \in N_1 \& x_1 = \langle i_1 - 1, x_0 \rangle \& \dots \& x_l = \langle i_l - 1, x_{l-1} \rangle \\ \& D_{i_1}(x_0) \cong 0 \& \dots \& D_{i_l}(x_{l-1}) \cong 0 \& x_l = x).$$

□

Corollary 1. *If* $N_1 \le_e \langle \mathfrak{B} \rangle$, *then* $Dom(\alpha) \le_e \langle \mathfrak{B} \rangle$.

Proposition 2. *For every* $1 \le i \le n$ *and* $y \in Dom(\alpha)$, $\alpha(f_i(y)) \cong \theta_i(\alpha(y))$.

Corollary 2. *Let* $\tau(Y)$ *be a term, and* $y \in Dom(\alpha)$. *Then*

$$\alpha(\tau_{\mathfrak{B}}(Y/y)) \cong \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Corollary 3. *If* $N_1 \le_e \langle \mathfrak{B} \rangle$, *then the normal pseudo-enumeration* $\langle \alpha, \mathfrak{B} \rangle$ *is an enumeration of the structure* $\mathfrak{A}$.

Proposition 3. *There exists an effective way for every* x *in* ω *to define* $y \in N_0$ *and a term* $\tau(Y)$ *such that* $x = \tau_{\mathfrak{B}^*}(Y/y)$.

We call a term $\tau(X_i)$ *standard* for x if $x = \tau_{\mathfrak{B}^*}(X_i/\mathbf{p_i})$ for some $\mathbf{p_i} \in N$.

Proposition 4. *There exists an effective way for every* x *in* ω *to define an element* $\mathbf{p_i}$ *and a standard term* $\tau(X_i)$ *such that* $x = \tau_{\mathfrak{B}^*}(X_i/\mathbf{p_i})$.

Proposition 5. *Let* $\tau(Y)$ *be a term,* $y \in \omega$, $\langle \alpha, \mathfrak{B} \rangle$ *be a normal pseudo-enumeration and* $\tau_{\mathfrak{B}^*}(Y/y) \in Dom(\alpha)$. *Then* $\tau_{\mathfrak{B}}(Y/y) \cong \tau_{\mathfrak{B}^*}(Y/y)$.

If $\langle \alpha, \mathfrak{B} \rangle$ is a normal pseudo-enumeration, then we shall use the notation

$$R_\alpha := \cup_{j=1}^{k} \{\langle j, x, z \rangle | \sigma_j(x) = z\} \cup \cup_{j=1}^{n} \{\langle j + k, x, z \rangle | D_j(x) = z\}.$$

It is clear that for every $W \subseteq \omega$, $W \le_e R_\alpha$ if and only if $W \le_e \langle \mathfrak{B} \rangle$, i.e. $R_\alpha \equiv_e \langle \mathfrak{B} \rangle$.

Proposition 6. *There exists an effective way for every natural* u *to define elements* $y_1, \dots, y_m \in N_0$ *and a termal predicate* $\Pi(Y_1, \dots, Y_m)$ *such that for every normal pseudo-enumeration* $\langle \alpha, \mathfrak{B} \rangle$:

$$u \in R_\alpha \iff \Pi_{\mathfrak{A}}(Y_1/\alpha(y_1), \dots, Y_m/\alpha(y_m)) \cong 0.$$

Proposition 7. *There exists an effective way for every code* v *of a finite set* E_v *to define elements* $y_1^v, \dots, y_{m_v}^v \in N_0$ *and a termal predicate* $\Pi_v(Y_1, \dots, Y_{m_v})$ *such that for every normal pseudo-enumeration* $\langle \alpha, \mathfrak{B} \rangle$:

$$E_v \subseteq R_\alpha \iff \Pi_{v,\mathfrak{A}}(Y_1/\alpha(y_1^v), \dots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

Therefore, there exists a recursive function γ such that

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(Y_1/\alpha(y_1^v), \ldots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

We call a termal predicate $\Pi^{\gamma(v)}$ *standard* for v in the pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$, if

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

Proposition 8. *There exists a recursive function γ such that for every code v of the finite set E_v to define elements $\mathbf{p_{j_1}}, \ldots, \mathbf{p_{j_{m_v}}}$ and a standard termal predicate $\Pi^{\gamma(v)}(X_{j_1}, \ldots, X_{j_{m_v}})$ such that for every normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$:*

$$E_v \subseteq R_\alpha \iff \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

Lemmas 1 and 2 and Proposition 9 below have analogous proofs, therefore we shall give only the proof of Proposition 9.

Lemma 1. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\tau(Y)$ be a term and $\varphi(y_1) \cong \tau_{\mathfrak{B}}(Y/y_1)$. Then $\langle G_\varphi \rangle \leq_e R_\alpha$.*

Lemma 2. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\tau^v(X_r)$ be a term with code v and $r' \in \omega$. Set $\varphi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \tau_{\mathfrak{B}}^v(X_r/x_r)$, if $r \leq r'$, and $\varphi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \tau_{\mathfrak{B}}^v(X_r/x_{r'})$, if $r > r'$. Then $\langle G_\varphi \rangle \leq_e R_\alpha$.*

Proposition 9. *Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration, $\Pi^v(X_1, \ldots, X_r)$ be a termal predicate or a condition with a code v and $r' \in \omega$. Set*

$$\pi(v, \langle x_1, \ldots, x_{r'} \rangle) \cong \begin{cases} \Pi_{\mathfrak{B}}^v(X_1/x_1, \ldots, X_r/x_r), & \text{if } r \leq r', \\ \Pi_{\mathfrak{B}}^v(X_1/x_1, \ldots, X_{r'}/x_{r'}, \ldots, X_r/x_{r'}), & \text{if } r > r'. \end{cases}$$

Then $\langle G_\pi \rangle \leq_e R_\alpha$.

Proof. We shall consider only the case when π_1 is obtained from C^v by projection, i.e. $C^v \iff \exists X_j \Pi^{\gamma_1(v)}$, where $\Pi^{\gamma_1(v)}$ is a termal predicate and γ_1 is a recursive function. For the sake of simplicity let $j = 1$. Let us assume that for the corresponding function π of $\Pi^{\gamma_1(v)}$ we have $\langle G_\pi \rangle \leq_e R_\alpha$. Then

$$\begin{aligned} \langle v, \langle x_2, \ldots, x_{r'} \rangle, y \rangle \in \langle G_{\pi_1} \rangle &\iff \exists x_1(\langle \gamma_1(v), \langle x_1, x_2, \ldots, x_{r'} \rangle, y \rangle \in \langle G_\pi \rangle) \\ &\iff \exists v_1(\langle \langle \gamma_1(v), \langle x_2, \ldots, x_{r'} \rangle, y \rangle, v_1 \rangle \in W \& E_{v_1} \subseteq \langle G_\pi \rangle), \end{aligned}$$

where

$$W = \{\langle \gamma_1(v), \langle x_2, \ldots, x_{r'} \rangle, y \rangle, v_1 \rangle | E_{v_1} = \{\langle \gamma_1(v), \langle x_1, x_2, \ldots, x_{r'} \rangle, y \rangle\} \text{ for some } x_1\}.$$

Therefore, $\langle G_{\pi_1} \rangle \leq_e \langle G_\pi \rangle \leq_e R_\alpha$. □

In this section we shall give necessary and sufficient conditions for a given unary partial structure to admit a least enumeration.

Let $\mathfrak{A} = \langle B; \theta_1, \dots, \theta_n; R_1, \dots, R_k \rangle$ be a unary partial structure. Let Π^v be a termal predicate with variables among $X_1, \dots, X_m$.

Type of a sequence $b_1, \dots, b_m$ of elements of B is called the set

$$\{v | \Pi^v_{\mathfrak{A}}(X_1/b_1, \dots, X_m/b_m) \cong 0\}.$$

The type of the sequence $b_1, \dots, b_m$ will be denoted by $[b_1, \dots, b_m]_{\mathfrak{A}}$. The type of an element a of B is the type of the sequence a.

Let C^v be a condition with free variables among $X_1, \dots, X_m$. *∃-type* of a sequence $b_1, \dots, b_m$ of elements of B is called the set

$$\{v | C^v(X_1/b_1, \dots, X_m/b_m) \cong 0\}.$$

The ∃-type of the sequence $b_1, \dots, b_m$ is denoted by $\exists[b_1, \dots, b_m]_{\mathfrak{A}}$.

∃-type could be defined in any partial structure. In the case of unary structures we can characterize the ∃-types by types of the elements of B and a fixed set of natural numbers. A condition is said to be *simple* if it does not contain free variables and it is in the form $\exists X_1 \Pi$, where Π is a termal predicate. Let $V_0^{\mathfrak{A}} = \{v | C^v_{\mathfrak{A}} \cong 0 \;\&\; C^v$ is a simple condition $\}$. It is easy to see that the following proposition is true:

Proposition 10. *Let* $\mathfrak{A}$ *be a unary partial structure. Then for any elements* $b_1, \dots, b_m$ *of* B, $\exists[b_1, \dots, b_m]_{\mathfrak{A}} \equiv_e [b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}$.

Lemma 3. *Let* $\mathfrak{A}$ *be a unary partial structure with degree* $\mathbf{a}$*. If there exists an universal set* U *for the family of all types of elements of* B*, then there exists an enumeration of* $\mathfrak{A}$ *which is normal pseudo-enumeration with e-degree* $\mathbf{a}$*.*

Proof. Let U be a universal set for the family of all types of elements of B with e-degree $\mathbf{a}$. By U_x we denote the set $\{v | (x, v) \in U\}$. In fact, for all x, U_x is a type of some element. We can assume that for every type $\mathbf{t}$ of an element of B there exist infinitely many x such that $\mathbf{t} = U_x$. Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal pseudo-enumeration of $\mathfrak{A}$, defined by a basis α^0 satisfying: $\alpha^0(\mathbf{p}_x) = a \iff [a]_{\mathfrak{A}} = U_x$ and $Ran(\alpha^0) = B$.

Then $Dom(\alpha^0) = \{\mathbf{p}_x | \exists v((x, v) \in U)\} \le_e \langle U \rangle = \{\langle x, v \rangle | (x, v) \in U\}$. According to Proposition 9, $\langle U \rangle \le_e R_\alpha$. Therefore, $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration. Furthermore, $\langle \mathfrak{B} \rangle \le_e \langle U \rangle \le_e R_\alpha$. Hence, $deg_e(R_\alpha) = \mathbf{a}$. □

Proposition 11. *Let* $\mathfrak{A}$ *be a unary partial structure. There exists a universal set* U *for the family of all types of elements of* B *with e-degree* $\mathbf{a}$ *iff there exists a universal set* U_1 *for the family of all ∃-types of sequences of elements of* B *with e-degree* $\mathbf{a}$*.*

Proof. Let us first assume that there exists a universal set U for the family of all types of elements of B with e-degree $\mathbf{a}$. According to Lemma 3, there exists a normal pseudo-enumeration $\langle \alpha, \mathfrak{B} \rangle$, which is an enumeration of $\mathfrak{A}$ such that $deg_e(R_\alpha) = \mathbf{a}$. Using Proposition 9, one can see in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ that the family of all $\exists$-types of sequences of elements in the structure $\mathfrak{B}$ has a universal set with degree $\mathbf{a}$ and it is universal set for the family of all $\exists$-types of sequences of B in the structure $\mathfrak{A}$.

To prove the converse, let U_1 be a universal for the set of all $\exists$-types of sequences of elements of B with e-degree $\mathbf{a}$. Then the set $U = \{(x, v) | (x, v) \in U_1 \& \Pi^v$ is a termal predicate with variable $X_1\}$ is universal for the types $[a]_{\mathfrak{A}}$ of all elements a of B and $deg_e(U) \leq \mathbf{a}$. To ensure that there exists a universal set with degree $\mathbf{a}$, let us define the set U' as follows: $U' = U \oplus (A \times U_{x_0})$, where U_{x_0} is a fixed type of an element of B and A is a set of naturals such that $deg_e(A) = \mathbf{a}$. It is obvious that $deg_e(U') = \mathbf{a}$ and U' is a universal set for the set of all types of elements of B. □

Proposition 12. *If $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$, then there exists a universal set U for the family of all types of elements of B with e-degree $\mathbf{a}$.*

Proof. Let $\langle \alpha, \mathfrak{B} \rangle$ be enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$ and $\langle \mathfrak{B} \rangle = \cup_{j=1}^{n} \{\langle j, x, z \rangle | f_j(x) = z\} \cup \cup_{j=1}^{k} \{\langle n + j, x, z \rangle | \sigma_j(x) = z\}$. Define the set U as follows:

$$(x, v) \in U \iff \exists u(\langle \langle x, v \rangle, u \rangle \in W_a \& E_u \subseteq \langle \mathfrak{B} \rangle),$$

where the set W_a is defined as follows:

$$W_a = \{\langle \langle x, v \rangle, u \rangle | \Pi^v = \mathbf{T}_{n_k - n}(\mathbf{f}_{n_{k-1}}(\dots \mathbf{f}_{n_0}(X_1) \dots)) \& E_u = \{\langle n_0, x, y_0 \rangle, \langle n_1, y_0, y_1 \rangle, \dots, \langle n_{k-1}, y_{k-1}, y_k \rangle, \langle n_k, y_k, 0 \rangle\}\}$$

for some $n_0, \dots, n_k, y_0, \dots, y_k$.

It is obvious that $U \leq_e \langle \mathfrak{B} \rangle$ and it is easy to see that U is a universal set for the family of all types of all elements of the structure $\mathfrak{B}$. Therefore, it is a universal set for the family of all types of all elements of $\mathfrak{A}$. As in the previous proposition, we may assume that $deg_e(U) = \mathbf{a}$. □

One can easily prove also the following corollaries.

Corollary 4. *If $\langle \alpha, \mathfrak{B} \rangle$ is an enumeration of the unary partial structure $\mathfrak{A}$ with e-degree $\mathbf{a}$, then there exists an enumeration of $\mathfrak{A}$ which is normal pseudo-enumeration with e-degree $\mathbf{a}$.*

Corollary 5. *If the unary partial structure $\mathfrak{A}$ admits a least enumeration, then it admits a least enumeration which is normal pseudo-enumeration.*

Corollary 6. *Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of the unary partial structure $\mathfrak{A}$. Then $deg_e(\langle \mathfrak{B} \rangle)$ is an upper bound of the family of e-degrees of all types ($\exists$-types) of the elements of B.*

Proof. Let $deg_e(R_\alpha) = \mathbf{a}$. Then, according to Proposition 12, there exists a universal set $U(U_1)$ for the family of all types ($\exists$-types) of $\mathfrak{A}$ and $deg_e(U_1) = deg_e(U) = \mathbf{a}$. It is obvious that for all $b_1, \dots, b_m \in B$, $deg_e([b_i]_{\mathfrak{A}}) \le_e \mathbf{a}$, $i = 1, \dots, m$ and $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}}) \le_e \mathbf{a}$. □

Theorem 1. *Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of an arbitrary partial structure $\mathfrak{A}$ and there do not exist elements $b_1, \dots, b_m$ of B such that $\langle \mathfrak{B}_0 \rangle \le_e \exists[b_1, \dots, b_m]_{\mathfrak{A}}$. Then there is a normal enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$ such that $\langle \mathfrak{B}_0 \rangle \not\le_e R_\alpha$.*

Proof. Let us first mention that this theorem is valid for arbitrary partial structure and we will not use in the proof that it is unary. We shall define the normal enumeration $\langle \alpha, \mathfrak{B} \rangle$ constructing a basis α^0 of N_0 onto B. The construction is step by step. At each step s we define a partial mapping α_s of N_0 into B such that:

(i) $\alpha_s \subseteq \alpha_{s+1}$;

(ii) $Dom(\alpha_s)$ is a finite subset of N_0.

At the end we take $\alpha^0 = \cup_{s=0}^{\infty} \alpha_s$.

With the even steps we ensure that α^0 is totally defined and that $Ran(\alpha^0) = B$, and with the odd steps we ensure that $\langle \mathfrak{B}_0 \rangle \not\le_e R_\alpha$.

Let a_0, a_1,... be an arbitrary enumeration of the set B and let $W = \langle \mathfrak{B}_0 \rangle$. We remind that

$$W \le_e R_\alpha \iff \exists e \forall x (x \in W \longleftrightarrow \exists v(\langle x, v \rangle \in W_e \& E_v \subseteq R_\alpha)) \iff$$

$$\exists e \forall x (x \in W \longleftrightarrow \exists v(\langle x, v \rangle \in W_e \& \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \dots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0))$$

for some standard termal predicate $\Pi^{\gamma(v)}(X_{j_1}, \dots, X_{j_{m_v}})$, some recursive function γ and some $\mathbf{p_{j_1}}, \dots, \mathbf{p_{j_{m_v}}}$. Hence,

$$W \not\le_e R_\alpha \iff$$

$$\forall e \exists x [(x \in W \& \forall v(\langle x, v \rangle \in W_e \to \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \dots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \not\cong 0))$$

$$\vee (x \notin W \& \exists v(\langle x, v \rangle \in W_e \& \Pi_{\mathfrak{A}}^{\gamma(v)}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \dots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0))].$$

In order that $W \not\le_e R_\alpha$ we need to satisfy at least one of the two disjunctive members. In case we are able to satisfy the second member, we do it and the construction on that step will be completed. Otherwise we shall see that the first member will be satisfied automatically.

Step s=-1. $Dom(\alpha_{-1}) = Ran(\alpha_{-1}) = \emptyset$.

Step s=2e. Let x be the least element of N_0 such that $x \notin Dom(\alpha_{s-1})$ and a be the first element in the sequence a_0, $a_1, \ldots$, such that $a \notin Ran(\alpha_{s-1})$. Set $\alpha_s(x) = a$ and $\alpha_s(y) = \alpha_{s-1}(y)$, for $y \in Dom(\alpha_{s-1})$.

Step s=2e+1. Let $Dom(\alpha_{s-1}) = \{x_0, \ldots, x_l\}$ and $c_i = \alpha_{s-1}(x_i)$, $i = 0, \ldots, l$. For every x we consider all v such that $\langle x, v\rangle \in W_e$.

There exists an effective way to find a standard termal predicate $\Pi^{\gamma(v)}(X_{j_1},\ldots,X_{j_{m_v}})$ such that

$$E_v \subseteq R_\alpha \iff \Pi^{\gamma(v)}_{\mathfrak{A}}(X_{j_1}/\alpha(\mathbf{p_{j_1}}), \ldots, X_{j_{m_v}}/\alpha(\mathbf{p_{j_{m_v}}})) \cong 0.$$

For the sake of simplicity, let us assume that $x_0 = \mathbf{p_0}, \ldots, x_l = \mathbf{p_l}$ and the list $X_0, \ldots, X_l, X_{l+1}, \ldots, X_{l+m}$ coincides with the list $X_{j_1}, \ldots, X_{j_{m_v}}$.

Then $E_v \subseteq R_\alpha \iff$

$$\Pi^{\gamma(v)}_{\mathfrak{A}}(X_0/\alpha(\mathbf{p_0}), \ldots, X_l/\alpha(\mathbf{p_l}), X_{l+1}/\alpha(\mathbf{p_{l+1}}), \ldots, X_{l+m}/\alpha(\mathbf{p_{l+m}}))) \cong 0.$$

Let $C^{v_1}_{\mathfrak{A}}(X_0, \ldots, X_l)$ be the condition $\exists X_{l+1} \ldots \exists X_{l+m}(\Pi^{\gamma(v)})$. We check whether there exist natural numbers $x \notin W$ and v, such that $\langle x, v\rangle \in W_e$ and $C^{v_1}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0$. If this is the case, we choose the least such v and find $b_1, \ldots, b_m$ such that

$$\Pi^{\gamma(v)}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l}), X_{l+1}/b_1, \ldots, X_{l+m}/b_m) \cong 0.$$

Set $\alpha_s(\mathbf{p_{l+j}}) = b_j$, $j = 1, \ldots, m$, $\alpha_s(y) = \alpha_{s-1}(y)$, for $y \in Dom(\alpha_{s-1})$. Otherwise, we do nothing, i.e. set $\alpha_s = \alpha_{s-1}$.

The construction is completed.

We continue proof of the theorem with a few auxiliary lemmas.

Lemma 4. *Let $C^{v_1}(X_0, \ldots, X_l)$ be the condition $\exists X_{l+1} \ldots \exists X_{l+m}(\Pi^{\gamma(v)})$ on step $s = 2e+1$ and there are no natural numbers $x \notin W$ and v, such that $\langle x, v\rangle \in W_e$ and $C^{v_1}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0$.*

Then there exists $x \in W$ such that for every v satisfying $\langle x, v\rangle \in W_e$ the conditional inequality

$$C^{v_1}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \ncong 0$$

holds.

Proof. Let us mention that

$$\forall x(\exists v(\langle x, v\rangle \in W_e \& C^{v_1}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0) \longrightarrow x \in W).$$

If we assume that

$$\forall x(x \in W \longrightarrow \exists v(\langle x, v\rangle \in W_e \& C^{v_1}_{\mathfrak{A}}(X_0/\alpha_{s-1}(\mathbf{p_0}), \ldots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0)),$$

then we would obtain that

$$\forall x(x \in W \longleftrightarrow \exists v(\langle x, v\rangle \in W_e \& C_{\mathfrak{A}}^{v_1}(X_0/\alpha_{s-1}(\mathbf{p_0}), \dots, X_l/\alpha_{s-1}(\mathbf{p_l})) \cong 0)).$$

Having in mind that we have obtained v_1 effectively from v, we conclude that $W \leq_e \exists[\alpha_{s-1}(\mathbf{p_0}), \dots, \alpha_{s-1}(\mathbf{p_l})]_{\mathfrak{A}}$ by index e, which contradicts the assumption of the theorem. □

The following lemma and corollary are obvious.

Lemma 5. *α^0 is a totally defined on N^0 surjective mapping.*

Corollary 7. *If all functions in the structure $\mathfrak{A}$ are total, then the normal enumeration $\langle \alpha, \mathfrak{B}\rangle$ is a totally defined surjective mapping.*

Let us assume now that $\langle \mathfrak{B}_0\rangle = W \leq_e R_\alpha$ by some index e. Then on step $s = 2e+1$ we have satisfied first or second disjunctive member of the right part of the non-equivalence $W \not\leq_e R_\alpha$, which contradicts the assumption. Theorem 1 is proved. □

The following corollary is obvious.

Corollary 8. *Let $\mathfrak{A}$ be a unary partial structure. If $\mathfrak{A}$ admits a least enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$, then there exist elements $b_1, \dots, b_m$ of B such that $\langle \mathfrak{B}_0\rangle \leq_e \exists[b_1, \dots, b_m]_{\mathfrak{A}}$.*

Theorem 2. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

Proof. Let us assume first that $\mathfrak{A}$ admits a least enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$. According to Corollary 8, there exist $b_1, \dots, b_m$ in B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B. By Proposition 12, there exists a universal set U of types, such that $deg_e(U) = deg_e(\langle \mathfrak{B}_0\rangle) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$.

Conversely, assume that there exist elements $b_1, \dots, b_m$ of B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types such that $deg_e(U) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$.

According to Lemma 3 there exists an enumeration $\langle \alpha_0, \mathfrak{B}_0\rangle$ of $\mathfrak{A}$ such that $deg_e(\langle \mathfrak{B}_0\rangle) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ and $\langle \alpha_0, \mathfrak{B}_0\rangle$ is the least enumeration of the structure $\mathfrak{A}$. □

Let us assume that $\mathfrak{A}$ is a unary partial structure and there exist elements $b_1, \dots, b_m$ of B such that $deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ is the least upper bound of the

e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e(\exists[b_1, \dots, b_m]_{\mathfrak{A}})$ and let us fix $A = \exists[b_1, \dots, b_m]_{\mathfrak{A}}$. Therefore, there exists an enumeration operator W_z such that $W_z(A) = \langle U \rangle$, i.e. for all natural x, u the following equivalence is true:

$$\langle x, u \rangle \in \langle U \rangle \iff \exists v(\langle \langle x, u \rangle\, v \rangle \in W_z \& E_v \subseteq A).$$

Using the S^m_n-theorem, we can find for a fixed z a recursive function h such that

$$\langle x, u \rangle \in \langle U \rangle \iff \exists v(\langle u, v \rangle \in W_{h(x)} \& E_v \subseteq A) \iff u \in W_{h(x)}(A),$$

i.e. the sequence $W_{h(0)}(\exists[b_1, \dots, b_m]_{\mathfrak{A}}), W_{h(1)}(\exists[b_1, \dots, b_m]_{\mathfrak{A}}), \dots$ is the sequence of all types of the elements of B. The converse is trivial. Thus we obtained the following

Corollary 9. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B and computable sequence of enumeration operators $W_{z_0}, W_{z_1}, \dots$ such that the family $\{W_{z_n}([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})\}_{n \in \omega}$ is the family of all types of elements of B.*

In order to formulate the corresponding corollaries for the case when the unary structures are total, we call *a type of some element* a the set $[a]_{\mathfrak{A}} \oplus (\omega \setminus [a]_{\mathfrak{A}})$, or equivalently $([a]_{\mathfrak{A}} \times \{0\}) \cup ((\omega \setminus [a]_{\mathfrak{A}}) \times \{1\})$, which is the graph of the characteristic function of the set $[a]_{\mathfrak{A}}$. Let us remind that a set A is total if and only if $A \equiv_e A \oplus (\omega \setminus A)$ and that an e-degree is total if it contains a total set. The following corollaries are obvious and we omit their proofs.

Corollary 10. *Let $\mathfrak{A}$ be a unary total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}))$ is a total e-degree which is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists universal function F for the characteristic functions of all types, such that $deg_e(F) = deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

Corollary 11. *Let $\mathfrak{A}$ be a unary total structure. Then $\mathfrak{A}$ admits a least total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if there exist elements $b_1, \dots, b_m$ of B and computable sequence of recursive operators $W_{z_0}, W_{z_1}, \dots$ such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}}) = deg_e(A)$ for some total set A and the function $\lambda n \lambda u. W^A_{z_n}(u)$ is universal function for the family of the characteristic functions of all types of elements of $\mathfrak{A}$.*

Corollary 12. *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits an effective enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if all $\exists$-types of the elements of B are computably enumerable and there exists r.e. universal set U of all types of elements of $\mathfrak{A}$.*

The following corollaries are related to [4, 5].

Corollary 13. *Let* $\mathfrak{A}$ *be a unary total structure. Then* $\mathfrak{A}$ *admits an effective total enumeration* $\langle \alpha_0, \mathfrak{B}_0 \rangle$ *if and only if all* $\exists$*-types of the elements of* B *are computably enumerable and there exists recursive universal function* F *of all types of elements of* $\mathfrak{A}$*.*

Corollary 14. *Let* $\mathfrak{A}$ *be a unary partial structure. Then* $\mathfrak{A}$ *admits an effective enumeration* $\langle \alpha_0, \mathfrak{B}_0 \rangle$ *if and only if all* $\exists$*-types of the elements of* B *are r.e. and there is a computable sequence of enumeration operators* $W_{z_0}, W_{z_1}, \ldots$ *such that the family* $\{W_{z_n}(\omega)\}_{n\in\omega}$ *is the family of all types of elements of* B*.*

Analogously to Theorem 1, one can prove the following

Theorem 3. *Let for every* $i = 1, \ldots, l$*,* $\langle \alpha_i, \mathfrak{B}_i \rangle$ *be an enumeration of an arbitrary partial structure* $\mathfrak{A}$*, and for every* $i = 1, \ldots, l$ *there do not exist elements* $b_1, \ldots, b_m$ *of* B *such that* $R_{\alpha_i} \leq_e \exists[b_1, \ldots, b_m]_{\mathfrak{A}}$*. Then there is an enumeration* $\langle \alpha, \mathfrak{B} \rangle$ *of* $\mathfrak{A}$ *such that for all* $i = 1, \ldots, l$*,* $R_{\alpha_i} \not\leq_e R_\alpha$*.*

Theorem 4. *Let for every* $i \in \omega$*,* $\langle \alpha_i, \mathfrak{B}_i \rangle$ *be an enumeration of an arbitrary partial structure* $\mathfrak{A}$*, and for every* $i \in \omega$ *there do not exist elements* $b_1, \ldots, b_m$ *of* B *such that* $R_{\alpha_i} \leq_e \exists[b_1, \ldots, b_m]_{\mathfrak{A}}$*. Then there is an enumeration* $\langle \alpha, \mathfrak{B} \rangle$ *of* $\mathfrak{A}$ *such that for all* $i \in \omega$*,* $R_{\alpha_i} \not\leq_e R_\alpha$*.*

Proof. We only sketch the proof: At even steps we will ensure the enumeration $\langle \alpha, \mathfrak{B} \rangle$ to be total and surjective. At steps of the kind $2\langle e, i \rangle + 1$ we will ensure, as in Theorem 1, that $R_{\alpha_i} \not\leq_e R_\alpha$ by index e. □

Corollary 15. *There doesn't exist a spectrum of a partial structure with denumerable minimal elements.*

Proof. Obvious. □

4. SOME CONSEQUENCES

As in [12], we can prove that for a unary partial structure $\mathfrak{A}$ the partial degree spectrum of $\mathfrak{A}$ is closed upward with respect to arbitrary e-degrees. As a special case we shall obtain that the degree spectrum is closed upward with respect to the total e-degrees, as well.

Proposition 13. *Let* $\langle \alpha, \mathfrak{B} \rangle$ *be an enumeration of the unary partial structure* $\mathfrak{A}$ *and* $deg_e(R_\alpha) \leq_e A$*. Then there exists an enumeration* $\langle \alpha_0, \mathfrak{B}_0 \rangle$ *of* $\mathfrak{A}$ *such that* $deg_e(\langle \mathfrak{B}_0 \rangle) = deg_e(A)$*.*

Proof. Let a be an element of B such that at least one function of $\mathfrak{A}$ is defined on A. Define an enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ which is a normal pseudo-enumeration as follows:

$$\alpha_0^0(\mathbf{p_i}) \cong \begin{cases} a, & \text{if } i \text{ is even } \& \frac{i}{2} \in A, \\ \alpha(\frac{i-1}{2}), & \text{if } i \text{ is odd}, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

It is easy to see that $deg_e(\langle \mathfrak{B}_0 \rangle) = deg_e(R_\alpha \oplus (A \oplus [a]_\mathfrak{A})) = deg_e(A)$. □

Analogously one can prove

Proposition 14. *Let $\langle \alpha, \mathfrak{B} \rangle$ be an enumeration of the unary partial structure $\mathfrak{A}$ such that $\theta_1, \dots, \theta_n$ are total and $deg_e(R_\alpha) \le_e A$, where A is a total set. Then there exists a total enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ of $\mathfrak{A}$ such that $deg_e(\langle \mathfrak{B}_0 \rangle) = deg_e(A)$.*

Proposition 15. *Let **a** be an arbitrary e-degree. Then there exists a unary partial structure $\mathfrak{A} = \langle B; \theta_1; R_1, R_2 \rangle$ with total function θ_1, such that $\mathfrak{A}$ has a least enumeration with e-degree **a**.*

Proof. Let A be an arbitrary set of natural numbers, such that $deg_e(A) = \mathbf{a}$. The idea of constructing such structure is the following. We take infinite disjoint copies of natural numbers with successor functions on all of them. Then on one of them we take copy of the set A and on the remaining infinite copies we ensure the codes of all existential formulas, which are true in the structure $\mathfrak{A}$ will be recursive and all types of those elements in that copies will be finite, hence recursive. Take $B = \{a_0, a_1, \dots, b_0^0, b_1^0, \dots, b_0^1, b_1^1, \dots\}$, where all $a_0, a_1, \dots, b_0^0, b_1^0, \dots, b_0^1, b_1^1, \dots$ are different. Set $\theta_1(a_n) = a_{n+1}$, $\theta_1(b_n^i) = b_{n+1}^i$ for all natural i, n; set $R_1(a_0) = R_1(b_0^i) = 0$ for all natural i while $R_1(a_n)$ and $R_1(b_n^i)$ are undefined for all natural i and positive n. Further,

$$R_2(a_n) \cong \begin{cases} 0, & \text{if } n \in A, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and $\Pi_\mathfrak{A}^v(X_j/b_0^v) \cong 0$ for all $v \in \omega$ such that the only predicate symbols and variables which occur in Π^v is a termal are $\mathbf{T}_2$ and X_j. Moreover, let R_2 be defined on the smallest finite subset of $\{b_0^v, b_1^v, \dots\}$ which guarantee that $\Pi_\mathfrak{A}^v(X_1/b_0^v) \cong 0$. Thus, the types $[b_i^j]_\mathfrak{A}$ will be finite sets and will ensure that the set of all $\exists$-types is recursive. Indeed, a closed condition of the type $\exists X_j \Pi^v$ is true on the structure $\mathfrak{A}$ if and only if $\Pi^v = \mathbf{T}_1(X_j)\&\Pi^{v'}$, where $\Pi^{v'}$ is an arbitrary termal predicate with predicate symbol $\mathbf{T}_2$ and variable X_j.

Since $deg(\mathfrak{A}) = \mathbf{a}$ it is easy to see that for all positive n

$[a_n]_\mathfrak{A} \equiv_e [a_0]_\mathfrak{A} \equiv_e \{m | R_1(a_0) = 0 \ \& \ R_2(\theta_1^m(a_0)) = 0\} \equiv_e A$. □

Proposition 16. *Let **a** be an arbitrary T-degree. Then there exists unary total structure $\mathfrak{A} = \langle B; \theta_1; R_1, R_2 \rangle$, such that $\mathfrak{A}$ has a least total enumeration with T-degree **a**.*

Proof. Let A be an arbitrary set of natural numbers, such that $deg_T(A) = \mathbf{a}$. The idea is the same as in the previous proposition: we take B and θ_1 to be the same as in the previous proposition; take $R_i(a) = 0$ whenever $R_i(a) = 0$ in the previous proposition and $R_i(a) = 1$ whenever in the previous proposition $R_i(a)$ is undefined, $i = 1, 2$. □

Analogously one can prove the following

Proposition 17. *Let* $\mathbf{A}$ *be a denumerable set of e-(T-)degrees. Then there exists a unary partial(total) structure* $\mathfrak{A} = \langle B; \theta_1; R_1, R_2 \rangle$ *with totally defined function* θ_1*, such that the set of the e-(T-)degrees of all types of* $\mathfrak{A}$ *coincides with the set* $\mathbf{A} \cup \{0\}$.

Proof. We consider only the case of T-degrees. Let $\mathbf{A} = \{a_i\}_{i \in I}$ for some countable index set I and A_i be an arbitrary set of natural numbers, such that $deg_e(A_i) = \mathbf{a_i}$ for any $i \in I$. Take $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i \rangle$ such that $deg_e(A_i) = \mathbf{a_i}$ for any $i \in I$ and all types of elements of B_i are finite or a_i. Assume that $B_i \cap B_j \neq \emptyset$ for all $i, j \in I$, $i \neq j$ and let $B = \cup_{i \in I} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in I$ and $j = 1, 2$. Then it is obvious that all type of B form the set $\mathbf{A} \cup \{0\}$. □

This proposition shows how to construct a various structures with or without degree. At the same time it shows that we can construct structures which contain different independent structures.

Proposition 18. *Let us consider the family of all recursive sets. There exists a unary total structure* $\mathfrak{A}_\mathbf{0} = \langle B; \theta_1; R_1, R_2 \rangle$*, such that the family of all types of elements of* B *coincides with the family of copies of all recursive sets (or with the characteristic functions of copies of all recursive sets).*

Proof. Let $A_0, A_1, \ldots$ be a sequence of all recursive sets. As above, for any recursive set A_i we take an independent copy $B_i = \{a_0^i, a_1^i, \ldots\}$ of the set of natural numbers and a total function successor θ_1^i such that $\theta_1^i(a_n^i) = a_{n+1}^i$ for all $i, n \in \omega$. Then we take $R_1^i(a_0^i) = 0$ and $R_1^i(a_n^i) = 1$ for all positive n;

$$R_2^i(a_n^i) \cong \begin{cases} 0, & \text{if } n \in A_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here R_1^i defines "zeros" and R_2^i defines a copy of the set A_i. Take the structure $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i \rangle$ for all i and assume that $B_i \cap B_j = \emptyset$ for all $i, j \in \omega$, $i \neq j$ and let $B = \cup_{i \in \omega} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in \omega$ and $j = 1, 2$. It is obvious that all types of elements of B of the structure $\mathfrak{A}_\mathbf{0} = \langle B; \theta_1; R_1, R_2 \rangle$ are recursive sets and are copies of all recursive sets. Moreover, the set $V_0^{\mathfrak{A}}$ is recursive. Therefore, the least upper bound of all degrees of $\exists$-types is 0. If we assume that the structure $\mathfrak{A}$ admits least enumeration, then we would obtain that the family of all recursive set has a universal recursive set. This is a contradiction, which shows that we cannot omit the condition with universal set (function) in Theorem 3. □

Question 1. What is $\mathbf{DS}(\mathfrak{A}_\mathbf{0})$?

The next definition belongs to Soskov [11]. Let W be a set of natural numbers. It is said that $d_e(W)$ is a quasi-degree of the structure $\mathfrak{A}$ if for all sets $A \subseteq \omega^m$ the following equivalence is true:

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Let us mention that this definition is not the original, but it is equivalent to the original one.

Proposition 19. *There exists a class of unary partial structures and sets of natural numbers W such that for all sets $A \subseteq \omega^m$ the following equivalence is true:*

$$A \text{ is } \exists\text{−definable in } \mathfrak{A} \iff A \leq_e W.$$

Proof. Let $\mathfrak{A}$ be a unary partial structure such that there exist elements $b_1, \dots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}})$ is the least upper bound of e-degrees of all types of elements of B and $W = [b_1]_{\mathfrak{A}} \oplus \dots \oplus [b_m]_{\mathfrak{A}}$. As in the previous propositions, take enough copies of natural numbers such that all types of the new elements to be finite and all (or recursive set of all) simple conditions to be true on those new elements and denote the new structure by $\mathfrak{A}'$. For the sake of simplicity let assume that $\mathfrak{A}' = \mathfrak{A}$. It is easy to see that $\mathfrak{A}'$ satisfies the required condition. Indeed, let A be $\exists$-definable in $\mathfrak{A}$, i.e. there exists recursive function γ of $m+1$ variables, having values in the set of all codes of conditions with free variables among $X_1, \dots, X_l$ such that for some elements $b'_1, \dots, b'_l$ the following equivalence is true:

$$(x_1, \dots, x_m) \in A \iff \exists n \in \omega(C^{\gamma(n,x_1,\dots,x_m)}(X_1/b'_1, \dots, X_l/b'_l) \cong 0).$$

Let us represent the condition $C^{\gamma(n,x_1,\dots,x_m)}(X_1, \dots, X_l) = C^{\gamma(n,\overline{x})}(X_1, \dots, X_l)$ in the form $\Pi^{\gamma_1(n,\overline{x})}(X_1)\& \dots \&\Pi^{\gamma_l(n,\overline{x})}(X_l)\&C^{\gamma_{l+1}(n,\overline{x})}$, where $C^{\gamma_{l+1}(n,\overline{x})}$ is a simple condition and all $\gamma_1, \dots, \gamma_{l+1}$ are recursive. Then, $[b'_i]_{\mathfrak{A}} \leq_e W$ and for some W_{z_i}, the following equivalence holds:

$$z \in [b'_i]_{\mathfrak{A}} \iff \exists v_i(\langle z, v_i\rangle \in W_{z_i}\&E_{v_i} \subseteq W), i = 1, \dots, m.$$

Therefore,

$$\overline{x} \in A \iff \exists n(\gamma_1(n,\overline{x}) \in [b'_1]_{\mathfrak{A}}\& \dots \&\gamma_l(n,\overline{x}) \in [b'_l]_{\mathfrak{A}}\&\gamma_{l+1}(n,\overline{x}) \in V_0^{\mathfrak{A}})$$

$$\iff \exists n(\gamma_1(n,\overline{x}) \in [b'_1]_{\mathfrak{A}}\& \dots \&\gamma_l(n,\overline{x}) \in [b'_l]_{\mathfrak{A}}) \iff$$

$$\exists n(\exists v_1(\langle\gamma_1(n,\overline{x}), v_1\rangle \in W_{z_1}\&E_{v_1} \subseteq W)\& \dots \&\exists v_l(\langle\gamma_l(n,\overline{x}), v_l\rangle \in W_{z_l}\&E_{v_l} \subseteq W))$$

$$\iff \exists v(\langle\langle\overline{x}\rangle, v\rangle \in W_z\&E_v \subseteq W),$$

where $\langle\langle\overline{x}\rangle, v\rangle \in W_z \iff$

$$\exists n(\exists v_1(\langle\gamma_1(n,\overline{x}), v_1\rangle \in W_{z_1})\& \dots \&\exists v_l(\langle\gamma_l(n,\overline{x}), v_l\rangle \in W_{z_1}\&E_v = E_{v_1} \cup \dots \cup E_{v_l}).$$

The converse, i.e. if $A \leq_e W$, then A is $\exists$-definable in $\mathfrak{A}$ is obvious. □

Proposition 20. *Let A be an arbitrary set of natural numbers with $deg_T(A) = \mathbf{a}$ and let us consider the family of all recursive in A sets. There exists a unary total structure $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2\rangle$, such that the family of all types of elements of B coincides with the family of copies of all recursive in A sets (or with the characteristic functions of copies of all recursive in A sets).*

Proof. Let $A_0, A_1, \dots$ be the sequence of all recursive in A sets. As above, for any recursive in A set A_i we take independent copy of set of natural numbers $B_i = \{a_0^i, a_1^i, \dots\}$ and a total function successor θ_1^i such that $\theta_1^i(a_n^i) = a_{n+1}^i$ for all $i, n \in \omega$. Then take $R_1^i(a_0^i) = 0$ and $R_1^i(a_n^i) = 1$, for all positive n;

$$R_2^i(a_n^i) \cong \begin{cases} 0, & \text{if } n \in A_i, \\ 1, & \text{otherwise.} \end{cases}$$

Here again R_1^i defines zero and R_2^i defines a copy of the set A_i. Take the structure $\mathfrak{A}_i = \langle B_i; \theta_1^i; R_1^i, R_2^i\rangle$ for all i and assume that $B_i \cap B_j = \emptyset$ for all $i, j \in \omega$, $i \neq j$ and let $B = \cup_{i\in\omega} B_i$. Then $\theta_1(a) = \theta_1^i(a)$ and $R_j(a) = R_j^i(a)$ if $a \in B_i, i \in \omega$ and $j = 1, 2$. Then it is obvious that all types of elements of B of the structure $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2\rangle$ are recursive in A sets and are copies of all recursive in A sets. Moreover, the set $V_0^{\mathfrak{A}}$ is recursive. Therefore the least upper bound of all degrees of $\exists$-types is $\mathbf{a}$. If we assume that the structure $\mathfrak{A}$ admits a least enumeration, then we would obtain that the family of all recursive sets in A has a universal recursive in A set. This is a contradiction, which shows that this structure $\mathfrak{A}_{\mathbf{a}}$ does not admit a least enumeration. At the same time it satisfies the condition of the previous proposition. Therefore, $\mathfrak{A}_{\mathbf{a}}$ has quasi-degree $\mathbf{a}$. □

Thus, we proved also the following

Corollary 16. *There exists a unary total structures $\mathfrak{A}_{\mathbf{a}} = \langle B; \theta_1; R_1, R_2\rangle$, such that $\mathfrak{A}_{\mathbf{a}}$ has a quasi-degree but does not have a least enumeration.*

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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ON INFINITE DIMENSIONAL HOMOGENEOUS SPACE

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In this paper we show that if G is a locally compact group with H closed and $H \leq G$ such that $\dim G/H < \infty$, then G/H contains a copy of $I^{\omega_0(G/H)}$, where $\omega_0(G/H) =$ weight of a connected component of G/H, except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ [13].

Keywords: Weight and local weight, homogeneous spaces of locally compact groups, Tychonoff cube

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1. INTRODUCTION

In this paper we investigate the existence of Tychonoff cubes of maximal weight in homogeneous spaces of locally compact groups of infinite covering dimension. We show that if G is a locally compact group with H closed$\leq G$ such that $G/H = \infty$, then G contains a copy of $I^{\omega_0(G/H)}$, where $\omega_0(G/H)$ = weight of a connected component of G/H except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ [13]. This result, except for the last exceptional case, was observed before [9, 16]. The proof for the locally compact case in [9, Theorem 4.2] is incorrect. The elegant proof in [16] contains a gap, we fix that proof here.

Throughout this paper we fix the following notations. If G is a locally compact group, G/G_0 compact, then $G = \varprojlim G_j$, G_j's finite dimensional Lie groups, $j \in J$ [14, p. 175]. Let $p_j : G \to G_j$ be the canonical map for all $j \in J$. We may assume that ker p_j is compact for all $j \in J$, hence $G = \varprojlim G/\ker p_j$.

The rest of this paper is divided into two sections. In Section 2 we collect some basic lemmas that are needed to establish our result. In Section 3 we prove our main theorem, Theorem 3.1.

2. SOME BASIC LEMMAS

Lemma 2.1. (see [9, Lemma 2.1]) *Let X be a topological space such that $X = \varprojlim\{X_i : i \in J\}$, where $\{X_i : i \in J\}$ is an inverse family of topological spaces, I cofimal $\subseteq J$. Then $\omega(X) \leq \max\{\mathrm{Card}\,(I),\ \omega(X_i) : i \in I\}$, where $\omega(*) =$ weight of the topological space $*$.*

Proof. Let B_i be a basis of X_i, $\mathrm{Card}\,(B_i) = \omega(X_i)$ for all $i \in I$. Then $\{p_i^{-1}(B_i) : i \in I\}$ is a basis of X, where $p_i : X \to X_i$ is the canonical map and

$$\mathrm{Card}\,(\{p_i^{-1}(B_i) \ : \ i \in I\}) \leq \sum_{i \in I} \mathrm{Card}\,(B_i) \leq \max\{\mathrm{Card}\,(I),\, \mathrm{Card}\,(B_i) \ : \ i \in I\}$$

(see [1, E III.49, Corollary 3]). □

Lemma 2.2. (generalizes [7, Theorem 8]) *Let G be a locally compact group, G/G_0 compact, H closed, non-open $\leq G$. Then:*

(i) $\omega(G/H) = 1.\omega(G/H)$ (= local weight of G/H);

(ii) $\omega(G/H) = \omega(G \cap \{gHg^{-1} : g \in G\}$;

(iii) (generalizes [9, Corollary 2.4 ii]) If G is connected and Y compact totally disconnected normal $\leq G$, then $\omega(G/H) = \omega(G/HY)$.

Proof. Let $K = \cap\{gHg^{-1} : g \in G\}$. Since $G/H = (G/K)/(H/K)$ and $\cap\{\overline{g}(H/K)\overline{g}^{-1} : \overline{g} \in G/K\} = 1$, and

$G/HY = (G/K)/(HY/K) = (G/K)/(H/K).(KY/K)$ for Y compact normal $\leq G$,

we may assume that $\bigcap\{gHg^{-1} : g \in H\} = 1$. Let $p : G \to G/H$ be the canonical map.

i. Choose $\{p(V_i) : i \in I\}$ a local basis at H in G/H such that $\mathrm{Card}\,(I) = 1.\omega(G/H) \geq \aleph_0$, since H is non-open, and for each $i \in I$, let $\ker p_i \subseteq V_i$. Then $\bigcap\{\ker p_i : i \in I\} \subseteq H$, hence $\bigcap\{\ker p_i : i \in I\} = 1$ and

$$G/H = \varprojlim\{G/H.\bigcap\{\ker p_i \ : \ i \in F \ \text{ finite } \ \subseteq I\}.$$

Since G is σ-compact, we get $\omega(G/H). \cap \{\ker p_i : i \in F \text{ finite } \subseteq I\}\} \leq \aleph_0$ and $\omega(G/H) \leq 1.\omega(G/H)$, by Lemma 2.1. Hence we have an equality.

ii. **Case 1:** H is compact.

Choose $\{U_j : j \in J\}$ a basis of G/H such that $\mathrm{Card}\,(J) = \omega(G/H)$ and let $\{z_s H : s \in S\}$ be dense $\subseteq G/H$ such that $\mathrm{Card}\,(S) \leq \omega(G/H)$. For all $z \in G$ let $\varphi_z : G \to G/H$ be defined by $\varphi_z(g) = g.z.H$ for all $g \in G$, then

$$\bigcap \{\varphi_{z_s}^{-1}(\overline{U}_j) : z_s H \in U_j,\, j \in J\} = z_s H z_s^{-1}$$

and

$$\begin{aligned}\bigcap \{z_s H z_s^{-1} : s \in S\} &= \bigcap \{\varphi_{z_s}^{-1}(z_s H) : s \in S\} \\ &= \bigcap \{\varphi_z^{-1}(zH) : z \in G\} \qquad [2, \mathrm{TGIII.12, Proposition 12}] \\ &= \bigcap \{zHz^{-1} : z \in G\} = 1.\end{aligned}$$

It follows by the compactness of H that the family of finite intersections of $\{\varphi_{z_s}^{-1}(\overline{U}_j) : z_s H \in U_j,\, j \in J,\, s \in S\}$ is a local basis at $1 \in G$, hence $\omega(G/H) \geq 1.\omega(G) = \omega(G)$, by part i., since G is non-discrete, and we get the desired equality.

Case 2: General case.

Let $\ker q$ be compact normal $\leq G$, $G/\ker q$ Lie group. Then

$$\begin{aligned}\omega(G/H) &= 1.\omega(G/H) && \text{by part } i \\ &= 1.\omega(G/(H \cap \ker q)) && \text{by virtue of the fiber bundle} \\ & && G/(H \cap \ker q) \to G/H \\ &= \omega(G/(H \cap \ker q)) && \text{by part i again} \\ &= \omega(G) && \text{by case 1.}\end{aligned}$$

iii. We have $Y \leq Z(G)$, $Z(G) \cap H = 1$ and since $HY \cong H \times Y$, we get $(\cap\{gHYg^{-1} : g \in G\})_0 \leq H$, hence $\cap\{gHYg^{-1} : g \in G\}$ is totally disconnected and therefore $\leq Z(G)$. It follows that $\cap\{gHYg^{-1} : g \in G\} = Y$ and

$$1 = \cap\{gHYg^{-1} : g \in G\}/Y = \cap\{\varphi(g)HY/Y\varphi(g^{-1}) : g \in G\},$$

where $\varphi : G \to G/Y$ is the canonical map.

Now $\omega(G/H) = \omega(G)$, since $\cap\{gHg^{-1} : g \in G\} = 1$ by part ii, and $\omega(G/HY) = \omega(G/Y)$, since $\cap\{\varphi(g)HY/Y\varphi(g^{-1}) : g \in G\} = 1$ by part ii again. Hence we may assume that $H = 1$.

Note that $\omega(G/Y) = \aleph_0 \Leftrightarrow \omega(G) = \aleph_0$, so we may assume that $\omega(G) > \aleph_0$. Let C be a maximal compact $\leq G$, then

$$\begin{aligned}\omega(G) &= \omega(C) && [12, \text{Theorem 13}],\ \text{since}\ \omega(C) > \aleph_0 \\ &= \omega(C/Y) && [8, \text{Proposition 12.26}] \\ &= \omega(G/Y) && [12, \text{Theorem 13}].\end{aligned}$$

The proof of Lemma 2.2 is complete. ◻

Lemma 2.3. ([17, Theorems 18, 19]) *Let G be a locally compact group, G/G_0 compact, H closed $\leq G$, G/H connected,* $\dim G/H < \infty$. *Let $j \in J$ be such that* $\dim(G/H \ker p_j) = \dim G/H$, *and assume that $\pi_1(G/H \ker p_j)$ is finitely generated. Then $\omega(G/H) \leq \aleph_0$.*

In particular, a connected locally compact finite dimensional group is of countable weight and a compact connected finite dimensional quotient of a locally compact group is of countable weight.

Proof. We have $\dim H \ker p_j/H = 0$ and $H \ker p_j/H \cong \ker p_j/H \cap \ker p_j$ compact. It follows that $\{K/H : H \leq K \text{ closed } \leq H \ker p_j, |H \ker p_j : K| < \infty\}$ is a fundamental system of neighborhoods of H in $H \ker p_j \cap H$.

Note that the function $\{K/H : H \leq K \text{ closed} \leq H \ker p_j, |H \ker p_j : K| < \infty\} \to \{\pi_1(G/H) \leq K \leq \pi_1(G/H \ker p_j) : |\pi_1(G/H \ker p_j) : K| < \infty\}$ defined by $K/H \to (q_K)_\#(\pi_1(G/K))$ is injective, where $q_K : G/K \to G/H \ker p_j$ is the canonical map: if $H \leq K_i$ closed$\leq H\ker p_j$, $|H \ker p_j : K_i| < \infty$, $i = 1, 2$, the exact sequence

$$1 \to \pi_1(G/K_1 \cap K_2) \xrightarrow{(q_{K_1 \cap K_2})_\#} \pi_1(G/H \ker p_j) \xrightarrow{\partial} H \ker p_j/K_1 \cap K_2 \to 1$$

gives $\partial^{-1}(K_i/K_1 \cap K_2) = (q_{K_i})_\#(\pi_1(G/K_i)$. Since $\pi_1(G/H \ker p_j)$ is finitely generated, $\{K \leq \pi_1(G/H \ker p_j) : |\pi_1(G/H \ker p_j) : K| < \infty\}$ is countable, hence $\omega(G/H) = 1.\omega(G/H$ by Lemma 2.2 part i) assuming that H is not open in G $\leq \max\{\aleph_0, 1.\omega(H \ker p_j/H)\} \leq \aleph_0$.

