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СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

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КНИГА 2 – ПРИЛОЖНА МАТЕМАТИКА И ИНФОРМАТИКА

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Aims and Scope. The *Annuaire* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer sciences. It is reviewed by *Zentralblatt für Mathematik*, *Mathematical Reviews* and the Russian *Referativnii Jurnal*. The *Annuaire* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the "St. Kliment Ohridski" University of Sofia, i.e., algebra, geometry and topology, analysis, mathematical logic, theory of approximations, numerical methods, computer sciences, classical, fluid and solid mechanics, and their fundamental applications.

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Professor Dr. Dimiter G. Skordev

On June 1, 1996, Professor Dimiter Skordev turned sixty. To mark this event, the Fourth Logical Biennial was dedicated to this anniversary. The Biennial took place in the "St. Kliment Ohridski" University House in Gjuletchitza from September 12 to September 15, 1996. Three invited lectures and seventeen contributed talks were presented during the meeting. The opening lecture of Dr. L. Ivanov was a review of the contributions of Prof. Skordev in the field of Algebraic Recursion Theory. This review, together with the scientific programme of the Colloquium and the full texts of most of the contributed talks, delivered there, are included in the present volume of the *Annuaire*.

Professor Skordev joined the Faculty of Mathematics at the "St. Kliment Ohridski" University of Sofia almost 40 years ago — in 1959, immediately after his graduation. He received his Ph. D. degree in 1967 with a thesis, devoted to some problems of the functional analysis. In 1986 he became a Doctor of Science in Mathematical Logic. Since 1972 he has been chairing the Department of Mathematical Logic at the Faculty of Mathematics and Informatics. The main research interests of Professor Skordev are in the areas of Logic and Computer Sciences. He is a founder of the Bulgarian School of Computability Theory. Eight graduate students have written their Ph. D. works under his supervision. During the years 1960–1996 Professor Skordev published more than seventy research papers and two monographs. For his achievements in Algebraic Recursion Theory he was awarded the Obreshkov Prize in 1981.

The Editorial Board of the *Annuaire* of the Faculty of Mathematics and Informatics, together with the numerous friends, colleagues, students (both former and present) of Professor Skordev use with great pleasure this occasion to wish him many years of fruitful work for the general benefit of the Bulgarian and the world mathematics.

The Editorial Board

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SKORDEV'S CONTRIBUTION TO RECURSION THEORY

Opening address at the Fourth Logical Biennial dedicated to the sixtieth anniversary of D. Skordev, September 12–14, 1996, Gyuletchitza

LYUBOMIR L. IVANOV

Ladies and gentlemen,

I would like first to thank the organizers of the Logical Biennial for the invitation to briefly share some reflections on the scientific deed of Professor Skordev. I feel greatly honoured by this invitation, indeed. At a jubilee like this it might be a permissible departure from the norm for a student to estimate his teacher's work rather than the opposite. I am not going to discuss so much specific results but rather concentrate on some methodological aspects of Skordev's contribution to Recursion Theory.

A cursory review of Skordev's past scientific activities reveals that a fairly major portion of his research and publications was devoted to Recursion Theory. Following