In particular, if G/H is compact, let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]). Note that $G^*/G^* \cap H \cong G^*H/H$ open$\subseteq G/H$ and if $\{a_i : i \in L\}$ is a left transversal of G^* in G, then $G/H = \bigoplus_{i \in L} a_i G^*/G^* \cap H$, so that $G/H = G^*/G^* \cap H$ and we may assume that G/G_0 is compact. Since $G/H \ker p_j$ is a compact manifold, we have $\pi_1(G/H \ker p_j)$ finitely generated, hence $\omega(G/H) \leq \aleph_0$. ◻

Corollary 2.4. (Generalized Wilcox Theorem [11, Theorem 7]) *Let G be a connected locally compact group such that for all $x \in G$, $< \overline{\chi >}$ is metrizable. Then G is metrizable if and only if $1.\omega(G) \leq \aleph_0$.*

Proof. Let $\ker p$ be a compact normal $\leq G$, $G/\ker p$ Lie group. Then $G/(\ker p)_0$ is finite dimensional, hence it is metrizable by Lemma 2.3. Mostert theorem [15] shows that we may assume that G is compact.

Claim 1. ([11, Lemma 1]) $(\mathbf{R}/\mathbf{Z})^{\omega_1} =< \overline{\chi >}$ *for some $x \in (\mathbf{R}/\mathbf{Z})^{\omega_1}$, where ω_1 is the first uncountable ordinal.*

Proof of Claim 1. Let $1 \in H$ be a Hamel basis of $\mathbf{R}$ over $\mathbf{Q}$, so that $\mathbf{R} = \bigoplus_{h \in H} \mathbf{Q}h$ and H is uncountable. Hence there exists $1 \notin \{h_\alpha : \alpha < \omega_1\} \subseteq H$. Now [3, TG VII.7, Corollary 2] shows that $x = (h_\alpha + \mathbf{Z}) \in (\mathbf{R}/\mathbf{Z})^{\omega_1}$ satisfies our claim. □

Case 1: G is abelian.

By [5, Lemma 5.2], there exists a continuous surjective homomorphism $\mathrm{a} : G \to (\mathbf{R}/\mathbf{Z})^{\omega(G)}$ and Claim 1 shows that $\omega(G) \leq \aleph_0$.

Case 2: General case.

If $\omega((Z(G))_0) = \omega(G)$, we are done by Case 1, so we may assume that $\omega((Z(G))_0) < \omega(G)$.

By [4, Theorem 4.2] we have $G/Z(G) = \prod_{i \in I} G_i$, where G_i is compact connected Lie group for all i. Taking a maximal torus in G_i for each $i \in I$, we get that there exists H closed $\leq G$ and a continuous surjective homomorphism $\mathrm{a} : H \to (\mathbf{R}/\mathbf{Z})^{\mathrm{Card}\,(I)}$. Again, as in Case 1, Claim 1 shows that we must have $\mathrm{Card}\,(I) \leq \aleph_0$. Now $\aleph_0 = \omega(G/Z(G)) = \omega(G)$ [4, Corollary 4.3]. □

Remark. (generalizes [10]) Let G be a locally compact group, H closed $\leq G$ such that $\mathrm{Card}\,(G/H) \leq 2^{\aleph_0}$. Then $1.\omega(G/H) \leq \aleph_0$ provided the following cardinal statement holds: $\aleph > \aleph_0 \;\Rightarrow\; 2^{\aleph} > 2^{\aleph_0}$.

Proof of Remark. Let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]), then $G^*/G^* \cap H \cong G^*H/H$ open$\subseteq G/H$ and we may assume that G/G_0 is compact. If $1.\omega(G/H) > \aleph_0$, then

$$\begin{aligned} 2^{\aleph_0} &\geq \mathrm{Card}\,(G/H) \\ &\geq 2^{l.\omega(G/H)} \qquad \text{by Čech–Pospíšil theorem [6, Theorem 3.12.11]}, \end{aligned}$$

which would contradict our hypothesis. □

3. MAIN THEOREM

Theorem 3.1. ([9, 16]) *Let G be a locally compact group, H closed $\leq G$. Then*

$$G/H \supseteq\cong \begin{cases} I^{\dim G/H}, & \text{if } \dim G/H < \infty, \\ I^{\omega_0(G/H)}, & \text{if } \dim G/H = \infty, \end{cases}$$

where $\omega_0(G/H)$ = weight of a connected component of G/H except perhaps when $\aleph_0 \leq \omega_0(G/H) \leq 2^{\aleph_0}$ and $\dim G/H) = \infty$. (In this case we can only guarantee that G/H contains a copy of $I^{\aleph_0}$).

Proof. If $\dim G/H < \infty$, let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]), then $G^*/G^* \cap H \cong G^*H/H$ open $\subseteq G/H$. Hence $\dim G/H =$

$\dim G^*H/H$ and we may assume that G/G_0 is compact. There exists $\ker p$ compact normal $\leq G$, $G/\ker p$ Lie group and $\dim G/H \ker p = \dim G/H$. The fiber bundle $G/H \to G/H \ker p$ proves our assertion in this case.

If $\dim G/H = \infty$, then $\dim G/(G_0H)^-) = 0$ by [2, TGIII.36, Corollary 1], hence $\dim((G_0H)^-/H) = \infty$ and since $\omega_0(G/H) = \omega((G_0H)^-/H)$ ([2, TGIII.36, Corollary 3]), we may assume that G/H is connected.

Let G^* be open $\leq G$, G^*/G_0 compact ([2, TGI.84] and [2, TGIII.36]). Note that if $\{a_j : j \in J\}$ is a complete system of representatives of the double coset decomposition $\{G^*xH : x \in G\}$ of G, then $G/H = \bigoplus_{j\in J} G^*a_jH/H$ and

$$G^*/G^* \cap a_jHa_j^{-1} \cong G^*a_jH/H \text{ open} \subseteq G/H,$$

so that $G^*/G^* \cap H \cong G^*H/H = G/H$ and we may further assume that G/G_0 is compact.

Let $K = \cap\{gHg^{-1} : g \in G\}$, then $G/H = (G/K)/(H/K)$ and we may assume in addition that $\cap\{gHg^{-1} : g \in G\} = 1$.

Let $\ker p$ be a compact normal $\leq G$ such that $G/\ker p$ be Lie group and suppose that $\omega(H(\ker p)_0/H) < \omega(G/H(\ker p)_0)$. Then

$$1.\omega(H(\ker p)_0/H) < 1.\omega(G/H(\ker p)_0)$$

by Lemma 2.2(i), and the fiber bundle $G/H \to G/H(\ker p)_0$ provided by Mostert theorem [15] shows that $1.\omega(G/H) = 1.\omega(G/H(\ker p)_0)$, hence, by Lemma 2.2(i) again, $\omega(G/H) = \omega(G/H(\ker p)_0)$. The fibration $G/(\ker p)_0 \to G/H(\ker p)_0$ induces a surjective map of the arc components $(G/(\ker p)_0)_a \to (G/H(\ker p)_0)_a$, and since $(G/(\ker p)_0)_a$ is Souslin [7, Theorem 7.2], it follows from the fibration $G \to G/H(\ker p)_0$ that $(G/H(\ker p)_0)_a$ is Souslin and dense in $G/H(\ker p)_0$, so the later space is separable. Therefore

$$\aleph_0 \leq \omega(H(\ker p)_0/H) < \omega(G/H(\ker p)_0) = \omega(G/H) \leq 2^{\aleph_0}$$

by [6, Theorem 1.5.7], and this is the exceptional case that should be avoided [13], so we may assume that $\omega(G/H(\ker p)_0) \leq \omega(H(\ker p)_0/H)$. Then the same argument as above shows that $\omega(G/H) = \omega(H(\ker p)_0/H)$, and since $(\ker p)_0/H \cap (\ker p)_0 \cong H(\ker p)_0/H$, we may assume further that G is compact connected.

Therefore we reduced our theorem just to the case of G compact connected group, H closed $\leq G$, $\dim G/H = \infty$ and $\cap\{gHg^{-1} : g \in G\} = 1$.

Let θ be the minimum ordinal such that $\operatorname{Card}\theta = 1.\omega(G)$ ([1, E III.87, Ex 10]). Let $\{U_\alpha : \alpha \in \theta\}$ be a fundamental system of open neighborhoods of $1 \in G$ and for all $\alpha \in \theta$, let $\ker p_{j_\alpha} \subseteq U_\alpha$. Define a well ordered system of compact normal subgroups of G under inclusion, $\{Y_\alpha : \alpha \in \theta\}$, by: $Y_0 = \ker p_{j_0}$, and for $0 < \alpha \in \theta$, $Y_\alpha = \cap\{\ker p_{j_\beta} : \beta < \alpha\}$ such that G/Y_0 is a non-trivial Lie group, $\cap\{Y_\alpha : \alpha \in \theta\} = 1$, $Y_\alpha/Y_{\alpha+1}$ Lie group. Therefore, we have a well-ordered inverse system $\{G/HY_\alpha : \alpha \in \theta\}$ and $G/H = \varprojlim G/HY_\alpha$. We have:

i. G/HY_0 is a non-trivial Euclidean manifold and $\aleph_0 \leq \omega(G/HY_\alpha)$, $\alpha \in \theta$;

ii. the canonical map $\varphi_{\alpha,\alpha+1} : G/HY_{\alpha+1} \to G/HY_\alpha$ is a fiber bundle with a compact Euclidean manifold as fiber, $\alpha \in \theta$;

iii. if $\alpha \in \theta$ has no predecessor, then $G/HY_\alpha = \varprojlim\{G/HY_\beta : \beta < \alpha\}$.

Suppose that $\operatorname{Card}\theta > \aleph_0$ and assume that there exists $\alpha \in \theta$ with $\omega(G/HY_\alpha) = \omega(G/H)$. Let $\alpha_0 = \min\{\alpha \in \theta : \omega(G/HY_\alpha) = \omega(G/H)\}$, then α_0 has no predecessor, since otherwise $\alpha_0 = \beta + 1$ and

$$\begin{aligned}\omega(G/HY_\beta) &= 1.\omega(G/HY_\beta) && \text{by Lemma 2.2(i)}\\ &= 1.\omega(G/HY_{\beta+1}) && \text{by condition ii. above}\\ &= \omega(G/HY_{\beta+1}) && \text{by Lemma 2.2(i) again.}\end{aligned}$$

Furthermore, $\operatorname{Card}\alpha_0 > \aleph_0$, since otherwise $\omega(G/HY_\alpha) = \aleph_0$ for $\alpha < \alpha_0$ and hence $\operatorname{Card}\theta = 1.\omega(G) = \omega(G) = \omega(G/H) = \omega(G/HY_\alpha) = \aleph_0$ by condition iii.

Applying the principle of transfinite induction ([1, E III.18, C59]) using conditions ii. and iii. and Lemmas 2.1 and 2.2, we get $\omega(G/HY_\alpha) \leq \max\{\aleph_0, \operatorname{Card}\alpha\}$ for $\alpha < \alpha_0$. Hence

$$\operatorname{Card}\theta = 1.\omega(G) = \omega(G) = \omega(G/H) = \omega(G/HY_\alpha) \leq \max\{\aleph_0, \operatorname{Card}\alpha_0\} = \operatorname{Card}\alpha_0,$$

and $\alpha_0 = \theta$. Therefore $\aleph_0 \leq \omega(G/HY_\alpha) < \omega(G/H)$ for all $\alpha \in \theta$, if $\operatorname{Card}\theta > \aleph_0$.

Claim 2. *There holds*

$$\{\alpha \in \theta : \dim(HY_\alpha/HY_{\alpha+1}) > 0\} \quad \textit{cofinal} \quad \subseteq \theta\,.$$

Proof. Assume the contrary, then there would exist $\gamma \in \theta$ such that for all $\gamma \leq \alpha \in \theta$, $|HY_\alpha/HY_{\alpha+1}| < \infty$ and $\dim(HY_\gamma/HY_\beta) = 0$ for all $\gamma \leq \beta \in \theta$ (since otherwise if $\gamma_0 = \min\{\gamma \leq \beta \in \theta : \dim(HY_\gamma/HY_\beta) > 0\}$, then γ_0 would have no predecessor and $HY_\gamma/HY_{\gamma_0} = \varprojlim\{HY_\gamma/HY_\beta : \gamma \leq \beta < \gamma_0\}$, hence $\dim(HY_\gamma/HY_{\gamma_0}) = 0$, which is absurd.

We have $HY_\gamma/H = \varprojlim\{HY_\gamma/HY_\beta : \gamma \leq \beta \in \theta\}$. Hence $\dim HY_\gamma/H = 0$. Since $\dim G/H = \infty$, we must have $\operatorname{Card}\theta > \aleph_0$. We have $(Y_\gamma)_0 \leq H$. Hence $(Y_\gamma)_0 = 1$ and Y_γ is totally disconnected. Lemma 2.2(iii) shows that $\omega(G/H) = \omega(G/HY_\gamma)$, which is absurd. □

Claim 3. *There holds*

$$\operatorname{Ord}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta/HY_{\beta+1}| < \infty\}) = \theta\,.$$

Proof. Since $\operatorname{Ord}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta / HY_{\beta+1}| < \infty\}) \leq \theta$, it suffices to show that $\operatorname{Card}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta / HY_{\beta+1}| < \infty\}) = \operatorname{Card}\theta$. If $\operatorname{Card}\theta = \aleph_0$, this is clear from Claim 2. If $\operatorname{Card}\theta > \aleph_0$, then

$$\begin{aligned}\operatorname{Card}\theta &\geq \operatorname{Card}(\theta_{-}(\alpha \in \theta : \alpha = \beta + 1, |HY_\beta / HY_{\beta+1}| < \infty\} \\ &\geq \operatorname{Card}(\{\alpha \in \theta : \alpha \text{ has no predecessor }\}) = \operatorname{Card}\theta,\end{aligned}$$

since $\theta = \bigcup_{n \geq 0} \{\alpha + n \in \theta : \alpha \text{ has no predecessor }\}$ (disjoint union). □

By Claims 2 and 3 we may further assume that $\dim(HY_\gamma / HY_{\gamma+1}) > 0$ for all $\gamma \in \theta$.

An application of the principle of transfinite induction ([1, E III.18, C 59]) shows that for all $\alpha \in \theta$, $G/HY_\alpha \supseteq I^\alpha$ such that $\alpha \leq \beta \in \theta$, $\varphi_{\alpha,\beta}| : I^\beta \to I^\alpha$ is equivalent to the projection map onto the first factor by virtue of conditions ii and iii. We get $G/H = G/HY_\theta \supseteq I^\theta$ as desired since $\operatorname{Card}\theta = 1.\omega(G) = \omega(G) = \omega(G/H)$. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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SOME PROPERTIES OF AN ALGEBRA OF ALL SETS OF NATURALS E-REDUCIBLE TO A FIXED SET

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In this paper we consider the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$, where A is an arbitrary fixed set of natural numbers, $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, $W_0, W_1, \ldots$ is the sequence of all computably enumerable sets, considered as e-operators, and Non is the predicate detecting non-emptiness. It is shown that for any set of natural numbers A the algebra $\mathfrak{N}^A$ has a least enumeration, admits equivalent representation with 3 operators and is finitely generated.

Keywords: Enumeration, enumeration degree, enumeration operator, degree of a structure, least degree of a structure, algebra.

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1. INTRODUCTION

In attempts to classify the family of all sets of naturals with respect to effective computability, different kinds of reducibilities have been introduced. In [8] Post first introduced the so-called "strong" reducibilities (m-,tt-,...) and later on in [9] – the Turing reducibility.

Every reducibility defines a pre-order. Thus in a natural way m-degrees, T-degrees, etc. have been introduced. Enumeration reducibility was introduced in 1959 by Friedberg and Rogers [5]. In [7] embedding of the semi-lattice of Turing degrees (T-degrees) into the semi-lattice of enumeration degrees (e-degrees) was found.This fact showed that two semi-lattices are closely related and any result or question about one of them triggered a question of validity for the other. In

1966 Sacks [12] and in 1967 Rogers [11] stated the basic question about T-degrees, namely whether there exist non-trivial automorphisms in the upper semi-lattice of T-degrees. In case that such non-trivial automorphisms do not exist, we say that the upper semi-lattice is rigid. The same question was stated for e-degrees, m-degrees, etc. This question is important because it is connected with definability in these semi-lattices. For m-degrees it was shown by Shore that there exist $2^{2^{\aleph_0}}$ automorphisms.

In 1977 Jockusch and Solovay [6] and in 1979 Richter [10] and Epstein [4] proved that for Turing degrees every automorphism is the identity on the cone above $0^{(3)}$. In 1986 Slaman and Woodin [13] improved the above result by showing that every automorphism is the identity on the cone above $0''$. Using the connections between both T- and e-jumps, Soskov and Ganchev [15] proved that for e-degrees every automorphism is the identity on the cone above $0^{(4)}$.

Since the upper semi-lattice of all e-degrees (e-degrees $\leq \mathbf{a}$) is defined by $\leq_e A$, in this paper for any fixed set of natural numbers A the algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ is considered. Here $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$ and $W_0, W_1, \ldots$ is the standard sequence of all computably enumerable (c.e.) sets, considered as e-operators and Non is the predicate for "non-emptiness". We would like to mention that the empty set plays a special role and we distinguish it from the other c.e. sets. We modify slightly the relation $\leq_e$ and show that the algebra $\mathfrak{N}^A$ has a least enumeration, admits equivalent representation with 3 operators ant is finitely generated. We use unary partial structures without equality [3, 2].

In Section 2 we give all necessary definitions, notions and propositions concerning normal and least enumerations of unary partial structures. Here we slightly modify the definitions of e-reducibility and e-operators, concerning the empty set. In Section 3 we prove our main result: The algebra $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ admits a least enumeration. Then we prove that this algebra is recursively equivalent to an algebra with only 3 operators, and that the latter algebra is finitely generated. At the end we see that among all algebras with different enumeration of all e-operators the standard one has a least enumeration.

2. PRELIMINARIES

In this paper we denote by ω the set of all natural numbers. By $Dom(f)$, $Ran(f)$ and G_f we denote the domain, the range and the graph of a function f, respectively; $\langle f\rangle$ or $\langle G_f\rangle$ stands for the set $\{\langle x_1, \ldots, x_n, y\rangle | (x_1, \ldots, x_n, y) \in G_f\}$, where $\langle ., \ldots, .\rangle$ is some fixed coding function for all finite sequences of natural numbers. We shall use $f(x)\downarrow$ to denote that $x \in Dom(f)$; also we say that $f(x)$ *is conditionally equal to* $g(x)$, or that the conditional equality $f(x) \cong g(x)$ is true if and only if

$$(f(x)\downarrow \& g(x)\downarrow \& f(x) = g(x)) \vee (\neg(f(x)\downarrow) \& \neg(g(x)\downarrow)).$$

$W_0, W_1, \ldots$ denotes the standard enumeration of all computably enumerable (c.e.) sets; $\{E_v\}_{v\in\omega}$ is an effective coding of the family of all finite subsets of ω.

If W is c.e. set, then we write $W_{[n]} = \{x | \langle n, x\rangle \in W\}$.

If A is an arbitrary subset of ω, then by $W(A)$ we denote the set

$$W(A) = \{x | \exists v(\langle x, v\rangle \in W \& E_v \neq \emptyset \& E_v \subseteq A)\}.$$

Notice that there is a slight deviation from the usual definition of the term e-operator. It concerns $\emptyset$.

We shall say that A is e-reducible to B ($A \leq_e B$) if there exists a c.e. set W such that $A = W(B)$; A is e-equivalent to B ($A \equiv_e B$) if $A \leq_e B \& B \leq_e A$; $\mathbf{d}_e(A) = \{B | A \equiv_e B\}$. Thus we obtain $\mathbf{0}_e$ — the family of all non-empty c.e. sets and $\mathbf{-1}_e = \{\emptyset\}$.

For two arbitrary sets A and B of naturals, set

$$A \oplus B := \{2x | x \in A\} \cup \{2x + 1 | x \in B\}.$$

If $A_0, A_1, \ldots$ is a sequence of sets of naturals, the notation $\oplus_{i\in\omega} A_i$ stands for the set $\{\langle i, x\rangle | x \in A_i\}$.

We recall some definitions from [14, 1].

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k\rangle$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1, \ldots, \theta_n$ are partial unary functions in B and $R_1, \ldots, R_k$ are unary partial predicates on B. We allow any of the sequences $\theta_1, \ldots, \theta_n$ and $R_1, \ldots, R_k$ to be infinite, as well. We call such structures unary. We identify the partial predicates with partial mapping taking values in $\{0, 1\}$, writing 0 for true and 1 for false.

Let $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k\rangle$ be a partial structure over the set ω. By $\langle\mathfrak{B}\rangle$ we denote the set $\langle\varphi_1\rangle \oplus \cdots \oplus \langle\varphi_n\rangle \oplus \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_k\rangle$ (in the case the when the set of functions or predicates is infinite we shall use the corresponding infinite version of $\bigoplus$).

Definition 1. *An enumeration of a structure* $\mathfrak{A}$ *is any ordered pair* $\langle\alpha, \mathfrak{B}\rangle$, *where* $\mathfrak{B} = \langle \omega; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k\rangle$ *is a partial unary structure on* ω *and* α *is a partial surjective mapping of* ω *onto* B *such that the following conditions hold:*

(i) $Dom(\alpha) \leq_e \langle\mathfrak{B}\rangle$;

(ii) $\alpha(\varphi_i(x)) \cong \theta_i(\alpha(x))$ *for every* $x \in \omega$, $1 \leq i \leq n$;

(iii) $\sigma_j(x) \cong R_j(\alpha(x))$ *for every* $x \in \omega$, $1 \leq j \leq k$.

An enumeration $\langle\alpha, \mathfrak{B}\rangle$ is said to be *total* if $Dom(\alpha) = \omega$.

Let $A \subseteq B$. The set A is called admissible in the enumeration $\langle\alpha, \mathfrak{B}\rangle$ if and only if there exists a set W of naturals such that $W \leq_e \langle\mathfrak{B}\rangle$ and for every $x \in \omega$, $x \in W \iff \alpha(x) \in A$.

A partial multiple-valued (p.m.v) function θ is called *admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a set $W \subseteq \omega^2$ such that $W \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$ and $t \in B$, the following equivalence is true:

$$t \in \theta(\alpha(x)) \iff \exists y((x, y) \in W \& \alpha(y) = t).$$

The above definition can be reformulated as follows: A p.m.v function θ is called *admissible* in the enumeration $\langle \alpha, \mathfrak{B} \rangle$ if there exists a p.m.v function φ in ω such that $\langle G_\varphi \rangle \leq_e \langle \mathfrak{B} \rangle$ and for every $x \in \omega$, $\alpha(\varphi(x)) = \theta(\alpha(x))$.

A set A or p.m.v function θ is called $\forall$-*admissible* in $\mathfrak{A}$ if it is admissible in every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$.

Let $\langle \alpha_0, \mathfrak{B}_0 \rangle$ be an enumeration of the structure $\mathfrak{A}$. We say that $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is a *least enumeration* of $\mathfrak{A}$ if for every enumeration $\langle \alpha, \mathfrak{B} \rangle$ of $\mathfrak{A}$, $\langle \mathfrak{B}_0 \rangle \leq_e \langle \mathfrak{B} \rangle$.

Let $\mathcal{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathcal{L}$ consists of n unary functional symbols $\mathbf{f_1}, \dots, \mathbf{f_n}$ and k unary predicate symbols $\mathbf{T_1}, \dots, \mathbf{T_k}$. We admit any of the sequences $\mathbf{f_1}, \dots, \mathbf{f_n}$ and $\mathbf{T_1}, \dots, \mathbf{T_k}$ to be infinite. Let us fix some denumerable set $X_1, X_2, \dots$ of variables. We use capital letters X, Y, Z and the same letters indexed to denote variables.

We use the standard definition of a term in the language $\mathcal{L}$: Every variable is a term; if τ is a term, then $\mathbf{f_i}(\tau)$ is a term. If τ is a term in the language $\mathcal{L}$, then we write $\tau(Y_1, \dots, Y_k)$ to denote that all variables which occur in the term τ are among $Y_1, \dots, Y_k$.

Termal predicate in the language $\mathcal{L}$ is defined by the following inductive clauses:

1) If $\mathbf{T} \in \{\mathbf{T_0}, \dots, \mathbf{T_k}\}$ and τ is a term, then $\mathbf{T}(\tau)$ and $\neg\mathbf{T}(\tau)$ are termal predicates.

2) If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Suppose that $\mathfrak{B}$ is a structure, $a_1, \dots, a_k$ are elements of B and $\tau(Y_1, \dots, Y_k)$ is a term. By $\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_k/a_k)$ we denote the value of the term τ in $\mathfrak{A}$ over the elements $a_1, \dots, a_k$, if it exists.

Let $\Pi(Y_1, \dots, Y_m)$ be a termal predicate whose variables are among $Y_1, \dots, Y_m$ and $a_1, \dots, a_m$ be elements of B. The value $\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ of Π over $a_1, \dots, a_r$ in $\mathfrak{A}$ is defined as follows:

If $\Pi = \mathbf{T}_j(\tau)$, $0 \leq j \leq k$, then $\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong R_j(\tau_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m))$.

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong \begin{cases} 1, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0, \\ 0, & \text{if } \Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong \begin{cases} \Pi^2_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m), & \text{if} \Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0, \\ 1, & \text{if} \Pi^1_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 1, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Formulae of the kind $\exists Y_1' \dots \exists Y_l'(\Pi)$, where Π is a termal predicate, are called *conditions*. Every variable which occurs in Π and is different from $Y_1', \dots, Y_l'$ is called free in the condition $\exists Y_1' \dots \exists Y_l'(\Pi)$.

Let $\exists Y_1' \dots \exists Y_l'(\Pi)$ be a condition, let all free variables in C be among $Y_1, \dots, Y_m$, and $a_1, \dots, a_m$ be elements of B. The value $C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m)$ is defined by the equivalence:

$$C_{\mathfrak{A}}(Y_1/a_1, \dots, Y_m/a_m) \cong 0 \iff$$
$$\exists t_1 \dots \exists t_l (\Pi_{\mathfrak{A}}(Y_1'/t_1, \dots, Y_l'/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

We assume that some effective codding of all terms, termal predicates and conditions of the language $\mathcal{L}$ is fixed. We shall use superscripts to denote the corresponding codes.

Let $A \subseteq \omega^r \times B^m$. The set A is said to be $\exists$-*definable* (or just *definable*) in the structure $\mathfrak{A}$ if and only if there exists a recursive function γ of $r+1$ variables such that for all $n, x_1, \dots, x_r$, $C^{\gamma(n,x_1,\dots,x_r)}$ is a condition with free variables among $Z_1, \dots, Z_l, Y_1, \dots, Y_m$ and for some fixed elements $t_1, \dots, t_l$ of B the following equivalence is true:

$$(x_1, \dots, x_r, a_1, \dots, a_m) \in A \iff$$
$$\exists n \in \omega (C_{\mathfrak{A}}^{\gamma(n,x_1,\dots,x_r)}(Z_1/t_1, \dots, Z_l/t_l, Y_1/a_1, \dots, Y_m/a_m) \cong 0).$$

If Π is a termal predicate and τ is a term, then $\exists Y_1' \dots \exists Y_l'(\Pi \supset \tau)$ is called a *conditional expression.*

Let $Q = \exists Y_1' \dots \exists Y_l'(\Pi \supset \tau)$ be a conditional expression with free variables among $X_1, \dots, X_a$, and $s_1, \dots, s_a \in B$. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \dots, X_a/s_a)$ of Q is the following subset of B:

$$\{\tau_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) | \Pi_{\mathfrak{A}}(Y_1'/p_1, \dots, Y_l'/p_l, X_1/s_1, \dots, X_a/s_a) \cong 0\}.$$

Let θ be a p.m.v. function in B. Then the function θ is called *definable in* $\mathfrak{A}$ if and only if for some c.e. set $\{Q^v\}_{v \in V}$ of conditional expressions with free variables among $X, Z_1, \dots, Z_r$ and for some fixed elements $t_1, \dots, t_r$ of B the following equivalence is true:

$$t \in \theta(s) \iff \exists v (v \in V \& t \in Q_{\mathfrak{A}}^v(Z_1/t_1, \dots, Z_r/t_r, X/s)).$$

In [14] Soskov has proved the following result.

Theorem 1. (Soskov [14]) *Let θ be a unary p.m.v. function in B. Then θ is $\forall$-admissible in $\mathfrak{A}$ if and only if θ is definable in $\mathfrak{A}$.*

Define $f_i(p) = \langle i-1, p \rangle$, $i = 1, \dots, n$ and $N_0 = \omega \setminus (Ran(f_1) \cup \dots \cup Ran(f_n))$. It is obvious that N_0 is an infinite recursive set and let $\{\mathbf{p_0}, \mathbf{p_1}, \dots\} = N_0$, where $\mathbf{p_i} < \mathbf{p_j}$ if $i < j$. In the case when the sequence f_i is infinite ($i \in \omega$) we can ensure N_0 to be infinite by taking for example $f_i(p) = \langle i-1, p, 0 \rangle$.

Next we recall the definition and some properties of normal enumerations [14] for the case of total enumerations. For every surjective mapping α^0 of N_0 onto B (called basis) we define a mapping α of ω onto B by the following inductive clauses:

(i) If $p \in N_0$, then $\alpha(p) = \alpha^0(p)$;

(ii) If $p = f_i(q)$, then $\alpha(q) = a$ and $\theta_i(a) = b$, then $\alpha(p) = b$.

Let $\sigma_1, \ldots, \sigma_k$ be the partial predicates, defined by $\sigma_j(x) \cong R_j(\alpha(x))$, $j = 1, \ldots, k$. Denote by $\mathfrak{B}$ the partial structure $\langle \omega; f_1, \ldots, f_n; \sigma_1, \ldots, \sigma_k \rangle$. It is well known [1, 14] that α is well defined and that the basis α^0 completely determines the normal enumeration $\langle \alpha, \mathfrak{B} \rangle$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a normal enumeration. We recall some obvious propositions for normal enumerations. Their proofs are the same as in [14].

Proposition 1. *For every $1 \leq i \leq n$ and $y \in \omega$, $\alpha(f_i(y)) = \theta_i(\alpha(y))$.*

Corollary 1. *Let $\tau(Y)$ be a term and $y \in \omega$. Then*

$$\alpha(\tau_{\mathfrak{B}}(Y/y)) = \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Proposition 2. *There exists an effective way for every x of ω to find $y \in N_0$ and a term $\tau(Y)$, such that $x = \tau_{\mathfrak{B}}(Y/y)$.*

If $\langle \alpha, \mathfrak{B} \rangle$ is a normal enumeration, we denote the set $\cup_{j=1}^{k} \{\langle j, x, z \rangle | \sigma_j(x) = z\}$ by R_α. In the general case we have to add some additional members, but in our situation the functions f_i are totally defined and no additional terms are needed. It is clear that for every $W \subseteq \omega$, $W \leq_e R_\alpha$ if and only if $W \leq_e \langle \mathfrak{B} \rangle$.

Proposition 3. *There exists an effective way for every natural u to find elements $y_1, \ldots, y_m \in N_0$ and a termal predicate $\Pi(Y_1, \ldots, Y_m)$ such that for every normal enumeration $\langle \alpha, \mathfrak{B} \rangle$,*

$$u \in R_\alpha \iff \Pi_{\mathfrak{A}}(Y_1/\alpha(y_1), \ldots, Y_m/\alpha(y_m)) \cong 0.$$

Proposition 4. *There exists an effective way for every code v of a finite set E_v to find elements $y_1^v, \ldots, y_{m_v}^v \in N_0$ and a termal predicate $\Pi^v(Y_1, \ldots, Y_{m_v})$ such that for every normal enumeration $\langle \alpha, \mathfrak{B} \rangle$,*

$$E_v \subseteq R_\alpha \iff \Pi^v_{\mathfrak{A}}(Y_1/\alpha(y_1^v), \ldots, Y_{m_v}/\alpha(y_{m_v}^v)) \cong 0.$$

To be precise, we have to mention that, for the sake of simplicity, in the above proposition we have used just Π^v instead of $\Pi^{\gamma(v)}$ with some recursive function γ.

Let $\mathfrak{A} = \langle B; \theta_1, \ldots, \theta_n; R_1, \ldots, R_k \rangle$ be a unary partial structure. *Type of the sequence* $b_1, \ldots, b_m$ of elements of B is called the set

$$\{v | \Pi^v_{\mathfrak{A}}(X_1/b_1, \ldots, X_m/b_m) \cong 0 \;\&\; \Pi^v \text{ is a termal predicate with variables} \in \{X_i\}_{i=1}^m\}.$$

The type of the sequence $b_1, \ldots, b_m$ is denoted by $[b_1, \ldots, b_m]_{\mathfrak{A}}$. The type of an element a of B is the type of the sequence a.

A condition is called *simple* if it does not contain free variables and it is in the form $\exists X_1 \Pi$, where Π is a termal predicate. Let $V_0^{\mathfrak{A}} = \{v | C_{\mathfrak{A}}^v \cong 0 \ \& \ C^v$ be a simple condition$\}$.

Definition 2. *Let $\mathcal{A}$ be a family of subsets of ω. A set $U \subseteq \omega^2$ is said to be universal for the family $\mathcal{A}$, if the following conditions hold:*

a) For every fixed $e \in \omega$, $\{x_1 | (e, x_1) \in U\} \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, then there exists e such that $A = \{x_1 | (e, x_1) \in U\}$.

Theorem 2. ([3]) *Let $\mathfrak{A}$ be a unary partial structure. Then $\mathfrak{A}$ admits a least partial enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ if and only if there exist elements $b_1, \ldots, b_m$ of B such that $deg_e([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$ is the least upper bound of e-degrees of all $\exists$-types of sequences of elements of B and there exists a universal set U of all types, such that $deg_e(U) = deg_e([b_1]_{\mathfrak{A}} \oplus \cdots \oplus [b_m]_{\mathfrak{A}} \oplus V_0^{\mathfrak{A}})$.*

3. THE MAIN RESULT

We shall consider the *standard structure* $\mathfrak{N} = \langle \mathcal{P}(\omega); W_0, W_1, \ldots; Non \rangle$, where $\mathcal{P}(\omega)$ is the family of all subsets of ω, $W_0, W_1, \ldots$ is a fixed sequence of all c.e. sets considered as functions (e-operators) and Non is the family of all non-empty sets of naturals. To be more precise, Non is a partial unary predicate defined as follows: $Non(A) = 0$, if $A \neq \emptyset$ and $Non(\emptyset) \uparrow$.

First we shall consider the structure $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non \rangle$, where $\mathcal{P}(\omega)^A = \{B | B \subseteq \omega \& B \leq_e A\}$, which we call standard as well. Let us mention that the functions $W_0, W_1, \ldots$ are totally defined as e-operators and we do not use the equality among the predicates. Let in addition $\mathbf{W}$ be the family of all c.e. sets considered as e-operators.

Let $\mathcal{L}^*$ be the first order language $\langle \mathbf{f_0}, \mathbf{f_1}, \ldots; \mathbf{T} \rangle$, containing a countable set of unary functional symbols $\mathbf{f_0}, \mathbf{f_1}, \ldots$ and a unary predicate symbol $\mathbf{T}$. We call $\overline{\mathfrak{A}}$ a *generalized structure* if $\overline{\mathfrak{A}} = \langle B; \mathbf{\Theta}; R \rangle$, where B is a denumerable set, $\mathbf{\Theta}$ – denumerable set of unary functions on B and R is a unary predicate on B. When we consider structures with finite functions and finite predicates, the considerations do not depend on the enumerations of the functions and the predicates. In the case when we consider denumerable set of functions the situation is different.

Enumeration of a family $\mathbf{\Theta}$ of functions is any sequence $\theta_0, \theta_1, \ldots$ such that $\mathbf{\Theta} = \{\theta_0, \theta_1, \ldots\}$. We do not require all members of the sequence $\theta_0, \theta_1, \ldots$ to be different.

Let us fix some enumeration $\theta_0^0, \theta_1^0, \ldots$ of the family $\mathbf{\Theta}$ and consider the structure $\mathfrak{A}_0 = \langle B; \theta_0^0, \theta_1^0, \ldots; R \rangle$.

We say that $\langle\alpha_0, \mathfrak{B}_0\rangle$ is a *least enumeration* of the generalized structure $\overline{\mathfrak{A}}$ if for every enumeration $\theta_0, \theta_1, \ldots$ of $\mathbf{\Theta}$ and every enumeration $\langle\alpha, \mathfrak{B}\rangle$ of $\mathfrak{A} = \langle B; \theta_0, \theta_1, \ldots; R\rangle$ the inequality $\langle\mathfrak{B}_0\rangle \leq_e \langle\mathfrak{B}\rangle$ holds.

Let us consider the structure $\mathfrak{N}^A = \langle\mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ for the language $\mathcal{L}^*$ and define the m.v.f. $\Phi^A : \mathcal{P}(\omega)^A \setminus \{\emptyset\} \to \mathcal{P}(\omega)^A \setminus \{\emptyset\}$ as follows: $\Phi^A(B) = \{C | C \leq_e B \& C \neq \emptyset\}$ for nonempty B.

Proposition 5. *The m.v.f. Φ^A is definable in the structure $\mathfrak{N}^A$.*

Proof. Let Q^n be the conditional expression $\mathbf{T}(X)\&\mathbf{T}(\mathbf{f_n}(X)) \supset \mathbf{f_n}(X)$. Notice that the sequence $\{Q^n\}_{n\in\omega}$ is c.e. and

$$C \in Q^n_{\mathfrak{N}^A}(X/B) \iff Non(B)\&Non(W_n(B))\&C = W_n(B).$$

Then

$$\begin{aligned}&C \in \Phi^A(B) \iff C \leq_e B\&C \neq \emptyset\&B \neq \emptyset \iff \\ &\exists n(W_n(B) = C\&C \neq \emptyset\&B \neq \emptyset) \iff \exists n(C \in Q^n_{\mathfrak{N}^A}(X/B)).\end{aligned}$$

Proposition 5 is proved. □

Let $L_A = \{\langle n, x\rangle | x \in W_n(A)\}$. The following lemma is well-known, its proof is a simple application of the S^m_n-theorem.

Lemma 1. *There exists a recursive function δ of two variables such that for all naturals m, n and a set C of naturals the following equality is true:*

$$W_m(W_n(C)) = W_{\delta(m,n)}(C).$$

Let us fix a function δ in Lemma 1 and define the pair $\langle\alpha_0, \mathfrak{B}_0\rangle$ as follows:

$\alpha_0(n) = W_n(A)$, $\mathfrak{B}_0 = \langle\omega; \varphi^0_0, \varphi^0_1, \ldots; \sigma^0\rangle$, where $\varphi^0_i(x) = \delta(i, x)$, $i, x \in \omega$, $\sigma^0(x) \cong 0 \iff W_x(A) \neq \emptyset$ and $\sigma^0(x)\uparrow$ if $W_x(A) = \emptyset$.

Lemma 2. *The pair $\langle\alpha_0, \mathfrak{B}_0\rangle$ is an enumeration of the structure $\mathfrak{N}^A$.*

Proof. $W_i(\alpha_0(x)) = W_i(W_x(A)) = W_{\delta(i,x)}(A) = \alpha_0(\delta(i, x)) = \alpha_0(\varphi^0_i(x))$.
$Non(\alpha_0(x)) \cong 0 \iff W_x(A) \neq \emptyset \iff \sigma^0(x) \cong 0$. □

Let $W_A = \{n | \exists x(\langle n, x\rangle \in L_A)\} = \{n | W_n(A) \neq \emptyset\} = \{n | \sigma^0(n) \cong 0\}$.

Proposition 6. $W_A \equiv_e A$.

Proof. Let n_0 be a fixed element of ω and define the set B by the following equivalence: $\langle\langle n, x\rangle, m\rangle \in B \iff \langle n, x\rangle \in L_A\&m = n_0$. Obviously, $B \leq_e L_A \equiv_e A$.

Therefore, using the S^m_n-theorem we obtain

$$\begin{aligned}\langle\langle n,x\rangle,m\rangle\in B \quad &\iff \exists v(\langle\langle\langle n,x\rangle,m\rangle,v\rangle\in W_a\&\emptyset\neq E_v\subseteq A)\\ &\qquad\qquad\text{(for some fixed natural } a)\\ &\iff \exists v(\langle\langle m,v\rangle,\langle n,x\rangle\rangle\in W_b\&\emptyset\neq E_v\subseteq A)\\ &\qquad\qquad\text{(for some fixed natural } b)\\ &\iff \exists v(\langle m,v\rangle\in W_{\gamma(\langle n,x\rangle)}\&\emptyset\neq E_v\subseteq A)\\ &\qquad\qquad\text{(for some fixed recursive function } \gamma)\\ &\iff m\in W_{\gamma(\langle n,x\rangle)}(A).\end{aligned}$$

We will show that $L_A\leq_m W_A$ by recursive function γ.

Let us assume $\langle n,x\rangle\in L_A$. Then $\langle\langle n,x\rangle,n_0\rangle\in B$, thus $n_0\in W_{\gamma(\langle n,x\rangle)}(A)$, i.e. $W_{\gamma(\langle n,x\rangle)}(A)\neq\emptyset$, hence $\gamma(\langle n,x\rangle)\in W_A$.

Let us suppose that $\gamma(\langle n,x\rangle)\in W_A$. Then $\exists m(m\in W_{\gamma(\langle n,x\rangle)}(A))$, thus $n_0\in W_{\gamma(\langle n,x\rangle)}(A)$. Therefore $\langle\langle n,x\rangle,n_0\rangle\in B$ and $\langle n,x\rangle\in L_A$.

We proved the equivalence $\langle n,x\rangle\in L_A \iff \gamma(\langle n,x\rangle)\in W_A$, i.e. $L_A\leq_m W_A$. Therefore, $L_A\leq_e W_A$.

Conversely,

$$\begin{aligned}n\in W_A \quad &\iff \exists x(\langle n,x\rangle\in L_A)\iff \exists x(x\in W_n(A))\\ &\iff \exists x\exists v(\langle x,v\rangle\in W_n\&\emptyset\neq E_v\subseteq A)\\ &\iff \exists v(\exists x(\langle n,v\rangle\in W_{\gamma_1(x)})\&\emptyset\neq E_v\subseteq A)\\ &\iff \exists v(\langle n,v\rangle\in W_a)\&\emptyset\neq E_v\subseteq A)\iff n\in W_a(A)\end{aligned}$$

for some fixed recursive function γ_1 and a fixed natural a. Hence, $W_A\leq_e A$. □

Lemma 3. *Let τ^v be the term with a code v. There exists a recursive function γ_0 such that for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v the equality $\tau^v_{\mathfrak{N}^A}(X/A)=W_{\gamma_0(v)}(A)$ holds.*

Proof. Decode $\tau^v(X)$ as a sequence of $f_{i_1},f_{i_2},\ldots,f_{i_p}$ and variable X. Then consider the composition of the operators $W_{i_1},W_{i_2},\ldots,W_{i_p}$ over A and use the recursive function δ. Thus there exists an effective way for any term $\tau^v(X)$ in the language $\mathcal{L}^*$ with variable X and code v to find a natural number n such that $\tau^v_{\mathfrak{N}^A}(X/A)=W_n(A)$. □

Lemma 4. $[A]_{\mathfrak{N}^A}\equiv_m W_A$.

Proof. Recall that $[A]_{\mathfrak{N}^A}=\{v|\tau^v_{\mathfrak{N}^A}(X/A)\neq\emptyset\}$. Let γ_0 be the recursive function from the previous lemma, then $v\in[A]_{\mathfrak{N}^A}\iff\tau^v_{\mathfrak{N}^A}(X/A)\neq\emptyset\iff W_{\gamma_0(v)}(A)\neq\emptyset\iff\sigma^0(\gamma_0(v))\cong 0\iff\gamma_0(v)\in W_A$. Thus, $[A]_{\mathfrak{N}^A}\leq_m W_A$.

Conversely, $n\in W_A\iff W_n(A)\neq\emptyset\iff$ the term $\mathbf{f}_n(X)$ with code $v(n)$ satisfies $(\mathbf{f}_n(X))^{v(n)}_{\mathfrak{N}^A}(X/A)\neq\emptyset$, i.e. $W_A\leq_m[A]_{\mathfrak{N}^A}$. □

Theorem 3. *The enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is the least enumeration of the structure $\mathfrak{N}^A$.*

Proof. According to Theorem 2, having in mind $W_A = V_0^{\mathfrak{N}^A}$, we need to show that all types of elements B such that B is a set of naturals and $B \leq_e A$ satisfy the condition $[B]_{\mathfrak{N}^A} \leq_e [A]_{\mathfrak{N}^A}$ and that there exists a universal set with e-degree $deg_e(A)$ for all types $[B]_{\mathfrak{N}^A}$.

Let $B \leq_e A$. Then there exists an e-operator W_n such that $W_n(A) = B$. Therefore, $v \in [B]_{\mathfrak{N}^A} \iff$ the code v_1 of the term $\mathbf{f_n}(\tau^v)$ belongs to $[A]_{\mathfrak{N}^A}$, thus $[B]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{N}^A}$. Further, using the type $[A]_{\mathfrak{N}^A}$, we define the set U^A by the equivalence: $(n, v) \in U^A \iff \exists v_1(\tau^{v_1} = \mathbf{f_n}(\tau^v) \& v_1 \in [A]_{\mathfrak{N}^A})$. Actually, we could define U^A by the equivalence: $(n, v) \in U^A \iff \langle n, v \rangle \in L_A$, as well. It is obvious that U^A is universal for the family of all types of the structure $\mathfrak{N}^A$. □

Let us consider the structure $\mathfrak{D}^A = \langle \mathcal{P}(\omega)^A; \Phi^A \rangle$. The following definition is natural, although it is not used because normally we do not consider structures with p.m.v. functions.

Definition 3. *Enumeration of the structure $\mathfrak{D}^A$ is called the pair $\langle \alpha, \mathfrak{B} \rangle$, where $\alpha : \omega \to \mathcal{P}(\omega)^A$, $\mathfrak{B} = \langle \omega; \varphi \rangle$ and φ is a partial m.v.f. in ω, such that for all natural n the equality $\alpha(\varphi(n)) = \Phi^A(\alpha(n))$ holds (here, we mean equality between sets).*

Proposition 7. *There exists an enumeration $\langle \alpha_0, \mathfrak{B}' \rangle$ of the structure $\mathfrak{D}^A$ such that $\langle \mathfrak{B}' \rangle \equiv_e A$.*

Proof. Let us recall that $\alpha_0(n) = W_n(A)$ and define the partial m.v.f. φ^0 as follows: $m \in \varphi^0(n) \iff \exists k(\sigma^0(m) \cong 0 \& \sigma^0(n) \cong 0 \& \delta(k, n) = m)$. It is clear that $\langle G_\varphi \rangle \leq_e A$. Then

$$
\begin{aligned}
& C \in \alpha_0(\varphi^0(n)) \iff \exists m(m \in \varphi(n) \& \alpha_0(m) = C) \iff \\
& \exists m(\exists k(\sigma^0(m) \cong 0 \& \sigma^0(n) \cong 0 \& \delta(k, n) = m) \& W_m(A) = C) \iff \\
& \exists m \exists k(W_m(A) = W_k(W_n(A)) \& C = W_m(A) \neq \emptyset \& W_n(A) \neq \emptyset) \iff \\
& \exists m(W_m(A) \leq_e W_n(A) \& C = W_m(A) \neq \emptyset \& W_n(A) \neq \emptyset) \iff \\
& \exists m(C = W_m(A) \in \Phi^A(W_n(A)) \iff C \in \Phi^A(\alpha_0(n)).
\end{aligned}
$$

Therefore $\langle \alpha_0, \mathfrak{B}' \rangle$ is an enumeration of $\mathfrak{D}^A$.

Further, let us fix some a such that $\alpha_0(a) = A$. Then $W_n(A) = W_n(W_a(A)) = W_{\delta(n,a)}(A)$ and hence

$$
\begin{aligned}
W_A &= \{n | W_n(A) \neq \emptyset\} \equiv_e \{\delta(n, a) | W_{\delta(n,a)}(A) \neq \emptyset\} \\
&= \{\delta(n, a) | \sigma^0(\delta(n, a)) \cong 0\} \equiv_e \{\delta(n, a) | \delta(n, a) \in \varphi^0(a)\} \leq_e \langle G_\varphi \rangle \equiv_e \langle \mathfrak{B}' \rangle .
\end{aligned}
$$

Proposition 7 is proved. □

Lemma 5. *There exist c.e. sets $V^{[n]}, n \in N, V', V^{[S]}$ such that the effective sequence of compositions $\{V^{[0]}(V^{[S]})^n V'\}_{n \in \omega}$ is recursively isomorphic to the sequence $\{W_n\}_{n \in \omega}$.*

Proof. Let us notice first that $V^{[0]}(V^{[S]})^n V'$ means the following:

$$V^{[0]}(V^{[S]})^0 V' = V^{[0]} V'; \quad V^{[0]}(V^{[S]})^{n+1} V' = ((V^{[0]}(V^{[S]})^n) V^{[S]}) V'.$$

Let us denote

$$V^{[n]} = \{\langle x, v\rangle | x \in \omega \& E_v = \{\langle n, x\rangle\}\}, \;\; V^{[S]} = \{\langle\langle n, x\rangle, v\rangle | n, x \in \omega \& E_v = \{\langle n+1, x\rangle\}\}.$$

Further, let $V = \{\langle n, x\rangle | x \in W_n\}$ and $V' = \{\langle\langle k, x\rangle, v\rangle | \langle k, \langle x, v\rangle\rangle \in V\}$. Then

$$\begin{aligned} x \in V^{[n]} V'(X) &\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \subseteq V'(X)) \\ &\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \& \langle n, x\rangle \in V'(X)) \\ &\iff \exists v(\langle\langle n, x\rangle, v\rangle \in V' \& \emptyset \neq E_v \subseteq X) \\ &\iff \exists v(\langle x, v\rangle \in V_{[n]} \& \emptyset \neq E_v \subseteq X) \iff x \in V_{[n]}(X), \end{aligned}$$

$$\begin{aligned} x \in V^{[n]} V^{[S]}(X) &\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \subseteq V^{[S]}(X)) \\ &\iff \exists v_1(\langle x, v_1\rangle \in V^{[n]} \& E_{v_1} = \{\langle n, x\rangle\} \& \langle n, x\rangle \in V^{[S]}(X)) \\ &\iff \exists v(\langle\langle n, x\rangle, v\rangle \in V^{[S]} \& E_v = \{\langle n+1, x\rangle\} \subseteq X) \\ &\iff \exists v(\langle x, v\rangle \in V^{[n+1]} \& E_v = \{\langle n+1, x\rangle\} \subseteq X) \\ &\iff x \in V^{[n+1]}(X). \end{aligned}$$

We shall prove by induction the equivalence

$$x \in V^{[0]}(V^{[S]})^n V'(X) \iff x \in V_{[n]}(X). \qquad (*)$$

Indeed, $x \in V^{[0]}(V^{[S]})^0 V'(X) \iff x \in V^{[0]} V'(X) \iff x \in V_{[0]}(X)$. Let us assume the equivalence $(*)$ is true. Then
$x \in V^{[0]}(V^{[S]})^{n+1} V'(X) \iff x \in V^{[n+1]} V'(X) \iff x \in V_{[n+1]}(X)$. □

The next two corollaries are obvious.

Corollary 2. *The structure* $\mathfrak{N}^A = \langle \mathcal{P}(\omega)^A; W_0, W_1, \ldots; Non\rangle$ *is equivalent to the structure* $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$, *where* $V^{[0]}, V^{[S]}, V'$ *is the c.e. sets from the previous lemma.*

Corollary 3. *For any set* A *of naturals the set* $\mathcal{P}(\omega)^A$ *is finitely generated in the structure* $\mathfrak{N}'^A = \langle \mathcal{P}(\omega)^A; V^{[0]}, V^{[S]}, V'\rangle$ *by the single element* A.

Proposition 8. *For any enumeration* $\{V_0, V_1, \ldots\}$ *of the family* $\boldsymbol{W}$ *the structure* $\mathfrak{M}^A = \langle \mathcal{P}(\omega)^A; V_0, V_1, \ldots; Non\rangle$ *admits a least enumeration* $\langle \alpha, \mathfrak{B}\rangle$ *such that* $A \leq_e \langle \mathfrak{B}\rangle$.

Proof. Let $\alpha^0 : N_0 \to \mathcal{P}(\omega)^A$ be defined as follows: $\alpha^0(\mathbf{p_n}) = V_n(A)$. Take α^0 as a basis of a normal enumeration $\langle \alpha, \mathfrak{B}\rangle$, where $\mathfrak{B} = \langle \omega; \varphi_0, \varphi_1, \ldots; \sigma\rangle$ and $\varphi_i(x)$ is a computable function of both variables i, x. According to Proposition 2, there exists an effective way for any x to find $y = \mathbf{p_n} \in N_0$ and a term τ such that $x = \tau_{\mathfrak{B}}(Y/y)$; thus $\alpha(x) = \tau_{\mathfrak{A}}(Y/\alpha(y)) = \tau_{\mathfrak{A}}(Y/\alpha^0(\mathbf{p_n})) = \tau_{\mathfrak{A}}(Y/V_n(A)) = \tau'_{\mathfrak{A}}(Y/A)$, where $\tau' = \tau(\mathbf{f}_n(Y))$.

Let us denote $V_A = \{n | \sigma(n) \cong 0\}$. Then, using the term τ' obtained above, $x \in V_A \iff \sigma(x) \cong 0 \iff \alpha(x) \neq \emptyset \iff \tau'_{\mathfrak{A}}(Y/A) \neq \emptyset \iff v' \in [A]_{\mathfrak{M}^A}$ for the code v' of the term τ'. Thus, having in mind that we can find v' effectively from x, we have proved that $V_A \leq_m [A]_{\mathfrak{M}^A}$.

Analogously, let $v' \in [A]_{\mathfrak{M}^A}$, $\tau^{v'} = \tau^{v'}(Y)$ and n be a fixed natural, such that $\alpha^0(\mathbf{p}_n) = V_n(A) = A$, where $y = \mathbf{p}_n \in N_0$. Then $\tau^{v'}_{\mathfrak{A}}(Y/A) = \alpha(\tau^{v'}_{\mathfrak{B}}(Y/y)) \neq \emptyset$ and let $x = \tau^{v'}_{\mathfrak{B}}(Y/y)$. Then $\sigma(x) \cong 0$ and $x \in V_A$. Therefore, $[A]_{\mathfrak{M}^A} \leq_m V_A$.

Hence, $[A]_{\mathfrak{M}^A} \equiv_m V_A$ and $\langle \mathfrak{B} \rangle \equiv_e [A]_{\mathfrak{M}^A} \equiv_e V_A$. □

Corollary 4. $W_A \leq_e V_A$.

Proof. Let $V_{i_0} = V^{[0]}$, $V_{i_1} = V^{[S]}$ and $V_{i_2} = V'$ and consider the sequence of terms $\tau^{v(n)}$, where $\tau^{v(n)} = \mathbf{f}_{i_0} \circ \mathbf{f}^n_{i_1} \circ \mathbf{f}_{i_2}(X)$. Here, $\mathbf{f}^n_{i_1}$ means n times the term $\mathbf{f}_{i_1}$. Then it is easy to check that $n \in [A]_{\mathfrak{N}^A} \iff v(n) \in [A]_{\mathfrak{M}^A}$. Thus we have proved that $[A]_{\mathfrak{N}^A} \leq_m [A]_{\mathfrak{M}^A}$, hence $W_A \leq_e V_A$. □

Corollary 5. *The enumeration $\langle \alpha_0, \mathfrak{B}_0 \rangle$ is the least for the generalized structure $\overline{\mathfrak{N}^A} = \langle \mathcal{P}(\omega)^A; \boldsymbol{W}; Non \rangle$.*

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UPPER ESTIMATES OF THE APPROXIMATION RATE OF COMBINATIONS OF ITERATES OF THE BERNSTEIN OPERATOR

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We present upper estimates of the approximation rate of combinations $\mathcal{B}_{r,n}$ of iterates of the Bernstein operator B_n, defined by $I-\mathcal{B}_{r,n} = (I-B_n)^r$, $r \in \mathbb{N}$. The treatment is based on (weighted) simultaneous approximation by the Bernstein operator. We give a sufficient condition on the smoothness of the function that implies approximation rate of n^{-r}.

Keywords: Bernstein polynomials, iterates, Jackson inequality, upper error estimate, simultaneous approximation, modulus of smoothness.

2000 Math. Subject Classification: Primary 41A10, Secondary 41A17, 41A25, 41A28, 41A35, 41A36

1. MAIN RESULTS

Probably the most investigated linear approximating operator is the Bernstein polynomial, defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is known (see [1, Chapter 10, § 7] and [5, Chapter 9]) that there exists $n_0 \in \mathbb{N}$ such that for all $f \in C[0,1]$ and $n \geq n_0$ there holds

$$\|B_n f - f\| \leq c\,\omega_\varphi^2(f, n^{-1/2}), \tag{1.1}$$

where $\| \circ \|$ stands for the uniform norm on the interval $[0,1]$, c is an absolute constant and $\omega_\varphi^2(f,t)$ is the Ditzian-Totik modulus of smoothness of second order with step-weight $\varphi(x) = \sqrt{x(1-x)}$, defined by (see [5, Chapter 1])

$$\omega_\varphi^2(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi}^2 f\|$$

and

$$\Delta_{h\varphi(x)}^2 f(x) = \begin{cases} f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x)), & x \pm h\varphi(x) \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

For $f \in AC_{loc}^1(0,1)$ and $n \in \mathbb{N}$ we have

$$\|B_n f - f\| \le \frac{c}{n} \|\varphi^2 f''\|. \tag{1.2}$$

Moreover, $B_n f$ cannot tend to f in $C[0,1]$ faster than n^{-1} unless f is a linear function, in which case we have $B_n f = f$ for all n (see e.g. [1, Chapter 10, § 5]).

One way to modify the Bernstein operator in order to get larger approximation rate is to form an appropriate linear combination of its iterates. Here we shall consider the bounded linear operator $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$, defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$. Our main objective is to establish the following upper estimate of the error of $\mathcal{B}_{r,n}$.

Theorem 1.1. *For $f \in C^{2r-2}[0,1]$ and $r \ge 2$, there holds*

$$\|\mathcal{B}_{r,n} f - f\| \le \frac{c}{n^{r-1}} \left(\omega_\varphi^2(\varphi^{2r-2} f^{(2r-2)}, n^{-1/2}) + \frac{1}{n}\|f^{(2r-2)}\| + \frac{1}{n}\|f^{(2)}\| \right).$$

The value of the constant c is independent of f and n.

The above implies a sufficient condition on the smoothness of the function, which yields an approximation order of n^{-r}.

Corollary 1.2. *Let $f \in C[0,1]$ and $n, r \in \mathbb{N}$ as $r \ge 2$. Then:*

(a) $$\|\mathcal{B}_{r,n} f - f\| \le \frac{c}{n^{r-1/2}} \left(\|\varphi^{2r-1} f^{(2r-1)}\| + \|f^{(2r-2)}\| + \|f^{(2)}\| \right), \quad f \in AC_{loc}^{2r-2}(0,1);$$

(b) $$\|\mathcal{B}_{r,n} f - f\| \le \frac{c}{n^r} \left(\|\varphi^{2r} f^{(2r)}\| + \|f^{(2r-2)}\| + \|f^{(2)}\| \right), \quad f \in AC_{loc}^{2r-1}(0,1).$$

The value of the constant c is independent of f and n.

In order to extend the estimates above for every continuous functions we can introduce the K-functional

$$K_r(f,t)=\inf_{g\in AC_{loc}^{2r-1}}\left\{\|f-g\|+t\big(\|\varphi^{2r}g^{(2r)}\|+\|g^{(2r-2)}\|+\|g^{(2)}\|\big)\right\}.$$

for $f\in C[0,1]$, $t>0$ and $r\in\mathbb{N}$ with $r\geq 2$. Standard considerations imply the following Jackson-type inequality from Corollary 1.2 (b).

Theorem 1.3. *Let $f\in C[0,1]$ and $n,r\in\mathbb{N}$ as $r\geq 2$. Then*

$$\|\mathcal{B}_{r,n}f-f\|\leq c\,K_r(f,n^{-r}).$$

The value of the constant c is independent of f and n.

Let us note that

$$K_r(f,t^{2r})\leq c\big(\omega^{2r}(f,t)+t^{2r}\|f\|\big),\quad f\in C[0,1],\quad t>0, \tag{1.3}$$

where $\omega^{\ell}(f,t)$ is the classical fixed-step modulus of smoothness of order ℓ, defined by

$$\omega^{\ell}(f,t)=\sup_{0<h\leq t}\|\Delta_h^{\ell}f\|$$

and Δ_h^{ℓ} is the ℓth symmetric finite difference

$$\Delta_h^{\ell}f(x)=\begin{cases}\displaystyle\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}f\left(x+\left(\frac{\ell}{2}-k\right)h\right), & x\pm\dfrac{\ell h}{2}\in[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

The inequality (1.3) follows from the embedding inequality

$$\|f^{(m)}\|\leq c\big(\|f\|+\|f^{(\ell)}\|\big),\quad m=0,\dots,\ell, \tag{1.4}$$

and the well-known result of Johnen (see e.g. [1, Chapter 6, Theorem 2.4])

$$\inf_{g\in AC^{\ell-1}[0,1]}\left\{\|f-g\|+t^{\ell}\|g^{(\ell)}\|\right\}\leq c\,\omega^{\ell}(f,t),\quad f\in C[0,1].$$

All estimates with the Ditzian-Totik modulus are established for $n\geq n_0$ with some absolute constant n_0. However, the assertions of Corollary 1.2 and Theorem 1.3 are valid for all n (see Remark 3.6 at the end).

We base our proof of Theorem 1.1 on upper estimates for simultaneous approximation by Bernstein polynomials. They are established in the next section. This approach lays stronger conditions on the function than necessary but provides us with a simple proof. We verify Theorem 1.1 (and its corollary) in the third and final section.

There is a simple method for deriving upper estimates for combinations of iterates of a linear operator by iterating the estimate for the operator (see [4, Theorem 10.2 and Corollary 10.3]). However, it is not applicable in the case of the Bernstein operator because it does not commute with the associated differential operator $Dg = \varphi^2 g''$. Another difficulty of a technical character lies with the fact that $\mathcal{B}_{r,n}$ is not generally a positive operator. In order to get round the latter, we shall establish upper estimates that are similar to (1.1) for simultaneous approximation. This will allow us to get the result about $\mathcal{B}_{r,n}$ still by a certain iteration. This approach has a shortcoming. It misses the point that $\mathcal{B}_{r,n}$ provides better approximation near the ends of the interval $[0,1]$ (it interpolates f at 0 and 1). The simultaneous approximation by B_n does not possess this property.

Our first result concerns the unweighted simultaneous approximation by B_n.

Theorem 2.1. *For $f \in C^s[0,1]$ there holds*

$$\|(B_n f - f)^{(s)}\| \le c\left(\omega_\varphi^2(f^{(s)}, n^{-1/2}) + \omega(f^{(s)}, n^{-1}) + \frac{1}{n}\|f^{(s)}\|\right).$$

The value of the constant c is independent of f and n.

Proof. The assertion is trivial for $n < s$. For $n \ge s$ it is known (see [14] or [1, Chapter 10, (2.3)], [5, p. 125]) that

$$(B_n f)^{(s)}(x) = \frac{n!}{(n-s)!} \sum_{k=0}^{n-s} \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{k}{n}\right) p_{n-s,k}(x), \tag{2.1}$$

where $\overrightarrow{\Delta}_h^s f(x) = \Delta_h^s f(x + sh/2)$ are the forward differences of order s.

Now, for $n = s$ the above formula immediately implies the assertion of the theorem. Let $n > s$. We set

$$\widetilde{D}_{s,n} f(x) = n^s \overrightarrow{\Delta}_{1/n}^{s} f\left(\frac{n-s}{n} x\right), \quad x \in [0,1].$$

Then by (2.1)

$$(B_n f)^{(s)}(x) = \frac{n!}{n^s (n-s)!} B_{n-s}(\widetilde{D}_{s,n} f)(x), \quad x \in [0,1]. \tag{2.2}$$

Hence

$$\left\| \frac{n^s (n-s)!}{n!} (B_n f)^{(s)} - B_{n-s}(f^{(s)}) \right\| \le \|\widetilde{D}_{s,n} f - f^{(s)}\|.$$

Consequently,

$$\begin{aligned} \|(B_n f - f)^{(s)}\| \le {} & \left(\frac{n^s (n-s)!}{n!} - 1\right) \|(B_n f)^{(s)}\| \\ & + \|\widetilde{D}_{s,n} f - f^{(s)}\| + \|B_{n-s}(f^{(s)}) - f^{(s)}\|. \end{aligned} \tag{2.3}$$

We shall estimate the three quantities on the right above separately.

First, due to (2.2), we have

$$\begin{aligned}\left(\frac{n^s(n-s)!}{n!}-1\right)\|(B_nf)^{(s)}\| &= \left(1-\frac{n!}{n^s(n-s)!}\right)\|B_{n-s}(\widetilde{D}_{s,n}f)\| \\ &\le \frac{c}{n}\,\|\widetilde{D}_{s,n}f\| \le \frac{c}{n}\,\|f^{(s)}\|.\end{aligned} \tag{2.4}$$

The finite forward difference of order s of $F \in AC^{s-1}[a,b]$ can be represented in the integral form

$$\overrightarrow{\Delta}_h^s F(x) = h^{s-1}\int_0^{sh} M_s(u/h)F^{(s)}(x+u)\,du, \quad x \in [a, b-sh], \tag{2.5}$$

where M_s is the s-fold convolution of the characteristic function of $[0,1]$ with itself (see e.g. [1, p. 45]). Consequently,

$$\widetilde{D}_{s,n}f(x) = n\int_0^{s/n} M_s(nu)f^{(s)}\left(\frac{n-s}{n}\,x+u\right)du, \quad x\in[0,1],$$

and

$$\begin{aligned}|\widetilde{D}_{s,n}f(x)-f^{(s)}(x)| &\le n\int_0^{s/n} M_s(nu)\left|f^{(s)}\left(\frac{n-s}{n}\,x+u\right)-f^{(s)}(x)\right|du \\ &\le c\,\omega(f^{(s)},n^{-1}), \quad x\in[0,1].\end{aligned} \tag{2.6}$$

Above we have used that

$$\int_0^s M_s(u)\,du = 1.$$

Finally, by (1.1) and [5, Theorem 4.1.2] we get that there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$\|B_{n-s}(f^{(s)})-f^{(s)}\| \le c\,\omega_\varphi^2(f^{(s)},(n-s)^{-1/2}) \le c\,\omega_\varphi^2(f^{(s)},n^{-1/2}). \tag{2.7}$$

Now, (2.3), (2.4), (2.6) and (2.7) imply the assertion of the theorem. □

Remark 2.2. Based on Ditzian [3], Jiang and Xie [11] (or see [12, (16)]) gave a pointwise generalization of

$$\left\|\frac{n^s(n-s)!}{n!}\,(B_nf)^{(s)}-f^{(s)}\right\| \le c\left(\omega_\varphi^2(f^{(s)},n^{-1/2})+\omega(f^{(s)},n^{-1})\right).$$

Theorem 2.1, the property of the moduli (see [5, Theorem 2.1.1] or [1, Chapter 6, Theorem 6.1])

$$\omega_\varphi^2(f,t) \le c\,t^2\|\varphi^2 f''\|, \quad f\in AC_{loc}^1(0,1),$$

and (1.4) imply the following estimate for the simultaneous approximation by the Bernstein polynomials (cf. [7]).

Corollary 2.3. *For $f \in C^{s+2}[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|(B_n f - f)^{(s)}\| \le \frac{c}{n}\left(\|f^{(s+2)}\| + \|f^{(s)}\|\right).$$

The value of the constant c is independent of f and n.

Let us mention that Gonska, Heilmann and Raşa [9] established a quantitative Voronovskaya-type theorem about simultaneous approximation by B_n. They also gave an account of other similar results.

Inequalities like the one in Theorem 2.1 but in terms of the classical moduli of smoothness were earlier established in [8] and [13].

A somewhat neater upper estimate holds in terms of the differential operator $\varphi^{2s}(d/dx)^{2s}$.

Theorem 2.4. *For $f \in C^{2s}[0,1]$ there holds*

$$\|\varphi^{2s}(B_n f - f)^{(2s)}\| \le c\left(\omega_\varphi^2(\varphi^{2s} f^{(2s)}, n^{-1/2}) + \frac{1}{n}\|f^{(2s)}\|\right).$$

The value of the constant c is independent of f and n.

Proof. The assertion is trivial for $n < 2s$. Let $n \ge 2s$. Using (2.1) we get

$$\begin{aligned}\varphi^{2s}(x)(B_n f)^{(2s)}(x) &= \sum_{k=s}^{n-s} \Delta_{1/n}^{2s} f\left(\frac{k}{n}\right) \frac{k!\,(n-k)!}{(k-s)!\,(n-k-s)!}\, p_{n,k}(x) \\ &= B_n(D_{s,n} f)(x),\end{aligned} \tag{2.8}$$

where we have set

$$D_{s,n} f(x_{n,k}) = \varphi_{s,n}(x_{n,k})\, n^{2s} \Delta_{1/n}^{2s} f(x_{n,k}), \quad x_{n,k} = \frac{k}{n}, \quad k = 0, 1, \dots, n,$$

and

$$\varphi_{s,n}(x) = \prod_{i=0}^{s-1}\left(x - \frac{i}{n}\right)\left(1 - x - \frac{i}{n}\right),$$

as $D_{s,n} f(x_{n,k})$ is defined to be 0 for $k = 0, \dots, s-1, n-s+1, \dots, n$.

Next, we get by means of (1.1) and (2.8) that for $n \ge n_0$ with some $n_0 \in \mathbb{N}$

$$\begin{aligned}&\|\varphi^{2s}(B_n f - f)^{(2s)}\| \\ &\quad\le \|B_n(\varphi^{2s} f^{(2s)}) - \varphi^{2s} f^{(2s)}\| + \|\varphi^{2s}(B_n f)^{(2s)} - B_n(\varphi^{2s} f^{(2s)})\| \\ &\quad\le c\left(\omega_\varphi^2(\varphi^{2s} f^{(2s)}, n^{-1/2}) + \max_{k=0,\dots,n} |D_{s,n} f(x_{n,k}) - \varphi^{2s}(x_{n,k}) f^{(2s)}(x_{n,k})|\right).\end{aligned}$$

For $k = 0$ and $k = n$, we have $D_{s,n}f(x_{n,k}) = \varphi^{2s}(x_{n,k}) = 0$. For $k = 1, \dots, s-1$, $n-s+1, \dots, n-1$, $s \geq 2$, we directly get

$$\begin{aligned}|D_{s,n}f(x_{n,k}) - \varphi^{2s}(x_{n,k})f^{(2s)}(x_{n,k})| &= \varphi^{2s}(x_{n,k})|f^{(2s)}(x_{n,k})| \\ &\leq \frac{c}{n^s}\|f^{(2s)}\|.\end{aligned}$$

Further, for $k = s, \dots, n-s$ we use the representation (see (2.5))

$$\begin{aligned}\Delta_h^{2s}f(x) &= h^{2s-1}\int_{-sh}^{sh} M_{2s}(u/h+s)f^{(2s)}(x+u)\,du \\ &= h^{2s-1}\int_0^{sh} M_{2s}(u/h+s)[f^{(2s)}(x+u)+f^{(2s)}(x-u)]\,du, \quad x \in [sh, 1-sh],\end{aligned}$$

to get for $x \in [s/n, 1-s/n]$

$$\begin{aligned}|D_{s,n}f(x) - \varphi^{2s}(x)f^{(2s)}(x)| &\leq n\int_0^{s/n} M_{2s}(nu+s)|\Delta_u^2(\varphi^{2s}f^{(2s)})(x)|\,du \\ &\quad + n\int_{-s/n}^{s/n} M_{2s}(nu+s)|\varphi_{s,n}(x) - \varphi^{2s}(x+u)|\,|f^{(2s)}(x+u)|\,du \\ &\leq c\left(\omega^2(\varphi^{2s}f^{(2s)}, n^{-1}) + \frac{1}{n}\|f^{(2s)}\|\right).\end{aligned}$$

Above we have also taken into account the trivial estimate

$$\begin{aligned}|\varphi_{s,n}(x) - \varphi^{2s}(x+u)| &\leq |\varphi_{s,n}(x) - \varphi^{2s}(x)| + |\varphi^{2s}(x) - \varphi^{2s}(x+u)| \\ &\leq \frac{c}{n} + c\,|u| \leq \frac{c}{n}, \quad x \in [0,1],\ u \in \left[-\frac{s}{n}, \frac{s}{n}\right].\end{aligned}$$

To complete the proof of the theorem, we apply [5, Theorem 3.1.1], which gives that there exists t_0 such that

$$\omega^2(F, t^2) \leq c\,\omega_\varphi^2(F, t), \quad 0 < t \leq t_0,$$

for every $F \in C[0,1]$. □

Just as in the unweighted case, but using the embedding inequality (see [6, Lemma 1])

$$\|\chi^{\alpha+m}f^{(m)}\| \leq c\left(\|\chi^\alpha f\| + \|\chi^{\alpha+\ell}f^{(\ell)}\|\right), \quad m = 0, \dots, \ell,$$

where $\chi(x) = x$ and $\alpha \in \mathbb{R}$, we derive the following estimate.

Corollary 2.5. *For $f \in C[0,1]$ such that $f \in AC_{loc}^{2s+1}(0,1)$ and $n \in \mathbb{N}$ there holds*

$$\|\varphi^{2s}(B_nf - f)^{(2s)}\| \leq \frac{c}{n}\left(\|\varphi^{2s+2}f^{(2s+2)}\| + \|f^{(2s)}\|\right).$$

The value of the constant c is independent of f and n.

3. PROOF OF THEOREM 1.1

The estimates of the error of $\mathcal{B}_{r,n}$ can now be quite straightforwardly established by means of the results on simultaneous approximation of the previous section.

Proof of Theorem 1.1. First, the estimate (1.2) implies

$$\|\mathcal{B}_{r,n}f - f\| = \|(B_n - I)^r f\| \le \frac{c}{n}\|\varphi^2[(B_n - I)^{r-1}f]''\|.$$

For $r = 2$ we estimate above the right side of this inequality by means of Theorem 2.4 and get the assertion in this case. For $r \ge 3$ we apply instead Corollary 2.5 and arrive at

$$\|\mathcal{B}_{r,n}f - f\| \le \frac{c}{n^2}\left(\|\varphi^4[(B_n - I)^{r-2}f]^{(4)}\| + \|[(B_n - I)^{r-2}f]^{(2)}\|\right).$$

Further, we estimate the first term on the right above by Corollary 2.5 and the second by Corollary 2.3 and continue in this way, applying also (1.4), until we get

$$\|\mathcal{B}_{r,n}f - f\| \le \frac{c}{n^{r-1}}\Big(\|\varphi^{2r-2}[(B_n - I)f]^{(2r-2)}\| + \|[(B_n - I)f]^{(2r-4)}\| + \|[(B_n - I)f]^{(2)}\|\Big).$$

Now, the assertion of the theorem follows from Theorem 2.4, Corollary 2.3 and (1.4). □

Proof of Corollary 1.2. Assertion (a) follows from Theorem 1.1 and the property (see [5, Theorems 2.1.1 and 4.1.3] or [1, Chapter 6, Theorem 6.1])

$$\omega_\varphi^2(f,t) \le c\,t\,\|\varphi f'\|, \quad f \in AC_{loc}(0,1), \quad 0 < t \le t_0.$$

Assertion (b) follows from Theorem 1.1 just as Corollary 2.5 follows from Theorem 2.4. □

Remark 3.6. Let us note that in all estimates with the Ditzian-Totik modulus we had to assume that $n \ge n_0$ with some absolute constant n_0 since (1.1) was proved under this restriction and some of the properties of the modulus we used are known only for t small enough. However, (1.2) as well as its analogue with $n^{-1/2}\|\varphi f'\|$ on the right are valid for all $n \in \mathbb{N}$ and hence all the corollaries as well as Theorem 1.3 are valid for all n.

NOTE ADDED IN PROOF. After submission I learned of the papers of H. Gonska and X.-l. Zhou [10], and of Ch. Ding and F. Cao [2], where results that are similar to and somewhat stronger than Theorem 1.3 were established. The techniques used there are different. Also, I learned of a paper by Sevy [15] who established upper estimates for the unweighted simultaneous approximation by such combinations of iterates of an operator, following just the same idea like the one used in the proof of Theorem 1.1. I am thankful to Prof. G. Tachev (University of Architecture, Civil Engineering and Geodesy, Sofia) for helping me find out those papers. In a subsequent publication I am going to show how the results proved in the present paper can be improved to include those in the above-mentioned works (in the univariate case).

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ON TURÁN'S INEQUALITY FOR ULTRASPHERICAL POLYNOMIALS

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We present a short proof of the Turán inequality for the ultraspherical polynomials. The proof makes use of the Hermite interpolation formula. A recent refinement of Turán's inequality for ultraspherical polynomials [8] is discussed and compared with the known results.

Keywords: Turán-type inequalities, Hermite interpolation formula, ultraspherical polynomials

2010 Math. Subject Classification: Primary 41A17, Secondary 33C45

1. INTRODUCTION

In the 40's of the last century, while studying the zeros of Legendre polynomials $P_n(x)$, P. Turán discovered the inequality

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \quad -1 \leq x \leq 1, \tag{1.1}$$

with equality only for $x = \pm 1$. Since the left-hand side of (1.1) is representable in determinant form,

$$\Delta_n(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}$$

$\Delta_n(x)$ is referred to as *Turán's determinant*.

The result of Turán inspired considerable interest, and by now there is a vast amount of publications on the so-called *Turán type inequalities*. G. Szegő [12] gave four different proof of (1.1). Soon after that, inequalities of similar nature were obtained for other classes of functions including ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions, etc. Let us briefly recall a general approach for derivation of Turán type inequalities

$$u_n^2(x) - u_{n-1}(x)u_{n+1}(x) \geq 0, \tag{1.2}$$

due to Skovgaard [9]. This approach is applicable to sequences of functions $\{u_n(x)\}$, which possess a generating function $F(x;z) =: F(z)$,

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z)\,,$$

and, in addition, the generating function $F(z)$ belongs to the Laguerre-Pólya class of entire functions. The latter class consists of the uniform limits on compact sets in the complex plane of algebraic polynomials having only real zeros. Every function from the Laguerre-Pólya class is representable in the form

$$F(z) = Ce^{-\alpha z^2+\beta z} z^r \prod_{m=1}^{\infty}(1-z/z_m)e^{z/z_m}\,, \tag{1.3}$$

where $\alpha \geq 0$, C, β and z_m are real numbers, and $\sum_{m=1}^{\infty} z_m^{-2} < \infty$.

The logarithmic differentiation of (1.3) yields

$$\frac{d}{dz}\Big(\frac{F'(z)}{F(z)}\Big) = -2\alpha - \frac{r}{z^2} - \sum_m \frac{1}{(z-z_m)^2},$$

and obviously the right-hand side is negative for every real z. Hence,

$$\frac{d}{dz}\Big(\frac{F'(z)}{F(z)}\Big) = \frac{F(z)F''(z)-(F'(z))^2}{F(z)^2} \leq 0, \quad z \in \mathbb{R}\,,$$

and therefore $(F'(z))^2 - F(z)F''(z) \geq 0$ for every $z \in \mathbb{R}$. Since the Lagguerre-Pólya class is invariant with respect to differentiation, it follows that for every $n \in \mathbb{N}$

$$(F^{(n)}(z))^2 - F^{(n-1)}(z)F^{(n+1)}(z) \geq 0, \quad z \in \mathbb{R}\,.$$

Now, by substituting $z = 0$ one immediately arrives at (1.2). The range of $x \in \mathbb{R}$ for which (1.2) is true is determined by the condition that $F(z) = F(x;z)$ belongs to the Lagguerre-Pólya class.

The approach described above is applicable to wide classes of orthogonal polynomials and other special functions. The history of case of Jacobi polynomials $P_n^{(\alpha,\beta)}$ is especially interesting. In 1960 S. Karlin and G. Szegő [6] posed the problem for characterizing the range of parameters $\{\alpha,\beta\}$, for which the normalized Jacobi polynomials $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ (so that $R_n^{(\alpha,\beta)}(1) = 1$) satisfy the Turán type inequality

$$\big(R_n^{(\alpha,\beta)}(x)\big)^2 - R_{n-1}^{(\alpha,\beta)}(x)R_{n+1}^{(\alpha,\beta)}(x) \geq 0, \quad x \in [-1,1]\,. \tag{1.4}$$

Szegő [13] proved that (1.4) is true when $\beta \geq |\alpha|$, $\alpha > -1$. In two subsequent papers G. Gasper [3, 4] improves consecutively Szegő's result, showing finally that (1.4) holds true if and only if $\beta \geq \alpha > -1$, thus solving the problem of Karlin and Szegő. The particular case $\alpha = \beta$ corresponds to the ultraspherical (or Gegenbauer) polynomials, which is the topic of this note. We recall below some well-known fact about ultraspherical polynomials. $P_n^{(\lambda)}(x)$ is the standard notation for the n-th ultraspherical polynomial, which is orthogonal in $[-1, 1]$ with respect to the weight function $w_\lambda(x) = (1 - x^2)^{\lambda - \frac{1}{2}}$. The standard normalization of $P_n^{(\lambda)}$ is $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$, but for Turán's type inequalities the appropriate normalization is

$$p_n^{(\lambda)}(x) := P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)\,. \tag{1.5}$$

With this notation, Turán's inequality for ultraspherical polynomials reads as

Theorem 1. ([9, 15, 16]) *For every $\lambda > -1/2$,*

$$\Delta_{n,\lambda}(x) := \left[p_n^{(\lambda)}(x)\right]^2 - p_{n-1}^{(\lambda)}(x)p_{n+1}^{(\lambda)}(x) \geq 0, \quad x \in [-1, 1], \tag{1.6}$$

and the equality occurs only for $x = \pm 1$.

For the sake of simplicity, if there is no danger of ambiguity, hereafter the superscript $^{(\lambda)}$ will be omitted, and we shall write $p_n(x)$ instead of $p_n^{(\lambda)}(x)$.

We refer the reader to two important recent papers and the literature cited therein. R. Szwarc [14] obtained rather general sufficient conditions for sequences of orthogonal (with respect to a measure μ with a finite support, say, $supp\,\mu = [-1, 1]$) polynomials to satisfy Turán's type inequality on the support of the measure. In [1], C. Berg and R. Szwarc studied the behavior of the normalized Turán determinants $\widetilde{\Delta}_n(x) := \Delta_n(x)/(1 - x^2)$, in particular conditions ensuring monotonicity of $\widetilde{\Delta}_n(x)$ are established. Both in [14] and [1] the conditions are expressed through the sequences of the coefficients in the three-term recurrence relation satisfied by the orthogonal polynomials.

In the next section we present a short proof of Theorem 1, based on the Hermite interpolation formula. In Section 3 a recent refinement of Theorem 1 obtained in [8] is presented and compared with the hitherto known results.

2. THEOREM 1 THROUGH HERMITE'S INTERPOLATION FORMULA

2.1. PRELIMINARIES

It is well-known that the classical orthogonal polynomials of Jacobi, Hermite and Laguerre satisfy second order ordinary differential equations. In particular, the n-th ultraspherical polynomial $P_n^{(\lambda)}$ satisfies the differential equation

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0\,, \quad y(x) = P_n^{(\lambda)}(x)\,. \tag{2.1}$$

Since the derivatives of the Jacobi, Hermite and Laguerre polynomials are also orthogonal polynomials, they satisfy certain first order difference-differential equations (DDEs). Here we shall need some DDEs satisfied by the ultraspherical polynomials $P_n^{(\lambda)}$. For easy reference, they are collected in the following lemma.

Lemma 1. *The ultraspherical polynomials satisfy the following identities:*

$$(n+1)P_{n+1}^{(\lambda)}(x) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x) = 2(n+\lambda)x\,P_n^{(\lambda)}(x)\,, \tag{2.2}$$

$$nP_n^{(\lambda)}(x) = x\,\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} - \frac{d}{dx}\big\{P_{n-1}^{(\lambda)}(x)\big\}\,, \tag{2.3}$$

$$(n+2\lambda)P_n^{(\lambda)}(x) = \frac{d}{dx}\big\{P_{n+1}^{(\lambda)}(x)\big\} - x\,\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\}\,, \tag{2.4}$$

$$(1-x^2)\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} = -nxP_n^{(\lambda)}(x) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x)\,, \tag{2.5}$$

$$(1-x^2)\frac{d}{dx}\big\{P_n^{(\lambda)}(x)\big\} = (n+2\lambda)xP_n^{(\lambda)}(x) - (n+1)P_{n+1}^{(\lambda)}(x)\,. \tag{2.6}$$

See [11], Eqs. (4.7.17), (4.7.28) and (4.7.27).

As was mentioned in the preceding section, we shall work with the renormalized ultraspherical polynomials $p_m(x)$, defined by $p_m(x) = \binom{m+2\lambda-1}{m}^{-1} P_m^{(\lambda)}(x)$ (the dependence of p_m on λ is suppressed, as $\lambda > -1/2$ is fixed). On using Lemma 1, it is easy to derive the analogous relations satisfied by $\{p_m\}$.

Lemma 2. *The polynomials* $\{p_m\} = \{p_m^{(\lambda)}\}$ *defined by* (1.5) *satisfy the following identities:*

$$(n+2\lambda)p_{n+1}(x) + n\,p_{n-1}(x) = 2(n+\lambda)x\,p_n(x)\,, \tag{2.7}$$

$$p'_{n-1}(x) = (n+2\lambda-1)\Big[\frac{x}{n}\,p'_n(x) - p_n(x)\Big]\,, \tag{2.8}$$

$$p'_{n+1}(x) = (n+1)\Big[p_n(x) + \frac{x}{n+2\lambda}\,p'_n(x)\Big]\,, \tag{2.9}$$

$$p_{n-1}(x) = \frac{1-x^2}{n}\,p'_n(x) + x\,p_n(x)\,, \tag{2.10}$$

$$p_{n+1}(x) = x\,p_n(x) - \frac{1-x^2}{n+2\lambda}\,p'_n(x)\,. \tag{2.11}$$

Let $\{x_k\}_{k=1}^n$ be the zeros of $p_n(x)$; they are all distinct and located in $(-1,1)$. For any function f defined in $[-1,1]$ and differentiable in $(-1,1)$, let $H_{2n+1}(f;x)$ be the Hermite interpolating polynomial satisfying the interpolatory conditions

$$\begin{aligned} &H_{2n+1}(f;-1) = f(-1), \quad H_{2n+1}(f;1) = f(1)\,, \\ &H_{2n+1}(f;x_k) = f(x_k), \quad\ H'_{2n+1}(f;x_k) = f'(x_k), \\ &\qquad\qquad\qquad\qquad\qquad\qquad (k=1,2,\dots,n)\,. \end{aligned} \tag{2.12}$$

Lemma 3. *If f is a function defined in $[-1,1]$ and differentiable in $(-1,1)$, which satisfies $f(-1)=f(1)=0$, then*

$$H_{2n+1}(f;x)=\sum_{k=1}^{n}\Big[\Phi_{k,0}(x)f(x_k)+\Phi_{k,1}(x)f'(x_k)\Big], \tag{2.13}$$

where, for $k=1,2,\dots,n$,

$$\Phi_{k,0}(x)=\frac{1-x^2}{1-x_k^2}\,\ell_k^2(x)\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right],$$

$$\Phi_{k,1}(x)=\frac{1-x^2}{1-x_k^2}\,\ell_k^2(x)\,(x-x_k)\,,$$

and

$$\ell_k(x):=\frac{p_n(x)}{(x-x_k)p_n'(x_k)}$$

is the k-th Lagrange basis polynomial for interpolation at the zeros of p_n.

Proof. All we need is to show that $\{\Phi_{k,0}(x)\}$ and $\{\Phi_{k,1}(x)\}$ are the Hermite basis polynomials for interpolation at the nodes $-1,x_1,x_1,x_2,x_2,\dots,x_n,x_n,1$.

Obviously, $\Phi_{k,j}(\pm1)=0$ for $j=0,1$, $\Phi_{k,1}(x_i)=0$ and $\Phi_{k,0}(x_i)=\delta_{i,k}$ for $i,k=1,2,\dots,n$, where $\delta_{i,k}=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$ is the Kronecker symbol. It remains to verify that $\Phi_{k,j}'(x_i)=\delta_{i,k}\,\delta_{j,1}$ for $i,k=1,2,\dots,n$ and $j=0,1$. The verification is straightforward in the case $i=k$, $j=1$, and the same applies to the case $i\neq k$, $j=0,1$, since in that case $\frac{d}{dx}\big\{\ell_k^2(x)\big\}_{|x=x_i}=2\ell_k(x_i)\ell_k'(x_i)=0$. Now we consider the case $i=k$, $j=0$. By the L'Hospital rule we have

$$\begin{aligned}\frac{d}{dx}\{\ell_k^2(x)\}|_{x=x_k}&=2\ell_k'(x_k)=2\,\frac{p_n'(x)(x-x_k)-p_n(x)}{(x-x_k)^2p_n'(x_k)}\Big|_{x=x_k}\\ &=\frac{2}{p_n'(x_k)}\lim_{x\to x_k}\frac{p_n'(x)+p_n''(x)(x-x_k)-p_n'(x)}{2(x-x_k)}=\frac{p_n''(x_k)}{p_n'(x_k)}\,.\end{aligned}$$

Taking into account that $p_n(x_k)=0$ and $y=p_n$ satisfies (2.1), we find

$$p_n''(x_k)=\frac{(2\lambda+1)x_k\,p_n'(x_k)}{1-x_k^2}\quad\Rightarrow\quad\frac{d}{dx}\big\{\ell_k^2(x)\big\}\big|_{x=x_k}=\frac{(2\lambda+1)x_k}{1-x_k^2}\,.$$

Hence,

$$\begin{aligned}\Phi_{k,0}'(x_k)=&\left(\frac{1-x^2}{1-x_k^2}\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right]\right)'\Bigg|_{x=x_k}\ell_k^2(x_k)\\ &+\frac{1-x^2}{1-x_k^2}\left[1+(1-2\lambda)\,\frac{x_k(x-x_k)}{1-x_k^2}\right]\Bigg|_{x=x_k}\frac{d}{dx}\{\ell_k^2(x)\}|_{x=x_k}\\ =&\left(\frac{-2x_k}{1-x_k^2}+\frac{(1-2\lambda)x_k}{1-x_k^2}\right)\cdot1+1\cdot\frac{(2\lambda+1)x_k}{1-x_k^2}=0\,.\end{aligned}$$

Lemma 3 is proved. □

By the uniqueness of the Hermite interpolating polynomial we immediately obtain

Corollary 1. *Assume that $P(x)$ is an algebraic polynomial of degree not exceeding $2n+1$, and $P(-1)=P(1)=0$. Then*

$$P(x)=\sum_{k=1}^{n}\Big[\Phi_{k,0}(x)P(x_k)+\Phi_{k,1}(x)P'(x_k)\Big]\,.$$

2.2. PROOF OF THEOREM 1

We observe that $\Delta_{n,\lambda}(x)=\big[p_n(x)\big]^2-p_{n-1}(x)p_{n+1}(x)$ satisfies the assumptions of Corollary 1. Indeed, $\Delta_{n,\lambda}(x)$ is a polynomial of degree $2n$, and since $p_m(1)=1$ and $p_m(-1)=(-1)^m$, it follows that $\Delta_{n,\lambda}(\pm1)=0$. By Corollary 1,

$$\Delta_{n,\lambda}(x)=\sum_{k=1}^{n}\Big[\Phi_{k,0}(x)\Delta_{n,\lambda}(x_k)+\Phi_{k,1}(x)\Delta'_{n,\lambda}(x_k)\Big]\,. \tag{2.14}$$

We apply Lemma 2 to represent $p_{n-1}(x_k)$, $p_{n+1}(x_k)$, $p'_{n-1}(x_k)$ and $p'_{n+1}(x_k)$ in terms of $p'_n(x_k)$. We obtain

$$p_{n-1}(x_k)=\frac{1}{n}(1-x_k^2)p'_n(x_k),\quad p_{n+1}(x_k)=-\frac{1}{n+2\lambda}(1-x_k^2)p'_n(x_k),$$

$$p'_{n-1}(x_k)=\frac{n+2\lambda-1}{n}\,x_k\,p'_n(x_k),\quad p'_{n+1}(x_k)=\frac{n+1}{n+2\lambda}\,x_k\,p'_n(x_k)\,.$$

Next, we express $\Delta_{n,\lambda}(x_k)$ and $\Delta'_{n,\lambda}(x_k)$ in terms of $p'_n(x_k)$:

$$\Delta_{n,\lambda}(x_k)=-p_{n-1}(x_k)p_{n+1}(x_k)=\frac{1}{n(n+2\lambda)}\,(1-x_k^2)\big[p'_n(x_k)\big]^2\,,$$

$$\begin{aligned}\Delta'_{n,\lambda}(x_k)&=-p'_{n-1}(x_k)p_{n+1}(x_k)-p_{n-1}(x_k)p'_{n+1}(x_k)\\&=\frac{2(\lambda-1)}{n(n+2\lambda)}\,x_k(1-x_k^2)\,\big[p'_n(x_k)\big]^2\,.\end{aligned}$$

Replacement of $\Delta_{n,\lambda}(x_k)$ and $\Delta'_{n,\lambda}(x_k)$ in (2.14) yields

$$\Delta_{n,\lambda}(x)=\frac{1-x^2}{n(n+2\lambda)}\,\sum_{k=1}^{n}\ell_k^2(x)(1-x_kx)\,\big[p'_n(x_k)\big]^2\,.$$

This accomplishes the proof of Theorem 1, since $1-x_kx>0$ for $x\in[-1,1]$. □

3. A REFINEMENT OF TURÁN'S INEQUALITY

The Turán determinant $\Delta_{n,\lambda}(x)$ vanishes at ± 1, and a fine result of Thiruvenkatachar and Nanjundiah [15] (see also [16]) states that in $(0,\infty)$ the normalized Turán determinant

$$\varphi_{n,\lambda}(x) := \frac{\Delta_{n,\lambda}(x)}{1-x^2}$$

is monotone increasing when $\lambda > 0$ and monotone decreasing when $-1/2 < \lambda < 0$. In particular,

$$c_{n,\lambda} \le \varphi_{n,\lambda}(x) \le C_{n,\lambda}\,, \quad x \in [-1,1], \tag{3.1}$$

with the sharp constants $0 < c_{n,\lambda} < C_{n,\lambda}$ given by

$$c_{n,\lambda} = p_n^2(0) - p_{n-1}(0)p_{n+1}(0), \quad C_{n,\lambda} = \frac{1}{2\lambda+1}\,, \quad \text{if} \quad \lambda > 0\,,$$

and with the interchanged formulae for $c_{n,\lambda}$ and $C_{n,\lambda}$ if $-1/2 < \lambda < 0$. That is to say, $c_{n,\lambda}$ and $C_{n,\lambda}$ are the best possible bounds for $\varphi_{n,\lambda}(x)$ in the "uniform sense", i.e., for the whole interval $[-1,1]$. However, for particular x's improvements are possible.

Recently, in a joint work with V. Pillwein [8] the author proved the following result:

Theorem 2. *Let $p_m = p_m^{(\lambda)}$ be the m-th ultraspherical polynomial normalized by $p_m(1) = 1$, $m \in \mathbb{N}_0$. If $\lambda \in (-1/2, 1/2]$, then for every $n \in \mathbb{N}$*

$$\widetilde{\Delta}_{n,\lambda}(x) := |x| p_n^2(x) - p_{n-1}(x)p_{n+1}(x) \ge 0 \quad \forall x \in [-1,1]. \tag{3.2}$$

The equality in (3.2) *is attained only for $x = \pm 1$ and, if n is even, for $x = 0$. Moreover, if $\lambda > 1/2$, then* (3.2) *fails for every $n \in \mathbb{N}$.*

A computer proof of the special case $\lambda = 1/2$ of Theorem 2 was given earlier by Gerhold and Kauers [5].

In view of Theorem 2, $\Delta_{n,\lambda}(x) = \widetilde{\Delta}_{n,\lambda}(x) + (1-|x|)\,p_n^2(x) \ge (1-|x|)\,p_n^2(x)$ for $\lambda \in (-1/2, 1/2]$, hence

$$\varphi_{n,\lambda}(x) \ge \frac{p_n^2(x)}{1+|x|} =: g_{n,\lambda}(x)\,, \quad \lambda \in (-1/2, 1/2]. \tag{3.3}$$

A result of a similar nature, due to O. Szász [10], asserts that

$$\varphi_{n,\lambda}(x) \ge \frac{\lambda(1-p_n^2(x))}{(n+\lambda-1)(n+2\lambda)} =: h_{n,\lambda}(x)(1-x^2)\,, \quad \lambda \in (0,1)\,. \tag{3.4}$$

In view of (3.1), (3.3) and (3.4), it is of interest to compare $\varphi_{n,\lambda}(x)$ with its lower bounds

- $g_{n,\lambda}(x)$, $h_{n,\lambda}(x)$ and $c_{n,\lambda} = \min_{x\in[-1,1]} \varphi_{n,\lambda}(x) = p_n^2(0) - p_{n-1}(0)p_{n+1}(0)$ in the case $0 < \lambda \le 1/2$;
- $g_{n,\lambda}(x)$ and $c_{n,\lambda} = \min_{x\in[-1,1]} \varphi_{n,\lambda}(x) = 1/(2\lambda+1)$ in the case $-1/2 < \lambda < 0$.

The graphs of $\varphi_{n,\lambda}(x)$, $g_{n,\lambda}(x)$ and $h_{n,\lambda}(x)$ in the Legendre case ($\lambda = 1/2$) are depicted in Fig. 1 for $n = 12$ (left) and $n = 13$ (right). The same graphs for the case $\lambda = 1/4$ are shown in Fig. 2.

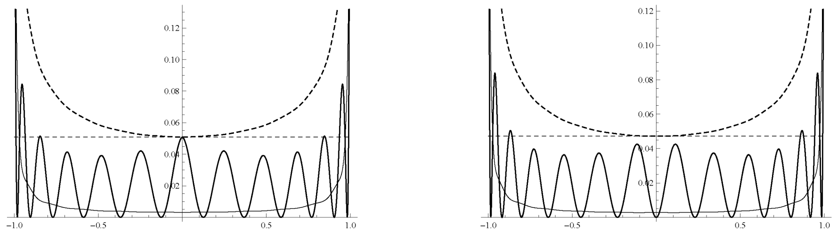

Fig. 1. Graphs of $g_{n,\lambda}(x)$ (thick), $h_{n,\lambda}(x)$ (thin) and $\varphi_{n,\lambda}(x)$ (dashed), $\lambda = 1/2$.

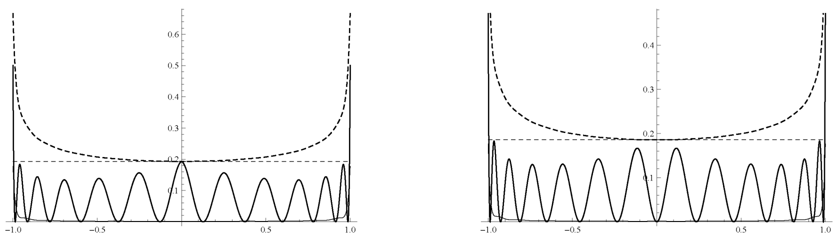

Fig. 2. Graphs of $g_{n,\lambda}(x)$ (thick), $h_{n,\lambda}(x)$ (thin) and $\varphi_{n,\lambda}(x)$ (dashed), $\lambda = 1/4$.

It is seen that the inequality $g_{n,\lambda}(x) \ge c_{n,\lambda}$ holds only near the endpoints of $[-1, 1]$, i.e., our pointwise lower bound $g_{n,\lambda}(x)$ for the normalized Turán determinant $\varphi_{n,\lambda}(x)$ improves upon the "uniform" lower bound $c_{n,\lambda}$ only on a subset of $[-1, 1]$ with a small measure. On the other hand, for most $x \in [-1, 1]$ our pointwise bound is better than the Szász one. This observation is typical for all $\lambda \in (0, 1/2)$.

The situation changes when λ is negative. Namely, in that case the inequality $g_{n,\lambda}(x) < c_{n,\lambda}$ holds only in some small neighborhoods of the zeros of p_n. That is to say, in the case $-1/2 < \lambda < 0$, $g_{n,\lambda}(x)$ provides better lower bounds than the "uniform" bound $c_{n,\lambda}$ except for a set of small measure in $[-1, 1]$. See Fig. 3 and Fig. 4 to compare the graphs of $\varphi_{n,\lambda}(x)$ and $g_{n,\lambda}(x)$ for $n = 12$ (left) and $n = 13$ (right) in the cases $\lambda = -1/4$ and $\lambda = -7/16$, respectively.

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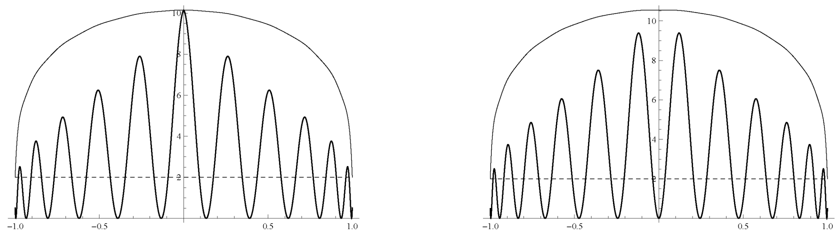

Fig. 3. Graphs of $g_{n,\lambda}(x)$ (thick) and $\varphi_{n,\lambda}(x)$ (thin), $\lambda = -\frac{1}{4}$ and $n = 12$ (left), $n = 13$ (right).

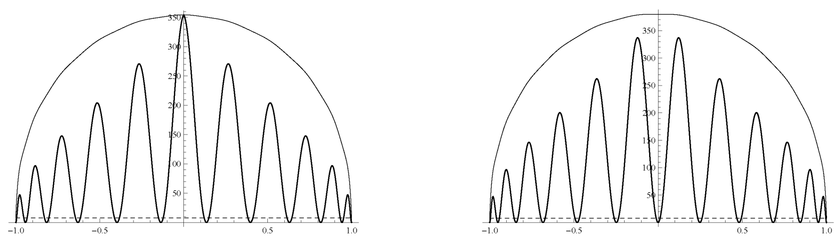

Fig. 4. Graphs of $g_{n,\lambda}(x)$ (thick) and $\varphi_{n,\lambda}(x)$ (thin), $\lambda = -\frac{7}{16}$ and $n = 12$ (left), $n = 13$ (right).

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 101

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

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A SIMPLE CHARACTERIZATION OF THE COMPUTABILITY OF REAL FUNCTIONS

DIMITER SKORDEV

The TTE-approach to computability of real functions uses infinitary names of the argument's and the function's values, computability being defined as the existence of some algorithmic procedure transforming the names of any argument's value into ones of the corresponding value of the function. Two ways to avoid using such names are considered in the present paper. At each of them, the corresponding characterization of computability of real functions is through the existence of an appropriate recursively enumerable set establishing some relation between rational approximations of the argument's value and rational approximations of the corresponding value of the function. The characterizations in question are derived from ones for computability of functions in metric and in topological spaces.

Keywords: Approximation, computable, continuous, enumeration, enumeration operator, real function, real number, recursive operator, recursively enumerable set, TTE

2010 Math. Subject Classification: 03D78

1. INTRODUCTION

The widely used TTE-approach to computability of real functions (cf. e.g. [6]) uses infinitary names of the argument's and the function's values, and computability is defined as the existence of some algorithmic procedure transforming all such names of any argument's value into ones of the corresponding value of the function. The standard TTE-computability of real functions[1] is a particular instance of

[1]I.e. the (ρ^p, ρ)-computability in the sense of [6] of partial functions from $\mathbb{R}^p$ to $\mathbb{R}$, and, more generally, the (ρ^p, ρ^q)-computability of partial functions from $\mathbb{R}^p$ to $\mathbb{R}^q$.

TTE-computability of functions in metric spaces, which, under some assumptions satisfied in this particular case, was characterized in [4] without using infinitary names. In the case in question, the corresponding characterization is through the existence of an appropriate recursively enumerable set establishing some relation between rational approximations of the argument's value and rational approximations of the corresponding value of the function. In [5, Example 3.10], a simpler similar characterization of the computability of real functions is given, and it is obtained by using the fact that the standard TTE-computability of real functions is a particular instance of TTE-computability of functions in topological spaces.[2] A somewhat more systematic consideration of these two characterizations is done in the present paper by introducing the notions of a metric approximation net and a topological approximation net for a real function. On the whole, the paper follows the slides of the author's talk at the 2013 Spring Scientific Conference of FMI[3], thus some details are omitted.

1.1. TWO CHARACTERIZATIONS OF THE COMPUTABILITY OF A REAL NUMBER

The two above-mentioned characterizations of computability of a real function can be regarded as analogs of the ones for the notion of computable real number which are indicated below.

Theorem 1. *For any real number y, the following three conditions are equivalent:*

A. The number y is computable.

B. A recursively enumerable set E of $\mathbb{Q} \times \mathbb{N}$ exists such that:

1. $\forall (b, n) \in E \left(|b - y| < \dfrac{1}{n+1} \right)$.

2. $\forall n \in \mathbb{N}\ \exists b \left((b, n) \in E \right)$.

C. The set $\left\{ (b, n) \in \mathbb{Q} \times \mathbb{N} \;\middle|\; |b - y| < \dfrac{1}{n+1} \right\}$ *is recursively enumerable.*

The proof of this theorem is straightforward.

Remark 1. *Of course, condition C is equivalent to the existence of a recursively enumerable subset E of $\mathbb{Q} \times \mathbb{N}$ such that*

$$|b - y| < \frac{1}{n+1} \Leftrightarrow (b, n) \in E$$

for any $b \in \mathbb{Q}$ and any $n \in \mathbb{N}$.

[2] TTE-computability in the topological case is considered, for instance, in [6, Section 3.2] and in [1,2,3,7].

[3] Held in Sofia on March 16, 2013.

Remark 2. *It can be non-constructively proved that Theorem 1 holds also with "recursive" instead of "recursively enumerable".*

1.2. SOME NOTATIONS, ASSUMPTIONS AND DEFINITIONS

For $z = (z_1, \dots, z_N) \in \mathbb{R}^N$, where $N \in \mathbb{N}^+$, we set

$$\|z\| = \max(|z_1|, \dots, |z_N|)$$

For $t \in \mathbb{N}$, we set $r_t = \dfrac{1}{t+1}$.

Throughout the paper, it will be supposed that

$$p, q \in \mathbb{N}^+,\ M \subseteq \mathbb{R}^p,\ f : M \to \mathbb{R}^q.$$

Two definitions follow. The notion introduced in the first one is a particular instance of a notion introduced in [4]. The second definition introduces a similar, but simpler notion. Some similarity can be observed between the conditions of these definitions and the conditions B and C in Theorem 1.

Definition 1. A metric approximation net *(abbr.* m.a.n.*) for the function f is a subset S of $\mathbb{Q}^p \times \mathbb{N} \times \mathbb{Q}^q \times \mathbb{N}$ such that the following conditions are satisfied for any $x \in M$:*

1. $\forall (a, m, b, n) \in S \big(\|a - x\| < r_m \Rightarrow \|b - f(x)\| < r_n\big)$.
2. $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall a \in \mathbb{Q}^p \big(\|a - x\| < r_m \Rightarrow \exists b \big((a, m, b, n) \in S\big)\big)$.

Definition 2. A topological approximation net *(abbr.* t.a.n.*) for the function f is a subset S of $\mathbb{Q}^p \times \mathbb{N} \times \mathbb{Q}^q \times \mathbb{N}$ such that*

$$\|b - f(x)\| < r_n \Leftrightarrow \exists a\, \exists m \big((a, m, b, n) \in S \,\&\, \|a - x\| < r_m\big) \tag{1.1}$$

for all $x \in M$, $b \in \mathbb{Q}^p$, $n \in \mathbb{N}$.

The two notions are different. The function f can be chosen so that a m.a.n. for f exists which is not a t.a.n. for it, and a t.a.n. for f exists which is not a m.a.n. for it.

Example 1. *Let $p = q = 1$, $M = \{0\}$, $f(0) = 0$, and let us set*

$$S_1 = \{(a, m, 0, n) \mid a \in \mathbb{Q},\ m, n \in \mathbb{N}\},$$
$$S_2 = \{(0, m, b, n) \mid b \in \mathbb{Q},\ m, n \in \mathbb{N},\ |b| < r_n\}.$$

Then S_1 is a m.a.n. for the function f without being a t.a.n. for it, and S_2 is a t.a.n. for the function f without being a m.a.n. for it.

Remark 3. *Definitions 1, 2 imply immediately that, whenever S is a t.a.n. for the function f, and some subset of S is a m.a.n. for it, the set S is also a m.a.n. for f.*

Despite the difference between the notions of m.a.n. and t.a.n., some essential properties of them are similar. The next theorem is a particular instance of a result from [4].

Theorem 2. *A m.a.n. for the function f exists if and only if f is continuous. Then the following set is a m.a.n. for f containing as subsets all such ones:*

$$\bigl\{(a,m,b,n)\in\mathbb{Q}^p\times\mathbb{N}\times\mathbb{Q}^q\times\mathbb{N}\,\big|\,\forall x\in M\bigl(\|a-x\|<r_m\ \Rightarrow\ \|b-f(x)\|<r_n\bigr)\bigr\}.$$

It is easily seen that Theorem 2 remains true after replacing m.a.n. with t.a.n. in its statement.

2. M.A.N., T.A.N. AND STANDARD TTE-COMPUTABILITY OF REAL FUNCTIONS

From now on, let $\alpha:\mathbb{N}\to\mathbb{Q}^p$ be a computable enumeration of $\mathbb{Q}^p$, and $\beta:\mathbb{N}\to\mathbb{Q}^q$ be a computable enumeration of $\mathbb{Q}^q$. In the terminology of [5], *an α-name* of an element x of $\mathbb{R}^p$ is any function $u:\mathbb{N}\to\mathbb{N}$ such that $\|\alpha(u(m))-x\|<r_m$ for all $m\in\mathbb{N}$, and similarly is defined what is a β-name of an element of $\mathbb{R}^q$. The function f is called (α,β)*-computable* if a recursive operator exists which transforms all α-names of any $x\in M$ into β-names of $f(x)$.

Clearly, the (α,β)-computability of f does not depend of the choice of the computable enumerations α and β, and it is equivalent to the (ρ^p,ρ^q)-computability of f.

The next theorem follows immediately from the main theorem in [4].

Theorem 3. *The function f is (ρ^p,ρ^q)-computable if and only if a recursively enumerable m.a.n. for f exists.*

In [5], another computability notion was considered besides (α,β)-computability. In the case considered here, it looks as follows. Suppose a computable bijective mapping of $\mathbb{N}^2$ of $\mathbb{N}$ is chosen, and let $\langle s,t\rangle$ denote the image of the pair (s,t) under this mapping. We consider the indexed base $\mathcal{U}=\{U_i\}_{i\in\mathbb{N}}$ of the space $\mathbb{R}^p$ and the indexed base $\mathcal{V}=\{V_j\}_{j\in\mathbb{N}}$ of the space $\mathbb{R}^q$, which are defined by means of the equalities

$$U_{\langle k,m\rangle}=\{x\in\mathbb{R}^p\,|\,\|\alpha(k)-x\|<r_m\},$$
$$V_{\langle l,n\rangle}=\{y\in\mathbb{R}^q\,|\,\|\beta(l)-y\|<r_n\}.$$

The function f is called $(\mathcal{U},\mathcal{V})$*-computable* if an enumeration operator exists which, for any $x\in M$, transforms the set $\{i\in\mathbb{N}\,|\,x\in U_i\}$ into the set $\{j\in\mathbb{N}\,|\,f(x)\in V_j\}$.

As seen from [5], standard TTE-computability and $(\mathcal{U},\mathcal{V})$-computability of f are equivalent.

In the general case studied in [5], some topological spaces $\mathbf{X}$ and $\mathbf{Y}$ with countable bases are considered instead of $\mathbb{R}^p$ and $\mathbb{R}^q$, and $\mathcal{U}=\{U_i\}_{i\in\mathbb{N}}$, $\mathcal{V}=\{V_j\}_{j\in\mathbb{N}}$ can be any indexed countable bases of these spaces. Under some assumptions, it is shown that the $(\mathcal{U},\mathcal{V})$-computability of f is equivalent to the existence of a recursively enumerable subset R of $\mathbb{N}^2$ with the following property:

$$\forall x\in M\ \forall j\in\mathbb{N}\,\big(\,f(x)\in V_j \Leftrightarrow \exists i\,\big(\,(i,j)\in R\ \&\ x\in U_i\,\big)\,\big) \tag{2.1}$$

(this is an improvement of a result from [3]).

The above-mentioned assumptions are satisfied in the case considered here thanks to the recursive enumerability of the sets

$$\{(a_1,a_2,r)\in\mathbb{Q}^p\times\mathbb{Q}^p\times\mathbb{Q}\mid \|a_1-a_2\|<r\},$$
$$\{(b_1,b_2,r)\in\mathbb{Q}^q\times\mathbb{Q}^q\times\mathbb{Q}\mid \|b_1-b_2\|<r\}$$

(these sets are even recursive). In this case, the property (2.1) is obviously equivalent to the following one:

$$\|\beta(l)-f(x)\|<r_n \Leftrightarrow \exists k,m\in\mathbb{N}\big((\langle k,m\rangle,\langle l,n\rangle)\in R\ \&\ \|\alpha(k)-x\|<r_m\big) \tag{2.2}$$

for any $x\in M$ and all $l,n\in\mathbb{N}$. Making use of (2.2), one easily gets the following result.

Theorem 4. *The function f is (ρ^p,ρ^q)-computable if and only if a recursively enumerable t.a.n. for f exists.*

Proof. Cf. Example 3.10 in [5]. ☐

2.1. SOME EXAMPLES OF RECURSIVELY ENUMERABLE T.A.N.'S

Example 2. *Let $p=q=1$, $M=\mathbb{R}\setminus\{0\}$, $f(x)=\dfrac{1}{x}$ for any $x\in M$, and let*

$$S=\left\{(a,m,b,n)\in\mathbb{Q}\times\mathbb{N}\times\mathbb{Q}\times\mathbb{N}\ \middle|\ r_m<|a|,\quad \left|b-\frac{1}{a}\right|+\frac{r_m}{|a|(|a|-r_m)}\le r_n\right\}.$$

We will show that S is a recursively enumerable t.a.n. for f. The recursive enumerability of this set is clear (it is even recursive). To prove that S is a t.a.n. for f, we have to show that, whenever $x\in\mathbb{R}\setminus\{0\}$, $b\in\mathbb{Q}$ and $n\in\mathbb{N}$, the inequality $\left|b-\dfrac{1}{x}\right|<r_n$ holds if and only if

$$r_m<|a|,\quad \left|b-\frac{1}{a}\right|+\frac{r_m}{|a|(|a|-r_m)}\le r_n,\quad |a-x|<r_m \tag{2.3}$$

for some $a \in \mathbb{Q}$ and some $m \in \mathbb{N}$. Let $x \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. If $a \in \mathbb{Q}$, $m \in \mathbb{N}$ and the inequalities (2.3) *hold, then $|x| > |a| - r_m$ and therefore*

$$\left|b - \frac{1}{x}\right| \le \left|b - \frac{1}{a}\right| + \frac{|x-a|}{|a|\,|x|} < \left|b - \frac{1}{a}\right| + \frac{r_m}{|a|(|a| - r_m)} \le r_n.$$

Suppose now that $\left|b - \frac{1}{x}\right| < r_n$. Then

$$r_m < |x|, \quad \left|b - \frac{1}{x}\right| + \frac{r_m}{|x|(|x| - r_m)} < r_n$$

for some $m \in \mathbb{N}$. At such a choice of m, the inequalities (2.3) *will be satisfied by any rational number a, which is sufficiently close to x.*

Remark 4. *It can be shown that $\{(a, m, b, n) \in S \mid ab = 1\}$ is a m.a.n. for f. Making use of Remark 3, we conclude that S is also a m.a.n. for f.*

Example 3. *Let $p = q = 1$, $M = \mathbb{R}$, $f(x) = \cos x$ for all $x \in M$. For any $k \in \mathbb{N}$, let S_k be the set of all $(a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N}$ satisfying the inequalities*

$$a^2 \le (2k+1)(2k+2), \quad |b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \le r_n, \tag{2.4}$$

where

$$\sigma_k(a) = (-1)^k \frac{a^{2k}}{2(2k)!} + \sum_{i<k} (-1)^i \frac{a^{2i}}{(2i)!}.$$

Let $S = \bigcup_{k=0}^{\infty} S_k$. The set S is recursively enumerable. We will show that it is a t.a.n. for the function f. Indeed, let $x \in M$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. We will prove that the equivalence (1.1) *holds. Suppose first that $(a, m, b, n) \in S$ for some a and m such that $|a - x| < r_m$. Then there exists some $k \in \mathbb{N}$ which satisfies the inequalities* (2.4)*, and, using it, we get*

$$\begin{aligned} |b - f(x)| &\le |b - \sigma_k(a)| + |\sigma_k(a) - \cos a| + |\cos a - \cos x| \\ &< |b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \le r_n\,. \end{aligned}$$

Conversely, let $|b - f(x)| < r_n$. Natural numbers k and m can be chosen which satisfy the inequalities

$$x^2 < (2k+1)(2k+2), \quad |b - f(x)| + \frac{x^{2k}}{(2k)!} + r_m < r_n,$$

and then

$$a^2 < (2k+1)(2k+2), \quad |b - f(a)| + \frac{a^{2k}}{(2k)!} + r_m < r_n, \quad |a - x| < r_m$$

for any rational number a sufficiently close to x. At such a choice of k, m and a, the quadruple (a, m, b, n) will belong to S_k, and therefore also to S, because then

$$|b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \leq |b - f(a)| + |\cos a - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m$$
$$\leq |b - f(a)| + \frac{a^{2k}}{(2k)!} + r_m < r_n.$$

Remark 5. *The same set S is shown in [4] to be a m.a.n. for f.*

Example 4. *Let $p = q = 1$, $M = \mathbb{R} \setminus \mathbb{Z}$, $f(x) = \lfloor x \rfloor$ for any $x \in M$. Then the recursive set*

$$S = \left\{ \left(k + \frac{1}{2}, 1, b, n \right) \middle| \, k \in \mathbb{Z} \,\&\, b \in \mathbb{Q} \,\&\, n \in \mathbb{N} \,\&\, |b - k| < r_n \right\}$$

is a t.a.s. for f. Indeed, let $x \in M$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. If $(a, m, b, n) \in S$ and $|a - x| < r_m$, then $a = k + \frac{1}{2}$, $|b - k| < r_n$ for some integer k, and $r_m = \frac{1}{2}$, thus $\left| k + \frac{1}{2} - x \right| < \frac{1}{2}$, hence $f(x) = k$ and therefore $|b - f(x)| < r_n$. Conversely, if $|b - f(x)| < r_n$ then $(a, m, b, n) \in S$ and $|a - x| < r_m$ for $a = f(x) + \frac{1}{2}$ and $m = 1$.

Remark 6. *The set S from Example 4 is not a m.a.n. for f, since condition 2 of Definition 1 is violated.*

Example 5. *Let p, q, M, f be the same as in Example 4, but S be the set of all $(a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N}$ which satisfy the inequalities*

$$a + r_m \leq \lfloor a - r_m \rfloor + 1, \quad |b - \lfloor a - r_m \rfloor| < r_n. \tag{2.5}$$

This set is recursive too. We will show that it is also a t.a.n. for f. Let $x \in M$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. If some $a \in \mathbb{Q}$ and $m \in \mathbb{N}$ satisfy the inequalities (2.5) *and the inequality $|a - x| < r_m$, then $f(x) = \lfloor a - r_m \rfloor$ and therefore $|b - f(x)| < r_n$. Conversely, if $|b - f(x)| < r_n$, then the inequalities* (2.5) *and the inequality $|a - x| < r_m$ can be satisfied by choosing some $m \in \mathbb{N}$ with $x - r_m > \lfloor x \rfloor$ and $x + r_m < \lfloor x \rfloor + 1$, and then choosing the rational number a sufficiently close to x.*

Remark 7. *The set S from Example 5 is a m.a.n. for f, since so is the set* $\{(a, m, b, n) \in S \,|\, b = \lfloor a - r_m \rfloor\}$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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weSPOT: INQUIRY-BASED SCIENCE EDUCATION APPROACH AND TECHNOLOGIES IN ACTION

KRASSEN S. STEFANOV, NIKOLINA N. NIKOLOVA, SVETOSLAV STAMENOV, TATIANA DIMITROVA, ELIZA P. STEFANOVA

Math, science and technology (MST) education in Europe is significantly based on experiments and problem solving. The innovative project weSPOT aims at enhancing MST education by emphasizing scientific inquiry, individualization of learning and the role of social interaction. To achieve this goal, students are supported by a set of software tools, helping them to plan, implement, comment and document their personal scientific experiments. The students are encouraged to share and discuss their experience and to learn how to conduct scientific research. They share their achievements in class as well as in their own social community (friends, family, scientists, etc.). This paper describes in details the scientific approach which is the base for the pilot experiments of the weSPOT methodology. It presents the first pilot of Inquiry-Based Science Education (IBSE) conducted in the First Privite Mathematical School and analyses applicability of the used software products.

Keywords: inquiry-based science education, technology-enhanced learning, weSPOT

2012 ACM Computing Classification: Education - collaborative learning, e-learning, mobile learning, collaborative and social computing - collaborative content creation, computer supported cooperative work, modeling and simulation - simulation tools, visualization toolkits, web services - mashups, ubiquitous and mobile computing systems and tools

1. INTRODUCTION

Modern conception of learning [1, 2] presents the aquiring of new knowledge and skills as a result of social interactions (conversations, discussions) and a practical solution of problems and tasks, the student interacts with the objects of reality, formulate their statements and assumptions seeks to justify and prove or refute. Unfortunately, the practice in most educational institutions do not comply with the requirements of the theory. Students at secondary schools and universities are mostly in a passive role in the classroom, and teachers are often in the role of mentors. Students are rarely motivated to take initiatives within their education and to expand it outside the school environment, provoked by curiosity.

One approach to solve the problem with the gap between theory and practice is the inquiry-based science education approach, in which students play the role of explorers and scientists as they try to address issues set by themselves, while finding answers to these questions is challenged by their own curiosity. This approach leverages a meaningful context for students to learn concepts by linking them with their personal experiences and insights. It leads to structured knowledge of the field of education and more skills to carry out effective research. In this way, students learn to explore, collaborate, be creative, use personal characteristics and identity, and have an impact in different environments and at different levels (e.g. individual, in the neighborhood, community, world).

Students can go through the process of inquiry-based learning at different levels of autonomy and complexity, respectively, with varying degrees of support [3]. At the lowest level, students are guided entirely by the teacher in defining the problem, choosing an appropriate method for studying it and finding a solution. At the highest level, called open inquiry, they drive the process of inquiry and analysis, make their own reasoning and explain the meaning of phenomena performing their activities individually or together, as well as sharing the knowledge representation they reached. The project weSPOT [4] aims at supporting the implementation of this approach through the design, development and testing of appropriate software tools that will enable students to:

- Customize their environment for inquiry-based education;
- Build, share and carry out research individually and/or in collaboration with their peers.

Thus, weSPOT aims to enable the connection of everyday life with training in subjects related to natural sciences in schools through the use of information and communication technologies (ICT).

From the perspective of European teachers, the weSPOT project will enable both teachers and students to apply an inquiry learning approach based on experiments, carried out in a real school environment. Such experiments can be supported by computer simulations and 3D images and video that will allow students to understand better the subject of natural sciences. This will make possible to develop new models of learning and teaching, which provoke students to research and create new bridges for the use of research results in business.

This article describes how some personal and social aspects of the research approach to learning can be accomplished through ICT support. It presents the currently deployed software tools, as well as a scenario for using them in conducting experiments along with its concrete implementation in a Bulgarian school and a brief analysis of the first results. In the Conclusion, further steps, which should be taken to support the technological support of the research-based training approach are drawn.

2. weSPOT INQUIRY–BASED SCIENCE EDUCATION

In the frame of the weSPOT project students are encouraged to go through the whole process of research, although the level of complexity of their research tasks can vary [5].

Visualization is the key to personal and social aspects of research training. Images are used for a description of the research process workflow in the training, which helps users to visualize better their research projects. Students can use images at different stages of their research to present their scientific reasoning, and/or integrate by images their questions, hypotheses, concepts, arguments, and data. As a strategy, the use of visualization aims to develop a knowledge map, through which students can link and articulate clearly their conceptual and procedural knowledge. Thus the visual language helps students to make a clear argument in order to generate comprehensible and clear plan of their documents.

In addition, weSPOT indicates when students have acquired a certain level of research expertise, so that their achievements are visible to others and can be used in their personal profiles in social networks.

weSPOT does not recommend a universal solution for all, but it starts from the pragmatic view that the optimal level of research actually is variable and it will likely vary for different students. It should reflect the key factors in the learning situation, including the content, context, the skills of the student, the teacher's knowledge and available materials. Students, compared to scientists, are beginners in research. When their knowledge on a given topic is limited, the open inquiry may not produce effective learning and may even hinder learning, adding internal cognitive tension. weSPOT model provides teachers and students with support and technological tools to grow step by step and reach competence, to progress and be able to find the optimum level of inquiry, which to meet their needs.

3. RELATED WORK

In the project weSPOT we plan and started to implement at the new level results from experience gained in other research projects. For example, the project Innovative Didactics for Web-Based Learning, IDWBL [6] considered five types of support for web-based learning: a web reference, web search, web research, e-mail project and collaboration. Students are guided by a teacher to experiment new methods and techniques. They share their inquires with teachers and each other, and thus achieve the learning objectives of the traditional classroom work. During the web-based learning projects teachers reported increased motivation for learning and development of mental processes in students.

The implementation of research-based education needs a practical methodology, approaches and tools that provide support for the daily practice. These needs are met by the methodology I*Teach [7] developed within the project Innovative Teacher. It is based on active, student-centered learning methods. The teacher is a partner in the process of realization of the didactic scenario, stimulating the development of creative talents of students [8]. The methodology was integrated into the projects TENCompetence [9] and ShareTEC [10] and in the training of teachers in vocational schools. Integration in ICT textbooks and methodological guides for teachers to use the research approach in training used in the project Fibonacci (http://www.fibonacci-project.eu/). In 2009 the I*Teach project was awarded as the most successful project of Leonardo da Vinci program.

Another relevant project is WebLabs - an European project aimed at developing a virtual learning environment (VLE) and a Weblabs learning model [11]. The learning environment allows students, teachers and researchers located in different geographical areas to be included in the overall process of research and education in mathematics and/or science. Understanding of the studied sciences is achieved through a partnership in the context of research. In addition, students accumulate social experience through collaboration and sharing of results [6].

Based on the experience gained in these projects, the conditions for a successful realization of the research-based approach in teaching students are formulated [12]: change the attitude of the teacher and provide strong support to students (micro level); support by the school management, providing necessary ICT infrastructure and building teacher teams to share experiences and good practices (meso level); the reformation of the curriculum, constantly offering training courses for teachers and a rich repository of resources based on national ICT infrastructure (macro level).

4. INFORMATION AND COMMUNICATION TECHNOLOGIES SUPPORTING PERSONAL AND SOCIAL ASPECTS OF INQUIRY

There is an abundance of software tools and services that can be used for the implementation of an ISBE approach. The main problem is to find out how they can be integrated and used together as much as possible in a meaningful and efficient way. weSPOT addresses this issue by providing a way to integrate data collected from a variety of research tools and services. It enables the integration of cognitive research tools and linking them to students' profiles as well as to their social and educational context. Individual and collaborative activities that engage students with various research tools will update the history and goals of student learning, and thus will enable them and their teachers to work in a unified learning environment in which they can monitor progress.

Testing the research tools offered by weSPOT with students and teachers in real-life scenarios in high school is essential for gathering requirements and feedback from end-users. The pilot project "Energy efficiency in buildings" was designed to implement the IBSE approach at guided level to help students identify the shortcomings of the building in which they are taught in terms of energy efficiency. Students are expected to predict (and provide evidence) what is the expected energy problems in the future. Working in teams, they will generate ideas to improve energy efficiency of future buildings. Teachers can help by asking questions such as:

- *What kind of new materials for new energy efficient building components, that conserve energy, to be used?*
- *What environmentally friendly technologies will provide high quality microclimate?*

In this way, students will come to better ideas and will develop skills in the field of education, but also will build new research skills and competences.

5. AIMS AND CONTENT OF THE PILOT EXPERIMENT

This pilot experiment is related to the Energy Efficiency in Buildings testbed. It is in the frame of an integrated training on the subject *Man and Nature* and *Information Technology* (IT) in 6th grade. It was led by three teachers (one teacher in *Man and Nature* and two IT teachers) and included 60 students aged 12-13 years from the First Private High School of Mathematics (PCHMG).

Domain competences, related to the subject *Human and nature*, which students should be able to develop during the pilot experiment were:

- Identifying processes, related to releasing and absorbing heat;
- Understanding that the current flow energy source gives consumers and describes the effects of thermal appliances;
- Calculating the cost of electricity used by household electrical appliances and finding ways to save it;
- Providing argumentation about the special role of man to protect and preserve energy;
- Observation and self-observation (of the objects in the nature and in the lab);
- Extracting information from graphs, tables and charts, and with IT;
- Forming attitude to the energy consumption.

The pilot experiment aimed to build also **inquiry skills** such as:

- Selecting among given questions and posing new scientific questions with guided support;
- Collecting certain data with guided support for what constitutes evidence;
- Formulating explanations from evidence with guided support;
- Linking areas and sources of scientific knowledge to clarify explanations;
- Communicating explanations based on scientific reasoning with guided support.

The scientific question of the experiment was: *What are the external factors influencing energy consumption and how the man can act to preserve it?*

Inquiry was organized as a completion between three 6 grade classes under the subtopic: **My classroom - the most energy efficient!**

Each class plays as a single team which search how much energy its classroom consumes and how it can save some energy without health risk. The main teams are divided into subteams according generated hypothesis and indicators chosen to measure. Each class collects and analyses data. At the end of the project each class presents its observations and conclusions in front of the other classes, parents and school managers. Each team shares ideas about increasing the energy efficiency of the building and provide argumentation based on its research.

The winner is the class providing most reasonable ideas.

Start and end date of inquiry are not accidental dates. They are especially selected: the start date, 17 November is the first day of the European Week for Waste Reduction (November 17-24); the end date, 5 June - the World Environment Day.

The environment for the inquiry is the classrooms of each of the three classes. They are located on a different floor of the building - basement, first and the second floor, and has different exposures. There are electric lights and air conditioners in each room. In some rooms joinery is old, while in the other - replaced with new one. Thermometers are placed on the wall of the rooms as well as on the outside wall.

6. PHASES OF THE EXPERIMENT

As a workflow of the pilot experiment students used the Mulholland et al model [13] with an inquiry cycle based on 8 phase octagonal (Fig 1). Realisation of each of these phases are presented shortly below.

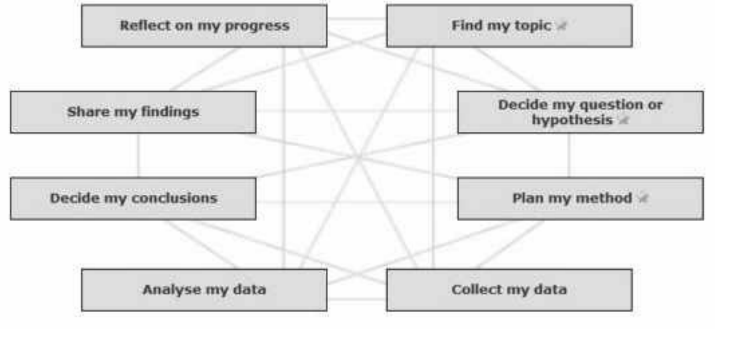

Figure 1: Inquiry Cycle [13]

During the first stage - finding the topic, the teacher in Human and nature subject, introduced students to the problem of energy consumption (Fig. 2). She presented data about the energy consumption of the school during the last year and how much it costed. After introducing the main idea, she set common problems to each of the three classes:

- *What measures should the school board take to reduce the energy use?*
- *Is there a place for alternative energy sources in the classroom?*

Figure 2: Introducing the topic

6.2. DECIDE MY QUESTION OR HYPOTHESIS

The next stage was to form a question or hypothesis. In this stage the brainstorming process took place. The students discussed the questions (Fig. 3) and found more specific questions they should answer in advance. In the process they came up with the ideas for observations.

Some examples of their questions were:

- *What are the energy consumers in the classroom?*
- *How much energy do they consume?*
- *Are there any dependencies between external climate characteristics (the temperature, wind, is it sunny or not, exposure, etc.) and the energy consumption and what they are?*
- *How long during the day the air-conditioner is working? At what degrees of temperature?*
- *How the students keep the energy of the classroom - if they open the windows, do they leave the door open during the breaks, do they keep the lights unnecessary switched on, etc.*

Figure 3: Discussing the energy consumers and climate conditions

The students chose which are the most important questions and stated their hypothesis based on the expected answers. After formulating the hypothesis, they formed 5 teams in each class to explore each question/hypothesis in parallel.

6.3. PLAN MY METHOD

During the third phase, according to the chosen questions the students in each team discussed what kind of information they need, how they can collect it, what they need to collect it and what they should and can measure in order to prove their hypotheses, formed in the previous phase.

Some examples:

- *To measure the temperature inside and outside the school building three times a day - early in the morning, in the middle of the school day and at lunch;*
- *To compare the temperature in the classroom and outside the building;*
- *To check at time of the measurement if the windows or doors are opened;*
- *To observe the external climate condition - is it sunny, windy, cloudy, etc.*
- *To check at what time the air conditioner is switched on and off;*
- *To calculate how much energy the air conditioner consumes per day;*
- *To do internet research about the particular model of the air conditioner and how it should be used in order to be the most efficient in working;*
- *To check at what time the lights are switched on and off; To calculate how much energy the lights consume per day;*
- *To check at what time during the day, the daylight in the classroom is enough for working properly;*
- *To provide informal interview with parents about the energy preservation at home.*

In this phase, the students composed a simple table (Fig. 4) for collecting the necessary data. They developed it on paper and put it on the wall of each classroom, so that in the next phase be able to fill in the data collected during the week. In order to not miss data from previous weeks, the students composed an electronic version of the table. It was made in the IT class. They used a spreadsheet to compose the desired table. They composed it, made validation rules, inserted a formula to calculate the dates automatically and put pictures for the weather conditions. They merged cells where it was necessary and put column and row headings. This was the first time where the IT and nature sciences were combined in the project.

room NO. 7 class VI^a FROM 19.11.2012 г. TO 23.11.2012 г.																
date	hour	t°		air-conditioner		factors				climate conditions						
		in	out	on/off	t°	number of children	opened door	opened windows	other	sun	clouds	wind	rain	snow		
19.11.2012 г.	07:30 - 08:00															
	10:30 - 11:00															
	13:00 - 13:30															
20.11.2012 г.	07:30 - 08:00															
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21.11.2012 г.	07:30 - 08:00															
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	13:00 - 13:30															
23.11.2012 г.	07:30 - 08:00															
	10:30 - 11:00															
	13:00 - 13:30															

Figure 4: The model of an electronic table for collecting data

The main goal was to encourage students to work in a team, not only in the class, but also to collaborate with other students from the other 6-th grades. Another goal was to put the students in an active position. In the second phase we reached this goal - they were very innovative, creative, defining many ideas and questions. During the process of composing the table they also worked together, shared opinions and knowledge. Some of our objectives were to develop their sense of civil position, to teach them to state an opinion and to defend it.

6.4. COLLECT MY DATA

During the fourth phase - collecting data, each team collected the data it has chosen to measure.

Before starting the collecting period, the students put thermometers in each classroom and outside the school. Fortunately the classrooms of the three classes are on different floors of the building - on the ground one, on the first and on the second. In each classroom they put a printed table on the wall for writing the measured data 3 times a day (Fig. 5).

Figure 5: Measuring the temperature in the classroom (left) and outside the school (right)

Right after measuring the observed factors students entered the collected data in the sheet. Once a week they entered the data from the wall sheets to the electronic spreadsheet during the classes of Information Technologies (Fig. 6).

Figure 6: Entering the data in a digital form

6.5. ANALYSE MY DATA

At this stage new teams were formed - every class should be presented at the final competition by one team. The new teams had to summarize the data from the whole class and to prepare diagrams and charts helping to find dependencies between observed external factors and energy consumption, human behavior and energy consumption, building condition and energy consumption, etc. Preparing diagrams and charts was in the classes of Information Technologies. During these lessons, the pupils learnt the basic chart types, what the main elements of a diagram are and how to create a diagram with the information, using spreadsheets.

6.6. DECIDE MY CONCLUSIONS

In the sixth stage, students made conclusions about their initial hypothesis. They discussed different scenarios. During the debate on the conclusions they made, each party provided its arguments and every student voted for one of the conclusions.

Based on these conclusions, the groups prepared a list of recommendations to the school management.

Some of the recommendations were in the following directions:

- *What time it is appropriate to switch on/off the lights according some found factors;*
- *What time it is appropriate to switch on/off the air conditioner according some found factors;*
- *Should the window frames or the doors be replaced;*
- *Is it possible to use some kind of alternative energy sources.*

6.7. SHARE MY FINDINGS

The seventh phase - sharing my finding, had two sub-phases.

The first sharing was after working on this pilot experiment for 2 months - at the end of January. Each of the teams in each class had to present the collected information on their topic in front of the other teams in the class. The presentation was about factors and conditions, which they had investigated. The students prepared the presentations. In order to be properly prepared for presenting their work, during the lessons of Information Technologies some of the presentational techniques were discussed.

The second sharing took place on the 5th of June - the World Environment Day. On this day the sharing was delivered in the form of competition between the three classes. In the competition not only students from the 6th grades took part, but also students from the 9th grade. The ninth graders organized the whole event. They were responsible for the preparation of the room, making the invitations for the jury, accompanying the guests, taking photos, etc. Every team from the 6th graders had a name and its own colour. Each team was suited in one of the Bulgarian flag's colours - white, green or red.

The competition was in 3 stages - an oral presentation, a practical workshop and solving PISA problems in natural sciences.

During the first stage every team shared in 10 minutes its conclusions and recommendations and provided evidence (data tables, diagrams, pictures, models)

Figure 7: A team shares the whole class findings

in order to prove it (Fig. 7). They also made concrete proposals for better energy consumption.

The second stage included a 12-minute practical workshop. Every team had a plastic box, which symbolized its classroom, a ruler, a pen, markers, a glue, a cardboard, an styrofoam, an aluminum foil and old newspapers. They had to isolate the box. The isolation was tasted by putting ice and a thermometer in every box (Fig. 8).

Figure 8: Experiments during the workshop

The last stage consisted of solving PISA problems. Every team had to solve problems and the first one had the right to answer, after pressing a button. When the team's answer was fully correct the team gained some points. Otherwise, the other two teams had the chance to answer in order to gain the points.

A jury formed by 3 university professors from Sofia University had to evaluate the presentation, conclusions and argumentation of each class. In front of the teachers, school managers and parents the jury decided which class was the winner. The decision was based upon the provided argumentations and answers of the questions from the audience.

There are a lot of criteria for evaluation. In addition to logical conclusions, based on the inquiry, there were also additional factors for rating the presentation,. Some of these factors are the organization of the presentation, its structure, the usage of graphics, the balance between text and graphics, spelling and grammatical errors, the appropriate design and the usage of presenting techniques. The main evaluation criteria for the practical task was the quality of the composed isolation and its aesthetics view. In the third stage, the first team with a correct answer to the problem gained the points. The winner was the team named Electra, that received the most points.

6.8. REFLECT ON MY PROGRESS

The last stage - the eighth one was the reflection on the progress. After the competition the classes participated in a group reflection with the three teachers. The reflection questions were:

- *Why the particular class is the winner?*
- *At what stages were they/we better? How did they/we achieve this?*

In each team reflection was concentrated on questions like:

- *How was the team work organized?*
- *Who had taken what responsibilities?*
- *How was the team communication performed?*
- *Were there "lazy" teammates and how did the team provoke them to work?*
- *How did the team members support each other?*

Each pupil was encouraged to think and share:

- *What were his own challenges during the project?*
- *What new did he learn?*

Finally, teachers summarised the conclusions and shared the recommendations for improvement in the next inquiry.

7. INFORMATION AND COMMUNICATION TECHNOLOGIES NEEDED TO SUPPORT IBSE IN THE EXPERIMENT

The use of the Mulholland et al model [13] in the experiment led us to the idea to use the integrated environment nQuiry [13], which allows a description and implementation of all phases of a chosen model. The web application nQuiry is based on Drupal and is especially developed to support IBSE.

The application supports tools for: selecting a model of research learning scenario (Fig. 9); describing of a scenario, associating user groups with it, data recording, sharing and dissemination of results.

Figure 9: Description of a research scenario for education in nQuiry

The experiment showed that although nQuiry provides a lot of utilities, it still needs additional options to be able to ensure an effective and efficient implementation of the research approach. For example, it is focused primarily on individual studies that can be monitored and controlled by the teacher. Although it has the tools to organize group work, it does not offers enough opportunities for visualizing and integration of the performance of the particular groups.

The pilot experiment showed that there is a need of additional tools in order to fully meet the needs of educational research, especially to support collaboration in a social environment.

Based on the observation during the pilot experiment, we think it will be perfect to have an integrated technological environment and tools to support activities in the different phases.

For the *first phase*, *finding topic*, it is good to have a shared place where preliminary notes by the teacher can be shared with all the students in the inquiry.

For the *phase of making hypothesis* - to have tools and a shared place for class brainstorming, tools for voting, place/option for subgroups (re)forming and working together. The technology environment should allow the subgroup work to be visible for other groups. In addition, tools should make possible for the subgroup to be able to share (if it likes) collected data outside of the main group. The teacher should be able to observe the brainstorming and to provide some directions if there is a need.

During the *phase of planing methods* the technological infrastructure should provide a common place for the whole class where each team can present its decision - what it will measure, observe, research and how. In addition, through technological tools the teacher should be able to provide separate feedback to each team as well as to the whole class. Moreover, the students should be able to have an open (shared) working space, supported by technology, to discuss what tools and resources are useful for the presented methods. They should have an opportunity (as well as the teacher) to recommend toolkits and resources for data collection and analysis.

In the *phase of collecting data* tools should be available to the teacher in order to be able to provide some pictograms, symbols, images or so, for the students to use to mark their observations in a common way. In the concrete experiment, it would be useful if the tool could provide them with graphical symbols for describing the weather (sunny, cloudy, rainy, etc.). A possibility to upload pictures directly from a mobile device and the system to mark the time of uploading is desirable. Using different tools students should be able to create and use a common dataset (for example an electronic spreadsheet) where each team to fill in its collected data. In the given case - a common electronic table is useful to measure at the same time on each date the internal and external temperature, the air conditioner settings, the whether characteristics etc. Tools should support students to organize different data sets according to the chosen method of research. The dataset should be visible to the other teams in the main group/class. It should be also shareable outside of the group.

During the *phase of analysing data* and *phase of making conclusions* the tools should support discussions within and between the teams, sharing of and voting on artefacts (photos, hypothesis, arguments), as well as feedback, including by the teacher. It will be useful to have an integration with mobile device applications, allowing to provide directly arguments or evidences for proposed conclusions, as well as immediate comments, notes and feedback.

In all of the phases tools should support rearrangement of the teams, as well as feedback or directions (if needed) from the teacher and other science experts.

These observations and recommendations were passed to the weSPOT developers in order to take them into account during the development of tools in the frame of the project. Some of them inspired the developers to create simulations, like the one shown in Fig. 10, which was developed by using the Elica software.

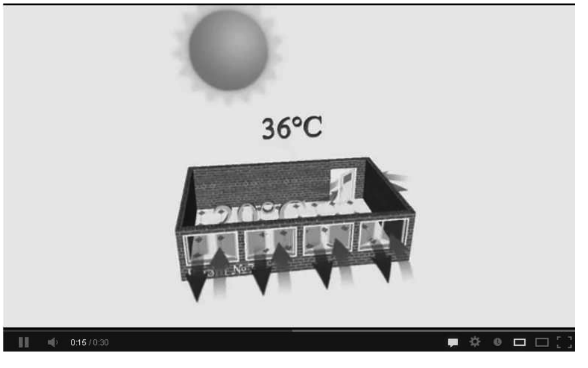

Figure 10: Classroom Energy Simulation (Summer) [14]

Looking on it the teacher of the Human and Nature subject reacted immediately "Great! Next pilot will take shorter time, because the children could experiment with a simulation and it will not be necessary to collect data for approximately six months. In addition, during the winter they could generate conclusions for the summer!"

8. THE IMPACT OF THE PILOT ON THE weSPOT INQUIRY-BASED LEARNING MODEL

Although the first weSPOT pilot was just finished, the process was closely monitored and used for the development of a new inqiury-based learning (IBL) model. The new model [15] (still not officially published) is simpler - it is based on 6 phases:

- Question/hypothesis
- Operationalisation (realisation of an idea with the aim to measure)
- Data collection
- Data analysis (processing)
- Interpretation/discussion
- Communication

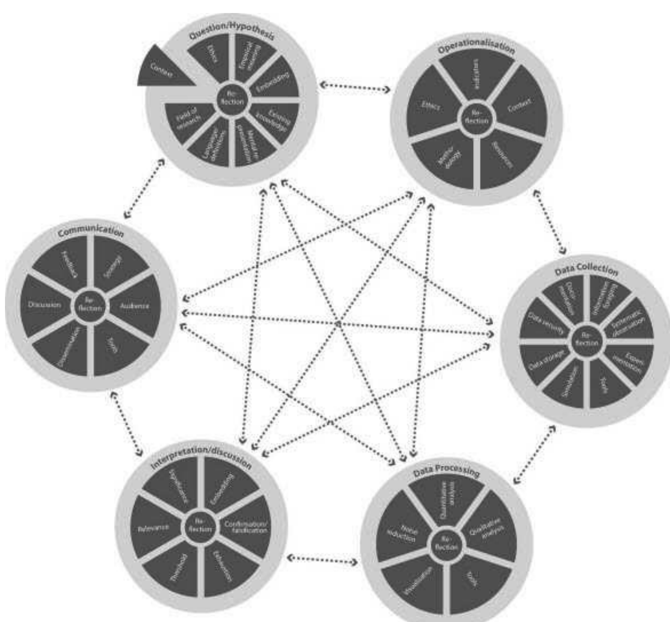

Figure 11: weSPOT IBL model [15]

The model shares many of the phases that Mulholland et al. [13] described (Fig. 11) [15], but it is more complex regarding the sub-phases providing a detailed description of things that teachers and students should consider when doing inquiry. It is mostly oriented to the inquiry competences that are developed during the process of education.

The weSPOT: *Information and communication technologies fit young researchers' learning needs* video [16] illustrates how the new model fits the pilot experiment phases. The pilot was very valuable for the weSPOT consortium also in identifying the needed technological tools which can provide the most effective way to teach according the IBL model. For example, a tool for real-time team forming and reforming, a tool for organization, implementation and summarization of brainstorming, shared data sheets, etc. and how these tools can improve the development of specific inquiry skills.

9. CONCLUSION

In this paper we presented the weSPOT methodology for science education. We showed how this methodology was applied in the First Private Mathematical School in Sofia, Bulgaria. This pilot experiment proves that Inquiry-Based Science Education (IBSE) is not only very interesting and innovative approach, but also that it is bringing higher motivation and is giving excellent results, especially when it is backed with the necessary software tools to support it.

This pilot also showed what improvements and further developments in these software tools are needed, which will be addressed in the next version of the weSPOT software framework. These new updated tools will be again applied and tested in a new set of pilot experiments not only in the secondary education, but also in the university courses as well.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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PARTIAL DIFFERENTIAL EQUATIONS OF TIME-LIKE WEINGARTEN SURFACES IN THE THREE-DIMENSIONAL MINKOWSKI SPACE

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We study time-like surfaces in the three-dimensional Minkowski space with diagonalizable second fundamental form. On any time-like W-surface we introduce locally natural principal parameters and prove that such a surface is determined uniquely (up to motion) by a special invariant function, which satisfies a natural non-linear partial differential equation. This result can be interpreted as a solution of the Lund-Regge reduction problem for time-like W-surfaces with real principal curvatures in Minkowski space. We apply this theory to the class of linear fractional time-like W-surfaces with respect to their principal curvatures and obtain the natural partial differential equations describing them.

Keywords: Time-like W-surfaces in Minkowski space, natural parameters on time-like W-surfaces in Minkowski space, natural PDE's of time-like W-surfaces in Minkowski space.

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1. INTRODUCTION

It has been known to Weingarten [21, 22], Eisenhart [4], Wu [23] that without changing the principal lines on a Weingarten surface in Euclidean space, one can find geometric coordinates in which the coefficients of the metric are expressed by the principal curvatures (or principal radii of curvature).

The geometric parameters on Weingarten surfaces were used in [23] to find the classes of Weingarten surfaces yielding "geometric" $\mathfrak{so}(3)$-scattering systems (real or complex) for the partial differential equations, describing these surfaces.

We have shown that the Weingarten surfaces in Euclidean space [5, 6] and space-like surfaces in Minkowski space [7] admit geometrically determined principal parameters (*natural principal parameters*), which have the following property: all invariant functions on W-surfaces can be expressed in terms of one function ν, which satisfies one *natural* partial differential equation. The Bonnet type fundamental theorem states that any solution to the natural partial differential equation determines a W-surface uniquely up to motion. Thus the description of any class of W-surfaces (determined by a given Weingarten relation) is equivalent to the study of the solution space of their natural PDE. This solves the Lund-Regge reduction problem [13] for W-surfaces in Euclidean space and space-like W-surfaces in Minkowski space.

The relationship between the solutions of certain types of partial differential equations and the determination of various kinds of surfaces of constant curvature has generated many results which have applications to the areas of both pure and applied mathematics. This includes the determination of surfaces of either constant mean curvature or Gaussian curvature. It has long been known that there is a connection between surfaces of negative constant Gaussian curvature in Euclidean $\mathbb{R}^3$ and the sine-Gordon equation. The fundamental equations of surface theory are found to yield a type of geometrically based Lax pair. For instance, given a particular solution of the sinh-Laplace equation, this Lax pair can be integrated to determine the three fundamental vector fields related to the surface. These are also used to determine the coordinate vector field of the surface.

Further results are obtained based on the fundamental equations of surface theory, and it is shown how specific solutions of this sinh-Laplace equation can be used to obtain the coordinates of a surface in either Minkowski $\mathbb{R}^3_1$ or Euclidean $\mathbb{R}^3$ space [9, 10].

In [3] Bracken introduces some fundamental concepts and equations pertaining to the theory of surfaces in three-space, and, in particular, studies a class of sinh-Laplace equation which has the form $\Delta u = \pm \sinh u$.

In this paper we study time-like surfaces with real principal curvatures in the three dimensional Minkowski space $\mathbb{R}^3_1$.

A time-like surface $\mathcal{M}$ with real principal curvatures ν_1 and ν_2 is a Weingarten surface (W-surface) [21, 22] if there exists a function ν on $\mathcal{M}$ and two functions (Weingarten functions) f, g of one variable, such that

$$\nu_1 = f(\nu), \quad \nu_2 = g(\nu).$$

A basic property of W-surfaces in Euclidean space is the following theorem of Lie [12]:

The lines of curvature of any W-surface can be found in quadratures.

This remarkable property is also valid for space-like and time-like W-surfaces in Minkowski space.

We use four invariant functions (two principal normal curvatures ν_1, ν_2 and two principal geodesic curvatures γ_1, γ_2) and divide time-like W-surfaces into two classes with respect to these invariants:

(1) the class of *strongly regular* time-like surfaces defined by

$$(\nu_1 - \nu_2)\,\gamma_1\,\gamma_2 \neq 0;$$

(2) the class of time-like surfaces defined by

$$\gamma_1 = 0, \quad (\nu_1 - \nu_2)\,\gamma_2 \neq 0.$$

The basic tool to investigate the relation between time-like surfaces and the partial differential equations describing them, is Theorem 2.1. This theorem is a reformulation of the fundamental Bonnet theorem for the class of strongly regular time-like surfaces in terms of the four invariant functions. Further, we apply this theorem to time-like W-surfaces.

In Section 3 we prove (Proposition 3.3) that any time-like W-surface admits locally special principal parameters (*natural principal parameters*).

Theorem 3.6 is the basic theorem for time-like W-surfaces of type (1):

Any strongly regular time-like W-surface is determined uniquely up to motion by the functions f, g and the function ν, satisfying the natural PDE (3.3)*.*

Theorem 3.7 is the baic theorem for time-like Weingarten surfaces of type (2):

Any time-like W-surface with $\gamma_1 = 0$ *is determined uniquely up to motion by the functions f, g and the function ν, satisfying the natural ODE* (3.8)*.*

In natural principal parameters the four basic invariant functions, which determine time-like W-surfaces uniquely up to motions in $\mathbb{R}^3_1$, are expressed by a single function, and the system of Gauss-Codazzi equations reduces to a single partial differential equation (the Gauss equation). Thus, the number of the four invariant functions, which determine time-like W-surfaces, reduces to one invariant function, and the number of Gauss-Codazzi equations reduces to one *natural* PDE. This result gives a solution to the Lund-Regge reduction problem [13] for the time-like W-surfaces in $\mathbb{R}^3_1$. The Lund-Regge reduction problem has been analyzed and discussed from several view points in the paper of Sym [18].

In Proposition 4.1 we prove that

The natural principal parameters of a given time-like W-surface $\mathcal{M}$ are natural principal parameters for all parallel time-like surfaces $\overline{\mathcal{M}}(a)$, $a = \mathrm{const} \neq 0$ of $\mathcal{M}$.

Theorem 4.2 states that (cf. [6, 7]):

The natural PDE of a given time-like W-surface $\mathcal{M}$ is the natural PDE of any parallel time-like surface $\overline{\mathcal{M}}(a)$, $a = \mathrm{const} \neq 0$, of $\mathcal{M}$.

In [14, 16] Milnor studies surface theory in Euclidean and Minkowski space, considering harmonic maps and various relations between the Gauss curvature K, the mean curvature H and the curvature $H' = \dfrac{\nu_1 - \nu_2}{2}$. In [15, 6] it is proved that any surface in $\mathbb{R}^3_1$, whose Gauss curvature K and mean curvature H satisfy the linear relation

$$\delta K = \alpha H + \gamma, \quad \alpha, \gamma, \delta - \text{constants}; \quad \alpha^2 + 4\gamma\delta \neq 0, \tag{1.1}$$

is parallel to a surface, satisfying one of the following conditions: $H = 0$, $K = 1$ or $K = -1$.

There arises the following question: what are the natural PDE's describing the surfaces, whose curvatures satisfy the relation (1.1)?

Since any time-like surface $\mathcal{M}$, whose invariants K and H satisfy the linear relation (1.1), is (locally) parallel to one of the following three types of basic surfaces: a surface with $H = 0$; a surface with $K = 1$; a surface with $K = -1$, from Theorem 4.2 it follows that

Up to similarity, the time-like surfaces, whose curvatures satisfy the linear relation (1.1)*, are described by the natural PDE's of the basic surfaces.*

A. Ribaucour [17] has proved that *a necessary condition for the curvature lines of the first and second focal surfaces of $\mathcal{M}$ to correspond to each other resp. to a conjugate parametric lines on $\mathcal{M}$ is $\rho_1 - \rho_2 = \mathrm{const}$ resp. $\rho_1 \, \rho_2 = \mathrm{const}$.*

Von Lilienthal [19] (cf. [20, 1, 2, 4]) has proved in $\mathbb{R}^3$ that a surface with a relation $\rho_1 - \rho_2 = \dfrac{1}{R}$, $R = \mathrm{const} \neq 0$, between its principal radii of curvature $\rho_1 = \dfrac{1}{\nu_1}$ and $\rho_2 = \dfrac{1}{\nu_2}$ has first and second focal surfaces $\widetilde{\mathcal{M}}$ of constant Gauss curvature $-R^2$ and vice versa. The involute surfaces $\overline{\mathcal{M}}(a)$, $a \in \mathbb{R}$ of $\widetilde{\mathcal{M}}$ are parallel surfaces of $\mathcal{M}$ with the property $\rho_1 - \rho_2 = \mathrm{const}$. This implies that the family $\overline{\mathcal{M}}(a)$ are integrable surfaces as a consequence of the integrability of $\widetilde{\mathcal{M}}$. The curvatures of the above surfaces $\mathcal{M}$ satisfy the relation $K = \beta \, H'$, $\beta = \mathrm{const} \neq 0$.

In $\mathbb{R}^3_1$ one can prove in a similar way the corresponding property: The first focal surface of a time-like surface with $K = \beta \, H'$, $\beta \neq 0$, is space-like of constant Gauss curvature $\beta^2/4$, and its second focal surface is time-like of constant Gauss curvature $-\beta^2/4$.

Obviously the time-like surfaces with $K = \beta \, H'$, $\beta = \mathrm{const} \neq 0$, are not included in the class characterized by (1.1).

These surfaces belong to the classes of time-like W-surfaces, defined by the following more general linear relation

$$\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \mathrm{constants}; \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0 \tag{1.2}$$

between the Gauss curvature K, the mean curvature H and the curvature H'. We denote this class by $\mathfrak{K}$.

We show that the class $\mathfrak{K}$ is the class of linear fractional time-like W-surfaces with respect to the principal curvatures (cf. [6, 7]). Furthermore, if $\mathcal{M}$ is a time-like surface in $\mathfrak{K}$, then its parallel surfaces $\overline{\mathcal{M}}(a)$, $a = \mathrm{const}$, belong to $\mathfrak{K}$ too.

In the main Theorem 5.3 in this paper we determine ten basic relations with respect to the constants in (1.2) and each of them generates a *basic subclass of surfaces* of $\mathfrak{K}$. Any time-like surface $\mathcal{M}$, whose invariants K, H and H' satisfy the linear relation (1.2) is (locally) parallel to one of these basic surfaces.

In [10] Hu has cleared up the relationship between the PDE's

$$\begin{aligned} &\alpha_{uu} - \alpha_{vv} = \pm \sin\alpha && (\sin - \text{Gordon PDE}),\\ &\alpha_{uu} - \alpha_{vv} = \pm \sinh\alpha && (\sinh - \text{Gordon PDE}),\\ &\alpha_{uu} + \alpha_{vv} = \pm \sin\alpha && (\sin - \text{Laplace PDE}),\\ &\alpha_{uu} + \alpha_{vv} = \pm \sinh\alpha && (\sinh - \text{Laplace PDE}) \end{aligned}$$

and the construction of various kinds of surfaces of constant curvature in $\mathbb{R}^3$ or $\mathbb{R}^3_1$.

In [11] by using Darboux transformations, from a known solution to the sinh-Laplace (resp. sin-Laplace) equation have been obtained explicitly new solutions to the sin-Laplace (resp. sinh-Laplace) equation.

Time-like surfaces with positive Gauss curvature and imaginary principal curvatures have been constructed in [8].

It is essential to note that the natural PDE's of the time-like W-surfaces from the class $\mathfrak{K}$ are expressed in the form $\delta\lambda = f(\lambda)$, where δ is one of the operators (cf. [6, 7]):

$$\begin{aligned} \Delta\lambda &:= \lambda_{xx} + \lambda_{yy}, & \bar{\Delta}\lambda &:= \lambda_{xx} - \lambda_{yy};\\ \Delta^*\lambda &:= \lambda_{xx} + (\lambda^{-1})_{yy}, & \bar{\Delta}^*\lambda &:= \lambda_{xx} - (\lambda^{-1})_{yy}. \end{aligned}$$

2. PRELIMINARIES

Let $\mathbb{R}^3_1$ be the three dimensional Minkowski space with the standard flat metric $\langle\, ,\, \rangle$ of signature $(2,1)$. We assume that the following orthonormal coordinate system $Oe_1e_2e_3 :\ e_1^2 = e_2^2 = -e_3^2 = 1,\ \langle e_i,\, e_j\rangle = 0,\ i \neq j$ is fixed and gives the orientation of the space.

Let $\mathcal{M} :\ z = z(u,v),\ (u,v) \in \mathcal{D}$ be a time-like surface in the three dimensional Minkowski space $\mathbb{R}^3_1$ and ∇ be the flat Levi-Civita connection of the metric $\langle\, ,\, \rangle$. The unit normal vector field to $\mathcal{M}$ is denoted by l and $E, F, G;\ L, M, N$ stand for the coefficients of the first and the second fundamental forms, respectively. Then we have

$$E = z_u^2 < 0, \quad F = z_u\, z_v, \quad G = z_v^2 > 0, \quad EG - F^2 < 0, \quad l^2 = 1.$$

The coefficients of the second fundamental form are given as follows:

$$L = l\, z_{uu} = -l_u\, z_u, \quad M = l\, z_{uv} = -l_u\, z_v = -l_v\, z_u, \quad N = l\, z_{vv} = -l_v\, z_v.$$

The linear Weingarten map γ is determined by the conditions

$$\gamma(z_u) = l_u, \quad \gamma(z_v) = l_v.$$

Then the mean curvature H and the Gauss curvature K of $\mathcal{M}$ are given in the standard way

$$H = -\frac{1}{2}\operatorname{tr}\gamma, \quad K = \det\gamma.$$

While the Weingarten map of a space-like surface satisfies the inequality $H^2 - K \geq 0$ and is always diagonalizable, the Weingarten map on a time-like surface can satisfy the inequalities $H^2 - K \geq 0$ or $H^2 - K < 0$.

Throughout this paper we deal with time-like surfaces satisfying the inequality $H^2 - K \geq 0$, i.e. time-like surfaces with real principal curvatures.

We suppose that the surfaces under consideration are free of points with $H^2 - K = 0$, i.e. satisfy the strong inequality

$$H^2 - K > 0 \tag{2.1}$$

and denote by H' the invariant curvature

$$H' = \sqrt{H^2 - K}.$$

Under the above condition the theory of time-like surfaces can be developed in a way similar to the theory of surfaces in Euclidean space or space-like surfaces in Minkowski space.

Time-like surfaces satisfying the condition (2.1) can be locally parameterized by principal parameters. Further we assume that the parametric net is principal, i.e.

$$F(u,v) = M(u,v) = 0, \quad (u,v) \in \mathcal{D}.$$

Then the principal curvatures ν_1, ν_2 and the principal geodesic curvatures (geodesic curvatures of the principal lines) γ_1, γ_2 are given by

$$\nu_1 = \frac{L}{E}, \quad \nu_2 = \frac{N}{G}; \qquad \gamma_1 = \frac{E_v}{2E\sqrt{G}}, \quad \gamma_2 = \frac{-G_u}{2G\sqrt{-E}}, \tag{2.2}$$

and ν_1, ν_2 satisfy the Rodrigues' formulas:

$$l_u = -\nu_1\, z_u, \quad l_v = -\nu_2\, z_v.$$

We consider the tangential frame field $\{X, Y\}$ determined by

$$X := \frac{z_u}{\sqrt{-E}}\,, \qquad Y := \frac{z_v}{\sqrt{G}}$$

and suppose that the moving frame field XYl is positive oriented.

The following Frenet type formulas for the frame field XYl are valid

$$\left|\begin{array}{lcll} \nabla_X X & = & & \gamma_1 Y - \nu_1 l, \\ \nabla_X Y & = & \gamma_1 X, & \\ \nabla_X l & = & -\nu_1 X, & \end{array}\right. \qquad \left|\begin{array}{lcll} \nabla_Y X & = & & -\gamma_2 Y, \\ \nabla_Y Y & = & -\gamma_2 X & \quad +\nu_2 l, \\ \nabla_Y l & = & & -\nu_2 Y. \end{array}\right. \tag{2.3}$$

The Codazzi equations have the form

$$\gamma_1 = \frac{-Y(\nu_1)}{\nu_1 - \nu_2} = \frac{-(\nu_1)_v}{\sqrt{G}(\nu_1 - \nu_2)}, \qquad \gamma_2 = \frac{-X(\nu_2)}{\nu_1 - \nu_2} = \frac{-(\nu_2)_u}{\sqrt{-E}\,(\nu_1 - \nu_2)}, \tag{2.4}$$

and the Gauss equation can be written as follows:

$$X(\gamma_2) + Y(\gamma_1) + \gamma_1^2 - \gamma_2^2 = -\nu_1\nu_2 = -K,$$

or

$$\frac{(\gamma_2)_u}{\sqrt{-E}} + \frac{(\gamma_1)_v}{\sqrt{G}} + \gamma_1^2 - \gamma_2^2 = -\nu_1\nu_2 = -K. \tag{2.5}$$

A time-like surface $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ parameterized by principal parameters is said to be *strongly regular* if (cf. [5, 6, 7])

$$(\nu_1(u,v) - \nu_2(u,v))\gamma_1(u,v)\gamma_2(u,v) \neq 0, \quad (u,v) \in \mathcal{D}.$$

The Codazzi equations (2.4) imply that

$$\gamma_1\gamma_2 \neq 0 \iff (\nu_1)_v(\nu_2)_u \neq 0.$$

Because of (2.4) the formulas

$$\sqrt{-E} = \frac{-(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{-(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} > 0 \tag{2.6}$$

are valid on strongly regular time-like surfaces.

Taking into account (2.6), for strongly regular time-like surfaces formulas (2.3) become

$$\left|\begin{array}{l} X_u = -\dfrac{\gamma_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,Y + \dfrac{\nu_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,l, \ Y_u = -\dfrac{\gamma_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,X, \ l_u = \dfrac{\nu_1\,(\nu_2)_u}{\gamma_2(\nu_1-\nu_2)}\,X; \\[2ex] X_v = \dfrac{\gamma_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,Y, \quad Y_v = \dfrac{\gamma_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,X - \dfrac{\nu_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,l, \quad l_v = \dfrac{\nu_2\,(\nu_1)_v}{\gamma_1(\nu_1-\nu_2)}\,Y. \end{array}\right. \tag{2.7}$$

Finding the compatibility conditions for the system (2.7), we reformulate the fundamental Bonnet theorem for strongly regular time-like surfaces in terms of the invariants of the surface.

Theorem 2.1. *Let the four functions $\nu_1(u,v)$, $\nu_2(u,v)$, $\gamma_1(u,v)$, $\gamma_2(u,v)$ be defined in a neighborhood $\mathcal{D}$ of (u_0,v_0) and satisfy the following conditions:*

1) $\quad (\nu_1 - \nu_2)\,\gamma_1\,(\nu_1)_v < 0, \quad (\nu_1 - \nu_2)\,\gamma_2\,(\nu_2)_u < 0,$

2.1) $\quad \left(\ln\dfrac{(\nu_1)_v}{\gamma_1}\right)_u = \dfrac{(\nu_1)_u}{\nu_1 - \nu_2}, \qquad \left(\ln\dfrac{(\nu_2)_u}{\gamma_2}\right)_v = -\dfrac{(\nu_2)_v}{\nu_1 - \nu_2},$

2.2) $\quad \dfrac{\nu_1 - \nu_2}{2}\left(\dfrac{(\gamma_2^2)_u}{(\nu_2)_u} + \dfrac{(\gamma_1^2)_v}{(\nu_1)_v}\right) - (\gamma_1^2 - \gamma_2^2) = \nu_1\nu_2.$

Let $z_0X_0Y_0l_0$ be an initial positive oriented orthonormal frame.

Then there exists a unique strongly regular time-like surface $\mathcal{M}:\ z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ ($(u_0,v_0) \in \mathcal{D}_0 \subset \mathcal{D}$) with prescribed invariants ν_1, ν_2, γ_1, γ_2 such that

$$z(u_0,v_0) = z_0,\ X(u_0,v_0) = X_0,\ Y(u_0,v_0) = Y_0,\ l(u_0,v_0) = l_0.$$

Formulas (2.3) imply explicit expressions for the curvature and the torsion of any principal line on the time-like surface $\mathcal{M}$.

Let $c_1 : z = z(s)$, $\mathcal{M} \in J$ be a line from the family $\mathcal{F}_1$ ($v = \text{const}$) parameterized by a natural parameter and κ_1, τ_1 be its curvature and torsion, respectively.

Since c_1 is an integral line of the unit time-like vector field X, then

$$z' = X, \quad z'' = \nabla_X X = \gamma_1 Y - \nu_1 l,$$

$$z''' = \nabla_X \nabla_X X = -X(\nu_1)\, l + X(\gamma_1)\, Y + (\nu_1^2 + \gamma_1^2)\, X,$$
$$\kappa_1^2 = \nu_1^2 + \gamma_1^2.$$

We use the formula

$$\tau = \frac{z'z''z'''}{z''^2}.$$

Since $\nu_1^2 + \gamma_1^2 > 0$ along c_1, we find

$$\tau_1 = \frac{\nu_1\, X(\gamma_1) - \gamma_1\, X(\nu_1)}{\nu_1^2 + \gamma_1^2} = \frac{\nu_1^2}{\kappa_1^2}\, X\left(\frac{\gamma_1}{\nu_1}\right).$$

Denoting $\sin\theta_1 = \frac{\gamma_1}{\kappa_1}$ and $\cos\theta_1 = \frac{\nu_1}{\kappa_1}$, we obtain

$$\tau_1 = X(\theta_1).$$

For the lines c_2 of the family $\mathcal{F}_2$ we obtain in a similar way the formulas

$$z' = Y, \quad z'' = \nabla_Y Y = -\gamma_2\, X + \nu_2\, l,$$

$$z''' = \nabla_Y \nabla_Y Y = Y(\nu_2)\, l - Y(\gamma_2)\, X + (\gamma_2^2 - \nu_2^2)\, Y,$$
$$\kappa_2^2 = \varepsilon_2\, z''^2 = \varepsilon_2\, (\nu_2^2 - \gamma_2^2), \quad \varepsilon_2 = \operatorname{sign} z''^2,$$

and in the case $z''^2 \neq 0$,

$$\tau_2 = \varepsilon_2\, \frac{\gamma_2\, Y(\nu_2) - \nu_2\, Y(\gamma_2)}{\kappa_2^2} = -\varepsilon_2\, \frac{\nu_2^2}{\kappa_2^2}\, Y\ \left(\frac{\gamma_2}{\nu_2}\right).$$

3. NATURAL PRINCIPAL PARAMETERS ON TIME-LIKE WEINGARTEN SURFACES

In this section we consider diagonalizable time-like Weingarten surfaces. For the sake of symmetry with respect to the principal curvatures ν_1 and ν_2 we use the following characterization of time-like Weingarten surfaces:

A diagonalizable time-like surface $\mathcal{M}: \; z = z(u,v), \; (u,v) \in \mathcal{D}$ *is Weingarten if there exist two real differentiable functions* $f(\nu), \; g(\nu), \; f(\nu) - g(\nu) \neq 0, \; f'(\nu)g'(\nu) \neq 0, \; \nu \in \mathcal{I} \subseteq \mathbb{R}$ *such that the principal curvatures of* $\mathcal{M}$ *at every point are given by* $\nu_1 = f(\nu), \; \nu_2 = g(\nu), \; \nu = \nu(u,v), \; (u,v) \in \mathcal{D}$.

The next statement gives a property of time-like Weingarten surfaces, which allows us to introduce special principal parameters on such surfaces.

Lemma 3.1. *Let* $\mathcal{M}: \; z = z(u,v), \; (u,v) \in \mathcal{D}$ *be a diagonalizable time-like Weingarten surface parameterized with principal parameters. Then the function*

$$\lambda = \sqrt{-E}\exp\left(\int \frac{f'd\nu}{f-g}\right)$$

does not depend on v*, while the function*

$$\mu = \sqrt{G}\exp\left(\int \frac{g'd\nu}{g-f}\right)$$

does not depend on u*.*

Proof. Taking into account (2.4) and (2.2), we find

$$\gamma_1 = \frac{-f'(\nu)Y(\nu)}{f(\nu)-g(\nu)} = Y(\ln\sqrt{-E}), \qquad \gamma_2 = \frac{-g'(\nu)\,X(\nu)}{f(\nu)-g(\nu)} = -X(\ln\sqrt{G}),$$

which imply that

$$Y\left(\int \frac{f'(\nu)\,d\nu}{f(\nu)-g(\nu))} + \ln\sqrt{-E}\right) = 0, \qquad X\left(\int \frac{g'(\nu)\,d\nu}{g(\nu)-f(\nu)} + \ln\sqrt{G}\right) = 0.$$

The last equalities mean that $\lambda_v = 0$ and $\mu_u = 0$. □

We define special principal parameters on a time-like Weingarten surface as follows:

Definition 3.2. *Let* $\mathcal{M}: \; z = z(u,v), \; (u,v) \in \mathcal{D}$ *be a diagonalizable time-like Weingarten surface parameterized with principal parameters. The parameters* (u,v)

are said to be natural *principal, if the functions* $\lambda(u)$ *and* $\mu(v)$ *from Lemma 3.1 are constants.*

Proposition 3.3. *Any diagonalizable time-like Weingarten surface admits locally natural principal parameters.*

Proof. Let $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ be a time-like Weingarten surface in the Minkowski space $\mathbb{R}^3_1$, parameterized with principal parameters. Then $\nu_1 = f(\nu)$, $\nu_2 = g(\nu), \ \nu = \nu(u,v)$ for some differentiable functions f, g and ν satisfying the conditions $(f(\nu) - g(\nu))\, f'(\nu)\, g'(\nu) \neq 0, \ (u,v) \in \mathcal{D}$.

Let $\mathfrak{a} = \text{const} \neq 0, \ \mathfrak{b} = \text{const} \neq 0, \quad (u_0, v_0) \in \mathcal{D}$ and $\nu_0 = \nu(u_0, v_0)$. We change the parameters $(u,v) \in \mathcal{D}$ with $(\bar{u}, \bar{v}) \in \bar{\mathcal{D}}$ by the formulas

$$\bar{u} = \mathfrak{a} \int_{u_0}^{u} \sqrt{-E} \exp\left(\int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right) du + \overline{u}_0, \quad \bar{u}_0 = \text{const},$$

$$\bar{v} = \mathfrak{b} \int_{v_0}^{v} \sqrt{G} \exp\left(\int_{\nu_0}^{\nu} \frac{g' d\nu}{g - f} \right) dv + \overline{v}_0, \quad \bar{v}_0 = \text{const}.$$

According to Lemma 3.1 it follows that $(\bar{u}, \bar{v})$ are again principal parameters and

$$\bar{E} = -\frac{1}{\mathfrak{a}^2} \exp\left(-2 \int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right), \quad \bar{G} = \frac{1}{\mathfrak{b}^2} \exp\left(-2 \int_{\nu_0}^{\nu} \frac{g' d\nu}{g - f} \right). \tag{3.1}$$

Then for the functions from Lemma 3.1 we find

$$\lambda(\bar{u}) = |\mathfrak{a}|^{-1}, \quad \mu(\bar{v}) = |\mathfrak{b}|^{-1}.$$

Furthermore $\mathfrak{a}^2\, \bar{E}(u_0, v_0) = -1, \ \mathfrak{b}^2\, \bar{G}(u_0, v_0) = 1.$ □

We assume now that the considered time-like Weingarten surface $\mathcal{M} : z = z(u,v), \ (u,v) \in \mathcal{D}$ is parameterized with natural principal parameters (u,v). It follows from the above proposition that the coefficients E and G (consequently L and N) are expressed by the invariants of the surface.

As an immediate consequence from Proposition 3.3 we get

Corollary 3.4. *Let* $\mathcal{M}$ *be a time-like Weingarten surface parameterized by natural principal parameters* (u,v)*. Then any natural principal parameters* $(\tilde{u}, \tilde{v})$ *on* $\mathcal{M}$ *are determined by* (u,v) *up to an affine transformation of the type*

$$\tilde{u} = a_{11}\, u + b_1, \quad \tilde{v} = a_{22}\, v + b_2, \quad a_{11} a_{22} \neq 0,$$

or of the type

$$\tilde{u} = a_{12}\, v + c_1, \quad \tilde{v} = a_{21}\, u + c_2, \quad a_{12} a_{21} \neq 0,$$

where a_{ij}, b_i, c_i; $i, j = 1, 2$ are constants.

Next we give a simple criterion principal parameters to be natural.

Proposition 3.5. *Let a time-like Weingarten surface $\mathcal{M}:\ z = z(u, v),\ (u, v) \in \mathcal{D}$ be parameterized with principal parameters. Then (u, v) are natural principal if and only if*

$$\sqrt{-EG}(\nu_1 - \nu_2) = \text{const} \neq 0. \tag{3.2}$$

Proof. The equality $\sqrt{-EG}\,(\nu_1 - \nu_2) = c\,\lambda\,\mu,\ c = \text{const} \neq 0$, and Lemma 3.1 imply the assertion. □

3.1. STRONGLY REGULAR TIME-LIKE W-SURFACES.

We consider strongly regular time-like W-surfaces, i.e. time-like W-surfaces, satisfying the condition

$$\nu_u(u, v)\nu_v(u, v) \neq 0, \qquad (u, v) \in \mathcal{D}.$$

Our main theorem for such surfaces is

Theorem 3.6. *Let $f(\nu)$, $g(\nu)$; $\nu \in \mathcal{I}$, be two differentiable functions satisfying $f(\nu) - g(\nu) \neq 0$, $f'(\nu)\,g'(\nu) \neq 0$, and let $\nu(u, v)$, $(u, v) \in \mathcal{D}$ be a differentiable function such that*

$$\nu_u\,\nu_v \neq 0, \quad \nu(u, v) \in \mathcal{I}.$$

Let $(u_0, v_0) \in \mathcal{D},\ \nu_0 = \nu(u_0, v_0)$ and $\mathfrak{a} \neq 0,\ \mathfrak{b} \neq 0$ be two constants. If

$$\begin{aligned} &\mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right] \\ &\quad +\mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g'd\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] = fg(f-g), \end{aligned} \tag{3.3}$$

then there exists a unique (up to a motion) strongly regular time-like Weingarten surface $\mathcal{M}:\ z = z(u, v),\quad (u, v) \in \mathcal{D}_0 \subset \mathcal{D}$ with invariants

$$\begin{aligned} &\nu_1 = f(\nu), \quad \nu_2 = g(\nu), \\ &\gamma_1 = \exp\left(\int_{\nu_0}^{\nu} \frac{g'd\nu}{g-f}\right)\frac{-\mathfrak{b}f'}{f-g}\,\nu_v, \quad \gamma_2 = \exp\left(\int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g}\right)\frac{-\mathfrak{a}g'}{f-g}\,\nu_u. \end{aligned} \tag{3.4}$$

Furthermore, (u,v) are natural principal parameters for $\mathcal{M}$.

Proof. Using Proposition 3.3, we obtain that the integrability conditions 2.1) and 2.2) in Theorem 2.2 reduce to (3.3), which proves the assertion. □

Introducing the functions

$$I := \int_{\nu_0}^{\nu} \frac{f'(\nu)\,d\nu}{f(\nu)-g(\nu)}\,, \qquad J := \int_{\nu_0}^{\nu} \frac{g'(\nu)\,d\nu}{g(\nu)-f(\nu)}\,, \tag{3.5}$$

we can write the PDE (3.3) in the form

$$\mathfrak{a}^2\, e^{2I}\, \left(J_{uu} + I_u\, J_u - J_u^2\right) - \mathfrak{b}^2\, e^{2J}\, \left(I_{vv} + I_v\, J_v - I_v^2\right) = -f\,g, \tag{3.6}$$

and the principal geodetic curvatures (3.4) in the form

$$\gamma_1 = -\mathfrak{b}\, e^J\, I_v, \quad \gamma_2 = \mathfrak{a}\, e^I\, J_u. \tag{3.7}$$

Hence, with respect to natural principal parameters every strongly regular time-like Weingarten surface possesses a *natural PDE* (3.3) (or equivalently (3.6)).

3.2. TIME-LIKE W-SURFACES WITH $\gamma_1 = 0$.

In this subsection we consider time-like W-surfaces in Minkowski space with first principal geodesic curvature $\gamma_1 = 0$ and prove the fundamental theorem of Bonnet type for this class.

Let $\mathcal{M}: \; z = z(u,v),\; (u,v) \in \mathcal{D}$ be a time-like W-surface, parameterized by natural principal parameters. Then we can assume

$$\mathfrak{a}\sqrt{E} = e^I, \qquad \mathfrak{b}\sqrt{G} = e^J,$$

where I and J are the functions (3.5) and $\mathfrak{a}$, $\mathfrak{b}$ are some positive constants. We note that under the condition $\gamma_1 = 0$ it follows that the function $\nu = \nu(u)$ does not depend on v.

Considering the system (2.3), we obtain that the compatibility conditions for this system reduce to only one - the Gauss equation, which has the form:

$$X(\gamma_2) - \gamma_2^2 = -f(\nu)\,g(\nu).$$

Thus we obtain the following Bonnet type theorem for time-like W-surfaces satisfying the condition $\gamma_1 = 0$:

Theorem 3.7. *Let $f(\nu)$, $g(\nu)$; $\nu \in \mathcal{I}$, be two differentiable functions satisfying $f(\nu) - g(\nu) \neq 0$, $f'(\nu)\,g'(\nu) \neq 0$ and let $\nu(u,v) = \nu(u)$, $(u,v) \in \mathcal{D}$ be a differentiable function such that*

$$\nu_u \neq 0, \quad \nu(u,v) \in \mathcal{I}.$$

Let $(u_0, v_0) \in \mathcal{D}$, $\nu_0 = \nu(u_0, v_0)$ *and* $\mathfrak{a} > 0$ *be a constant. If*

$$\mathfrak{a}^2 e^{2I} (J_{uu} + I_u J_u - J_u^2) = -f(\nu)\, g(\nu), \tag{3.8}$$

then there exists a unique (up to a motion) time-like W-surface $\mathcal{M}: \ z = z(u, v)$, $(u, v) \in \mathcal{D}_0 \subset \mathcal{D}$ *with invariants*

$$\begin{aligned} \nu_1 &= f(\nu), \quad &\nu_2 &= g(\nu), \\ \gamma_1 &= 0, \quad &\gamma_2 &= \mathfrak{a}\, e^I\, (J)_u. \end{aligned} \tag{3.9}$$

Furthermore, (u, v) *are natural principal parameters on* $\mathcal{M}$.

Hence, with respect to natural principal parameters every time-like Weingarten surface with $\gamma_1 = 0$ possesses a *natural ODE* (3.8).

4. PARALLEL TIME-LIKE SURFACES IN MINKOWSKI SPACE AND THEIR NATURAL PDE'S

Let $\mathcal{M}: \ z = z(u, v), \ (u, v) \in \mathcal{D}$ be a time-like surface, parameterized by principal parameters and $l(u, v)$, $l^2 = 1$ be the unit normal vector field of $\mathcal{M}$. The parallel surfaces of $\mathcal{M}$ are given by

$$\overline{\mathcal{M}}(a): \ \bar{z}(u, v) = z(u, v) + a\, l(u, v), \quad a = \text{const} \neq 0, \quad (u, v) \in \mathcal{D}. \tag{4.1}$$

We call the family $\{\overline{\mathcal{M}}(a), \ a = \text{const} \neq 0\}$ the *parallel family* of $\mathcal{M}$.

Taking into account (4.1), we find

$$\bar{z}_u = (1 - a\, \nu_1)\, z_u, \quad \bar{z}_v = (1 - a\, \nu_2)\, z_v. \tag{4.2}$$

Excluding the points, where $(1 - a\,\nu_1)(1 - a\,\nu_2) = 0$, we obtain that the corresponding unit normal vector fields $\bar{l}$ to $\overline{\mathcal{M}}(a)$ and l to $\mathcal{M}$ satisfy the equality $\bar{l} = \varepsilon\, l$, where $\varepsilon := \text{sign}\,(1 - a\,\nu_1)(1 - a\,\nu_2)$. In view of (4.2) it follows that $\bar{E} < 0$ and $\bar{G} > 0$. Hence, the parallel surfaces $\overline{\mathcal{M}}(a)$ of a time-like surface $\mathcal{M}$ are also time-like surfaces.

The relations between the principal curvatures $\nu_1(u, v)$, $\nu_2(u, v)$ of $\mathcal{M}$ and $\bar{\nu}_1(u, v)$, $\bar{\nu}_2(u, v)$ of its parallel time-like surface $\overline{\mathcal{M}}(a)$ are

$$\bar{\nu}_1 = \varepsilon\, \frac{\nu_1}{1 - a\,\nu_1}, \quad \bar{\nu}_2 = \varepsilon\, \frac{\nu_2}{1 - a\,\nu_2}; \quad \nu_1 = \frac{\varepsilon\,\bar{\nu}_1}{1 + a\,\varepsilon\,\bar{\nu}_1}, \quad \nu_2 = \frac{\varepsilon\,\bar{\nu}_2}{1 + a\,\varepsilon\,\bar{\nu}_2}. \tag{4.3}$$

Let $K = \nu_1\,\nu_2$, $H = \dfrac{1}{2}(\nu_2 + \nu_2)$, $H' = \dfrac{1}{2}(\nu_2 - \nu_2)$ be the three invariants of the time-like surface $\mathcal{M}$. The equalities (4.3) imply the relations between the invariants $\bar{K}$, $\bar{H}$ and $\bar{H}'$ of $\overline{\mathcal{M}}(a)$ and the corresponding invariants of $\mathcal{M}$:

$$K = \frac{\bar{K}}{1 + 2a\,\varepsilon\bar{H} + a^2\bar{K}}, \quad H = \frac{\varepsilon\,\bar{H} + a\bar{K}}{1 + 2a\,\varepsilon\bar{H} + a^2\bar{K}}, \quad H' = \frac{\varepsilon\,\bar{H}'}{1 + 2a\,\varepsilon\,\bar{H} + a^2\bar{K}}. \tag{4.4}$$

Now let $\mathcal{M}: \ z = z(u,v), \ (u,v) \in \mathcal{D}$ be a time-like Weingarten surface with Weingarten functions $f(\nu)$ and $g(\nu)$. We suppose that (u,v) are natural principal parameters for $\mathcal{M}$. We show that (u,v) are also natural principal parameters for any parallel time-like surface $\overline{\mathcal{M}}(a)$.

Proposition 4.1. *The natural principal parameters (u,v) of a given time-like W-surface $\mathcal{M}$ are natural principal parameters for all parallel time-like surfaces $\overline{\mathcal{M}}(a), \ a = \text{const} \neq 0$ of $\mathcal{M}$.*

Proof. Let $(u,v) \in \mathcal{D}$ be natural principal parameters for $\mathcal{M}$, (u_0,v_0) be a fixed point in $\mathcal{D}$ and $\nu_0 = \nu(u_0,v_0)$. The coefficients E and G of the first fundamental form of $\mathcal{M}$ are given by (3.1). The corresponding coefficients $\bar{E}$ and $\bar{G}$ of $\overline{\mathcal{M}}(a)$ in view of (4.2) are

$$\bar{E} = (1 - a\,\nu_1)^2\,E, \quad \bar{G} = (1 - a\,\nu_2)^2\,G. \tag{4.5}$$

Equalities (4.3) imply that $\overline{\mathcal{M}}(a)$ is again a Weingarten surface with Weingarten functions

$$\bar{\nu}_1(u,v) = \bar{f}(\nu) = \frac{\varepsilon f(\nu)}{1 - af(\nu)}\,, \quad \bar{\nu}_2(u,v) = \bar{g}(\nu) = \frac{\varepsilon g(\nu)}{1 - ag(\nu)}. \tag{4.6}$$

Using (4.6), we compute

$$\bar{f} - \bar{g} = \frac{\varepsilon(f-g)}{(1 - a\,f)(1 - a\,g)}\,,$$

which shows that $\operatorname{sign}(\bar{f} - \bar{g}) = \operatorname{sign}(f - g)$.

Further, we denote by $f_0 := f(\nu_0), \ g_0 := g(\nu_0)$ and taking into account (3.2) and (4.5), we compute

$$\sqrt{-\bar{E}\,\bar{G}}\,(\bar{f} - \bar{g}) = \sqrt{-E\,G}\,(f - g) = \text{const} \neq 0,$$

which proves the assertion. □

Using the above statement, we prove the following theorem.

Theorem 4.2. T*he natural PDE of a given time-like W-surface $\mathcal{M}$ is the natural PDE of any parallel time-like surface $\overline{\mathcal{M}}(a), \ a = \text{const} \neq 0$, of $\mathcal{M}$.*

Proof. We have to express equation (3.3) in terms of the Weingarten functions of the parallel time-like surface $\overline{\mathcal{M}}(a)$.

Putting

$$\bar{E}_0 = (1 - a\,\nu_1(u_0,v_0))^2\,E_0 = -\mathfrak{a}^{-2}\,(1 - a\,f_0)^2 =: -\bar{\mathfrak{a}}^{-2},$$

$$\bar{G}_0 = (1 - a\,\nu_2(u_0,v_0))^2\,G_0 = \mathfrak{b}^{-2}\,(1 - a\,g_0)^2 =: \bar{\mathfrak{b}}^{-2},$$

we obtain

$$\begin{aligned}
&\bar{\mathfrak{a}}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{\bar{f}'d\nu}{\bar{f}-\bar{g}}\right)\left[\bar{g}'\nu_{uu} + \left(\bar{g}'' - \frac{2\bar{g}'^2}{\bar{g}-\bar{f}}\right)\nu_u^2\right] \\
&+ \bar{\mathfrak{b}}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{\bar{g}'d\nu}{\bar{g}-\bar{f}}\right)\left[\bar{f}'\nu_{vv} + \left(\bar{f}'' - \frac{2\bar{f}'^2}{\bar{f}-\bar{g}}\right)\nu_v^2\right] - \bar{f}\,\bar{g}(\bar{f}-\bar{g}) \\
&= \mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f'd\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right] \\
&+ \mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g'd\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] - f\,g(f-g).
\end{aligned}$$

Hence, the natural PDE of $\overline{\mathcal{M}}(a)$ in terms of the Weingarten functions $\bar{f}(\nu)$, $\bar{g}(\nu)$ coinsides with the natural PDE of $\mathcal{M}$ in terms of the Weingarten functions $f(\nu)$ and $g(\nu)$. □

5. TIME-LIKE SURFACES WHOSE CURVATURES SATISFY A LINEAR RELATION

We now consider time-like W-surfaces, whose three invariants K, H and H' satisfy a linear relation:

$$\delta K = \alpha\, H + \beta\, H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}, \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0. \tag{5.1}$$

A time-like W-surface with principal curvatures ν_1 and ν_2 is said to be *linear fractional* if

$$\nu_1 = \frac{A\nu_2 + B}{C\nu_2 + D}, \quad BC - AD \neq 0. \tag{5.2}$$

We exclude the case $A = D$, $B = C = 0$, which characterizes the points with $H^2 - K = 0$, and show that the classes of surfaces with characterizing conditions (5.1) and (5.2), respectively, coincide.

Lemma 5.1. *Any surface whose invariants* $K = \nu_1\,\nu_2$, $H = \dfrac{1}{2}(\nu_1 + \nu_2)$ *and* $H' = \dfrac{1}{2}(\nu_1 - \nu_2)$ *satisfy the linear relation* (5.1) *is a linear fractional time-like Weingarten surface determined by* (5.2), *and vice versa.*

The relations between the constants $\alpha, \beta, \gamma, \delta$ in (5.1) and A, B, C, D in (5.2) are given by the equalities:

$$\alpha = A - D, \quad \beta = -(A + D), \quad \gamma = B, \quad \delta = C. \tag{5.3}$$

We denote by $\mathfrak{K}$ the class of all time-like surfaces with $H^2 - K > 0$, whose curvatures satisfy (5.1) or equivalently (5.2).

The aim of our study is to classify all natural PDE's of the surfaces from the class $\mathfrak{K}$.

The parallelism between two surfaces given by (4.1) is an equivalence relation. On the other hand, Theorem 4.2 shows that the surfaces from an equivalence class have one and the same natural PDE. Hence, it is sufficient to find the natural PDE's of the equivalence classes. For any equivalence class, we use a special representative, which we call *a basic class*. Thus the classification of the natural PDE's of the surfaces in the class $\mathfrak{K}$ reduces to the natural PDE's of the basic classes.

In view of Theorem 4.2, we prove the following classification theorem.

Theorem 5.2. *Up to similarity, the time-like surfaces in Minkowski space, whose curvatures K, H and H' satisfy the linear relation*

$$\delta K = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}; \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0,$$

are described by the natural PDE's of the following basic surfaces:

(1) $H = 0: \quad \nu = e^\lambda, \quad \bar{\Delta}\lambda = e^\lambda;$

(2) $H = \dfrac{1}{2}: \quad \nu = \dfrac{1}{2}(1 - e^\lambda), \quad \bar{\Delta}\lambda = \sinh\lambda;$

(3) $H' = 1: \quad \bar{\Delta}^*(e^\nu) = 2\,\nu\,(\nu + 2);$

(4) $H = \beta\, H'\ (\beta^2 > 1): \qquad \bar{\Delta}^*(\nu^\beta) = 2\,\dfrac{\beta\,(\beta+1)}{(\beta-1)^2}\,\nu;$

(5) $H = \beta\, H'\ (\beta^2 < 1): \qquad \Delta^*(\nu^\beta) = 2\,\dfrac{\beta\,(\beta+1)}{(\beta-1)^2}\,\nu;$

(6) $\left|\begin{array}{l} H = \beta H' + 1 \\ \beta^2 > 1 \end{array}\right. : \ \nu = \dfrac{(\beta-1)\lambda+2}{2}, \ \bar{\Delta}^*(\lambda^\beta) = \dfrac{\beta((\beta-1)\lambda+2)((\beta+1)\lambda+2)}{2(\beta-1)\lambda};$

(7) $\left|\begin{array}{l} H = \beta\, H' + 1 \\ \beta^2 < 1 \end{array}\right. : \ \nu = \dfrac{(\beta-1)\,\lambda+2}{2}, \ \Delta^*(\lambda^\beta) = \dfrac{\beta((\beta-1)\lambda+2)((\beta+1)\lambda+2)}{2(\beta-1)\,\lambda};$

(8) $K = -1: \quad \nu = \tan\lambda, \quad \Delta\lambda = -\sin\lambda;$

(9) $K = 2\,H': \quad \nu = \dfrac{\lambda-4}{\lambda-2}, \quad \bar{\Delta}^*(e^\lambda) = 2;$

$$(10)\ K = \beta H' + \gamma\ (\beta \neq 0,\ \gamma < 0): \quad \left| \begin{array}{l} \nu = \lambda + \dfrac{\beta}{2}, \quad \mathcal{I} = \dfrac{1}{\sqrt{-\gamma}}\ \arctan \dfrac{\lambda}{\sqrt{-\gamma}}\,, \\ \bar{\Delta}^*(e^{\beta\,\mathcal{I}}) = -\dfrac{\beta\,\gamma}{2}\,\dfrac{\lambda\,(\beta\,\lambda + 2\,\gamma)}{\lambda^2 - \gamma}. \end{array} \right.$$

Proof. According to the constant C in (5.2), the linear fractional time-like W-surfaces are divided into two classes: linear fractional time-like W-surfaces, determined by the condition $C = 0$ and linear fractional time-like W-surfaces, determined by the condition $C \neq 0$.

I. Linear fractional time-like Weingarten surfaces with $C = 0$.

This class is determined by the equality

$$\alpha\,H + \beta\,H' + \gamma = 0, \quad (\alpha, \gamma) \neq (0, 0), \quad \alpha^2 - \beta^2 \neq 0. \tag{5.4}$$

For the invariants of the time-like parallel surface $\overline{\mathcal{M}}(a)$ of $\mathcal{M}$, because of (4.4), we get the relation

$$\varepsilon\,(\alpha + 2\,a\,\gamma)\,\bar{H} + \varepsilon\,\beta\,\bar{H}' + \gamma = -a\,(\alpha + a\,\gamma)\,\bar{K}. \tag{5.5}$$

Let $\eta := \operatorname{sign}(\alpha^2 - \beta^2)$. Each time choosing appropriate values for the constants $\mathfrak{a}$, $\mathfrak{b}$ and ν_0 in (3.3), we consider the following subclasses and their natural PDE's:

1) $\alpha = 0,\ \beta \neq 0,\ \gamma \neq 0$. Assuming that $\gamma = 1$, the relation (5.4) becomes

$$\beta\,H' + 1 = 0.$$

The natural PDE for these W-surfaces is

$$(e^{-\beta\,\nu})_{uu} - (e^{\beta\,\nu})_{vv} = \frac{2}{\beta}\,\nu\,(\beta\,\nu - 2). \tag{5.6}$$

Up to similarities these time-like W-surfaces are generated by the basic class $H' = 1$ with the natural PDE

$$(e^{\nu})_{uu} - (e^{-\nu})_{vv} = 2\,\nu\,(\nu + 2), \tag{5.6*}$$

which is the case (3) in the statement of the theorem.

2) $\alpha \neq 0,\ \gamma = 0$. Assuming that $\alpha = 1$, the relation (5.4) becomes

$$H + \beta\,H' = 0.$$

2.1) $\beta \neq 0,\ \eta = -1\ (\beta^2 - 1 > 0)$. Choosing $\mathfrak{b}^2\,\dfrac{\beta-1}{\beta+1}\,\nu_0^{-(\beta+1)} = 1,\ \mathfrak{a}^2\,\nu_0^{\beta-1} = 1$, the natural PDE becomes

$$\left(\nu^{-\beta}\right)_{uu} - \left(\nu^{\beta}\right)_{vv} = 2\,\frac{\beta(\beta-1)}{(\beta+1)^2}\,\nu, \tag{5.7}$$

which is the case (4) in the statement of the theorem.

2.2) $\beta \neq 0,\ \eta = 1\ (\beta^2 - 1 < 0)$. Choosing $\mathfrak{b}^2\, \dfrac{\beta-1}{\beta+1} \nu_0^{-(\beta+1)} = -1,\ \mathfrak{a}^2\, \nu_0^{\beta-1} = 1$, the natural PDE becomes

$$\left(\nu^{-\beta}\right)_{uu} + \left(\nu^{\beta}\right)_{vv} = 2\, \frac{\beta(\beta-1)}{(\beta+1)^2}\, \nu, \tag{5.8}$$

which is the case (5) in the statement of the theorem.

2.3) $\beta = 0$. Putting $\nu = e^{\lambda}$, we get the natural PDE for time-like surfaces with $H = 0$:

$$\lambda_{uu} - \lambda_{vv} = e^{\lambda}, \tag{5.9}$$

which is the case (1) in the statement of the theorem.

3) $\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0$. Assuming that $\alpha = 1$, the relation (5.4) becomes

$$H + \gamma = 0.$$

Putting $|H|\, e^{\lambda} := H - \nu = H' > 0$, we get the one-parameter system of natural PDE's for CMC time-like surfaces with $H = -\gamma$:

$$\lambda_{uu} - \lambda_{vv} = 2\,|H|\, \sinh \lambda. \tag{5.10}$$

Up to similarities these time-like W-surfaces are generated by the basic class $|H| = \frac{1}{2}$ with the natural PDE

$$\lambda_{uu} - \lambda_{vv} = \sinh \lambda, \tag{5.10*}$$

which is the case (2) in the statement of the theorem.

4) $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$. Assuming that $\alpha = 1$ we have

$$H + \beta\, H' + \gamma = 0, \quad \beta^2 - 1 \neq 0.$$

Let $\lambda := 2\, H' = \dfrac{-2}{\beta+1}\, (\nu + \gamma) > 0$.

4.1) If $\eta = -1\ (\beta^2 - 1 > 0)$ and choosing

$$\mathfrak{b}^2 = \frac{\beta+1}{\beta-1}\, \left(\frac{-2}{\beta+1}(\nu_0 + \gamma)\right)^{\beta+1}, \ \mathfrak{a}^2 = \left(\frac{-2}{\beta+1}(\nu_0 + \gamma)\right)^{-(\beta-1)},$$

the natural PDE becomes

$$\left(\lambda^{-\beta}\right)_{uu} - \left(\lambda^{\beta}\right)_{vv} = \frac{\beta}{2\,(\beta+1)}\, \frac{((\beta+1)\lambda + 2\,\gamma)((\beta-1)\lambda + 2\,\gamma)}{\lambda}. \tag{5.11}$$

Up to similarities these time-like W-surfaces are generated by the basic class $H = \beta\, H' + 1,\ \beta^2 > 1$ with the natural PDE

$$\left(\lambda^{\beta}\right)_{uu} - \left(\lambda^{-\beta}\right)_{vv} = \frac{\beta}{2\,(\beta-1)}\, \frac{((\beta+1)\lambda + 2)((\beta-1)\lambda + 2)}{\lambda}, \tag{5.11*}$$

which is the case (6) in the statement of the theorem.

4.2) If $\eta = 1$ ($\beta^2 - 1 < 0$) and choosing

$$\mathfrak{b}^2 = -\frac{\beta+1}{\beta-1}\left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{\beta+1}, \ \mathfrak{a}^2 = \left(\frac{-2}{\beta+1}(\nu_0+\gamma)\right)^{-(\beta-1)},$$

the natural PDE becomes

$$\left(\lambda^{-\beta}\right)_{uu} + \left(\lambda^{\beta}\right)_{vv} = \frac{\beta}{2\,(\beta+1)}\,\frac{((\beta+1)\lambda+2\,\gamma)((\beta-1)\lambda+2\,\gamma)}{\lambda}. \quad (5.12)$$

Up to similarities these time-like W-surfaces are generated by the basic class $H = \beta\,H' + 1$, $\beta^2 < 1$ with the natural PDE

$$\left(\lambda^{\beta}\right)_{uu} + \left(\lambda^{-\beta}\right)_{vv} = \frac{\beta}{2\,(\beta-1)}\,\frac{((\beta+1)\lambda+2)((\beta-1)\lambda+2)}{\lambda}, \quad (5.12^*)$$

which is the case (7) in the statement of the theorem.

II. Linear fractional time-like Weingarten surfaces with $C \neq 0$.

Let $C = 1$. The equality (5.1) gets the form

$$K = \alpha\,H + \beta\,H' + \gamma. \quad (5.13)$$

The corresponding relation for the parallel surface $\overline{\mathcal{M}}(a)$ is

$$\varepsilon(\alpha + 2\,a\,\gamma)\,\bar{H} + \varepsilon\,\beta\,\bar{H}' + \gamma = (1 - a\,\alpha - a^2\,\gamma)\,\bar{K}. \quad (5.14)$$

Each time choosing appropriate values for the constants $\mathfrak{a}$, $\mathfrak{b}$ and ν_0 in (3.3), we consider the following subclasses and their natural PDE's:

5) $\alpha = \gamma = 0$, $\beta \neq 0$. The relation (5.13) becomes

$$K = \beta H' \quad \Longleftrightarrow \quad \rho_1 - \rho_2 = -\frac{2}{\beta},$$

where $\rho_1 = \dfrac{1}{\nu_1}$, $\rho_2 = \dfrac{1}{\nu_2}$ are the principal radii of curvature of $\mathcal{M}$.

Putting $\lambda := 4\,\dfrac{\nu-\beta}{2\,\nu-\beta}$, the natural PDE of these time-like surfaces gets the form

$$\left(e^{\lambda}\right)_{uu} - \left(e^{-\lambda}\right)_{vv} - \frac{\beta^4}{8} = 0. \quad (5.15)$$

Up to similarities these time-like W-surfaces are generated by the basic class $K = 2\,H'$ with the natural PDE

$$\left(e^{\lambda}\right)_{uu} - \left(e^{-\lambda}\right)_{vv} - 2 = 0, \quad (5.15^*)$$

which is the case (9) in the statement of the theorem.

6) $(\alpha, \gamma) \neq (0,0)$, $\alpha^2 + 4\gamma \geq 0$. The relation (5.14) implies that there exists a time-like surface $\overline{\mathcal{M}}(a)$, parallel to $\mathcal{M}$, which satisfies the relation (5.4). Hence the natural PDE of $\mathcal{M}$ is one of the PDE's (5.6) - (5.12).

7) $\alpha^2 + 4\,\gamma < 0$. It follows that $\gamma < 0$. The relation (5.14) implies that there exists a time-like surface $\overline{\mathcal{M}}(a)$ parallel to $\mathcal{M}$, which satisfies the relation

$$K = \beta H' + \gamma. \tag{5.16}$$

7.1) $\beta = 0$. The relation (5.16) becomes $K = \gamma < 0$, i.e. $\overline{\mathcal{M}}$ is of constant negative sectional curvature γ. Putting $\lambda := 2\,\arctan \dfrac{\nu}{\sqrt{-\gamma}}$, we get the natural PDE of this surface

$$\lambda_{uu} + \lambda_{vv} = -K^2\,\sin\lambda. \tag{5.17}$$

Up to similarities these time-like W-surfaces are generated by the basic class $K = -1$ with the natural PDE

$$\lambda_{uu} + \lambda_{vv} = -\sin\lambda, \tag{5.17*}$$

which is the case (8) in the statement of the theorem.

7.2) $\beta \neq 0$, $\gamma < 0$. Choosing $\nu_0 = \dfrac{\beta}{2}$, the natural PDE of $\overline{\mathcal{M}}$ becomes

$$(\exp{(\beta\,\mathcal{I})})_{uu} - (\exp{(-\beta\,\mathcal{I})})_{vv} = -\frac{\beta\,\gamma}{2}\,\frac{\lambda\,(\beta\,\lambda + 2\,\gamma)}{\lambda^2 - \gamma}\,, \tag{5.18}$$

where

$$\mathcal{I} = \frac{1}{\sqrt{-\gamma}}\,\arctan\frac{\lambda}{\sqrt{-\gamma}}, \quad \lambda := \nu - \frac{\beta}{2},$$

which is the case (10) in the statement of the theorem.

The proof of Theorem 5.2 is complete. □

6. SUMMARY

Summarizing the results in [6, 7] and in the present paper, we obtain the following parallel between the natural PDE's describing linear fractional W-surfaces in $\mathbb{R}^3$, linear fractional space-like and time-like W-surfaces in $\mathbb{R}^3_1$, respectively.

(i) The natural PDE for a Weingarten surface in Euclidean space is of the type:

$$\begin{aligned}&\mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu}\frac{f'd\nu}{f-g}\right)\left[g'\nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right)\nu_u^2\right]\\ - \quad &\mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu}\frac{g'd\nu}{g-f}\right)\left[f'\nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right)\nu_v^2\right] = -fg(f-g),\end{aligned}$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\, (J_{uu} + I_u\, J_u - J_u^2) + \mathfrak{b}^2\, e^{2J} (I_{vv} + I_v\, J_v - I_v^2) = f(\nu)\, g(\nu).$$

(ii) The natural PDE for a space-like Weingarten surface in Minkowski space is of the type:

$$a^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f' d\nu}{f-g}\right) \left[g' \nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right) \nu_u^2\right]$$
$$-b^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g' d\nu}{g-f}\right) \left[f' \nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right) \nu_v^2\right] = fg(f-g),$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\, (J_{uu} + I_u\, J_u - J_u^2) + \mathfrak{b}^2\, e^{2J} (I_{vv} + I_v\, J_v - I_v^2) = -f(\nu)\, g(\nu).$$

(iii) The natural PDE for a time-like Weingarten surface with real principal curvatures in Minkowski space is of the type:

$$\mathfrak{a}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{f' d\nu}{f-g}\right) \left[g' \nu_{uu} + \left(g'' - \frac{2g'^2}{g-f}\right) \nu_u^2\right]$$
$$+\mathfrak{b}^2 \exp\left(2\int_{\nu_0}^{\nu} \frac{g' d\nu}{g-f}\right) \left[f' \nu_{vv} + \left(f'' - \frac{2f'^2}{f-g}\right) \nu_v^2\right] = fg(f-g),$$

or, equivalently,

$$\mathfrak{a}^2\, e^{2I}\, \left(J_{uu} + I_u\, J_u - J_u^2\right) - \mathfrak{b}^2\, e^{2J}\, \left(I_{vv} + I_v\, J_v - I_v^2\right) = -f(\nu)\, g(\nu).$$

Therefore for the corresponding basic linear fractional surfaces in $\mathbb{R}^3$ and $\mathbb{R}^3_1$ we obtain the correspondence between their natural PDE's.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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COMPOSITION OF INVERSE PROBLEMS WITH A GIVEN LOGICAL STURCTURE

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The paper presents a method for obtaining problems whose conclusions contain disjunctive propositions. These problems constitute a version of inverse problems with a given logical structure. The logical models in the groups of problems studied have been interpreted comprehensively. Equivalent problems have been given by keeping or not keeping the condition of homogeneity in their conclusion.

Keywords: Inverse problems, composition of problems with given logical structures.

2000 Math. Subject Classification: Primary 51F20, Secondary 51M15

1. INTRODUCTION

In mathematical logic a propositional calculus (also called sentential calculus or sentential logic) is a formal system in which formulas of a formal language may be interpreted to represent propositions. A system of rules and logical statements allows certain formulas to be derived. These derived formulas may be interpreted to be true propositions. Usually in Truth-functional propositional logic, formulas are interpreted as having either a truth value of *true* or a truth value of *false*.

Using the sentential logic in this paper we propose a composing technology of new problems as an interpretation of specific logical models. Our aim is to give suitable logical models for formulation of *equivalent* problems and *generating* problems of a given problem.

2. PRELIMINARIES

In logic, a set of symbols is commonly used to express logical representations. Let us recall the basic symbols and logical representations we shall deal with.

Let p and q be two statements.

i) $p \wedge q$ denotes *logical conjunction* (should be read as "*p and q*"). The statement $p \wedge q$ is *true* if p and q are both true; else it is *false*.

ii) $p \vee q$ denotes *logical disjunction* (should be read as "*p or q*"). The statement $p \vee q$ is *true* if p or q (or both) are true; if both are false, the statement is *false*.

iii) $p \veebar q$ denotes *exclusive disjunction* (should be read as "*either p or q*"). The statement $p \veebar q$ is *true* when either p or q, but not both, are true.

iv) $\neg p$ denotes *negation* (should be read as "*not p*"). The statement $\neg p$ is *true* if and only if p is false.

v) $p \rightarrow q$ denotes *logical implication* (should be read as "*if p then q*"). The statement $p \rightarrow q$ is *true* just in the case that either p is false or q is true, or both. The statements p and q aren't necessarily related comprehensively to each other.

vi) $p \Rightarrow q$ denotes *material implication* (should be read as "*p implies q*" or "*q follows p*"). The relation $p \Rightarrow q$ means that if p is true then q is also true; if p is false then nothing is said about q. The statements p and q are related comprehensively to each other.

vii) $p \leftrightarrow q$ denotes *logical equivalence* (should be read as "*p if and only if q*"). The statement $p \leftrightarrow q$ is *true* just in case either both p and q are false, or both p and q are true. The statements p and q aren't necessarily related comprehensively to each other.

viii) $p \Leftrightarrow q$ denotes *material equivalence* (should be read as "*q is necessary and sufficient for p*"). The relation $p \Leftrightarrow q$ means that $p \Rightarrow q$ and $q \Rightarrow p$. The statements p and q are related comprehensively to each other.

3. THEORETICAL BASIS OF THE PROPOSED METHOD FOR GENERATING PROBLEMS

In this section we describe in detail the theoretical basis of the method for generating problems with a given logical structure. In what follows p_1, p_2; t, p, q, r will stand for statements.

In this paper we deal with a generalization of the formal logical rule [6]

$$(p_1 \to r) \wedge (p_2 \to r) \Leftrightarrow (p_1 \vee p_2 \to r). \tag{$*$}$$

Semantic rules connected with the material implication correspond to the formal derivation rules used in the proofs below. By semantic interpretations the formal derivation rules are called *consequence rules* [1].

This correspondence allows us to formulate and comprehensively use the proposition below.

Proposition 3.1. *The following equivalence is true:*

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow t \wedge (p \vee q) \to r. \tag{1}$$

Proof. Let statement p_1 in (*) have structure $t \wedge p$ and statement p_2 in (*) have structure $t \wedge q$. Then

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow (t \wedge p) \vee (t \wedge q) \to r \Leftrightarrow t \wedge (p \vee q) \to r,$$

i. e. the conjunction of the problems

$$t \wedge p \to r \tag{2}$$

and

$$t \wedge q \to r \tag{3}$$

is equivalent to the problem

$$t \wedge (p \vee q) \to r. \tag{4}$$

□

Any true proposition could have more than one inverse proposition. However, not every inverse proposition is a true statement. The truth value of an inverse proposition of a given true proposition depends essentially on its composition principle.

According to [6], if a given proposition has the logical structure $p_1 \wedge p_2 \to r$, then each one of the following propositions could be considered to be its inverse: $r \to p_1 \wedge p_2$, $p_1 \wedge r \to p_2$ and $r \wedge p_2 \to p_1$.

The most interesting and important inverse propositions are those that are true as well as independent from the other possible inverse propositions, i. e. the *strongest* inverse propositions.

Equivalence (1) formally describes a method for composing new problems with a given logical structure and for formulating their inverse problems.

According to Proposition 3.1 problems with logical structures (2) and (3) generate a problem with a logical structure (4).

In this paper we consider only problems inverse to problems of type (4) with structure

$$t \wedge r \;\rightarrow\; p \vee q. \tag{5}$$

Problems with logical structures (2) and (3) are said to be *generating* problems with structure (4) and their inverse problems with structure (5).

To change the logical structure in the conclusion of the inverse problem from *logical disjunction* to *exclusive disjunction* we need a dichotomic decomposition of the considered set of geometric objects with respect to any remarkable property and its negation. Such a decomposition guarantees the *homogeneity* [4] of the statements (based on one and the same equivalence relation) in the conclusion of the problem.

Proposition 3.2. *The following equivalence is true:*

$$t \wedge r \;\rightarrow\; p \vee q \;\Leftrightarrow\; t \wedge r \;\rightarrow\; p \veebar (\neg p \wedge q). \tag{6}$$

Proposition 3.2 gives the equivalence between problems with a logical structure (5) and problems with a logical structure

$$t \wedge r \;\rightarrow\; p \veebar (\neg p \wedge q). \tag{7}$$

Any problem with a logical structure (7) satisfies the condition of *homogeneity in the conclusion.*

4. APPLICATION OF THE METHOD TO SPECIFIC GROUPS OF PROBLEMS

We discuss four groups of problems to illustrate the described generating method. In each of the groups we formulate suitable *generating* problems for the corresponding equivalent and inverse problems.

The problems in each of the proposed groups are comprehensively related to each other.

4.1. PROBLEMS OF GROUP I

The statements used for the formulation of the problems in this group are

t:={*The straight line* AD, $D \in BC$, *is a median in* $\triangle ABC$.}

p:= $\{AC = AB\}$

q:=$\{\angle BAC = 90^0\}$

r:=$\{\angle DAC + \angle ABC = 90^0\}$

First we formulate and solve the *generating* problems.

Problem 4.1. *Let the straight line AD, $D \in BC$, be a median in $\triangle ABC$. Prove that if $AC = AB$, then $\angle DAC + \angle ABC = 90^0$.*

This problem has a logical structure $t \wedge p \rightarrow r$. Its proof follows immediately from Fig. 1.

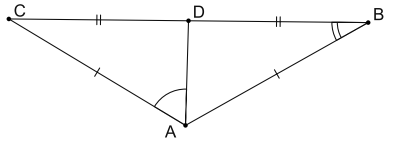

Fig. 1.

Problem 4.2. *Let the straight line AD, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle BAC = 90^0$, then $\angle DAC + \angle ABC = 90^0$.*

Problem 4.2 has a logical structure $t \wedge q \rightarrow r$. The proof follows easily from Fig. 2.

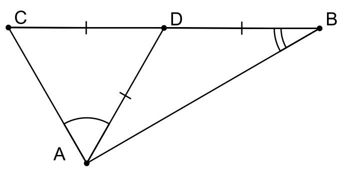

Fig. 2.

According to the logical structures of Problems 4.1 and 4.2 and in view of Proposition 3.1, we construct the following inverse problem with logical structure $t \wedge r \rightarrow p \vee q$.

Problem 4.3. ([3, Problem 3]) *Let the straight line AD, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$, then $AC = AB$ or $\angle BAC = 90^0$.*

The next two problems are equivalent to Problem 4.3.

Problem 4.4. ([3, Problem 2]) *Let the straight line AD, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$ and $\angle BAC \neq 90^0$, then $AC = AB$.*

Problem 4.5. ([3, Problem 1]) *Let the straight line AD, $D \in BC$, be a* *median in* $\triangle ABC$*. Prove that if* $\angle DAC + \angle ABC = 90^0$ *and* $AC \neq AB$*, then* $\angle BAC = 90^0$*.*

Another version of Problem 4.5 is Problem 246, p. 211 in [10].

In view of Proposition 3.2, Problem 4.3 can be reformulated as follows by keeping the condition of homogeneity in its conclusion (compare also with [3, Problem 4]; [9, p. 24, Problem 6]; [8, p. 22, Problem 1]; [11, p. 265, Problem 312]).

Problem 4.6. *Let the straight line* AD*,* $D \in BC$*, be a median in* $\triangle ABC$*. Prove that if* $\angle DAC + \angle ABC = 90^0$*, then* $\triangle ABC$ *is either isosceles* ($AC = AB$)*, or not isosceles but right-angled* ($\angle BAC = 90^0$)*.*

4.2. PROBLEMS OF GROUP II

The statements used for the formulation of the problems in this group are

t:=$\{$*In* $\triangle ABC$ *the straight line* AA_1*,* $A_1 \in BC$*, is the bisector of* $\angle CAB$*, the straight line* BB_1*,* $B_1 \in AC$*, is the bisector of* $\angle CBA$ *and* $AA_1 \cap BB_1 = J$*.*$\}$

$p := \{AC = BC\}$

$q := \{\angle ACB = 60^0\}$

$r := \{JA_1 = JB_1\}$

First we formulate and solve the *generating* problems.

Problem 4.7. *Let in* $\triangle ABC$ *the straight line* AA_1*,* $A_1 \in BC$*, be the bisector of* $\angle CAB$*, the straight line* BB_1*,* $B_1 \in AC$*, be the bisector of* $\angle CBA$ *and* $AA_1 \cap BB_1 = J$*. Prove that if* $AC = BC$*, then* $JA_1 = JB_1$*.*

Problem 4.7 has a logical structure $t \wedge p \rightarrow r$.

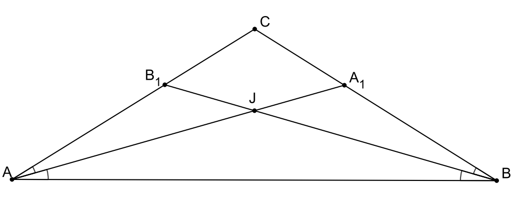

Fig. 3.

Proof. Since $AC = BC$, then $\angle CAB = \angle CBA$ and hence $\angle A_1AB = \angle B_1BA$ (Fig. 3). From the Criteria for congruence of triangles we have $\triangle A_1AB \cong \triangle B_1BA$.

As a consequence it follows that $AA_1 = BB_1$, $\triangle AJB$ is isosceles, $AJ = BJ$ and $JA_1 = JB_1$. □

Problem 4.8. *Let in $\triangle ABC$ the straight line AA_1, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line BB_1, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $\angle ACB = 60^0$, then $JA_1 = JB_1$.*

This problem has a logical structure $t \wedge q \rightarrow r$.

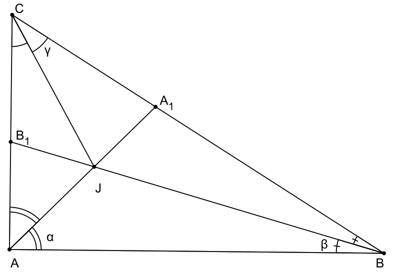

Fig. 4.

Proof. Let us denote $\angle BAA_1 = \angle CAA_1 = \alpha$, $\angle ABB_1 = \angle CBB_1 = \beta$ (Fig. 4). Since J is the intersection point of the bisectors AA_1 and BB_1 of $\triangle ABC$, then CJ is the bisector of $\angle ACB$ and $\angle JCA = \angle JCB = \gamma = 30^0$. Since $\alpha+\beta+\gamma = 90^0$, then $\alpha + \beta = 60^0$, $\angle AJB = 120^0$ and the quadrilateral CA_1JB_1 can be inscribed in a circle. Hence, $JA_1 = JB_1$ as chords corresponding to equal angles (arcs) in a circle. □

According to the logical structures of Problems 4.7 and 4.8 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \rightarrow p \vee q$.

Problem 4.9. ([3, Problem 7]) *Let in $\triangle ABC$ the straight line AA_1, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line BB_1, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$, then $AC = BC$ or $\angle ACB = 60^0$.*

The next two problems are equivalent to Problem 4.9.

Problem 4.10. ([3, Problem 5]) *Let in $\triangle ABC$ the straight line AA_1, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line BB_1, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$ and $AC \neq BC$, then $\angle ACB = 60^0$.*

Problem 4.11. ([3, Problem 6]) *Let in $\triangle ABC$ the straight line AA_1, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line BB_1, $B_1 \in AC$, be the bisector of*

$\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$ and $\angle ACB \neq 60^0$, then $AC = BC$.

In view of Proposition 3.2, Problem 4.9 can be reformulated by keeping the condition of homogeneity in its conclusion.

Problem 4.12. ([3, Problem 8]) *Let in $\triangle ABC$ the straight line AA_1, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line BB_1, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$, then $\triangle ABC$ is either isosceles $(CA = CB)$ or not isosceles but $\angle ACB = 60^0$.*

4.3. PROBLEMS OF GROUP III

The statements used for the formulation of the problems in this group are

$t :=$ {*The straight line CH, $H \in AB$, is the altitude and the straight line CM, $M \in AB$, is the median of $\triangle ABC$.*}

$p := \{AC = BC\}$

$q := \{\angle ACB = 90^0\}$

$r := \{\angle ACM = \angle BCH\}$

First we formulate and solve the *generating* problems.

Problem 4.13. *Let the straight line CH, $H \in AB$, be the altitude and the straight line CM, $M \in AB$, be the median of $\triangle ABC$. Prove that if $AC = BC$, then $\angle ACM = \angle BCH$.*

Problem 4.13 has a logical structure $t \wedge p \rightarrow r$.

Proof. In any isosceles triangle the altitude and the median to its base are congruent. Hence, $M \equiv H$ and $\angle ACM = \angle BCH$. □

Problem 4.14. *Let the straight line CH, $H \in AB$, be the altitude and the straight line CM, $M \in AB$, be the median of $\triangle ABC$. Prove that if $\angle ACB = 90^0$, then $\angle ACM = \angle BCH$ (and also $\angle ACH = \angle BCM$).*

This problem has a logical structure $t \wedge q \rightarrow r$.

Proof. In the right-angled not isosceles $\triangle ABC$ the location of the collinear points B, H and M is either H/BM or M/BH. Let, for instance, H/BM (Fig. 5). Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. Then $\alpha + \beta = 90^0$. Since $AM = MC$ $(= MB)$, then $\triangle AMC$ is isosceles and $\angle ACM = \alpha$. In the right-angled $\triangle BHC$ we have $\angle BCH = 90^0 - \beta = \alpha$. Hence, $\angle ACM = \angle BCH$ (and also $\angle ACH = \angle BCM$).

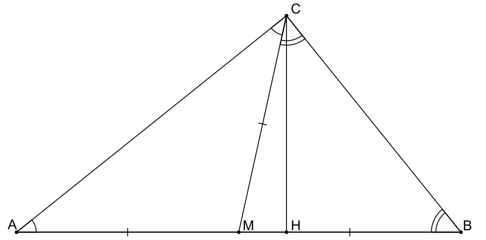

Fig. 5.

For a right-angled isosceles triangle see Problem 4.13. □

According to the logical structures of problems 4.13 and 4.14 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \rightarrow p \vee q$.

Problem 4.15. *Let the straight line CH, $H \in AB$, be the altitude and the straight line CM, $M \in AB$, be the median of $\triangle ABC$. Prove that if $\angle ACM = \angle BCH$, then $AC = BC$ (i. e. $\triangle ABC$ is isosceles) or $\angle ACB = 90^0$ (i. e. $\triangle ABC$ is right-angled).*

Proof. Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. In any triangle at least two of the angles must be acute angles. Hence, in $\triangle ABC$ at least one of the angles α and β is acute. Let, for instance, $\beta < 90^0$. If we assume that $\alpha \geq 90^0$ then the location of the collinear points A, H and M is either A/HM, or $A \equiv H$ (Fig. 6). Then for

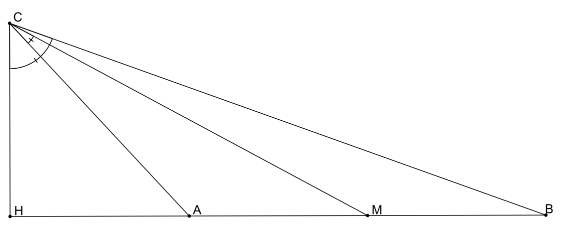

Fig. 6.

the right-angled $\triangle BCH$ is valid $\angle ACM < \angle BCH$, which contradicts the given condition $\angle ACM = \angle BCH$. Hence, $\alpha < 90^0$ and the points H and M lie between the points A and B.

There are two possibilities for the points H and M - they either coincide or not.

(i) Let $H \equiv M$. In this case the median CM in $\triangle ABC$ coincides with the altitude CH, i. e. $\triangle ABC$ is isosceles. If in addition $\angle ACB = 90^0$, then $\triangle ABC$ is isosceles right-angled.

(ii) Let $H \neq M$ and H/BM (the considerations in the case M/BH are analogous). In the considered case $\alpha < \beta$ (Fig. 7).

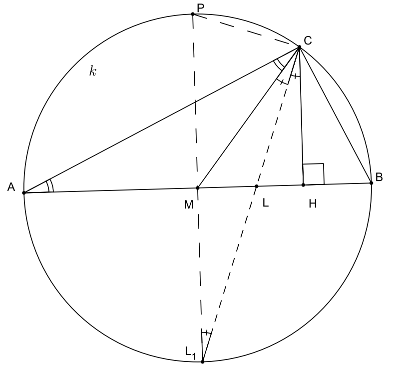

Fig. 7.

Let CL, $L \in AB$, be the bisector of $\angle ACB$. It follows that CL is also the bisector of $\angle MCH$ (see also [2, p. 184, problem 29]; [5, p. 41, problem 2.32]).

Let k be the circumscribing circle of $\triangle ABC$ and $CL \cap k = L_1$. The point L_1 is the middle point of the arc $\widehat{AL_1B}$. The points C and L_1 lie on alternate sides of AB. The perpendicular projection of L_1 onto the chord AB is the middle point M. Then the straight line L_1M is the perpendicular bisector of AB.

The straight line CL_1 cuts the parallel lines CH ($CH \perp AB$) and L_1M ($L_1M \perp AB$) and hence the alternate angles $\angle HCL$ and $\angle ML_1L$ are equal, i. e. $\triangle CML_1$ is isosceles. Thus the point M also lies on the the perpendicular bisector of the chord CL_1.

Since the perpendicular bisectors of any two non parallel chords of a circle cut at its center, the point M is the center of k, the chord AB is a diameter of k and $\angle ACB = 90^0$.

Remark 4.16. Let $P = ML_1 \cap k$. Then PL_1 is a diameter of k and $\angle PCL_1 = 90^0$. It is easily seen that $\triangle MPC$ is isosceles and the point M is the center of k. □

We reformulate Problem 4.15 by keeping the condition of homogeneity of the conclusion.

Problem 4.17. *Let the straight line CH, $H \in AB$, be the altitude and the straight line CM, $M \in AB$, be the median of $\triangle ABC$. Prove that if $\angle ACM = \angle BCH$, then $\triangle ABC$ is either isosceles ($AC = BC$), or not isosceles but right-angled ($\angle ACB = 90^0$).*

The statements used for the formulation of the problems in this group are

$t :=$ {*The middle points of the sides BC, CA and AB of $\triangle ABC$ are F, D, and E respectively.*}

$p := \{AC = BC\}$

$q := \{\angle ACB = 60^0\}$

$r :=$ { *The center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$*}.

First we formulate and solve the *generating* problems.

Problem 4.18. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if $AC = BC$, then the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$.*

This problem has a logical structure $t \wedge p \rightarrow r$.

Proof. The median CE of the isosceles $\triangle ABC$ is the perpendicular bisector of AB and DF and the bisector of $\angle ACB$. Hence, the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$. □

Problem 4.19. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if $\angle ACB = 60^0$, then the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$.*

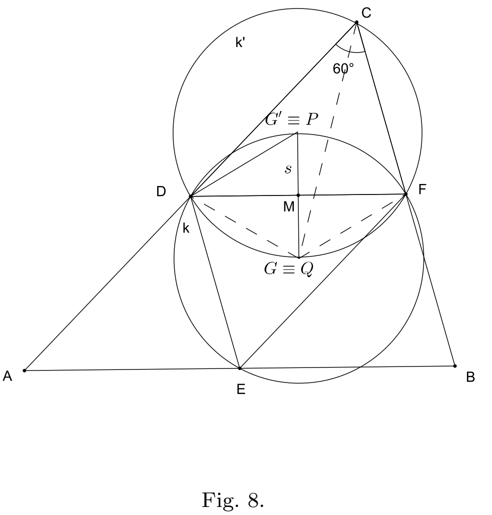

Fig. 8.

This problem has a logical structure $t \wedge q \ \rightarrow\ r$.

Proof. The quadrilateral $EFCD$ (Fig. 8) is a parallelogram with $\angle DCF = 60^0$. Hence, $\triangle EFD \cong \triangle CDF$ and the circumscribing circles k and k' of $\triangle EFD$ and $\triangle CDF$ respectively have equal radii. The centers G and G' of these circles lie on the perpendicular bisector s of DF.

Let $P = s \cap k,\ Q = s \cap k'$. It is easy to be seen that the quadrilateral $FPDQ$ is a rhombus with $\angle PDQ = 60^0$ and $QD = QP = QF$, i. e. the point Q coincides with the center G of k. Consequently, the point P coincides with the center G' of k'.

The point Q is also the middle point of the arc $\widehat{DQF}$ of k' and then lies on the bisector of $\angle DCF \equiv \angle ACB$. □

According to the logical structures of problems 4.18 and 4.19 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \ \rightarrow\ p \vee q$ (a formulation with a different logical structure is given in [7, Problem 12]):

Problem 4.20. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$, then $AC = BC$ or $\angle ACB = 60^0$.*

The next two problems are equivalent to Problem 4.20.

Problem 4.21. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$ and $BC \neq AC$, then $\angle ACB = 60^0$.*

Problem 4.22. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$ and $\angle ACB \neq 60^0$, then $BC = AC$.*

We reformulate Problem 4.20 by keeping the condition of homogeneity of the conclusion.

Problem 4.23. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D, and E respectively. Prove that if the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$, then the $\triangle ABC$ is either isosceles ($AC = BC$), or not isosceles but $\angle ACB = 60^0$.*

Proof. Let G' be the center of the circumscribing circle k' of $\triangle FDC$ (Fig. 9). In view of the Criteria for congruence of triangles we get that $\triangle FDE \cong \triangle DFC$. It follows that the circumscribing circles k and k' of $\triangle FDE$ and $\triangle DFC$ respectively have equal radii.

Let M be the middle point of DF and $L = GM \cap k'$. The point G' lies on the perpendicular bisector GM of DF. Hence, the point L is the middle point of the arc $\widehat{DLF}$ of k' and CL is the bisector of $\angle DCF \equiv \angle ACB$.

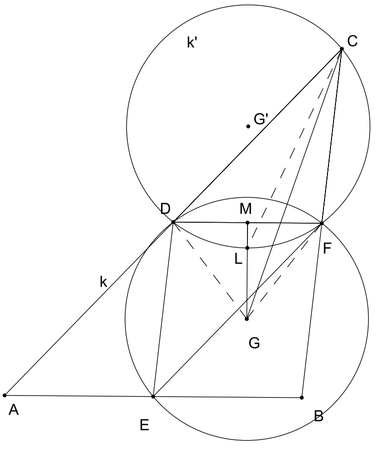

Fig. 9.

Since the center G of k lies on the bisector CL (according to the condition of the Problem), then the straight lines CL and GM either cut at G (have no other common points), or coincide (all of their points are common).

(i) Let $CL \cap GM = L \equiv G$.

In this case $G' \in k$ (Fig. 8) and $\triangle G'DG$ is equilateral, the central $\angle DG'F$ of k' has a measure 120^0 and hence $\angle ACB = 60^0$.

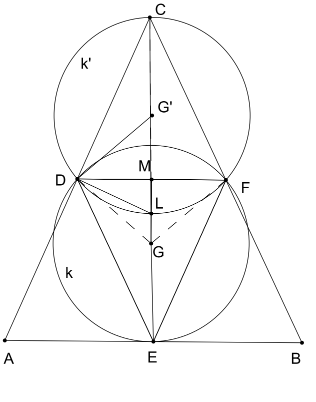

Fig. 10.

(ii) Let $CL \equiv GM$ (Fig. 10).

In this case the bisector CL of $\angle DCF$ coincides with the perpendicular bisector of DF. Then $\triangle DCF$ and also $\triangle ABC$ are isosceles, i. e. $AC = BC$. □

5. SUMMARY

In this section we formulate a new problem whose proof emphasizes the importance and significance of the described method for generating problems.

The similar conclusions of Problems 4.9 and 4.20 lead to

Problem 5.1. *Let the middle points of the sides BC, CA and AB of $\triangle ABC$ be F, D and E respectively. Let further the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$, respectively, and let $AA_1 \cap BB_1 = J$.*

Prove that the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$ if and only if $JA_1 = JB_1$.

Proof. (i) Let the center G of the circumscribing circle k of $\triangle FDE$ lie on the bisector of $\angle ACB$.

From Problem 4.20 it follows that $AC = BC$ or $\angle ACB = 60^0$.

- If $AC = BC$ then from Problem 4.7 it follows that $JA_1 = JB_1$.

- If $\angle ACB = 60^0$ then from Problem 4.8 it follows that $JA_1 = JB_1$.

(ii) Let $JA_1 = JB_1$. From Problem 4.9 it follows that either $AC = BC$ or $\angle ACB = 60^0$.

- If $AC = BC$ then from the generating Problem 4.18 it follows that the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$.

- If $\angle ACB = 60^0$ then from the generating Problem 4.19 it follows that the center G of the circumscribing circle k of $\triangle FDE$ lies on the bisector of $\angle ACB$.

□

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ON THE EXTENDABILITY OF GRIESMER ARCS

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We prove a new sufficient condition for the extendability of Griesmer arcs with certain parameters.

Keywords: Finite projecive geometries, arcs, blocking sets, divisible arcs, quasi-divisible arcs, the griesmer bound, Griesmer codes, Griesmer arcs, extendable arcs, minihypers

2000 Math. Subject Classification: Primary 51A20, 51A21, 51A22; Secondary 94B65

1. INTRODUCTION

The geometric nature of certain problems in coding theory has been long known. In this paper we present a new result on the extendability of arcs in finite projective spaces which translates in a natural way into a result about the extendability of linear codes.

It is a well-known fact that adding a parity check to a binary $[n, k, d]$-code of odd minimum distance d increases the minimum distance of the codes, i.e. the resulting codes have parameters $[n + 1, k, d + 1]$. This result has been generalized by Hill and Lizak in [4,5]. They showed that if all weights in an $[n, k, d]_q$ code are congruent to 0 or $d \pmod q$, with $(d, q) = 1$, then it can be extended to an $[n+1, k, d+1]_q$-code. This fact has a natural explanation in terms of blocking sets containing a hyperplane. It was proved independently in [6] and [9] that the theorem of Hill and Lizak can be obtained from the well-known Bose-Burton theorem for blocking sets in $\mathrm{PG}(k-1, q)$. This result was further generalized in [7] by using a result of Beutelspacher and Heim on the size of the minimal non-trivial (i.e. not containing a hyperplane) blocking set in a finite projective geometry.

In a series of papers, Maruta obtained further results [9,10,11,12,13] on the extendability of linear codes. He introduced the notion of diversity of a linear code with spectrum (A_i) as the pair (Φ_0, Φ_1), where

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i\neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv 0, d(q)} A_i.$$

Maruta proved that for various values of the diversity the investigated codes are indeed extendable. In particular, he showed that a linear $[n, k, d]$-code over $\mathbb{F}_q$, with $q \geq 5$, $d \equiv -2 \pmod q$, having all non-zero weights congruent to $-2, -1$, and 0 modulo q is extendable.

Dodunekov and Simonis proved in [3] that linear $[n,k,d]_q$-codes of full length and $(n,n-d)$-arcs in $\mathrm{PG}(k-1,q)$ are in some sense equivalent objects. With each linear code one can associate an arc (possibly in an non-unique way) so that semilinearly isomorphic codes give rise to equivalent arcs and vice versa. Arcs associated with codes meeting the Griesmer bound are called Griesmer arcs.

This paper deals with the question of the extendability of arcs associated with codes meeting the Griesmer bound. The results translate in an obvious way for linear codes over finite fields. In section 2, we give some basic definitions and introduce the important notion of t-quasidivisibility modulo q. In section 3, we define a special arc $\widetilde{\mathcal{K}}$ in the dual geometry and relate the extendability property for $\mathcal{K}$ with the existence of a hyperplane in the support of $\widetilde{\mathcal{K}}$. Section 4 contains the main theorem stating that a t-quasidivisible Griesmer arc with divisor q, $t < \sqrt{q}$, which has an additional numerical condition on the parameters, is t-times extendable.

2. BASIC DEFINITIONS

Let $\mathcal{P}$ be the set of points of the projective geometry $\mathrm{PG}(k-1,q)$. Every mapping $\mathcal{K}:\mathcal{P}\to\mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in $\mathrm{PG}(k-1,q)$. This mapping is extended additively to the subsets of $\mathcal{P}$: for every $\mathcal{Q}\subseteq\mathcal{P}$, $\mathcal{K}(\mathcal{Q})=\sum_{P\in\mathcal{Q}}\mathcal{K}(P)$. The integer $n:=\mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. For every set of points $\mathcal{Q}\subset\mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P)=\begin{cases}1 & \text{if } P\in\mathcal{Q},\\ 0 & \text{otherwise.}\end{cases}$$

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-multiarc (or simply (n,w)-arc) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\le w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-blocking set with respect to the hyperplanes (or (n,w)-minihyper) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\ge w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$.

An (n,w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called t-extendable, if there exists an $(n+t,w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\ge\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. An arc is called simply extendable if it is 1-extendable. Similarly, an (n,w)-blocking set $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called reducible, if there exists an $(n-1,w)$-blocking set $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\le\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. A blocking set is called irreducible if it is not reducible.

Given a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$, we denote by a_i the number of hyperplanes H with $\mathcal{K}(H)=i$. The sequence (a_i) is called the spectrum of $\mathcal{K}$. An (n,w)-arc $\mathcal{K}$ with spectrum (a_i) is said to be divisible with divisor $\Delta>1$ if $a_i=0$ for all $i\not\equiv n \pmod{\Delta}$. The (n,w)-arc $\mathcal{K}$ with $w\equiv n+t \pmod{q}$ is called t-quasidivisible with divisor $\Delta>1$ (or t-quasidivisible modulo Δ) if $a_i=0$ for all $i\not\equiv n,n+1,\ldots,n+t \pmod{\Delta}$, $1\le t\le q-1$. The result of Hill and Lizak says that every 1-quasidivisible arc with divisor q is extendable; Maruta's theorem from [11] claims that for q odd every 2-quasidivisible arc with divisor q is extendable.

3. THE CONNECTION BETWEEN QUASIDIVISIBILITY AND EXTENDABILITY OF GRIESMER ARCS

As already noted, there exists a one-to-one correspondence between the classes of isomorphic $[n,k,d]_q$-codes and the classes of projectively equivalent $(n,n-d)$-arcs in $\mathrm{PG}(k-1,q)$ [3]. With every multiset $\mathcal{K}$ we can associate many isomorphic linear codes. Fix arbitrarily one of these codes and denote it by $C_{\mathcal{K}}$. If $C_{\mathcal{K}}$ is a Griesmer code then we call $\mathcal{K}$ a Griesmer arc.

Let $\mathcal{K}$ be a t-quasidivisible (n,w)-arc with divisor q in $\Sigma = \mathrm{PG}(k-1,q)$, $t < q$. Set $d = n - w$. This is a typical situation when one investigates the existence of Griesmer arcs with given parameters.

Define a new multiset $\widetilde{\mathcal{K}}$ in the dual geometry $\widetilde{\Sigma}$ by

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \rightarrow \quad \{0,1,\ldots,t\} \\ H & \rightarrow \quad \widetilde{\mathcal{K}}(H) \equiv n + t - \mathcal{K}(H) \pmod q, \end{cases} \tag{3.1}$$

where $\mathcal{H}$ is the set of all hyperplanes in Σ, i.e. the set of all ponts in $\widetilde{\Sigma}$. In other words, hyperplanes of multiplicity congruent to $n+a \pmod q$ become $(t-a)$-points in the dual geometry. The following result is straightforward.

Theorem 1. *Let $\mathcal{K}$ be an (n,w)-arc in $\Sigma = \mathrm{PG}(k-1,q)$, which is t-quasidivisible modulo q with $t < q$. Let $\widetilde{\mathcal{K}}$ be defined by (3.1). If*

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{H}_i} + \widetilde{\mathcal{K}}'$$

for some multiset $\mathcal{K}'$ and c not necessarily different hyperplanes $\widetilde{H}_1, \ldots, \widetilde{H}_c$, then $\mathcal{K}$ is c-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support, then $\mathcal{K}$ is extendable.

Proof. Since maximal hyperplanes correspond to 0-points in the dual geometry, the condition of the theorem is that there exist points in Σ of total multiplicity c that are not incident with maximal hyperplanes. □

By Theorem 1, the extendability of t-quasidivisible arcs is linked with the structure of the multiset $\widetilde{\mathcal{K}}$ defined in the dual geometry. It turns out that this multiset is highly divisible.

Theorem 2. *Let $\mathcal{K}$ be an (n,w)-arc in $\Sigma = \mathrm{PG}(k-1,q)$ which is t-quasidivisible modulo q with $t < q$. For every subspace $\widetilde{S}$ of $\widetilde{\Sigma}$ with $\dim \widetilde{S} \geq 1$,*

$$\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod q.$$

Proof. Let $\widetilde{S}$ be a line in the dual geometry $\widetilde{\Sigma}$. It corresponds to a subspace S of codimension 2 in Σ. Denote by H_i, $i = 0, \ldots, q$, the set of all hyperpalnes through S. We have

$$n = \sum_{i=0}^{q} \mathcal{K}(H_i) - q\mathcal{K}(S).$$

Reducing both sides modulo q and using the fact that $\mathcal{K}(H_i) + \widetilde{\mathcal{K}}(H_i) \equiv n + t \pmod q$, one gets

$$(q+1)(n+t) - \sum_{i=0}^{q} \widetilde{\mathcal{K}}(H_i) \equiv n \pmod q,$$

whence

$$\widetilde{\mathcal{K}}(\widetilde{S}) = \sum_{i=0}^{q} \widetilde{\mathcal{K}}(H_i) \equiv t \pmod q.$$

For subspaces of larger dimension, we can use the fact that the multiplicity of each line in $\widetilde{S}$ is t modulo q. Then we sum the multiplicities of all lines through a fixed 0-point in $\widetilde{S}$. □

By the above theorem, the multiset $\widetilde{\mathcal{K}}$ has the following properties:

- the multiplicity of each point is at most t;

- the multiplicity of each subspace of dimension r, $1 \le r \le k-1$, is at least tv_r.

Here we use the conventional notation $v_r = (q^r-1)/(q-1)$. Let us note that in the general case the cardinality of $\widetilde{\mathcal{K}}$ is not known.

For $t = 1$, the arc $\mathcal{K}$ is always extendable. In fact, this is another formulation of the theorem by Hill and Lizak. A plane arc with the above properties for $t = 1$ turns out to be projective. Then every line is 1- or $(q+1)$-line, the arc is either a line or the complete plane. More generally, in higher dimensions such an arc is either a hyperplane or the complete space. The second case does not occur since a maximal hyperplane maps to a 0-point. Therefore every 1-quasidivisible arc $\mathcal{K}$ is extendable by Theorem 1.

For $t = 2$ and odd $q \ge 5$, the arcs $\widetilde{\mathcal{K}}$ were characterized by Maruta [11]. He proved that in this case, the arc $\widetilde{\mathcal{K}}$ contains a hyperplane without 0-points, which implies that the arc $\mathcal{K}$ is again extendable.

The next theorem relates the extendability of $\mathcal{K}$ with the spectrum of a maximal hyperplane of Σ with respect to $\mathcal{K}$.

Theorem 3. *Let $\mathcal{K}$ be a Griesmer t-quasidivisible modulo q arc with parameters (n, w) in* $\mathrm{PG}(k-1, q)$, *where $w = n - d$. For a fixed hyperplane H_0 of multiplicity w, denote by (a_i) the spectrum of the arc $\mathcal{K}|_{H_0}$, the restriction of $\mathcal{K}$ to the hyperplane H_0. Let A be the largest integer such that a $(tv_{k-1}+A, tv_{k-2})$-minihyper contains a hyperplane in its support. If*

$$qa_{w-\lceil d/q\rceil-1} + 2qa_{w-\lceil d/q\rceil-2} + \ldots + (t-2)qa_{w-\lceil d/q\rceil-t+2}(t-1)q \sum_{u\le w-\lceil d/q\rceil-t+1} a_u \le A,$$

then $\mathcal{K}$ is extendable.

Proof. By the fact that $\mathcal{K}$ is a Griesmer arc, we have that

$$n = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil, \qquad w = \sum_{i=1}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

By straightforward counting, one gets that the maximal multiplicity of a subspace of codimension 2 contained in H_0 is

$$w' = w - \lceil \frac{d}{q} \rceil = \sum_{i=2}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

Let $\widetilde{\mathcal{K}}$ be the arc in $\widetilde{\Sigma}$ defined earlier in this section. The point $\widetilde{P} = H_0$ is a 0-point in $\widetilde{\Sigma}$. Denote by $\widetilde{L}_i$ all lines in $\widetilde{\Sigma}$ through $\widetilde{P}$. They correspond to the hyperlines δ_i in H_0, i.e. the subspaces of codimension 2 that are contained in H_0.

Consider a fixed line $\widetilde{L} = \delta$, where $\mathcal{K}(\delta) = w' - \lambda$, $\lambda \in \{0, \ldots, t-1\}$. Denote by $H_0, H_1, \ldots, H_q$ all hyperplanes through δ. Set

$$\mathcal{K}(H_i) = w - \alpha_i q - \beta_i, \ \ \beta_i \in \{0, \ldots, t\}.$$

Since $\mathcal{K}(H_i) + \widetilde{\mathcal{K}}(H_i) \equiv n + t \equiv w \pmod{q}$, we get that $\widetilde{\mathcal{K}}(H_i) = \beta_i$. Now we have

$$\begin{aligned} n &= \sum_{i=0}^{q} \mathcal{K}(H_i) - q(w' - \lambda) \\ &= \sum_{i=0}^{q} (w - \alpha_i q - \beta_i) - q(w' - \lambda) \\ &= w - q\sum_{i=0}^{q} \alpha_i - \sum_{i=0}^{q} \beta_i + q\lceil \frac{d}{q} \rceil + q\lambda, \end{aligned}$$

whence

$$\sum_{i=0}^{q}\beta_i = q\lceil\frac{d}{q}\rceil + q\lambda - d - q\sum_{i=0}^{q}\alpha_i.$$

Since $d \equiv -t \pmod q$, we have $q\lceil\frac{d}{q}\rceil - d = t$. This gives an upper bound on the multiplicity of $\widetilde{L}$ with respect to $\widetilde{\mathcal{K}}$:

$$\widetilde{\mathcal{K}}(\widetilde{L}) = \sum_i \widetilde{\mathcal{K}}(H_i) = \sum_{i=0}^{q}\beta_i = t + q\lambda - q\sum_{i=0}^{q}\alpha_i \le t + q\lambda.$$

Now summing up the multiplicities of all lines $\widetilde{L}$ through $\widetilde{P}$ and taking into account that $\widetilde{\mathcal{K}}(\widetilde{P}) = 0$, one gets for the cardinality of $\widetilde{\mathcal{K}}$ the following estimate:

$$\begin{aligned}
|\widetilde{\mathcal{K}}| &= \sum_i \widetilde{\mathcal{K}}(\widetilde{L}_i) \\
&\le a_{w'}t + a_{w'-1}(t+q) + \ldots + a_{w'-(t-2)}(t+(t-2)q) + \sum_{u\le w'-(t-1)} a_u(t+(t-1)q) \\
&= \left(\sum_{u\le w'} a_u\right) t + a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q \\
&= v_{k-1}t + a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q.
\end{aligned}$$

Here we use the fact that for lines $\widetilde{L} = \delta$ with $\mathcal{K}(\delta) \le w' - (t-1)$, one has $\widetilde{\mathcal{K}}(\widetilde{L}) \le t + (t-1)q$. If

$$a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q \le A$$

we have that $|\widetilde{\mathcal{K}}| \le tv_{k-1} + A$. This implies that $\widetilde{\mathcal{K}}$ contains a hyperplane without 0-points. Hence, by Theorem 1, $\mathcal{K}$ is extendable. □

The idea of Theorem 3 can be used to restrict the spectrum not only of the maximal hyperplanes, but also of hyperplanes with a smaller multiplicity. Unfortunately, the value of A is not known in general. Partial results for the plane case were proved in [1] and [2].

4. A THEOREM ON THE EXTENDABILITY OF GRIESMER ARCS

In this section we prove our main extendability result for Griesmer arcs. Consider a Griesmer t-quasidivisible arc $\mathcal{K}$, $t < q$, with parameters (n, w) in $\mathrm{PG}(k-1, q)$. Set $d = n - w$ and let $C_{\mathcal{K}}$ be a linear code associated with $\mathcal{K}$. The code $C_{\mathcal{K}}$ has parameters $[n, k, d]_q$. Write d as

$$d = sq^{k-1} - \sum_{i=0}^{k-2}\varepsilon_i q^i, \quad 0 \le \varepsilon_i < q. \tag{4.1}$$

Then we have $\lceil d/q^j\rceil = sq^{k-j-1} - \sum_{i=j}^{k-2}\varepsilon_i q^i$, which implies

$$n = sv_k - \sum_{i=0}^{k-2}\varepsilon_i v_{i+1}. \tag{4.2}$$

Let us note that with this notation $t = \varepsilon_0$, since $n + \varepsilon_0 \equiv w \pmod q$. Denote by w_j the maximal multiplicity of a subspace S of codimension j of $\mathrm{PG}(k-1,q)$: $w_j = \max_{\operatorname{codim} S = j} \mathcal{K}(S)$, $j = 1, \ldots, k-1$. We have

$$w_j = \sum_{i=j}^{k-1} \lceil \frac{d}{q^i} \rceil = s v_{k-j} - \sum_{i=j}^{k-2} \varepsilon_i v_{i-j+1}. \tag{4.3}$$

By convention, $w_0 = n$.

In the next lemmas we establish some important properties of the arc $\widetilde{\mathcal{K}}$.

Lemma 1. *Let $\mathcal{K}$ be a t-quasidivisible $(n, n-d)$-Griesmer arc with d given by (4.1). Let S be a subspace of codimension 2 contained in the hyperplane H_0 with $\mathcal{K}(H_0) = w_1 - aq$, where $a \geq 0$ is an integer.*

(i) If $\mathcal{K}(S) = w_2 - a - b$, $0 \leq b \leq t-2$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + bq$;

(ii) If $\mathcal{K}(S) = w_2 - a - b$, $b \geq t-1$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + (t-1)q$.

Proof. (i) Denote by H_i the hyperplanes through S in Σ. Set $\mathcal{K}(H_i) = w_1 - \alpha_i$, $i = 1, \ldots, q$. Note that $\widetilde{\mathcal{K}}(H_i) \equiv n + t - w_1 + \alpha_i \equiv \alpha_i \pmod q$, since $n + t \equiv w_1 \pmod q$. Thus $\widetilde{\mathcal{K}}(H_i) \leq \alpha_i$. Furthermore, we have

$$\begin{aligned} n &= \sum_{i=0}^{q} \mathcal{K}(H_i) - q\mathcal{K}(S) \\ &= (q+1)w_1 - \sum_{i=1}^{q} \alpha_i - aq - q(w_2 - a - b) \\ &= n + t - \sum_{i=1}^{q} \alpha_i + bq. \end{aligned}$$

This implies that $\sum \alpha_i = t + bq$. On the other hand,

$$\begin{aligned} \widetilde{\mathcal{K}}(\widetilde{S}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{H}_i) \\ &\leq \sum_{i=1}^{q} \alpha_i \pmod q \\ &= t + bq. \end{aligned}$$

(ii) This follows by the facts that $\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod q$, each point is of multiplicity at most t and the line $\widetilde{S}$ is incident with the 0-point $\widetilde{H}_0$. □

Lemma 2. *Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be as in Lemma 1. Let T be a subspace of codimension 3 in* $\mathrm{PG}(k-1,q)$ *with $\mathcal{K}(T) = w_3$. Then*

$$\widetilde{\mathcal{K}}(\widetilde{T}) \leq t(q+1) + \varepsilon_1 q.$$

Proof. Denote by S_i, $i = 0, \ldots, q$, the subspaces of codimension 2 through T in a maximal hyperplane H. Set $\mathcal{K}(S_i) = w_2 - \alpha_i$. We have that

$$\begin{aligned} \mathcal{K}(H) = w_1 &= \sum_{i=0}^{q} \mathcal{K}(S_i) - q\mathcal{K}(T) \\ &= (q+1)w_2 - \sum_{i=0}^{q} \alpha_i - qw_3 \\ &= (q+1)(sv_{k-2} - \varepsilon_{k-2}v_{k-3} - \ldots - \varepsilon_3 v_2 - \varepsilon_2 v_1) - \\ &\qquad q(sv_{k-3} - \varepsilon_{k-2}v_{k-4} - \ldots - \varepsilon_3 v_1) - \sum_{i=0}^{q} \alpha_i. \end{aligned}$$

Since $(q+1)v_{j-1} - qv_{j-2} = v_j$, this simplifies to

$$\begin{aligned} w_1 &= sv_{k-1} - \varepsilon_{k-2}v_{k-2} - \ldots - \varepsilon_3 v_3 - (q+1)\varepsilon_2 v_1 - \sum_{i=0}^{q} \alpha_i \\ &= sv_{k-1} - \varepsilon_{k-2}v_{k-2} - \ldots - \varepsilon_3 v_3 - \varepsilon_2 v_2 - \sum_{i=0}^{q} \alpha_i \\ &= w_1 + \varepsilon_1 v_1 - \sum_{i=0}^{q} \alpha_i. \end{aligned}$$

This implies that $\sum_{i=0}^{q} \alpha_i = \varepsilon_1 v_1 = \varepsilon_1 < q$. By Lemma 1, $\widetilde{\mathcal{K}}(\widetilde{S}_i) \leq t + \alpha_i q$, whence

$$\begin{aligned} \widetilde{\mathcal{K}}(\widetilde{T}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{S}_i) - q\widetilde{K}(\widetilde{H}) \\ &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{S}_i) \\ &\leq \sum_{i=0}^{q} (t + \alpha_i q) \\ &= t(q+1) + q\sum_{i=0}^{q} \alpha_i \\ &\leq t(q+1) + \varepsilon_1 q. \end{aligned}$$

□

Lemma 3. *Let $\mathcal{K}$ be t-quasidivisible Griesmer (n, w)-arc in $\mathrm{PG}(k-1, q)$, $q \geq 3$ with $d = n - w$ given by (4.1). Let $\widetilde{\mathcal{K}}$ be defined by (3.1). Let further $\varepsilon_0, \varepsilon_1 \leq \sqrt{q}$. For every maximal subspace T of codimension 3 in $\mathrm{PG}(k-1, q)$, i.e. a subspace with $\mathcal{K}(T) = w_3$, there holds*

$$\widetilde{\mathcal{K}}(\widetilde{T}) = t(q+1).$$

Proof. We have that $\widetilde{T}$ is a plane in $\widetilde{\mathrm{PG}}(k-1, q)$. By Lemma 2, $\widetilde{\mathcal{K}}(\widetilde{T}) \leq \varepsilon_0(q+1) + \varepsilon_1 q$. Set $\widetilde{\mathcal{K}}(\widetilde{T}) = \varepsilon_0(q+1) + \varepsilon_1' q$, where $0 \leq \varepsilon_1' \leq \varepsilon_1$.

Assume $\varepsilon_1' > 0$. Set $\widetilde{\mathcal{F}} = \widetilde{\mathcal{K}}|_{\widetilde{T}}$, i.e. $\widetilde{\mathcal{F}}$ is the restriction of $\widetilde{\mathcal{K}}$ to the plane $\widetilde{T}$ in the dual geometry. Define a dual plane arc $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ by

$$\mathcal{F}(\widetilde{L}) = i \quad \text{iff} \quad \widetilde{\mathcal{F}}(L) = t + iq.$$

Denote by (A_i) the spectrum of $\widetilde{\mathcal{F}}$. We have

$$\begin{aligned} \sum A_{t+iq} &= q^2 + q + 1, \\ \sum (t+iq) A_{t+iq} &= (\varepsilon(q+1) + \varepsilon_1' q)(q+1) \end{aligned}$$

for some $\varepsilon_1' \leq \varepsilon_1$. This implies $\sum_i iA_{t+iq} = \varepsilon_1'(q+1) + \varepsilon_0$.

Now let us denote by B_i the number of lines L with $\widetilde{\mathcal{F}}(L) = t + iq$ through a fixed point P of multiplicity $c \geq 0$. Then

$$\begin{aligned} \sum B_{t+iq} &= q+1\,, \\ \sum (t+iq) B_{t+iq} &= (q+1)\varepsilon_0 + \varepsilon_1' q + cq\,, \end{aligned}$$

which implies $\sum iB_{t+iq} = \varepsilon_1' + c$. Hence $\mathcal{F}$ is a $(\varepsilon_0'(q+1)+\varepsilon_0, \varepsilon')$-blocking set.

From $\varepsilon_0, \varepsilon_1 < \sqrt{q}$ and $q \geq 3$ we get that $\varepsilon_0 + \varepsilon_1' < \sqrt{\varepsilon_1' q} + 1$ and, consequently, $\varepsilon'(q+1)+\varepsilon_0 < \varepsilon' q + \sqrt{\varepsilon' q} + 1$. By a well-known result by Ball [1] and De Beule-Storme-Metsch [2], $\mathcal{F}$ contains a line. Going back to $\widetilde{\mathcal{F}}$, this implies that all lines L_i in $\widetilde{T}$ through P have multiplicity at least $t + q = \varepsilon_0 + q$. Now we have

$$\begin{aligned}
\varepsilon_0(q+1) + \varepsilon_1 q \geq \widetilde{\mathcal{K}}(\widetilde{T}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(L_i) - q\widetilde{\mathcal{K}}(P) \\
&\geq (q+1)(\varepsilon_0 + q) - q\widetilde{\mathcal{K}}(P) \\
&\geq \varepsilon_0(q+1) + q(q+1) - \varepsilon_0 q.
\end{aligned}$$

This implies $q+1 \leq \varepsilon_0 + \varepsilon_1 < 2\sqrt{q}$, i.e. $(\sqrt{q}-1)^2 < 0$, which is a contradiction. Therefore $\varepsilon_1' = 0$, which proves the lemma. □

Lemma 4. *Let $\mathcal{K}$ be t-quasidivisible Griesmer (n,w)-arc in* $\mathrm{PG}(k-1,q)$*, $q \geq 3$ with $d = n - w$ given by (4.1). Let $\widetilde{\mathcal{K}}$ be defined by (3.1). Let U be a subspace in* $\mathrm{PG}(k-1,q)$ *with* $\operatorname{codim} U = r$*, $1 \leq r \leq k$, which is of maximal multiplicity w_r (if* $\operatorname{codim} U = k$*, $U = \varnothing$). If $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-2} < \sqrt{q}$, then*

$$\widetilde{\mathcal{K}}(\widetilde{U}) = \varepsilon_0 v_{r-1}.$$

Proof. Assume that the result is proved for all subspaces of codimension up to $r-1$. Note that $\widetilde{U}$ is an $(r-1)$-dimensional subspace of $\widetilde{\mathrm{PG}}(k-1,q)$.

Let $U \subset S$ be maximal subspaces of codimensions r and $r-2$, respectively. Denote by T_i, $i = 0, \ldots, q$, the subspaces through U of codimension $r-1$ that are contained in S. Then at most ε_{r-2} of the subspaces T_i are not of maximal multiplicity, i.e. at least $q+1-\varepsilon_{r-2}$ of them are of multiplicity w_{r-1}. Indeed, if the number of the maximal subspaces among the T_i's is denoted by γ, then we have $w_{r-2} \leq (q+1)w_{r-1} - qw_r - \gamma$, i.e.

$$\begin{aligned}
\gamma &\leq -w_{r-2} + (q+1)w_{r-1} - qw_r \\
&= \varepsilon_{r-1}v_2 + \varepsilon_{r-2}v_1 - (q+1)\varepsilon_{r-1}v_1 \\
&= \varepsilon_{r-2}.
\end{aligned}$$

Since U is a subspace of maximal multiplicity, there exists a maximal hyperplane H containing U. Hence $\widetilde{U}$ contains a 0-point with respect to $\widetilde{\mathcal{K}}$, say $\widetilde{P}$. In the case of codimension k, we can take as $\widetilde{P}$ any 0-point in $\widetilde{\mathrm{PG}}(k-1,q)$.

Consider a projection φ from $\widetilde{P}$ onto some hyperplane $\widetilde{V}$ in $\widetilde{U}$ disjoint from $\widetilde{P}$. We have $\widetilde{V} \cong \mathrm{PG}(r-2,q)$. Define a new arc

$$\mathcal{F} = \frac{1}{q}(\widetilde{\mathcal{K}}^{\varphi} - \varepsilon_0).$$

For every point $X \in \widetilde{V}$ we have $0 \leq \mathcal{F}(X) \leq \varepsilon_0 - 1$. $\varphi(\widetilde{U})$ is a subspace of dimension $r-2$, $\varphi(\widetilde{T}_i)$ are hyperplanes in $\varphi(\widetilde{U})$ (dimension $r-3$), and $\varphi(\widetilde{S})$ is a subspace of dimension $r-4$ contained in all $\varphi(\widetilde{T}_i)$. By the induction hypothesis $\mathcal{F}(\varphi(\widetilde{T}_i)) = 0$ for T_i of maximal multiplicity, i.e. $\mathcal{K}(T_i) = w_{r-1}$. Without loss of generality T_i, $i = \varepsilon_{r-2}, \ldots, q$, are maximal. So, the points $X \in \widetilde{V}$ with $\mathcal{F}(X) > 0$ are contained in the subspaces $\varphi(\widetilde{T}_j)$ with $j \in \{0, \ldots, \varepsilon_{r-2}-1\}$.

We can repeat the argument from the last two paragraphs to another subspace S' of codimension $r-2$ containing U. We get that the points $X \in \widetilde{V}$ are contained in another ε_{r-2} subspaces of $\varphi(\widetilde{U})$, say $\varphi(\widetilde{T}_j')$ with $j \in \{0, \ldots, \varepsilon_{r-2}-1\}$. So the non-zero points of $\mathcal{F}$ are contained $\varphi(\widetilde{T}_i) \cap \varphi(\widetilde{T}_j')$, where $i, j \in \{0, \ldots, \varepsilon_{r-2}-1\}$. Hence the number of points X with $\mathcal{F}(X) > 0$ does not exceed

$$\varepsilon_{r-2}^2 v_{r-3} \leq q v_{r-3} = v_{r-2} - 1.$$

Let $X \in \widetilde{V}$ with $\mathcal{F} = c \geq 0$. Every point in $\varphi(\widetilde{U})$ is incident with v_{r-2} lines. Thus, there is a line $L \in \widetilde{V}$ through X which contains apart from X just 0-points. This line is the image of a plane π which has q-lines of multiplicity ε_0 and one line of multiplicity $\varepsilon_0 + cq$, where $c \leq \varepsilon_0 - 1$ (with respect to $\widetilde{\mathcal{K}}$). Thus, $\widetilde{\mathcal{K}}(\pi) = \varepsilon_0(q+1) + cq$ and, by Lemma 3, we should have $c = 0$.

Thus, $\mathcal{F}(X) = 0$ for all $X \in \widetilde{V}$ and all lines through P in $\widetilde{U}$ are t-lines. This proves the lemma. □

Now we can prove our main theorem.

Theorem 4. *Let $\mathcal{K}$ be a Griesmer $(n, n-d)$-arc which is t-quasidivisible modulo q with d given by (4.1). Let $t = \varepsilon_0, \ldots \varepsilon_{k-2} < \sqrt{q}$. Then $\mathcal{K}$ is t-extendable.*

Proof. By Lemma 4, $\widetilde{\mathcal{K}}$ is a (tv_{k-1}, tv_{k-2})-minihyper. By Corollary 3.5 from [8], every (xv_{k-1}, xv_{k-2}) minihyper in $\mathrm{PG}(k-1, q)$ with $x \leq q - \frac{q}{p}$ is the sum of hyperplanes. Since $t < \sqrt{q}$, the result follows. □

We conclude with an example illustrating our approach to the extendability of incomplete caps. Let $\mathcal{K}$ be a (q^2+1-t)-cap in $\mathrm{PG}(3, q)$ with $t < \sqrt{q}$. Assume the largest hyperplane (plane) has multiplicity $q+1$. This is obviously always the case for odd q. The code $C_{\mathcal{K}}$ associated with $\mathcal{K}$ has parameters $[q^2+1-t, 4, q^2-q-t]_q$ and $d = q^2 - q - t = q^3 - (q-1)q^2 - q - t$, i.e. $s = 1$, $\varepsilon_2 = q-1$, $\varepsilon_1 = 1$, $\varepsilon_0 = t < \sqrt{q}$. The admissible multiplicities of planes are $q+1, \ldots, q+1-t, 1$ and 0. Since $\varepsilon_2 \geq \sqrt{q}$, we cannot apply Theorem 4 directly. We can state only that if L is a 2-line, then $\widetilde{\mathcal{K}}(\widetilde{L}) = t$. Nevertheless, we can prove the t-extendability of $\mathcal{K}$.

At first, we prove that every point of $\mathcal{K}$ is incident with an 1-plane. Consider a projection from such 1-point P onto a plane π not incident with P. The induced arc $\mathcal{K}^{\varphi}$ is a (q^2-t, q)-arc and its complement is a $(q+1-t, 1)$-blocking set. Since $t < \sqrt{q}$, it contains a line L and the plane $\langle L, P\rangle$ is an 1-plane in $\mathrm{PG}(3, q)$. Now, by Lemma 1, $\widetilde{\mathcal{K}}(\widetilde{L}) = t$.

Now consider an 1-line L_0 and assume it is incident only with planes of multiplicity at least $q+1-t$. Consider one such plane π with $\mathcal{K}(\pi) = q-b$, $b \leq t-1$. Let P be the 1-point on L_0 and denote the other 1-lines in π by $L_1, \ldots, L_b$. One of them is on the 1-plane through P. Consider the plane $\widetilde{P}$ in the dual geometry. Now $\widetilde{\pi}$ is a 0-point and the $q+1-b$ of the lines through it are t-lines, while the remaining b lines are t or $(t+q)$-lines. This implies that $\widetilde{\mathcal{K}}(\widetilde{P}) \leq t(q+1) + bq$ and, by Lemma 3, we have $b = 0$. So, we have proved that for every 1-line L, $\widetilde{\mathcal{K}}(\widetilde{L}) = t$.

Now consider a 1-plane π. Let L be a 0-line in π which is contained in another 1-plane (different from π). Counting the multiplicities of the planes through L, we get $\widetilde{\mathcal{K}}(\widetilde{L}) = t$. There are $q^2 - t$ such lines. Hence $q^2+q+1-t$ of the lines through $\widetilde{\pi}$ are t-lines, and the remaining lines have multiplicity t or $t+q$. Now $|\widetilde{\mathcal{K}}| = t(q+1) + tq$ and, again by Lemma 3, $|\widetilde{\mathcal{K}}| = t(q+1)$. This implies that $\widetilde{\mathcal{K}}$ is a sum of planes and $\mathcal{K}$ is t-extendable.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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A LICHNEROWICZ–TYPE RESULT ON A SEVEN–DIMENSIONAL QUATERNIONIC CONTACT MANIFOLD

ALEXANDER PETKOV

In this paper we establish an analogue of the classical Lichnerowicz' theorem giving a sharp lower bound of the first non-zero eigenvalue of the sub-Laplacian on a compact seven-dimensional quaternionic contact manifold, assuming a lower bound of the qc-Ricci tensor, torsion tensor and its distinguished covariant derivatives.

Keywords: Quaternionic Contact Structures, Sub-Laplacian, First Eigenvalue, Lichnerowicz Inequality, 3-Sasakian.

2000 Math. Subject Classification: 53C21, 58J60, 53C17, 35P15, 53C25

1. INTRODUCTION

The aim of this paper is to prove a seven-dimensional version of the main result established in [22]. Namely, we give a sharp lower bound of the first non-zero eigenvalue of the sub-Laplacian on a compact seven-dimensional quaternionic contact (abbr. QC) manifold, assuming some condition on the qc-Ricci tensor, torsion tensor and its derivatives. We pay attention to the fact that a similar result has been established in our resent paper [23], in which it is concerned the so called P-function and its non-negativity for any eigenfunction.

The problem concerning the sharp estimation of the first eigenvalue of the sub-Laplacian arises from the classical Lichnerowicz' theorem [33], giving a sharp lower bound of the first eigenvalue of the (Riemannian) Laplacian on a compact Riemannian manifold, assuming some a-priori estimate on the Ricci tensor. More precisely, it was shown in [33] that for every compact Riemannian manifold (M, g) of dimension n for which the a-priori estimate

$$Ric(X, Y) \geq (n-1)g(X, Y) \tag{1.1}$$

holds true, the first positive eigenvalue λ_1 of the Laplacian satisfies the sharp estimate

$$\lambda_1 \geq n. \tag{1.2}$$

The above estimate is sharp in the sense that the equality is attained on the round unit n-dimensional sphere $S^n(1)$.

In a natural way, a similar question arises in the sub-Riemannian geometry. Recently, a number of Lichnerowicz-type results have been established in the CR case. All of them are provoked by the Greenleaf's work [17], in which it is obtained a Lichnerowicz-type result for a $(2n+1)$-dimensional CR manifold, $n \geq 3$. Subsequently, the above result was extended to the case $n = 2$ in [34], where the authors have used Greenleaf's method. Another, more restrictive result can be found in [1]. In the quaternionic contact geometry a sharp estimate of the first eigenvalue of the sub-Laplacian is established in [22] for the $(4n+3)$-dimensional QC manifolds, $n \geq 2$.

The situation is more delicate in the lowest dimensions in the CR geometry and the QC geometry. The reason that this happens is that in the low-dimensional geometries appear some additional difficulties, which require a different geometric analysis, see [18, 20] for the QC case. In the CR, as well as in the QC low-dimensional geometries it is necessary to be involved some different methods in comparison with these in the bigger dimensions. An exception to the rule is the conformal flatness problem, where there are no differences between the seven and the bigger dimensional cases in the QC geometry, in contrast to the CR geometry, see [6, 12, 30, 25]. In the three-dimensional CR geometry a sharp estimate is obtained in [13], where, in contrast to the higher dimensions, the author involves the CR-Paneitz operator and imposes the additional assumption for its non-negativity (some related results in the CR geometry appear in [7, 8, 9, 10] and [11]). In the seven-dimensional QC geometry a similar result has been established in [23], where the authors introduce a non-linear C operator, motivated by the Paneitz operators, which appear in the Riemannian and the CR geometries. Precisely, the next theorem holds.

Theorem 1.1. [23] *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven. Suppose there is a positive constant k_0 such that the qc-Ricci tensor Ric and the torsion tensor T^0 satisfy the Lichnerowicz type inequality*

$$Ric(X, X) + 6T^0(X, X) \geq k_0 g(X, X) \tag{1.3}$$

for every horizontal vector field X. If, in addition, the $P-$function of any eigenfunction of the sub-Laplacian is non-negative, then for any eigenvalue λ of the *sub-Laplacian* $\triangle$ *we have the inequality*

$$\lambda \geq \frac{1}{3} k_0. \tag{1.4}$$

Another proof of the main result in [22] is given in [23] via the (established) non-negativity of the P-function in the higher dimensions.

Another Lichnerowicz-type result in the 3D CR geometry is proved in [34], where the Ricci tensor, the torsion tensor and some its covariant derivatives partake in the a-priori condition. The main result of the present paper is namely a QC analog of the upper result.

Our main result follows.

Theorem 1.2. *Let* $(M, g, \mathbb{Q})$ *be a seven-dimensional compact quaternionic contact manifold. Suppose there exists a positive constant* k_0 *such that the qc-Ricci tensor* Ric *and the torsion tensor* T^0 *satisfy the Lichnerowicz type inequality*

$$Ric(X,X) - 2T^0(X,X) - \frac{36}{k_0} A(X) \geq k_0 g(X,X) \tag{1.5}$$

for any horizontal vector field X, *where*

$$A(X) \overset{def}{=} \sum_{s=1}^{3} \Big[\frac{1}{6}(I_s X)^2 S + 2|T(\xi_s, X)|^2 - \frac{2}{9} I_s X\Big((\nabla_{e_a} T^0)(e_a, I_s X)\Big) \\ + \frac{1}{6} I_s X\Big((\nabla_{e_a} T)(\xi_u, e_a, I_t X) - (\nabla_{e_a} T)(\xi_t, e_a, I_u X)\Big) - (\nabla_{\xi_s} T)(\xi_s, X, X)\Big].$$

Then for the first nonzero eigenvalue λ *of the sub-Laplacian the next sharp estimate holds true*

$$\lambda \geq \frac{1}{3} k_0. \tag{1.6}$$

The torsion tensor T^0, the QC-Ricci tensor Ric and the normalized QC-scalar curvature S are defined in (2.6) and (2.11). See Convention 1.4 for the summation rules in the definition of the function $A(X)$.

Another natural question that arises from the Riemannian geometry is studying the case of equality in the estimate (1.6) of Theorem 1.2. The corresponding problem in the Riemannian case was considered by Obata [36]. More precisely, as a consequence of his general result it can be stated that the equality in (1.2) is attained if and only if the Riemannian manifold (M, g) is isometrical to the unit sphere $S^n(1)$ endowed with the round metric, as (1.1) holds. This result has provoked a similar question in the sub-Riemannian geometry and in particular in the CR geometry, where the problem is successfully solved, see [28, 29, 35].

The corresponding question in the QC geometry is completely resolved for higher dimensions ($dimM \geq 11$) in [24], but it remains still open in the seven-dimensional case, except of the 3−Sasakian case [23, Corollary 1.2], where it was shown that the minimal possible eigenvalue of the sub-Laplacian is attained only on the standard unit 3−Sasakian sphere (up to a QC-equivalence).

In [21] the authors describe explicitly the eigenfunctions corresponding to the first eigenvalue of the sub-Laplacian on the standard unit 3−Sasakian sphere.

In connection with the studying of the equality cases in the estimates (1.4) and (1.6) we get as a simple consequence from Theorem 1.1 and Theorem 1.2 the following

Corollary 1.3. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven and f be an arbitrary eigenfunction of the first eigenvalue λ of the sub-Laplacian. Assume that some of the next a-priori conditions holds:*

a) The inequality (1.3) *is satisfied and* $T^0(\nabla f, \nabla f) \geq 0$ *(resp.* $T^0(\nabla f, \nabla f) \leq 0$*).*

b) The inequality (1.5) *is satisfied and* $2T^0(\nabla f, \nabla f) - \frac{36}{k_0} A(\nabla f) \geq 0$ *(resp.* $2T^0(\nabla f, \nabla f) - \frac{36}{k_0} A(\nabla f) \leq 0$*).*

If, in addition, λ takes its minimal possible value, $\lambda = \frac{1}{3} k_0$, then the sharp estimate

$$S \leq \frac{k_0}{6} \quad (\textit{resp.} \quad S \geq \frac{k_0}{6}) \tag{1.7}$$

holds true.

In order to simplify the exposition, we state the following

Convention 1.4. *Throughout this paper we shall suppose that:*

a) *X, Y, Z, U denote horizontal vector fields, i.e. $X, Y, Z, U \in \Gamma(H)$, while A, B, C, D denote arbitrary vector fields, i.e. $A, B, C, D \in \Gamma(TM)$;*

b) *$\{e_1, \dots, e_{4n}\}$ stands for a local orthonormal basis of the horizontal distribution H;*

c) *if two equal vectors from the basis $\{e_1, \dots, e_{4n}\}$ appear in a given formula, then we have summation over them. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means $k = \sum_{a,b=1}^{4n} P(e_b, e_a, e_a, e_b)$;*

d) *the triples (i, j, k) and (s, t, u) denote cyclic permutations of* $(1, 2, 3)$*;*

e) *s is a number from the set* $\{1, 2, 3\}$*, $s \in \{1, 2, 3\}$.*

2. PRELIMINARIES ON THE QUATERNIONIC CONTACT GEOMETRY

The quaternionic contact structures were introduced by O. Biquard [4]. One can think these are quaternionic analogues of the CR structures. We refer the reader to [18], [25] and [27] for comprehensive exposition and further results.

Definition 2.1. *A quaternionic contact (QC) structure on a* $(4n+3)$*-dimensional manifold* M *is the data of co-dimension three distribution* H *on* M *(which is called horizontal space), locally given as the kernel of a* 1*-form* $\eta = (\eta_1, \eta_2, \eta_3)$ *(the contact form) with values in* $\mathbb{R}^3$, $H = Ker(\eta)$, *which satisfy:*

1. H *is equipped with an* $Sp(n)Sp(1)$*-structure, i.e. there exist a Riemannian metric* g *on* H *and a rank three bundle* $\mathbb{Q}$ *consisting of endomorphisms on* H, *locally generated by the three almost complex structures* $I_s : H \to H$, $s = 1, 2, 3$, *satisfying the quaternionic identities:* $I_1^2 = I_2^2 = I_3^2 = -id_{|H}$, $I_1 I_2 = -I_2 I_1 = I_3$, *and which are Hermitian compatible with the metric:* $g(I_s\cdot, I_s\cdot) = g(\cdot, \cdot)$;
2. *the compatibility conditions*
$$2g(I_s X, Y) = d\eta_s(X, Y), \quad s = 1, 2, 3,$$
hold.

A manifold M, *endowed with a QC structure, is called a quaternionic contact (QC) manifold, and is denoted by* $(M, g, \mathbb{Q})$ *(or* $(M, g, \mathbb{Q}, \eta)$*).*

Note that given a QC structure generates a 2-sphere bundle Q of almost complex structures on H, locally given by $Q = \{aI_1 + bI_2 + cI_3 | a^2 + b^2 + c^2 = 1\}$. As Biquard shows in [4], given a contact form η on M determines in a unique way the metric and the quaternionic structure on the horizontal space H (if they exist). Moreover, the rotation of the contact form and the quaternionic structure (i.e. the almost complex structures I_1, I_2 and I_3) by the same rotation gives again a contact form and an almost complex structures, satisfying the above conditions (the metric is unchanged). Another essential fact is that given a horizontal distribution and a metric on it determine at most one 2-sphere bundle of associated contact forms and a corresponding 2-sphere bundle of almost complex structures [4].

Basic (and essential) examples of QC manifolds are the quaternionic Heisenberg group $\mathbf{G}(\mathbb{H})$ (the flat model), endowed with the corresponding QC structure, and the 3-Sasakian manifolds, see [27].

On a quaternionic contact manifold with a fixed horizontal distribution H and a metric g on it there exists a canonical connection, the *Biquard connection*, defined in [4]. Precisely, the following theorem holds.

Theorem 2.2. [O. Biquard, [4]] *Let* $(M, g, \mathbb{Q})$ *be a QC manifold of dimension* $4n+1 > 7$ *with a fixed horizontal distribution* H *and a metric* g *on it. Then there exist a unique connection* ∇ *on* M *with torsion tensor* T *and a unique supplementary distribution* V *to* H *in* TM, *such that the following conditions hold:*

1. ∇ *preserves the decomposition* $H \oplus V$ *and the* $Sp(n)Sp(1)$*-structure on* H, *i.e.* $\nabla g = 0$ *and* $\nabla\sigma \in \Gamma(\mathbb{Q})$ *for any section* $\sigma \in \Gamma(\mathbb{Q})$;

2. *the restriction of the torsion on H is given by $T(X,Y) = -[X,Y]_{|V}$ and for any vector field $\xi \in \Gamma(V)$ the torsion endomorphism $T_\xi(\cdot) := T(\xi,\cdot)_{|H}$ of H lies in $(sp(n) \oplus sp(1))^\perp \subset gl(4n)$;*

3. *the connection on V is generated by the natural identification φ of V with the subspace $sp(1) := span\{I_1, I_2, I_3\}$ of the endomorphisms on H, or in other words, $\nabla\varphi = 0$.*

Throughout this paper we shall denote by ∇ only the Biquard connection. Note that in condition (2) of Theorem 2.2 the inner product $< \cdot, \cdot >$ of the endomorphisms on H is given by

$$< \Phi, \Psi >:= \sum_{a=1}^{4n} g(\Phi(e_a), \Psi(e_a)), \quad \Phi, \Psi \in End(H).$$

In [4] Biquard explicitly describes the supplementary subspace V (the *vertical space*) on the QC-manifolds of dimension bigger than seven. Namely, V is locally generated by the three vector fields ξ_1, ξ_2 and ξ_3 (called *Reeb vector fields*), i.e. $V = span\{\xi_1, \xi_2, \xi_3\}$, satisfying the conditions:

$$\eta_s(\xi_t) = \delta_{st}, \qquad (\xi_s \lrcorner d\eta_t)_{|H} = -(\xi_t \lrcorner d\eta_s)_{|H}, \qquad (\xi_s \lrcorner d\eta_s)_{|H} = 0, \tag{2.1}$$

where $\lrcorner$ means the interior multiplication of a vector field and a differential form.

In the seven dimensional case the Biquard's theorem is not always true. However, Duchemin [14] shows that if we assume the existence of the Reeb vector fields, satisfying conditions (2.1), then Theorem 2.2 holds true. Because of this, throughout this paper we shall assume that a QC structure in the 7D case satisfies conditions (2.1).

The Riemannian metric g on H can be extended to a metric on the entire TM (i.e. to a Riemannian metric on M) by the requirements $H \perp V$ and $g(\xi_s, \xi_t) = \delta_{st}$. Note that the extended metric (which we shall again denote by g) is invariant under the rotations in V, i.e. the action of the group $SO(3)$ on V, and of course is parallel with respect to ∇, $\nabla g = 0$.

The *fundamental* 2*-forms* ω_s of the quaternionic structure $(\mathbb{Q}, g)$ on H are defined in a standard way by

$$\omega_s(X,Y) := g(I_s X, Y), \quad s = 1,2,3,$$

and can be extended to 2-forms on M by the requirement $\xi \lrcorner \omega_s = 0, \quad \xi \in \Gamma(V)$.

The covariant derivatives of the quaternionic structure and the Reeb vector fields with respect to the Biquard connection are given by

$$\nabla I_i = -\alpha_j \otimes I_k + \alpha_k \otimes I_j, \quad \nabla \xi_i = -\alpha_j \otimes \xi_k + \alpha_k \otimes \xi_j, \tag{2.2}$$

where $\alpha_s, \quad s = 1,2,3$, are the $sp(1)$-connection 1-forms of the Biquard connection.

The orthonormal frame

$$\{e_1, e_2 = I_1 e_1, e_3 = I_2 e_1, e_4 = I_3 e_1, \dots, e_{4n} = I_3 e_{4n-3}, \xi_1, \xi_2, \xi_3\}$$

of TM is called a *QC-normal frame* at a given point $p \in M$, if the connection 1-forms of the Biquard connection vanishes at p. The existence of a QC-normal frame at any point of M is provided by Lemma 4.5 in [18].

2.2. INVARIANT DECOMPOSITIONS OF THE ENDOMORPHISMS OF H

Any endomorphism $\Psi : H \to H$ of H can be decomposed in a unique way into four $Sp(n)$-invariant parts with respect to the quaternionic structure $(\mathbb{Q}, g)$ as follows:

$$\Psi = \Psi^{+++} + \Psi^{+--} + \Psi^{-+-} + \Psi^{--+},$$

where Ψ^{+++} commutes with all three I_i, Ψ^{+--} commutes with I_1 and anti-commutes with the others two, etc. Further, we can regard Ψ as decomposed into two $Sp(n)Sp(1)$-invariant parts with respect to $(\mathbb{Q}, g)$, $\Psi = \Psi_{[3]} + \Psi_{[-1]}$, where $\Psi_{[3]} = \Psi^{+++}$, $\Psi_{[-1]} = \Psi^{+--} + \Psi^{-+-} + \Psi^{--+}$. Note that in the above decomposition the lower indices [3] and [−1] arise from the fact that $\Psi_{[3]}$ and $\Psi_{[-1]}$ appear the projections of Ψ on the eigenspaces of the Casimir operator

$$\Upsilon = \ I_1 \otimes I_1 \ + \ I_2 \otimes I_2 \ + \ I_3 \otimes I_3,$$

corresponding, respectively, to the eigenvalues 3 and −1, see [5].

In the case $n = 1$ an important fact is that the space of the symmetric endomorphisms of H, commuting with all three almost complex structures I_s, is one-dimensional. Consequently, the [3]-component $\Psi_{[3]}$ of any symmetric endomorphism Ψ of H is proportional to the identity operator $Id_{|H}$ of H, explicitly, $\Psi_{[3]} = -\frac{tr\Psi}{4} Id_{|H}$.

2.3. THE TORSION AND THE CURVATURE OF BIQUARD CONNECTION

The torsion tensor T of Biquard connection is defined as usually by

$$T(A, B) = \nabla_A B - \nabla_B A - [A, B].$$

The corresponding tensor of type $(0, 3)$ via the metric g is obtained in a standard way and is denoted by the same letter, $T(A, B, C) = g(T(A, B), C)$. The restriction of the torsion to the horizontal space H has the expression

$$T(X, Y) = -[X, Y]_{|V} = 2 \sum_{s=1}^{3} \omega_s(X, Y)\xi_s,$$

see [27]. For an arbitrary but fixed vertical vector field $\xi \in \Gamma(V)$ one obtains an endomorphism T_ξ on H, defined by

$$T_\xi(\cdot) := T(\xi, \cdot)_{|H} : H \to H.$$

The torsion endomorphism T_ξ is completely trace-free [4], i.e. $trT_\xi = tr(T_\xi \circ I_s) = 0$, or explicitly

$$T(\xi, e_a, e_a) = T(\xi, e_a, I_s e_a) = 0. \tag{2.3}$$

We shall need the identities

$$T(\xi_i, \xi_k, \xi_i) = T(\xi_i, \xi_j, \xi_i) = 0, \tag{2.4}$$

see e.g. [27, Eqn. (4.34)]. The torsion endomorphism T_ξ can be decomposed in a standard way into a symmetric T^0_ξ and a skew-symmetric b_ξ parts, $T_\xi = T^0_\xi + b_\xi$, and the symmetric part enjoys the properties

$$\begin{aligned} T^0_{\xi_i} I_i = -I_i T^0_{\xi_i}, \quad I_2(T^0_{\xi_2})^{+--} = I_1(T^0_{\xi_1})^{-+-}, \\ I_3(T^0_{\xi_3})^{-+-} = I_2(T^0_{\xi_2})^{--+}, \quad I_1(T^0_{\xi_1})^{--+} = I_3(T^0_{\xi_3})^{+--}. \end{aligned} \tag{2.5}$$

For a fixed Reeb vector field ξ_i the skew-symmetric part b_{ξ_i} of T_{ξ_i} can be represented as $b_{\xi_i} = I_i U$, where U is a traceless symmetric endomorphism of H, which commutes with all three almost complex structures I_s, $s = 1, 2, 3$. As a consequence in the case $n = 1$ one obtains that the tensor U vanishes identically, $U = 0$, (see the end of Subsection) and the torsion endomorphism T_ξ is a symmetric tensor, $T_\xi = T^0_\xi$.

Ivanov et al. have introduced [18] the two $Sp(n)Sp(1)$-invariant symmetric and traceless tensors T^0 and U on H, defined by

$$T^0(X,Y) = g((T^0_{\xi_1} I_1 + T^0_{\xi_2} I_2 + T^0_{\xi_3} I_3)X, Y) \quad \text{and} \quad U(X,Y) = g(UX, Y). \tag{2.6}$$

These tensors satisfy the equalities

$$\begin{aligned} T^0(X,Y) + T^0(I_1X, I_1Y) + T^0(I_2X, I_2Y) + T^0(I_3X, I_3Y) = 0, \\ U(X,Y) = U(I_1X, I_1Y) = U(I_2X, I_2Y) = U(I_3X, I_3Y). \end{aligned} \tag{2.7}$$

The symmetric part $T^0_{\xi_s}$ of T_{ξ_s} enjoys the property [25, Proposition 2.3]

$$4T^0(\xi_s, I_sX, Y) = T^0(X,Y) - T^0(I_sX, I_sY), \tag{2.8}$$

where as usually $T^0(\xi, X, Y) = g(T^0(\xi, X), Y)\Big(= g(T^0_\xi(X), Y)\Big)$. As a corollary of (2.7) and (2.8) we obtain the equality

$$\begin{aligned} T(\xi_s, I_sX, Y) &= T^0(\xi_s, I_sX, Y) + g(I_sUI_sX, Y) \\ &= \frac{1}{4}\Big[T^0(X,Y) - T^0(I_sX, I_sY)\Big] - U(X,Y). \end{aligned} \tag{2.9}$$

As a consequence of (2.7) and (2.9) we get

$$\sum_{s=1}^{3} T(\xi_s, I_s X, Y) = T^0(X,Y) - 3U(X,Y). \tag{2.10}$$

The curvature tensor R of Biquard connection is defined in a standard way by

$$R(A,B,C) = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C.$$

The corresponding tensor of type $(0,4)$ with respect to the metric g is denoted by the same letter, $R(A,B,C,D) := g(R(A,B,C),D)$.

There are several tensors, arising from the curvature tensor, which play crucial role in the QC geometry. The *QC-Ricci tensor* Ric, the *QC-scalar curvature* $Scal$, the *normalized QC-scalar curvature* S, the *QC-Ricci forms* ρ_s and the *Ricci-type tensors* ζ_s of the Biquard connection are defined, respectively, by the following formulas.

$$\begin{aligned} &Ric(A,B) = R(e_b, A, B, e_b),\ Scal = R(e_b, e_a, e_a, e_b),\ 8n(n+2)S = Scal,\\ &\rho_s(A,B) = \frac{1}{4n} R(A,B,e_a,I_s e_a), \quad \zeta_s(A,B) = \frac{1}{4n} R(e_a, A, B, I_s e_a). \end{aligned} \tag{2.11}$$

Some significant relations between the upper objects and the torsion tensors are established in [18] (see also [20, 25]). Namely, the following formulas hold true.

$$\begin{aligned} Ric(X,Y) &= (2n+2)T^0(X,Y) + (4n+10)U(X,Y) + 2(n+2)Sg(X,Y),\\ \zeta_s(X, I_s Y) &= \frac{2n+1}{4n} T^0(X,Y) + \frac{1}{4n} T^0(I_s X, I_s Y)\\ &\quad + \frac{2n+1}{2n} U(X,Y) + \frac{S}{2} g(X,Y),\\ T(\xi_i,\xi_j) &= -S\xi_k - [\xi_i,\xi_j]_{|H}, \qquad S = -g(T(\xi_1,\xi_2),\xi_3),\\ g(T(\xi_i,\xi_j),X) &= -\rho_k(I_i X, \xi_i) = -\rho_k(I_j X, \xi_j) = -g([\xi_i,\xi_j],X). \end{aligned} \tag{2.12}$$

In the seven dimensional case ($n = 1$) the above formulas are valid with $U = 0$.

An important class of QC structures consists of the *QC-Einstein structures*, defined as follows.

Definition 2.3. *A QC structure is called QC-Einstein, if the horizontal restriction of the QC-Ricci tensor is proportional to the metric, i.e.*

$$Ric(X,Y) = 2(n+2)Sg(X,Y). \tag{2.13}$$

A manifold endowed with a QC-Einstein structure is called *QC-Einstein manifold*. The first equality in (2.12) implies that the QC-Einstein condition (the equation (2.13)) is equivalent to the vanishing of the torsion endomorphism, i.e.

$T^0 = U = 0$. An established in [18] result asserts that a QC-Einstein structure of dimension greater than seven has constant QC-scalar curvature, and the vertical distribution is integrable. The corresponding result in the seven-dimensional case was established recently in [19].

Note that the vanishing of the horizontal restriction of the $sp(n)$-connection 1-forms $\alpha_s, s = 1, 2, 3$, implies the vanishing of the torsion endomorphism T_ξ of the Biquard connection, see [18].

Examples of QC-Einstein manifolds are the 3-Sasakian manifolds, since they have zero torsion endomorphism. The converse is also true in a local sense, namely, any QC-Einstein manifold with positive QC-scalar curvature is locally 3-Sasakian [18] (see [26] for the case of negative QC-scalar curvature).

2.4. THE HORIZONTAL DIVERGENCE THEOREM AND THE SUB-LAPLACIAN

On a QC manifold $(M, g, \mathbb{Q})$ of dimension $4n + 3$ the *horizontal divergence* of a horizontal 1-form (or a horizontal vector field) $\omega \in \Lambda^1(H)$ is defined by

$$\nabla^*\omega = -tr|_H \nabla\omega = -\nabla\omega(e_a, e_a).$$

If $\eta = (\eta_1, \eta_2, \eta_3)$ is a fixed local contact form of the QC manifold then for an arbitrary $s \in \{1, 2, 3\}$ the form $Vol_\eta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \omega_s^{2n}$ is locally defined volume form, which is independent of the choice of s and the local 1-forms η_1, η_2 and η_3. Consequently, Vol_η is globally defined volume form on $(M, g, \mathbb{Q})$. If the QC manifold is compact, the integration by parts over M is possible due to the next divergence formula:

$$\int_M (\nabla^*\omega)\ Vol_\eta \ = \ 0,$$

see [18], [37].

For a smooth function f on M the *horizontal Hessian* $\nabla^2 f(\cdot, \cdot) : \Gamma(H) \times \Gamma(H) \to \Lambda^0(M)$ and the *sub-Laplacian* $\Delta f \in \Lambda^0(M)$ are defined in a standard way by

$$\nabla^2 f(X, Y) = (\nabla_X df)(Y) \quad \text{and} \quad \Delta f = \nabla^* df = -\nabla^2 f(e_a, e_a).$$

By definition, the *horizontal gradient* of f is the vector field ∇f, s.t.

$$g(\nabla f, X) = df(X), \quad X \in \Gamma(H).$$

Any (non-zero) smooth function f satisfying the equation $\Delta f = \lambda f$ for some constant λ is called *eigenfunction*, corresponding to the *eigenvalue* λ of Δ. In the case of compact M the last equation and the divergence formula yield the non-negativity of the spectrum of the sub-Laplacian.

3. SOME BASIC IDENTITIES

In this section we list some identities which we shall use in the proof of the main results. We shall need the following *Ricci identities* [18, 27]

$$
\begin{aligned}
&\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = -2\sum_{s=1}^{3}\omega_s(X,Y)df(\xi_s),\\
&\nabla^2 f(X,\xi_s) - \nabla^2 f(\xi_s,X) = T(\xi_s,X,\nabla f),\\
&\nabla^3 f(\xi_i,X,Y) = \nabla^3 f(X,Y,\xi_i) - \nabla^2 f\left(T\left(\xi_i,X\right),Y\right) - \nabla^2 f\left(X,T\left(\xi_i,Y\right)\right)\\
&\qquad\qquad - df\left(\left(\nabla_X T\right)\left(\xi_i,Y\right)\right) - R(\xi_i,X,Y,\nabla f).
\end{aligned}
\tag{3.1}
$$

As a consequence of the first identity in (3.1) we get

$$
g(\nabla^2 f,\omega_s) = \nabla^2 f(e_a,I_s e_a) = -4n df(\xi_s). \tag{3.2}
$$

The next basic formula we shall need is a representation of the curvature tensor [25, 27]

$$
\begin{aligned}
R(\xi_i,X,Y,Z) =& -(\nabla_X U)(I_iY,Z) + \omega_j(X,Y)\rho_k(I_iZ,\xi_i) - \omega_k(X,Y)\rho_j(I_iZ,\xi_i)\\
& - \omega_j(X,Z)\rho_k(I_iY,\xi_i) + \omega_k(X,Z)\rho_j(I_iY,\xi_i)\\
& - \omega_j(Y,Z)\rho_k(I_iX,\xi_i) + \omega_k(Y,Z)\rho_j(I_iX,\xi_i)\\
& - \frac{1}{4}\Big[(\nabla_Y T^0)(I_iZ,X) + (\nabla_Y T^0)(Z,I_iX)\Big]\\
& + \frac{1}{4}\Big[(\nabla_Z T^0)(I_iY,X) + (\nabla_Z T^0)(Y,I_iX)\Big],
\end{aligned}
\tag{3.3}
$$

where the *Ricci 2-forms* are given by (see [25] or [27])

$$
\begin{aligned}
6(2n+1)\rho_s(\xi_s,X) =& (2n+1)X(S) + \frac{1}{2}(\nabla_{e_a}T^0)[(e_a,X) - 3(I_s e_a,I_s X)]\\
& - 2(\nabla_{e_a}U)(e_a,X),\\
6(2n+1)\rho_i(\xi_j,I_kX) =& -6(2n+1)\rho_i(\xi_k,I_jX)\\
=& (2n-1)(2n+1)X(S) - \frac{4n+1}{2}(\nabla_{e_a}T^0)(e_a,X)\\
& - \frac{3}{2}(\nabla_{e_a}T^0)(I_ie_a,I_iX) - 4(n+1)(\nabla_{e_a}U)(e_a,X).
\end{aligned}
\tag{3.4}
$$

By the well-known formula for the relation between two metric connections, we obtain the next one in the case of the Biquard connection ∇ and the Levi-Civita connection ∇^g of the extended Riemannian metric g:

$$
g(\nabla_A B,C) - g(\nabla^g_A B,C) = \frac{1}{2}\Big(T(A,B,C) - T(B,C,A) + T(C,A,B)\Big). \tag{3.5}
$$

4. PROOF OF THEOREM 1.2

Let λ is the first (non-zero) eigenvalue of the sub-Laplacian and f is a smooth function on M that satisfies the equalities

$$\Delta f = \lambda f \quad \text{and} \quad \int_M f^2 \, Vol_\eta = 1. \tag{4.1}$$

Note that the second equality in (4.1) can be always obtained by a suitable constant rescaling of f. The proof of Theorem 1.2 depends on a number of lemmas, which we formulate and prove below. We start with the following

Lemma 4.1. *Let $(M, g, \mathbb{Q})$ be a compact quaternionic contact manifold of dimension seven. Then the following integral inequality holds true*

$$\int_M \Big[Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{3}{4}\lambda |\nabla f|^2 - 12 \sum_{s=1}^{3} \Big(df(\xi_s) \Big)^2 \Big] Vol_\eta \leq 0. \tag{4.2}$$

Proof. Following [34], we start with the Bochner-type formula, established in our previous paper [22, Eqn. (3.3)]

$$\begin{aligned} -\frac{1}{2}\triangle|\nabla f|^2 =& |\nabla^2 f|^2 - g\left(\nabla(\triangle f), \nabla f\right) + Ric(\nabla f, \nabla f) \\ &+ 2\sum_{s=1}^{3} T(\xi_s, I_s\nabla f, \nabla f) + 4\sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f). \end{aligned} \tag{4.3}$$

Similarly to the case of higher dimensions, this formula is a crucial ingredient of the proof of the desired estimate. The next basic formula is [23, Eqn. (3.3)]

$$\sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s X) = \frac{1}{4n} \sum_{s=1}^{3} \nabla^3 f(I_s X, I_s e_a, e_a) - \sum_{s=1}^{3} T(\xi_s, I_s X, \nabla f). \tag{4.4}$$

Integrating over M the both sides of (4.4) for $n = 1$ and $X = \nabla f$ and using the integral identity

$$\int_M \sum_{s=1}^{3} \nabla^3 f(I_s \nabla f, I_s e_a, e_a) \, Vol_\eta = -16 \int_M \sum_{s=1}^{3} \Big(df(\xi_s) \Big)^2 \, Vol_\eta \tag{4.5}$$

and (2.10), we obtain

$$\int_M \sum_{s=1}^{3} \nabla^2 f(\xi_s, I_s \nabla f) \, Vol_\eta = -\int_M \Big[4\sum_{s=1}^{3} \Big(df(\xi_s) \Big)^2 + T^0(\nabla f, \nabla f) \Big] \, Vol_\eta. \tag{4.6}$$

It should be pointed out that in our calculations for getting (4.5) we have used (3.2), an integration by parts and the $Sp(n)Sp(1)-$invariance of the expression $\sum_{s=1}^{3} \nabla^3 f(I_s \nabla f, I_s e_a, e_a)$, which allows us to work in a QC-normal frame.

Further, we take the next inequalities for the $Sp(n)Sp(1)$-invariant parts of the horizontal Hessian, [22, Eqs. (4.6) and (4.7)],

$$|(\nabla^2 f)_{[-1]}|^2 \geq 4n \sum_{s=1}^{3} \Big(df(\xi_s)\Big)^2, \quad |(\nabla^2 f)_{[3]}|^2 \geq \frac{1}{4n}(\triangle f)^2,$$

which in the seven-dimensional case ($n = 1$) give the next inequality for the norm of the horizontal Hessian:

$$|\nabla^2 f|^2 = |(\nabla^2 f)_{[-1]}|^2 + |(\nabla^2 f)_{[3]}|^2 \geq 4 \sum_{s=1}^{3} \Big(df(\xi_s)\Big)^2 + \frac{1}{4}(\Delta f)^2. \tag{4.7}$$

Taking into account the divergence formula, we get the integral identity

$$\int_M (\triangle f)^2 \, Vol_\eta = \lambda \int_M |\nabla f|^2 \, Vol_\eta. \tag{4.8}$$

Finally, integrating (4.3) over M and using (2.10), (4.6), (4.7) and (4.8), we obtain (4.2). □

Our next goal is to find a suitable estimate of the term $\int_M \sum_{s=1}^3 \Big(df(\xi_s)\Big)^2 Vol_\eta$ which appears in (4.2). The aim of the following results is to establish one such estimate.

Lemma 4.2. ["Vertical Bochner formula"] *Let ϕ be a smooth function on a QC manifold $(M, g, \mathbb{Q})$ of dimension $4n+3$. Then the following formula holds true:*

$$\sum_{s=1}^{3} \Delta(\xi_s\phi)^2 = 2\sum_{s=1}^{3} \Big[-|\nabla(\xi_s\phi)|^2 + d\phi(\xi_s)\xi_s(\Delta\phi) - d\phi(\xi_s)R(\xi_s, e_a, e_a, \nabla\phi) - d\phi(\xi_s)(\nabla_{e_a}T)(\xi_s, e_a, \nabla\phi) - 2d\phi(\xi_s)g\Big(T_{\xi_s}, \nabla^2\phi\Big)\Big]. \tag{4.9}$$

Proof. First, it should be noted that the tensor T_{ξ_s} appearing in the last term of the right-hand side of (4.9) is assumed to be the tensor of type $(0, 2)$, corresponding to the torsion endmorphism T_{ξ_s} via g. The left-hand side of the desired equality (4.9) is an $Sp(n)Sp(1)$-invariant and hence we can carry out our computations in a QC-normal frame. Using the first and the third Ricci identity in (3.1) and the properties of the torsion endomorphism, after some standard calculations we obtain

$$\begin{aligned}\sum_{s=1}^{3}\Delta(\xi_s\phi)^2 =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2+d\phi(\xi_s)\Delta(\xi_s\phi)\Big]\\
=&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2-d\phi(\xi_s)\nabla^3\phi(e_a,e_a,\xi_s)\Big]\\
=&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2-d\phi(\xi_s)\Big(\nabla^3\phi(\xi_s,e_a,e_a)+\nabla^2\phi(T(\xi_s,e_a),e_a)\\
&+\nabla^2\phi(e_a,T(\xi_s,e_a))+d\phi((\nabla_{e_a}T)(\xi_s,e_a))+R(\xi_s,e_a,e_a,\nabla\phi)\Big)\Big]\\
=&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s\phi)|^2+d\phi(\xi_s)\xi_s(\Delta\phi)-d\phi(\xi_s)R(\xi_s,e_a,e_a,\nabla\phi)\\
&-d\phi(\xi_s)(\nabla_{e_a}T)(\xi_s,e_a,\nabla\phi)-2d\phi(\xi_s)g\Big(T_{\xi_s},\nabla^2\phi\Big)\Big],\end{aligned}$$

which completes the proof of Lemma 4.2. □

Applying (4.9) to the case of a seven-dimensional QC manifold and an eigenfunction f on it, we obtain the next lemma.

Lemma 4.3. *On a QC manifold $(M,g,\mathbb{Q})$ of dimension seven the following formula holds true:*

$$\begin{aligned}\sum_{s=1}^{3}\Delta(\xi_s f)^2 =&2\sum_{s=1}^{3}\Big[-|\nabla(\xi_s f)|^2+\lambda\Big(df(\xi_s)\Big)^2-\frac{2}{3}df(\xi_s)dS(I_s\nabla f)\\
&-\frac{2}{3}df(\xi_s)\Big((\nabla_{e_a}T^0)(\xi_u,e_a,I_t\nabla f)-(\nabla_{e_a}T^0)(\xi_t,e_a,I_u\nabla f)\Big)\\
&+\frac{8}{9}df(\xi_s)(\nabla_{e_a}T^0)(e_a,I_s\nabla f)-2df(\xi_s)e_a\Big(T(\xi_s,e_a,\nabla f)\Big)\Big].\end{aligned}\tag{4.10}$$

Proof. As in the proof of the previous lemma, we can work in a QC-normal frame. Using the properties of the torsion tensor, listed in Subsection 2.3, we get

$$\begin{aligned}&\sum_{s=1}^{3}df(\xi_s)(\nabla_{e_a}T)(\xi_s,e_a,\nabla f)\\
&\qquad=-\frac{1}{4}\sum_{s=1}^{3}df(\xi_s)\Big[(\nabla_{e_a}T^0)(\nabla f,I_s e_a)+(\nabla_{e_a}T^0)(I_s\nabla f,e_a)\Big].\end{aligned}\tag{4.11}$$

Next we use (3.3) and the properties of the torsion tensor to obtain

$$\begin{aligned}
&\sum_{s=1}^{3} df(\xi_s) R(\xi_s, e_a, e_a, \nabla f) \\
&\quad = \sum_{s=1}^{3} df(\xi_s) \Big[-\frac{1}{4}\Big((\nabla_{e_a} T^0)(I_s \nabla f, e_a) + (\nabla_{e_a} T^0)(\nabla f, I_s e_a) \Big) \\
&\qquad\qquad - 2\omega_t(e_a, \nabla f)\rho_u(I_s e_a, \xi_s) + 2\omega_u(e_a, \nabla f)\rho_t(I_s e_a, \xi_s) \Big].
\end{aligned} \tag{4.12}$$

We use representations (3.4) for the Ricci 2-forms that appear in (4.12) to obtain

$$\begin{aligned}
\rho_u(I_s e_a, \xi_s) &= -\frac{1}{6} dS(I_u e_a) + \frac{5}{36}(\nabla_{e_b} T^0)(e_b, I_u e_a) - \frac{1}{12}(\nabla_{e_b} T^0)(I_u e_b, e_a), \\
\rho_t(I_s e_a, \xi_s) &= -\frac{1}{6} dS(I_t e_a) + \frac{5}{36}(\nabla_{e_b} T^0)(e_b, I_t e_a) - \frac{1}{12}(\nabla_{e_b} T^0)(I_t e_b, e_a).
\end{aligned} \tag{4.13}$$

Substituting (4.11), (4.12) and (4.13) in the right-hand side of (4.9) and using the properties of the torsion tensor, we get (4.10) after a number of standard computations. □

An integral equality, which is one of the main instruments for derivation of the needed sharp estimate for the term $\int_M \sum_{s=1}^3 \Big(df(\xi_s)\Big)^2 Vol_\eta$ appearing in (4.2), is given in the next lemma.

Lemma 4.4. *On a seven-dimensional compact QC manifold $(M, g, \mathbb{Q})$ the following integral formula holds true:*

$$\begin{aligned}
&\int_M \sum_{s=1}^{3} |\nabla(\xi_s f)|^2 \, Vol_\eta \\
&\quad = \int_M \sum_{s=1}^{3} \Big[2|T(\xi_s, \nabla f)|^2 + \frac{1}{6}(I_s \nabla f)^2 S - \frac{2}{9} I_s \nabla f \Big((\nabla_{e_a} T^0)(e_a, I_s \nabla f) \Big) \\
&\qquad + \frac{1}{6} I_s \nabla f \Big((\nabla_{e_a} T)(\xi_u, e_a, I_t \nabla f) \Big) - \frac{1}{6} I_s \nabla f \Big((\nabla_{e_a} T)(\xi_t, e_a, I_u \nabla f) \Big) \\
&\qquad - (\nabla_{\xi_s} T)(\xi_s, \nabla f, \nabla f) + \lambda \Big(df(\xi_s) \Big)^2 \Big] \, Vol_\eta.
\end{aligned} \tag{4.14}$$

Proof. We begin with integrating over M the both sides of (4.10). We shall work as before in a QC-normal frame in view of the $Sp(n)Sp(1)$-invariance of the tensors under consideration. Having in mind the divergence formula, we shall simplify some of the terms that appear under the integral.

Using (3.2) and integration by parts, after some standard calculations we get the identities

$$\int_M \sum_{s=1}^{3} df(\xi_s) dS(I_s\nabla f)\, Vol_\eta = -\frac{1}{4}\int_M \sum_{s=1}^{3} (I_s\nabla f)^2 S\, Vol_\eta, \tag{4.15}$$

$$\begin{aligned}\int_M \sum_{s=1}^{3} df(\xi_s)(\nabla_{e_a} T^0)(e_a, I_s\nabla f)\, Vol_\eta \\ = -\frac{1}{4}\int_M \sum_{s=1}^{3} I_s\nabla f\Big((\nabla_{e_a} T^0)(e_a, I_s\nabla f)\Big)\, Vol_\eta,\end{aligned} \tag{4.16}$$

$$\begin{aligned}\int_M \sum_{s=1}^{3} df(\xi_s)(\nabla_{e_a} T)(\xi_u, e_a, I_t\nabla f)\, Vol_\eta \\ = -\frac{1}{4}\int_M \sum_{s=1}^{3} I_s\nabla f\Big((\nabla_{e_a} T)(\xi_u, e_a, I_t\nabla f)\Big)\, Vol_\eta.\end{aligned} \tag{4.17}$$

In order to transform the term $\int_M \sum_{s=1}^{3} df(\xi_s) e_a\Big(T(\xi_s, e_a, \nabla f)\Big)\, Vol_\eta$, let us introduce some auxiliary notation and facts. We denote by div^{∇} and div^{∇^g} the divergences corresponding to the Biquard connection ∇ and to the Levi-Civita connection ∇^g, respectively. For any vertical vector field ξ on a QC manifold of dimension $4n+3$ we have

$$\begin{aligned} div^{\nabla^g}(\xi) &= \sum_{a=1}^{4n} g(\nabla^g_{e_a}\xi, e_a) + \sum_{s=1}^{3} g(\nabla^g_{\xi_s}\xi, \xi_s) \\ &= \sum_{a=1}^{4n} g(\nabla_{e_a}\xi, e_a) + \sum_{s=1}^{3} g(\nabla_{\xi_s}\xi, \xi_s) \\ &= div^{\nabla}(\xi), \end{aligned} \tag{4.18}$$

where for the second equality we have used (3.5) and the properties of the torsion tensor (2.3) and (2.4). Since the volume form Vol_η differs from the Riemannian volume form $d\mu^g$ by a constant multiplier C, $Vol_\eta = C.d\mu^g$, we get by the Riemannian divergence formula and (4.18)

$$\int_M div^{\nabla}(\xi)\, Vol_\eta = C\int_M div^{\nabla}(\xi)\, d\mu^g = C\int_M div^{\nabla^g}(\xi)\, d\mu^g = 0. \tag{4.19}$$

We have

$$
\begin{aligned}
\int_M \sum_{s=1}^{3} df(\xi_s) e_a \Big(T(\xi_s, e_a, \nabla f)\Big)\, Vol_\eta \\
&= - \int_M \sum_{s=1}^{3} \nabla^2 f(e_a, \xi_s) T(\xi_s, e_a, \nabla f)\, Vol_\eta \\
&= - \int_M \sum_{s=1}^{3} \Big[T(\xi_s, e_a, \nabla f) T(\xi_s, e_a, \nabla f) \\
&\qquad + \nabla^2 f(\xi_s, e_a) T(\xi_s, e_a, \nabla f)\Big]\, Vol_\eta \\
&= - \int_M \sum_{s=1}^{3} \Big[|T(\xi_s, \nabla f)|^2 - df(e_a)\xi_s \Big(T(\xi_s, e_a, \nabla f)\Big)\Big]\, Vol_\eta \\
&= \int_M \sum_{s=1}^{3} \Big[- |T(\xi_s, \nabla f)|^2 + \frac{1}{2} (\nabla_{\xi_s} T)(\xi_s, \nabla f, \nabla f)\Big]\, Vol_\eta,
\end{aligned}
\tag{4.20}
$$

where we have used integration by parts for the first equality in the above chain, next we took into account the second Ricci identity in (3.1) to obtain the second one, and finally, in order to get the third and the fourth equalities , we have used (4.19) for the vertical vector field $\xi := T(\xi_s, \nabla f, \nabla f)\xi_s$.

Now, substituting (4.15), (4.16), (4.17) and (4.20) in the integrated over M equality (4.10), we get (4.14). □

An important role for obtaining the desired estimate plays the integral equality

$$
\int_M \sum_{s=1}^{3} \Big(df(\xi_s)\Big)^2 Vol_\eta = \frac{1}{4} \int_M \sum_{s=1}^{3} df(I_s e_a) d(\xi_s f)(e_a)\, Vol_\eta, \tag{4.21}
$$

which follows easily by (3.2) and an integration by parts. We have:

$$
\begin{aligned}
\sum_{s=1}^{3} \int_M \lambda \Big(df(\xi_s)\Big)^2 Vol_\eta &= \sum_{s=1}^{3} \int_M \frac{\lambda}{4} df(I_s e_a) d(\xi_s f)(e_a)\, Vol_\eta \\
&\le \sum_{s=1}^{3} \Big[\int_M \frac{\lambda^2}{16} \Big(df(I_s e_a)\Big)^2 Vol_\eta\Big]^{\frac{1}{2}} \Big[\int_M \Big(d(\xi_s f)(e_a)\Big)^2 Vol_\eta\Big]^{\frac{1}{2}} \\
&\le \frac{1}{2} \sum_{s=1}^{3} \Big[\int_M \frac{\lambda^2}{16} \Big(df(I_s e_a)\Big)^2 Vol_\eta + \int_M \Big(d(\xi_s f)(e_a)\Big)^2 Vol_\eta\Big] \\
&= \frac{3\lambda^2}{32} \int_M |\nabla f|^2\, Vol_\eta + \frac{1}{2} \sum_{s=1}^{3} \int_M |\nabla(\xi_s f)|^2\, Vol_\eta.
\end{aligned}
\tag{4.22}
$$

For the above chain we have used (4.21) to obtain the first equality and the Cauchy-Schwarz inequality for the integral scalar product to get the first inequality. The second inequality is obtained in an obvious manner.

Using the notation $A(X)$ from the statement of Theorem 1.2, the equality (4.14) takes the form

$$\int_M \sum_{s=1}^3 |\nabla(\xi_s f)|^2 \, Vol_\eta = \int_M \Big[A(\nabla f) + \sum_{s=1}^3 \lambda\Big(df(\xi_s)\Big)^2\Big] \, Vol_\eta,$$

which, combined with (4.22), gives the next integral inequality

$$\sum_{s=1}^3 \int_M |\nabla(\xi_s f)|^2 \, Vol_\eta \le \int_M \Big[2A(\nabla f) + \frac{3\lambda^2}{16}|\nabla f|^2\Big] \, Vol_\eta. \tag{4.23}$$

For any constant $b > 0$ we have the following chain of relations:

$$\begin{aligned} \sum_{s=1}^3 \int_M \Big(df(\xi_s)\Big)^2 \, Vol_\eta &= \sum_{s=1}^3 \int_M \frac{\sqrt{b}}{4} df(I_s e_a) \frac{1}{\sqrt{b}} d(\xi_s f)(e_a) \, Vol_\eta \\ &\le \sum_{s=1}^3 \Big[\frac{b}{16}\int_M \Big(df(I_s e_a)\Big)^2 \, Vol_\eta\Big]^{\frac12} \Big[\frac{1}{b}\int_M \Big(d(\xi_s f)(e_a)\Big)^2 \, Vol_\eta\Big]^{\frac12} \\ &\le \frac{3b}{32}\int_M |\nabla f|^2 \, Vol_\eta + \frac{1}{2b}\sum_{s=1}^3 \int_M |\nabla(\xi_s f)|^2 \, Vol_\eta, \end{aligned} \tag{4.24}$$

where we have used (4.21) to obtain the equality and the Cauchy-Schwarz inequality for the integral scalar product to get the first inequality. The second inequality is obvious. Combining (4.23) and (4.24), we get the next key inequality

$$\sum_{s=1}^3 \int_M \Big(df(\xi_s)\Big)^2 \, Vol_\eta \le \int_M \Big[\frac{3b}{32}|\nabla f|^2 + \frac{1}{b}A(\nabla f) + \frac{3\lambda^2}{32b}|\nabla f|^2\Big] \, Vol_\eta. \tag{4.25}$$

Substituting (4.25) in (4.2), we obtain

$$\int_M \Big[Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{12}{b}A(\nabla f) + (-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b})|\nabla f|^2\Big] \, Vol_\eta \le 0. \tag{4.26}$$

Taking into account the a-priori condition

$$Ric(X,X) - 2T^0(X,X) - \frac{12}{b}A(X) \ge k_0 g(X,X) \quad \text{for any} \quad X \in \Gamma(H),$$

we deduce from (4.26)

$$\int_M \Big(-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b} + k_0\Big)|\nabla f|^2 \, Vol_\eta \le 0.$$

The last inequality implies

$$-\frac{3}{4}\lambda - \frac{9b}{8} - \frac{9\lambda^2}{8b} + k_0 \le 0,$$

which after choosing $b = \frac{k_0}{3}$ becomes

$$(3\lambda - k_0)(9\lambda + 5k_0) \geq 0. \tag{4.27}$$

Since $9\lambda + 5k_0 > 0$, the inequality (4.27) gives the estimate

$$\lambda \geq \frac{k_0}{3}, \tag{4.28}$$

which completes the proof of Theorem 1.2. □

5. PROOF OF COROLLARY 1.2

In [23, Remark 4.1] the authors give the identity

$$10T^0(\nabla f, \nabla f) + 6S|\nabla f|^2 = k_0|\nabla f|^2, \tag{5.1}$$

which holds for the extremal eigenfunction f in the case of equality in Theorem 1.1, i.e. $\lambda = \frac{1}{3}k_0$. Assuming the condition a) in Corollary 1.3 and taking account (5.1), we obtain (1.7).

In a similar way, the case of equality in Theorem 1.2, i.e. $\lambda = \frac{1}{3}k_0$, together with the a-priori condition (1.5) and (4.26) imply the identity

$$Ric(\nabla f, \nabla f) - 2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) = k_0|\nabla f|^2,$$

which holds for the extremal eigenfunction f. Using the first formula in (2.12), the upper identity can be rewritten as

$$6S|\nabla f|^2 + 2T^0(\nabla f, \nabla f) - \frac{36}{k_0}A(\nabla f) = k_0|\nabla f|^2. \tag{5.2}$$

Now, obviously the assumption of the condition b) in Corollary 1.3 yields the desired estimate (1.7), which completes the proof of Corollary 1.2. □

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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AN APPROACH FOR DERIVATION OF MARKOV–TYPE INEQUALITIES IN L_2 NORMS

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An approach for derivation of Markov-type inequalities in L_2 norms proposed in [9] is applied to the classical case of a constant weight function. According to a result of E. Schmidt, the sharp constant in this inequality is asymptotically equal to $\frac{n^2}{\pi}$. We obtain upper and lower bounds for the best constant.

Keywords: Markov type inequality, ultraspherical polynomials, quadratic forms.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Throughout this paper, π_n will mean the class of algebraic polynomials of degree not exceeding n.

A classical result in Approximation Theory, the inequality of the brothers Markov [5], [6], asserts that for any $f \in \pi_n$

$$\|f^{(k)}\| \le \|T_n^k\| \|f\| \quad \text{for } k = 1, \dots, n,$$

where $\|\cdot\|$ stands for the uniform norm in $[-1, 1]$ and $T_n(x) := \cos n \arccos x$ is the Chebyshev polynomial of the first kind.

The topic of this paper is Markov type inequalities in the L_2-norms, i.e., norms of the type

$$\|f\| := \Big(\int_a^b w(x) |f(x)|^2 dx \Big)^{1/2},$$

where $w(x)$ is a weight function on the finite or infinite interval [a,b] (i.e., $w(x)$ is non-negative and integrable on [a,b] with all moments finite). It is well-known that (see, e.g., [4] or [8]) there exists a constant $c_n = c_n(a,b,w)$ such that

$$\|f'\| \le c_n \, \|f\| \quad \text{for every} \quad f \in \pi_n. \tag{1.1}$$

The sharp constant c_n in (1.1) is known to be the largest singular value of a certain matrix (see, e.g., [3] or [7, Theorems 1.6.3 and 1.6.5]). Despite of this simple characterization, not much is known about the exact constants even in the classical cases of weight function of Hermite, Laugerre and Gegenbauer. Schmidt [10] has found that in the case of Hermite weight function ($a = -b = \infty, w(x) = \exp(-x^2)$) the best constant is $c_n = \sqrt{2n}$, and the Hermite polynomial H_n is the extremal polynomial. Turán [12] has proven that the best constant in the case of Laguerre weight function ($a = 0, b = \infty,\ w(x) = \exp(-x)$) is

$$c_n = \Big(\sin\frac{\pi}{4n+2}\Big)^{-1}$$

In the case $[a,b] = [-1,1]$, $w(x) = 1$, E. Schmidt [10] found the best constant asymptotically, proving that for $n \ge 5$,

$$c_n = \frac{(2n+3)^2}{4\pi}\Big(1 - \frac{\pi^2-3}{3(2n+3)^2} + \frac{16R}{(2n+3)^4}\Big)^{-1}, \quad \text{where} \quad -6 < R < 13. \tag{1.2}$$

The proof of this asymptotic estimate runs in a paper of about 40 pages.

G. Nikolov [9] has studied Markov-type inequalities in the L_2-norm induced by the Gegenbauer weight function

$$w_\lambda(x) := (1-x^2)^{\lambda-1/2}, \quad \lambda > -1/2, \quad x \in (-1,1).$$

The notation $\|\cdot\|_\lambda$ will stand for the $L_2[-1,1]$ norm induced by w_λ, i.e.,

$$\|f\|_\lambda := \Big(\int_{-1}^{1} w_\lambda(x)|f(x)|^2 dx\Big)^{1/2}.$$

Specifically, in [9] are proven Markov-type inequalities in the L_2-norms induced by the Chebyshev weight functions $w_0(x) = (1-x)^{-1/2}$ and $w_1(x) = (1-x)^{1/2}$.

Theorem A. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_0 \le 0.478849(n+2)^2\|f\|_0. \tag{1.3}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that $\|f'\|_0 \ge 0.472135\, n^2\|f\|_0$.

Theorem B. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_1 \le 0.256861(n+5/2)^2\|f\|_1. \tag{1.4}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that $\|f'\|_1 \ge 0.248549n^2\|f\|_1$.

Let us mention that, although the constants in (1.3) and (1.4) are not sharp, the supplementary inequalities in Theorems A and B show that they overestimate the best constants by a factor not exceeding 1.0142 and 1.0334, respectively.

Here, we apply the approach proposed in [9] to obtain an elementary proof of L_2 Markov inequality associated with a constant weight function, i.e., $w_{1/2}(x) = 1$. Our result reads as follows:

Theorem 1.1. *For every $n \in \mathbb{N}$ and $f \in \pi_n$, the following inequality holds true:*

$$\|f'\|_{1/2} \le 0.325779(n+1.6)^2\,\|f\|_{1/2}. \tag{1.5}$$

Moreover, for every $n \in \mathbb{N}$ there exists $f \in \pi_n$ such that

$$\|f'\|_{1/2} \ge 0.317837\,(n+1/2)^2\,\|f\|_{1/2}. \tag{1.6}$$

2. REQUISITES

In this section we introduce some results from [9] which will be needed for the proof of Theorem 1.1.

The notation $|\cdot|$ will stand for the Euclidean norm, i.e., if $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$, then $|\mathbf{t}| = (t_1^2 + \cdots + t_m^2)^{1/2}$. The unit sphere in $\mathbb{R}^m$ is denoted by S_m,

$$S_m := \{\mathbf{t} \in \mathbb{R}^m : |\mathbf{t}| = 1\}.$$

By S_m^+ (resp. $\mathbb{R}_+^m$) we shall mean the subsets of S_m (resp. $\mathbb{R}^m$) with non-negative coordinates.

For the Markov inequality in the L_2-norm corresponding to $w_\lambda(x)$ we need some facts about the associated orthogonal polynomials. The latter are the ultraspherical polynomials (also called Gegenbauer polynomials) $\{C_m^\lambda(x)\}_{m=0}^\infty$. It is well known that (see [11]), for $\lambda \ne 0$

$$\int_{-1}^{1} w_\lambda(x) C_j^\lambda(x) C_k^\lambda(x)\,dx = \delta_{jk} h_k^2 \quad j, k = 0, 1, \dots,$$

with δ_{jk} being the Kronecker symbol and

$$h_k = h_{k,\lambda} := \Big(\frac{2^{1-2\lambda}\pi\Gamma(k+2\lambda)}{k!(k+\lambda)\Gamma^2(\lambda)}\Big)^{1/2}.$$

For $\mathbf{t} \in \mathbb{R}^m$, we introduce the following positive definite quadratic forms:

$$P_m(\mathbf{t}) := \sum_{k=1}^{m} \Big(\sum_{j=k}^{m} (2k+\lambda-1)\frac{h_{2k-1}}{h_{2j}}\,t_j\Big)^2 \tag{2.1}$$

and

$$Q_m(\mathbf{t}) := \sum_{k=1}^{m} \Big(\sum_{j=k}^{m} (2k+\lambda-2) \frac{h_{2k-2}}{h_{2j-1}}\, t_j \Big)^2. \tag{2.2}$$

The best constants in the Markov-type inequalities in $\|\cdot\|_\lambda$-norm, $\lambda \geq 0$ and the quadratic forms $P_m(\mathbf{t})$ and $Q_m(\mathbf{t})$ are related through the following

Theorem 2.1. ([9]) *If $\lambda \geq 0$, then*

$$\sup_{f\in\pi_n, f\neq 0} \frac{\|f'\|_\lambda^2}{\|f\|_\lambda^2} = \begin{cases} 4 \sup\limits_{t\in S_m^+} P_m(\boldsymbol{t}), & \text{if} \quad n = 2m, \\ \\ 4 \sup\limits_{t\in S_m^+} Q_m(\boldsymbol{t}), & \text{if} \quad n = 2m-1. \end{cases}$$

The next lemma provides upper bounds for the supremum over S_m of positive definite quadratic forms like P_m and Q_m.

Lemma 2.1. ([9]) *Given positive a_{kj} $(1 \leq k \leq m, \quad k \leq j \leq m)$, set*

$$K(\boldsymbol{t}) := \sum_{k=1}^{m} \Big(\sum_{j=k}^{m} a_{kj}\, t_j \Big)^2.$$

Then, for every $\boldsymbol{p} = (p_1, \dots, p_m)$, $(p_k > 0,\ k = 1, \dots, m)$,

$$\sup_{t\in S_m} K(\boldsymbol{t}) \leq \max_{1\leq k\leq m} A_k(\boldsymbol{p}), \tag{2.3}$$

where

$$A_k(\boldsymbol{p}) := \frac{1}{p_k} \sum_{i=1}^{k} a_{ik} \Big(\sum_{j=i}^{m} p_j a_{ij} \Big).$$

The equality in (2.3) *occurs only if $A_1(\boldsymbol{p}) = A_2(\boldsymbol{p}) = \cdots = A_m(\boldsymbol{p})$.*

We shall use a familiar property of the trapezium and the midpoint quadratures

$$Q_{m+1}^{Tr}[f] = \frac{h}{2}[f(x_0) + f(x_m)] + h \sum_{k=1}^{m-1} f(x_k), \quad Q_m^{Mi}[f] = h \sum_{k=1}^{m} f(x_{k-1/2}),$$

where $x_j := a + jh$ and $h = (b-a)/m$.

Lemma 2.2. *a) If f is convex in $[a, b]$, then*

$$Q_m^{Mi}[f] \leq \int_a^b f(x)dx \leq Q_{m+1}^{Tr}[f].$$

b) If $f'' \geq 0$ and f'' is convex in $[a,b]$, then

$$Q_m^{Mi}[f] \geq \int_a^b f(x)dx - \frac{h^2}{24}[f'(b)-f'(a)], \quad Q_{m+1}^{Tr}[f] \leq \int_a^b f(x)dx + \frac{h^2}{12}[f'(b)-f'(a)].$$

3. PROOF OF THEOREM 1.1: THE CASE OF EVEN n, $n = 2m$

According to Theorem 2.1, we have

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4 \sup_{\mathbf{t}\in S_m^+} P_m(\mathbf{t}), \tag{3.1}$$

and in our particular case $\lambda = 1/2$ the quadratic form P defined by (2.2) becomes

$$P_m(\mathbf{t}) = \sum_{k=1}^{m} \left(\sum_{j=k}^{m} \frac{1}{2}\sqrt{(4k-1)(4j+1)}\, t_j \right)^2. \tag{3.2}$$

3.1. AN UPPER BOUND

We apply Lemma 2.1 to $K = P_m$, the quadratic form given by (3.2), i.e., with $a_{kj} = \frac{1}{2}\sqrt{(4k-1)(4j+1)}$. We obtain

$$4 \sup_{\mathbf{t}\in S_m^+} P_m(\mathbf{t}) = 4 \sup_{\mathbf{t}\in S_m} P_m(\mathbf{t}) \leq 4 \max_{1\leq k\leq m} A_k(\mathbf{p}) = \max_{1\leq k\leq m} 4\, A_k(\mathbf{p}),$$

where

$$\begin{aligned}
A_k(\mathbf{p}) &= \frac{1}{p_k} \sum_{i=1}^{k} \frac{1}{2}\sqrt{(4i-1)(4k+1)} \left(\sum_{j=i}^{m} \frac{1}{2}\sqrt{(4i-1)(4j+1)}\, p_j \right) \\
&= \frac{1}{4p_k} \sum_{i=1}^{k} \sqrt{(4i-1)(4k+1)} \left(\sum_{j=i}^{m} \sqrt{(4i-1)(4j+1)}\, p_j \right) \\
&= \frac{\sqrt{4k+1}}{4p_k} \sum_{i=1}^{k} (4i-1) \left(\sum_{j=i}^{m} \sqrt{4j+1}\, p_j \right),
\end{aligned}$$

and $\mathbf{p} = (p_1, \ldots, p_m)$ is an arbitrary m-tuple of positive numbers. Let us choose

$$p_j = \frac{(4j+3)^\alpha - (4j-1)^\alpha}{\sqrt{4j+1}}, \; j = 1, \ldots, m,$$

where $\alpha \in (3,4)$ will be specified later. In view of inequality

$$(4k+3)^\alpha - (4k-1)^\alpha \ge 4\alpha(4k+1)^{\alpha-1}, \quad k \in \mathbb{N},$$

we get

$$\begin{aligned}
4A_k(\mathbf{p}) &= \frac{4k+1}{(4k+3)^\alpha - (4k-1)^\alpha} \sum_{i=1}^{k} (4i-1) \sum_{j=i}^{m} \Big((4j+3)^\alpha - (4j-1)^\alpha \Big) \\
&\le \frac{4k+1}{4\alpha(4k+1)^{\alpha-1}} \sum_{i=1}^{k} \Big[(4i-1)(4m+3)^\alpha - (4i-1)^{\alpha+1} \Big] \qquad (3.3)\\
&= \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[(2k^2+k)(4m+3)^\alpha - \sum_{i=1}^{k} (4i-1)^{\alpha+1} \right].
\end{aligned}$$

We estimate from below the latter sum with the help of Lemma 2.2 b). We have

$$\begin{aligned}
\sum_{i=1}^{k} (4i-1)^{\alpha+1} &\ge \int_{1/2}^{k+1/2} (4x-1)^{\alpha+1}\, dx - \frac{4(\alpha+1)}{24} \Big[(4k+1)^\alpha - 1 \Big] \\
&= \frac{1}{4(\alpha+2)} \Big[(4k+1)^{\alpha+2} - 1 \Big] - \frac{\alpha+1}{6} \Big[(4k+1)^\alpha - 1 \Big] \\
&\ge \frac{1}{4(\alpha+2)} \, (4k+1)^{\alpha+2} - \frac{\alpha+1}{6} \, (4k+1)^\alpha
\end{aligned}$$

(for the latter inequality we used that $\frac{\alpha+1}{6} - \frac{1}{4(\alpha+2)} > 0$, since $\alpha \in (3,4)$). Applying this estimation to (3.3) and performing further estimation we obtain

$$\begin{aligned}
4A_k(\mathbf{p}) &\le \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[(2k^2+k)(4m+3)^\alpha - \frac{1}{4(\alpha+2)} \, (4k+1)^{\alpha+2} + \frac{\alpha+1}{6} \, (4k+1)^\alpha \right] \\
&= \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[\frac{(4k+1)^2 - 1}{8} \, (4m+3)^\alpha - \frac{1}{4(\alpha+2)} \, (4k+1)^{\alpha+2} + \frac{\alpha+1}{6} \, (4k+1)^\alpha \right] \\
&\le \frac{(4k+1)^{2-\alpha}}{4\alpha} \left[\frac{(4k+1)^2(4m+3)^\alpha}{8} - \frac{1}{4(\alpha+2)} \, (4k+1)^{\alpha+2} + \Big(\frac{\alpha+1}{6} - \frac{1}{8} \Big) (4m+1)^\alpha \right] \\
&= \frac{(4k+1)^{4-\alpha}}{32\alpha} \left[(4m+3)^\alpha - \frac{2(4k+1)^\alpha}{\alpha+2} \right] + \frac{4\alpha+1}{96\alpha} (4k+1)^{2-\alpha} (4m+1)^\alpha \\
&\le \frac{(4k+1)^{4-\alpha}}{32\alpha} \left[(4m+3)^\alpha - \frac{2(4k+1)^\alpha}{\alpha+2} \right] + \frac{4\alpha+1}{96\alpha} (4m+1)^2 .
\end{aligned}$$

For the first summand in the last expression we need an upper bound which does not depend on k. The function

$$h(x) := \frac{x^{4-\alpha}}{32\alpha} \left[M^\alpha - \frac{2x^\alpha}{\alpha+2} \right], \quad (M \in \mathbb{N}, \ 0 < x < M, \ \ \alpha \in (3,4))$$

has a derivative

$$h'(x) = \frac{x^{3-\alpha}}{32\alpha} \left[(4-\alpha) M^\alpha - \frac{8}{\alpha+2} x^\alpha \right],$$

hence under the above assumptions $h(x)$ has a unique critical point x_0 in $(0, M)$,

$$x_0 = \left(\frac{(4-\alpha)(a+2)M^\alpha}{8}\right)^{\frac{1}{\alpha}} = \left(\frac{(4-\alpha)(a+2)}{8}\right)^{\frac{1}{\alpha}} M.$$

Since $h'(x) > 0$ in $(0, x_0)$ and $h'(x) < 0$ in (x_0, M), it follows that x_0 is a point of an absolute maximum for $h(x)$ in the interval $(0, M)$. For the maximal value of $h(x)$ in $(0, M)$ we obtain

$$\max_{x\in(0,M)} h(x) = \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} M^4.$$

Going back to the estimation of $4A_k(\mathbf{p})$, substituting $M = 4m+3$ and $x = 4k+1$, we get

$$4A_k(\mathbf{p}) \le \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} (4m+3)^4 + \frac{4\alpha+1}{96\alpha}(4m+1)^2,$$

and the latter inequality holds true for $k = 1, 2, \ldots, m$. Hence,

$$\begin{aligned}\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} &\le \max_{1\le k\le m} 4A_k(\mathbf{p}) \\ &\le \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}} (4m+3)^4 + \frac{4\alpha+1}{96\alpha}(4m+1)^2 .\end{aligned}$$

The above inequality holds for every value of the parameter $\alpha \in (3, 4)$, and we exploit this fact to minimize with respect to α the coefficient of $(4m+3)^4$. With the help of Wolfram's *MATHEMATICA*, we find that the minimum value of the function

$$\psi(\alpha) := \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}}, \quad \alpha \in (3, 4),$$

is equal to $\psi(\alpha_*) = 0.006633243689\ldots$, where $\alpha_* = 3.23308\ldots$ satisfies $\alpha_* \in (3, 4)$. We obtain

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} \le 0.006633244(4m+3)^4 + \frac{4\alpha_*+1}{96\alpha_*}(4m+1)^2 . \tag{3.4}$$

It is easy to see that for every $m \in \mathbb{N}$ we have

$$0.006633244(4m+3)^4 + \frac{4\alpha_*+1}{96\alpha_*}(4m+1)^2 \le 0.006633244(4m+3.2)^4, \quad m \in \mathbb{N}. \tag{3.5}$$

Indeed, the expression

$$\frac{(4m+3.2)^4 - (4m+3)^4}{(4m+1)^2}$$

is an increasing function of m, and it suffices to verify (3.5) for $m = 1$ only.

Combining (3.4) and (3.5), we obtain

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} \le 0.006633244\,(4m+3.2)^4 = 0.106131904\,(2m+1.6)^4$$
$$\le 0.325778919^2\,(2m+1.6)^4\,,$$

which implies

$$\sup_{f\in\pi_{2m}, f\neq 0} \frac{\|f'\|_{1/2}}{\|f\|_{1/2}} \le 0.325779\,(2m+1.6)^2\,.$$

Thus, inequality (1.5) is proven for $n = 2m$.

3.2. A LOWER BOUND

To prove inequality (1.6), we observe that every even polynomial $f \in \pi_{2m}$ can be written as a linear combination of Legendre polynomials with even indices $\{P_{2k}(x)\}$ (written below as polynomials of Gegenbauer with a parameter $\lambda = 1/2$ in order to avoid confusion with the quadratic forms P). If

$$f(x) = \sum_{k=1}^{m} t_k C_{2k}^{1/2}(x)\,, \tag{3.6}$$

then

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4\frac{P_m(\mathbf{t})}{|\mathbf{t}|^2}\,,$$

and it suffices to find a vector of coefficients $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ in the expression (3.6), such that $4\frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} \ge 0.317837^2(2m+1/2)^4$.

For an arbitrary $\beta \in (3, 3.5)$ (its value will be specified later), we choose

$$t_j := \frac{(4j+3)^\beta - (4j-1)^\beta}{\sqrt{4j+1}}\,, \quad j = 1, \ldots, m\,.$$

With this choice of $\mathbf{t}$ we shall find a lower bound for the value of the quadratic form $4P_m(\mathbf{t})$ and an upper bound for $|\mathbf{t}|^2$. This will imply a lower bound for $4P_m(\mathbf{t})/|\mathbf{t}|^2$ (depending on the parameter β).

For the value of the quadratic form $4P_m(\mathbf{t})$ we obtain

$$\begin{aligned} 4P_m(\mathbf{t}) &= \sum_{k=1}^{m} (4k-1)\left[\sum_{j=k}^{m} \Big((4j+3)^\beta - (4j-1)^\beta\Big)\right]^2 \\ &= \sum_{k=1}^{m} (4k-1)\Big[(4m+3)^\beta - (4k-1)^\beta\Big]^2 \qquad (3.7) \\ &= (2m^2+m)(4m+3)^{2\beta} - 2(4m+3)^\beta \sum_{k=1}^{m}(4k-1)^{\beta+1} + \sum_{k=1}^{m}(4k-1)^{2\beta+1}. \end{aligned}$$

Now we estimate from below $4P_m(\mathbf{t})$. We estimate from above the first sum of the last line of (3.7) using Lemma 2.2 a):

$$\sum_{k=1}^{m}(4k-1)^{\beta+1} \le \int_{1/2}^{m+1/2}(4x-1)^{\beta+1}dx < \frac{1}{4(\beta+2)}(4m+1)^{\beta+2}.$$

A lower bound for the second sum in the last line of (3.7) is obtained with the help of Lemma 2.2 b):

$$\begin{aligned}\sum_{k=1}^{m}(4k-1)^{2\beta+1} &\ge \int_{1/2}^{m+1/2}(4x-1)^{2\beta+1}dx - \frac{1}{24}\Big[4(2\beta+1)(4m+1)^{2\beta} - 4(2\beta+1)\Big]\\ &= \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} + \frac{2\beta+1}{6} - \frac{1}{8(\beta+1)}\\ &> \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta}\end{aligned}$$

(for the later inequality we used that $\frac{2\beta+1}{6} - \frac{1}{8(\beta+1)} > 0$).

Substituting the above lower bounds in (3.7), we obtain

$$\begin{aligned}4P_m(\mathbf{t}) >&\frac{1}{8}\Big[(4m+1)^2-1\Big](4m+3)^{2\beta} - \frac{1}{2(\beta+2)}(4m+3)^{\beta}(4m+1)^{\beta+2}\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta}\\ =&\frac{1}{8}(4m+1)^2(4m+3)^{2\beta} - \frac{1}{2(\beta+2)}(4m+3)^{\beta}(4m+1)^{\beta+2}\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}\\ =&(4m+3)^{\beta}\Big[\frac{1}{8}(4m+1)^2(4m+3)^{\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta+2}\Big]\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}.\end{aligned}$$

A further lower bound is obtained from the inequality

$$(4m+3)^{\beta} > (4m+1)^{\beta} + 2\beta(4m+1)^{\beta-1}$$

(which follows from Maclaurin's formula $(1+x)^{\beta} = 1 + \beta x + \frac{\beta(\beta-1)}{2}x^2(1+\xi)^{\beta-1}$ with $x = \frac{2}{4m+1}$ and $0 < \xi < x$):

$$\begin{aligned}4P_m(\mathbf{t}) >&\Big[(4m+1)^{\beta} + 2\beta(4m+1)^{\beta-1}\Big]\\ &\times\Big[\frac{1}{8}(4m+1)^{\beta+2} + \frac{\beta}{4}(4m+1)^{\beta+1} - \frac{1}{2(\beta+2)}(4m+1)^{\beta+2}\Big]\\ &+ \frac{1}{8(\beta+1)}(4m+1)^{2\beta+2} - \frac{2\beta+1}{6}(4m+1)^{2\beta} - \frac{1}{8}(4m+3)^{2\beta}\end{aligned}$$

$$
\begin{aligned}
=&\frac{\beta^2}{8(\beta+1)(\beta+2)}(4m+1)^{2\beta+2}+\frac{\beta^2}{2(\beta+2)}(4m+1)^{2\beta+1}\\
&+\frac{(\beta-1)(3\beta+1)}{6}(4m+1)^{2\beta}-\frac{1}{8}(4m+3)^{2\beta}.
\end{aligned}
$$

The expression in the last line is positive when $m \geq 2$ and $\beta \in (3, 3.5)$, and therefore can be neglected. Indeed, to prove the inequality

$$
\frac{4(\beta-1)(3\beta+1)}{3} > \left(\frac{4m+3}{4m+1}\right)^{2\beta},
$$

we observe that its right-hand side is less than $\left(\frac{11}{9}\right)^7$ while its left-hand side is greater than $\frac{8.10}{3} = \frac{80}{3}$, and $\frac{80}{3} - \left(\frac{11}{9}\right)^7 > 0$.

Hence,

$$
4P_m(\mathbf{t}) > \frac{\beta^2}{8(\beta+1)(\beta+2)}\Big[(4m+1)^{2\beta+2}+4(\beta+1)(4m+1)^{2\beta+1}\Big]. \tag{3.8}
$$

Our next task is to obtain an upper bound for the norm of $\mathbf{t}$. For the purpose we estimate all of its components

$$
t_j = \frac{(4j+3)^\beta - (4j-1)^\beta}{\sqrt{4j+1}}, \quad j = 1, \dots, m,
$$

bearing in mind that $\beta \in (3, 3.5)$. On using the Maclaurin series, we obtain

$$
\begin{aligned}
(1+x)^\beta - (1-x)^\beta =&2\beta x + \frac{\beta(\beta-1)(\beta-2)}{3}x^3\\
&+\frac{\beta(\beta-1)(\beta-2)(\beta-3)}{24}x^4\Big[(1+\theta_1 x)^{\beta-4} - (1-\theta_2 x)^{\beta-4}\Big],
\end{aligned}
$$

where $\theta_1, \theta_2 \in (0,1)$. For $3 < \beta < 4$ and $0 < x < 1$ the expression in the square brackets is negative, therefore for such β and x we have

$$
(1+x)^\beta - (1-x)^\beta < 2\beta x + \frac{\beta(\beta-1)(\beta-2)}{3}x^3. \tag{3.9}
$$

Applying this inequality with $x = \frac{2}{4j+1}$ $(x \in (0,1))$, we get an upper bound for t_j:

$$
\begin{aligned}
t_j &< 4\beta(4j+1)^{\beta-3/2} + \frac{8}{3}\beta(\beta-1)(\beta-2)(4j+1)^{\beta-3\frac{1}{2}}\\
&= 4\beta(4j+1)^{\beta-3/2}\Big[1+\frac{2}{3}(\beta-1)(\beta-2)\frac{1}{(4j+1)^2}\Big]\\
&< 4\beta(4j+1)^{\beta-3/2}\Big[1+\frac{5}{2}\frac{1}{(4j+1)^2}\Big].
\end{aligned}
$$

Consequently,

$$\begin{aligned} t_j^2 &< 16\beta^2(4j+1)^{2\beta-3}\Big[1+5\frac{1}{(4j+1)^2}+\frac{25}{4}\frac{1}{(4j+1)^4}\Big] \\ &\le 16\beta^2(4j+1)^{2\beta-3}\Big[1+\frac{21}{4}\frac{1}{(4j+1)^2}\Big], \end{aligned}$$

and thus

$$t_j^2 < 16\beta^2(4j+1)^{2\beta-3} + 84\beta^2(4j+1)^{2\beta-5}, \quad j=1,\ldots,m. \tag{3.10}$$

To obtain an upper bound for $|\mathbf{t}|^2 = t_1^2+t_2^2+\cdots+t_m^2$, we shall use (3.10) and the fact that for $\beta \in (3, 3.5)$ the functions $g_1(x) = (4x+1)^{2\beta-3}$ and $g_2(x) = (4x+1)^{2\beta-5}$ are convex and have convex second derivatives in the interval $[0,m]$. This enables us to apply Lemma 2.2 b) to estimate the sums which appear. With Q_m^{tr} being the $(m+1)$-point trapezium quadrature formula for the interval $[0,m]$, we have

$$\begin{aligned} \sum_{j=1}^{m}(4j+1)^{2\beta-3} &= -\frac{1}{2}+\frac{1}{2}(4m+1)^{2\beta-3}+Q_m^{tr}[g_1] \\ &< \frac{1}{2}(4m+1)^{2\beta+3}+\int_0^m(4x+1)^{2\beta-3}dx+\frac{4(2\beta-3)}{12}\Big[(4m+1)^{2\beta-4}-1\Big] \\ &< \frac{1}{8(\beta-1)}(4m+1)^{2\beta-2}+\frac{1}{2}(4m+1)^{2\beta-3}+\frac{2\beta-3}{3}(4m+1)^{2\beta-4}, \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^{m}(4j+1)^{2\beta-5} &= -\frac{1}{2}+\frac{1}{2}(4m+1)^{2\beta-5}+Q_m^{tr}[g_2] \\ &< \frac{1}{2}(4m+1)^{2\beta-5}+\int_0^m(4x+1)^{2\beta-5}dx+\frac{4(2\beta-5)}{12}\Big[(4m+1)^{2\beta-6}-1\Big] \\ &< \frac{1}{8(\beta-2)}(4m+1)^{2\beta-4}+\frac{1}{2}(4m+1)^{2\beta-5}+\frac{2\beta-5}{3}(4m+1)^{2\beta-6}. \end{aligned}$$

We use (3.10) and these two estimations in order to obtain an upper bound for $|\mathbf{t}|^2$:

$$\begin{aligned} |\mathbf{t}|^2 &< 16\beta^2\sum_{j=1}^{m}(4j+1)^{2\beta-3}+84\beta^2\sum_{j=1}^{m}(4j+1)^{2\beta-5} \\ &< \frac{2\beta^2}{\beta-1}(4m+1)^{2\beta-2}+8\beta^2(4m+1)^{2\beta-3}+\frac{16\beta^2(2\beta-3)}{3}(4m+1)^{2\beta-4} \\ &\quad +\frac{21\beta^2}{2(\beta-2)}(4m+1)^{2\beta-4}+41\beta^2(4m+1)^{2\beta-5}+\frac{84\beta^2(2\beta-5)}{3}(4m+1)^{2\beta-6} \\ &= \frac{2\beta^2}{\beta-1}(4m+1)^{2\beta-2}+\beta^2(4m+1)^{2\beta-3} \\ &\quad\times\left[8+\left(\frac{16(2\beta-3)}{3}+\frac{21}{2(\beta-2)}\right)\frac{1}{4m+1}+\frac{41}{(4m+1)^2}+\frac{84(2\beta-5)}{3}\frac{1}{(4m+1)^3}\right]. \end{aligned}$$

With $m \geq 2$ and $\beta \in (3, 3.5)$ we estimate the expression in the square brackets as follows:

$$8 + \left(\frac{16(2\beta - 3)}{3} + \frac{21}{2(\beta - 2)} \right) \frac{1}{4m+1} + \frac{41}{(4m+1)^2} + \frac{84(2\beta - 5)}{3} \frac{1}{(4m+1)^3}$$
$$< 8 + \left(\frac{64}{3} + 7 \right) \cdot \frac{1}{9} + \frac{41}{9^2} + \frac{168}{3} \cdot \frac{1}{9^3} < 12\,.$$

Hence for $\beta \in (3, 3.5)$ and $m \geq 2$ we have

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta - 1} \left[(4m+1)^{2\beta - 2} + 6(\beta - 1)(4m+1)^{2\beta - 3} \right].$$

This inequality combined with (3.8) yields, for $\beta \in (3, 3.5)$ and $m \geq 2$,

$$4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta - 1}{16(\beta + 1)(\beta + 2)} (4m+1)^4 \, \frac{1 + \frac{4(\beta+1)}{4m+1}}{1 + \frac{6(\beta-1)}{4m+1}}$$
$$> \frac{\beta - 1}{(\beta + 1)(\beta + 2)} (2m + 1/2)^4.$$

Since the last inequality holds true for every $\beta \in (3, 3.5)$, we can optimize our choice, searching for the maximum of the function

$$\varphi(\beta) = \frac{\beta - 1}{(\beta + 1)(\beta + 2)}\,, \quad \beta \in (3, 3.5).$$

The zeros of φ' are $\beta_1 = 1 - \sqrt{6}$ and $\beta_2 = 1 + \sqrt{6}$; only $\beta_2 = 1 + \sqrt{6} = 3,44949\ldots$ is in $(3, 3.5)$, and $\beta = \beta_2$ is a point of a global maximum for $\varphi(\beta)$ in this interval. We have

$$\varphi(1 + \sqrt{6}) = \frac{\sqrt{6}}{(2 + \sqrt{6})(3 + \sqrt{6})} = \frac{\sqrt{6}}{12 + 5\sqrt{6}} = \frac{1}{5 + 2\sqrt{6}} = 5 - 2\sqrt{6} = (\sqrt{3} - \sqrt{2})^2.$$

Therefore for $\beta = \beta_2$ and $n = 2m,\ m \geq 2$, we have

$$4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} > (\sqrt{3} - \sqrt{2})^2 (n + 1/2)^4\,.$$

The last inequality means that for the polynomial $f(x) = \sum_{k=1}^{m} t_k C_{2k}^{1/2}(x)$ we have

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4 \frac{P_m(\mathbf{t})}{|\mathbf{t}|^2} (\sqrt{3} - \sqrt{2})^2 (n + 1/2)^4\,.$$

Since $\sqrt{3} - \sqrt{2} = 0.317837245\ldots$, this proves the lower bound (1.6) in Theorem 1.1 for $n = 2m, \quad m \geq 2$.

4. PROOF OF THEOREM 1.1: THE CASE OF AN ODD n, $n = 2m-1$

According to Theorem 2.1, we have

$$\sup_{f\in\pi_{2m-1}, f\neq 0} \frac{\|f'\|^2_{1/2}}{\|f\|^2_{1/2}} = 4 \sup_{\mathbf{t}\in S_m^+} Q_m(\mathbf{t}), \tag{4.1}$$

where, in our particular case $\lambda = 1/2$, the quadratic form Q_m defined by (2.3) becomes

$$Q_m(\mathbf{t}) = \sum_{k=1}^{m}\left(\sum_{j=k}^{m} \frac{1}{2}\sqrt{4k-3}\sqrt{4j-1}\, t_j\right)^2. \tag{4.2}$$

4.1. AN UPPER BOUND

For any $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{R}^m_+$, Lemma 2.1 applied to $K = Q_m$ implies

$$4 \sup_{\mathbf{t}\in S_m^+} Q_m(\mathbf{t}) = 4 \sup_{\mathbf{t}\in S_m} Q_m(\mathbf{t}) \le 4 \max_{1\le k\le m} A_k(\mathbf{p}) = \max_{1\le k\le m} 4\, A_k(\mathbf{p}),$$

where

$$\begin{aligned} A_k(\mathbf{p}) &= \frac{1}{p_k}\sum_{i=1}^{k}\frac{1}{2}\sqrt{4i-3}\,\sqrt{4k-1}\Big(\sum_{j=i}^{m}\frac{1}{2}\sqrt{4i-3}\sqrt{4j-1}p_j\Big) \\ &= \frac{\sqrt{4k-1}}{4\,p_k}\sum_{i=1}^{k}(4i-3)\Big(\sum_{j=i}^{m}\sqrt{4j-1}p_j\Big). \end{aligned}$$

For some $\alpha \in (3,4)$, which will be specified later, we choose

$$p_j = \frac{(4j+1)^\alpha - (4j-3)^\alpha}{\sqrt{4j-1}}\,,\ j = 1, \dots, m.$$

For any such α we have the inequality

$$(4k+1)^\alpha - (4k-3)^\alpha \ge 4\alpha(4k-1)^{\alpha-1}, \quad k \in \mathbb{N},$$

and we apply it to obtain

$$\begin{aligned} 4A_k(\mathbf{p}) &= \frac{4k-1}{(4k+1)^\alpha - (4k-3)^\alpha}\sum_{i=1}^{k}(4i-3)\sum_{j=i}^{m}\left((4j+1)^\alpha - (4j-3)^\alpha\right) \\ &\le \frac{4k-1}{4\alpha(4k-1)^{\alpha-1}}\sum_{i=1}^{k}\Big[(4i-3)(4m+1)^\alpha - (4i-3)^{\alpha+1}\Big] \\ &= \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[(2k^2-k)(4m+1)^\alpha - \sum_{i=1}^{k}(4i-3)^{\alpha+1}\right] \\ &= \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[\frac{(4k-1)^2-1}{8}(4m+1)^\alpha - \sum_{i=2}^{k}(4i-3)^{\alpha+1} - 1\right]. \end{aligned} \tag{4.3}$$

For the last sum appearing in the right-hand side of (4.3) we apply Lemma 2.2 b) to obtain

$$\begin{aligned}\sum_{i=2}^{k}(4i-3)^{\alpha+1} &\geq \int_{3/2}^{k+1/2}(4x-3)^{\alpha+1}\,dx - \frac{4(\alpha+1)}{24}\Big[(4k-1)^{\alpha}-3^{\alpha}\Big]\\ &= \frac{1}{4(\alpha+2)}\Big[(4k-1)^{\alpha+2}-3^{\alpha}\Big] - \frac{\alpha+1}{6}\Big[(4k-1)^{\alpha}-3^{\alpha}\Big]\\ &\geq \frac{1}{4(\alpha+2)}(4k-1)^{\alpha+2} - \frac{\alpha+1}{6}(4k-1)^{\alpha}\end{aligned}$$

(for the latter inequality we have used that $\frac{\alpha+1}{6}3^{\alpha} - \frac{3^{\alpha}}{4(\alpha+2)} > 0$, since $\alpha \in (3,4)$). Substitution of this bound in (4.3) and a further estimation yield

$$\begin{aligned}4A_k(\mathbf{p}) &\leq \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[\frac{(4k-1)^2-1}{8}(4m+1)^{\alpha} - \frac{1}{4(\alpha+2)}(4k-1)^{\alpha+2} + \frac{\alpha+1}{6}(4k-1)^{\alpha}-1\right]\\ &\leq \frac{(4k-1)^{2-\alpha}}{4\alpha}\left[\frac{(4k-1)^2-1}{8}(4m+1)^{\alpha} - \frac{1}{4(\alpha+2)}(4k-1)^{\alpha+2} + \Big(\frac{\alpha+1}{6}-\frac{1}{8}\Big)(4m-1)^{\alpha}\right]\\ &= \frac{(4k-1)^{4-\alpha}}{32\alpha}\left[(4m+1)^{\alpha} - \frac{2(4k-1)^{\alpha}}{\alpha+2}\right] + \frac{4\alpha+1}{96\alpha}(4k-1)^{2-\alpha}(4m-1)^{\alpha}\\ &\leq \frac{(4k-1)^{4-\alpha}}{32\alpha}\left[(4m+1)^{\alpha} - \frac{2(4k-1)^{\alpha}}{\alpha+2}\right] + \frac{4\alpha+1}{96\alpha}(4m-1)^{2}.\end{aligned}$$

From the analysis in the case $(n=2m)$ we know that the function

$$h(x) := \frac{x^{4-\alpha}}{32\alpha}\left[M^{\alpha} - \frac{2x^{\alpha}}{\alpha+2}\right]$$

has a unique global maximum in the interval $(0,M)$ for $\alpha \in (3,4)$. Repeating the argument from Section 3.1, substituting $M = 4m+1$ and $x = 4k-1$, we obtain

$$4\,A_k(\mathbf{p}) \leq \frac{1}{128}\left(\frac{(4-\alpha)(\alpha+2)}{8}\right)^{\frac{4-\alpha}{\alpha}}(4m+1)^4 + \frac{4\alpha+1}{96\alpha}(4m-1)^2, \quad 1 \leq k \leq m.$$

Minimization of the major term in the right-hand side with respect to α yields

$$\sup_{f\in\pi_{2m-1},f\neq 0}\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} \leq \max_{1\leq k\leq m} 4\,A_k(\mathbf{p}) \leq 0.10613184(n+1.6)^4\,.$$

Inequality (1.5) is proven in the case $n = 2m-1$, $m \geq 2$.

4.2. A LOWER BOUND

Every odd polynomial $f \in \pi_{2m-1}$ can be expressed as a linear combination of the Legendre polynomials with odd indices $\{P_{2k-1}\}$, which we write again as polynomials of Gegenbauer with a parameter $\lambda = 1/2$. If

$$f(x) = \sum_{k=1}^{m} t_k C_{2k-1}^{1/2}(x)\,, \tag{4.4}$$

then

$$\frac{\|f'\|_{1/2}^2}{\|f\|_{1/2}^2} = 4\frac{Q_m(\mathbf{t})}{|t|^2} \quad .$$

We will find a suitable vector of the coefficients $\mathbf{t} = (t_1, t_2, \ldots, t_m) \in \mathbb{R}_+^m$ in (4.4), such that $4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} \geq 0.317837^2(n+1/2)^4$.

For a $\beta \in (3, 3.5)$, which will be specified later, we choose

$$t_j := \frac{(4j+1)^\beta - (4j-3)^\beta}{\sqrt{4j-1}} \quad .$$

As it was done in Section 3.2, we estimate from below the quadratic form $4Q_m(\mathbf{t})$ and from above $|\mathbf{t}|^2$, thus obtaining a lower bound for $4Q_m(\mathbf{t})/|\mathbf{t}|^2$. For this choice of $\mathbf{t}$ we have

$$\begin{aligned} 4Q_m(\mathbf{t}) &= \sum_{k=1}^m (4k-3)\Big(\sum_{j=k}^m (4j+1)^\beta - (4j-3)^\beta\Big)^2 \\ &= \sum_{k=1}^m (4k-3)\Big[(4m+1)^\beta - (4k-3)^\beta\Big]^2 \qquad (4.5) \\ &= (2m^2-m)(4m+1)^{2\beta} - 2(4m+1)^\beta \sum_{k=1}^m (4k-3)^{\beta+1} + \sum_{k=1}^m (4k-3)^{2\beta+1}. \end{aligned}$$

For the first of the sums above we apply Lemma 2.2 a) to obtain

$$\begin{aligned} \sum_{k=1}^m (4k-3)^{\beta+1} &= 1 + \sum_{k=1}^{m-1} (4k+1)^{\beta+1} < 1 + \int_{1/2}^{m-1/2} (4x+1)^{\beta+1} dx \\ &= 1 + \frac{1}{4(\beta+2)}\Big[(4m-1)^{\beta+2} - 3^{\beta+2}\Big] < \frac{1}{4(\beta+2)}(4m-1)^{\beta+2}, \end{aligned}$$

where for the last inequality we have used that $1 - \frac{3^{\beta+2}}{4(\beta+2)} < 0$.

Lemma 2.2 b) applied to the second sum of the last line of (4.5) yields

$$\begin{aligned} \sum_{k=1}^m (4k-3)^{2\beta+1} &= \sum_{k=0}^{m-1} (4k+1)^{2\beta+1} = 1 + \sum_{k=1}^{m-1} (4k+1)^{2\beta+1} \\ &> 1 + \int_{1/2}^{m-1/2} (4x+1)^{2\beta+1} dx - \frac{2\beta+1}{6}\Big[(4m-1)^{2\beta} - 3^{2\beta}\Big] \\ &= \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta} + 1 + \frac{2\beta+1}{6}3^{2\beta} - \frac{9}{8(\beta+1)}3^{2\beta} \\ &> \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta}, \end{aligned}$$

where for the last line we have used that $1 + 3^{2\beta}\Big(\frac{2\beta+1}{6} - \frac{9}{8(\beta+1)}\Big) > 0$.

Substitution of the bounds for these sums in (4.5) implies

$$\begin{aligned}
4Q_m(\mathbf{t}) >& \frac{1}{8}\Big[(4m-1)^2-1\Big](4m+1)^{2\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta}(4m-1)^{\beta+2} \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \frac{1}{8}(4m-1)^2(4m+1)^{2\beta} - \frac{1}{2(\beta+2)}(4m+1)^{\beta}(4m-1)^{\beta+2} \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& (4m+1)^{\beta}\Big[\frac{1}{8}(4m-1)^2(4m+1)^{\beta} - \frac{1}{2(\beta+2)}(4m-1)^{\beta+2}\Big] \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta}.
\end{aligned}$$

Furthermore, from $(4m+1)^{\beta} > (4m-1)^{\beta} + 2\beta(4m-1)^{\beta-1}$ we get

$$\begin{aligned}
4Q_m(\mathbf{t}) >& \Big[(4m+1)^{\beta} + 2\beta(4m-1)^{\beta-1}\Big] \\
&\times \left(\frac{1}{8}(4m-1)^2\Big[(4m+1)^{\beta} + 2\beta(4m-1)^{\beta-1}\Big] - \frac{1}{2(\beta+2)}(4m-1)^{\beta+2}\right) \\
&+ \frac{1}{8(\beta+1)}(4m-1)^{2\beta+2} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \Big[\frac{1}{8} - \frac{1}{2(\beta+2)} + \frac{1}{8(\beta+1)}\Big](4m-1)^{2\beta+2} + \Big(\frac{\beta}{2} - \frac{\beta}{\beta+2}\Big)(4m-1)^{2\beta+1} \\
&+ \frac{\beta^2}{2}(4m-1)^{2\beta} - \frac{1}{8}(4m+1)^{2\beta} - \frac{2\beta+1}{6}(4m-1)^{2\beta} \\
=& \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2} + \frac{\beta^2}{2(\beta+2)}(4m-1)^{2\beta+1} \\
&+ \frac{(3\beta+1)(\beta-1)}{6}(4m-1)^{2\beta} - \frac{1}{8}(4m+1)^{2\beta}.
\end{aligned}$$

For $m \geq 2$ and $\beta \in (3, 3.5)$ the expression in the last line is positive, and therefore can be neglected. Indeed, in the inequality

$$\frac{4(\beta-1)(3\beta+1)}{3} > \Big(\frac{4m+1}{4m-1}\Big)^{2\beta}$$

the right-hand side is $< \left(\frac{9}{7}\right)^7$, the left-hand side is $> \frac{80}{3}$, and also $\frac{80}{3} - \left(\frac{9}{7}\right)^7 > 0$. Therefore,

$$\begin{aligned}
4Q_m(\mathbf{t}) &> \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2} + \frac{\beta^2}{2(\beta+2)}(4m-1)^{2\beta+1} \\
&= \frac{\beta^2}{8(\beta+1)(\beta+2)}(4m-1)^{2\beta+2}\Big[1 + 4(\beta+1)\frac{1}{4m-1}\Big].
\end{aligned} \tag{4.6}$$

Next, we find an upper bound for the norm of $\mathbf{t}$. For this purpose we estimate all of its components

$$t_j = \frac{(4j+1)^\beta - (4j-3)^\beta}{\sqrt{4j-1}}\,, \quad j = 1, \dots, m\,,$$

using that $\beta \in (3, 3.5)$. Inequality (3.9) applied with $x = \frac{2}{4j-1}$ yields an upper bound for t_j:

$$\begin{aligned} t_j &< 4\beta(4j-1)^{\beta-3/2} + \frac{8}{3}\beta(\beta-1)(\beta-2)(4j-1)^{\beta-3\frac{1}{2}} \\ &= 4\beta(4j-1)^{\beta-3/2}\Big[1 + \frac{2}{3}(\beta-1)(\beta-2)\frac{1}{(4j-1)^2}\Big] \\ &< 4\beta(4j-1)^{\beta-3/2}\Big[1 + \frac{5}{2}\frac{1}{(4j-1)^2}\Big]. \end{aligned}$$

Since $j \geq 1$, we have

$$\begin{aligned} t_j^2 &< 16\beta^2(4j-1)^{2\beta-3}\Big[1 + 5\frac{1}{(4j-1)^2} + \frac{25}{4}\frac{1}{(4j-1)^4}\Big] \\ &\leq 16\beta^2(4j-1)^{2\beta-3}\Big[1 + 5\frac{1}{(4j-1)^2} + \frac{25}{36}\frac{1}{(4j-1)^2}\Big] \\ &= 16\beta^2(4j-1)^{2\beta-3}\Big[1 + \frac{205}{36}\frac{1}{(4j-1)^2}\Big]\,. \end{aligned}$$

Thus,

$$t_j^2 < 16\beta^2(4j-1)^{2\beta-3} + \frac{820}{9}\beta^2(4j-1)^{2\beta-5}\,, \quad j = 1, \dots, m\,. \tag{4.7}$$

To estimate from above $|\mathbf{t}|^2$, we make use of (4.7) and the fact that for $\beta \in (3, 3.5)$ the functions $h_1(x) = (4x-1)^{2\beta-3}$ and $h_2(x) = (4x-1)^{2\beta-5}$ are convex and have convex second derivatives in the interval $[1, m]$. Let Q_{m-1}^{tr} be the m-point trapezium quadrature formula for the interval $[1, m]$. By Lemma 2.2 b) we have

$$\begin{aligned} \sum_{j=1}^{m}(4j-1)^{2\beta-3} &= \frac{3^{2\beta-3}}{2} + \frac{(4m-1)^{2\beta-3}}{2} + Q_{m-1}^{tr}[h_1] \\ &< \frac{3^{2\beta-3}}{2} + \frac{(4m-1)^{2\beta+3}}{2} + \int_1^m (4x-1)^{2\beta-3}dx + \frac{2\beta-3}{3}\Big[(4m-1)^{2\beta-4} - 3^{2\beta-4}\Big] \\ &= \frac{1}{8(\beta-1)}(4m-1)^{2\beta-2} + \frac{1}{2}(4m-1)^{2\beta-3} + \frac{2\beta-3}{3}(4m-1)^{2\beta-4} \\ &\quad + \Big[\frac{1}{2} - \frac{2\beta-3}{9}\frac{3}{8(\beta-1)}\Big]3^{2\beta-3}\,, \end{aligned}$$

$$\sum_{j=1}^{m}(4j-1)^{2\beta-5} = \frac{3^{2\beta-5}}{2} + \frac{(4m-1)^{2\beta-5}}{2} + Q_{m-1}^{tr}[h_2]$$

$$\le \frac{3^{2\beta-5}}{2} + \frac{(4m-1)^{2\beta-5}}{2} + \int_1^m (4x-1)^{2\beta-5}dx + \frac{2\beta-5}{3}\Big[(4m-1)^{2\beta-6} - 3^{2\beta-6}\Big]$$

$$= \frac{1}{8(\beta-2)}(4m-1)^{2\beta-4} + \frac{1}{2}(4m-1)^{2\beta-5} + \frac{2\beta-5}{3}(4m-1)^{2\beta-6}$$

$$+ \Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big]3^{2\beta-5}.$$

Using these two estimations we obtain

$$|\mathbf{t}|^2 < 16\beta^2\sum_{j=1}^{m}(4j-1)^{2\beta-3} + \frac{820}{9}\sum_{j=1}^{m}\beta^2(4j-1)^{2\beta-5}$$

$$= \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + 16\beta^2\Big[\frac{1}{2}(4m-1)^{2\beta-3} + \frac{2\beta-3}{3}(4m-1)^{2\beta-4}\Big]$$

$$+ \frac{205\beta^2}{18(\beta-2)}(4m-1)^{2\beta-4} + \frac{820}{9}\beta^2\Big[\frac{1}{2}(4m-1)^{2\beta-5} + \frac{2\beta-5}{3}(4m-1)^{2\beta-6}\Big]$$

$$+ \left(16\Big[\frac{1}{2} - \frac{2\beta-3}{9} - \frac{3}{8(\beta-1)}\Big] + \frac{820}{81}\Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big]\right)\beta^2 3^{2\beta-3}.$$

Let us show that the expression in the last line is negative. Set

$$\psi(\beta) = 16\Big[\frac{1}{2} - \frac{2\beta-3}{9} - \frac{3}{8(\beta-1)}\Big] + \frac{820}{81}\Big[\frac{1}{2} - \frac{2\beta-5}{9} - \frac{3}{8(\beta-2)}\Big],$$

where $\beta \in (3, 3.5)$. Since

$$\psi'(\beta) = -\frac{4232}{729} + \frac{6}{(\beta-1)^2} + \frac{205}{54(\beta-2)^2}$$

is a decreasing function in the interval $(3, 3.5)$, therein we have

$$\psi'(\beta) < \psi'(3) = -\frac{4232}{729} + \frac{3}{2} + \frac{205}{54} < 0,$$

so $\psi(\beta)$ decreases in the interval $(3, 3.5)$, and therefore $\psi(\beta) \le \psi(3) < 0$.

Thus, we obtain

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + 8\beta^2(4m-1)^{2\beta-3} + \Big[\frac{16\beta^2(2\beta-3)}{3} + \frac{205\beta^2}{18(\beta-2)}\Big](4m-1)^{2\beta-4}$$

$$+ \frac{410}{9}\beta^2(4m-1)^{2\beta-5} + \frac{820\beta^2(2\beta-5)}{27}(4m-1)^{2\beta-6}$$

$$= \frac{2\beta^2}{\beta-1}(4m-1)^{2\beta-2} + \beta^2(4m-1)^{2\beta-3}D(\beta, m),$$

where

$$D(\beta,m) := 8 + \left(\frac{16(2\beta-3)}{3} + \frac{205}{18(\beta-2)}\right)\frac{1}{4m-1} + \frac{410}{9(4m-1)^2} + \frac{820(2\beta-5)}{27(4m-1)^3}\,.$$

An crude estimation reveals that $D(\beta,m) < 14$ for $m \geq 2$ and $\beta \in (3, 3.5)$. Therefore, for these β and m we have

$$|\mathbf{t}|^2 < \frac{2\beta^2}{\beta-1}\Big[1 + \frac{7(\beta-1)}{4m-1}\Big].$$

By (4.6), for $\beta \in (3, 3.5)$ and $m \geq 2$ we also have

$$4Q_m(\mathbf{t}) > \frac{\beta^2}{8(\beta+1)(\beta+2)}\Big[1 + \frac{4(\beta+1)}{(4m+1)}\Big]\,,$$

whence

$$4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta-1}{16(\beta+1)(\beta+2)}(4m-1)^4\,\frac{1+\frac{4(\beta+1)}{4m-1}}{1+\frac{7(\beta-1)}{4m-1}}.$$

Since $4(\beta+1) > 7(\beta-1)$ for $\beta \in (3, 3.5)$, the above inequality implies

$$4\frac{Q_m(\mathbf{t})}{|\mathbf{t}|^2} > \frac{\beta-1}{16(\beta+1)(\beta+2)}(4m-1)^4 = \frac{\beta-1}{(\beta+1)(\beta+2)}(n+1/2)^4\,.$$

Repeating our final argument from Section 3.2, we maximize the coefficient of $(n+1/2)^4$ with respect to β to obtain inequality (1.6) for $n = 2m-1$, $m \geq 2$.

The proof of Theorem 1.1 is complete, but (1.6) is shown for $n \geq 3$ only, due to our assumption $m \geq 2$. This restriction is easily removed, see the next section.

5. FINAL REMARKS

1. The proof of (1.6) in the cases $n = 2m$ and $n = 2m-1$ was accomplished under the assumption that $m \geq 2$. In fact, for $n \leq 8$ inequality (1.6) is verified with $f = P_n$ - the n-th Legendre polynomial. We have

$$\|P_n\| = \sqrt{\frac{2}{2n+1}},$$

and to evaluate $\|P_n'\|$, we exploit the fact that P_n is orthogonal to π_{n-1} and other well-known properties of P_n such as $P_n(1) = 1$, $P_n(-1) = (-1)^n$ and $P_n'(1) = n(n+1)/2$:

$$\begin{aligned}\|P_n'\|^2 &= \int_{-1}^{1} P_n'(x)\,dP_n(x) = P_n(1)P_n'(1) - P_n(-1)P_n'(-1) - \int_{-1}^{1} P_n(x)P_n''(x)dx \\ &= 2P_n'(1) = n(n+1)\,,\end{aligned}$$

i.e., $\|P_n'\| = n(n+1)$. The inequality (1.6) with $f = P_n$ is equivalent to

$$\sqrt{n(n+1)} > (\sqrt{3}-\sqrt{2})\frac{(n+1/2)^2}{\sqrt{n+1/2}}.$$

It is easy to see that the last inequality is true for $n \le 8$.

2. With more elaborate estimations of P_m, Q_m and $\mathbf{t}$ (including a Taylor series expansion up to ninth term), and using *MATHEMATICA*, inequality (1.6) could be improved to

$$\|f'\|_{1/2} \ge 0.317837(n+3/2)^2\,\|f\|_{1/2}.$$

We however decided to skip the derivation of this slightly better inequality.

3. In view of (1.2), the overestimation of the best constant in Markov's L_2 inequality, given by (1.5), is asymptotically equal to

$$\frac{0.325779}{1/\pi} = 1.02346\ldots.$$

On the other hand,

$$\frac{1/\pi}{\sqrt{3}-\sqrt{2}} = 1.00149\ldots,$$

which shows that the lower bound for the best constant in Markov's L_2 inequality, given by (1.6), is rather satisfactory.

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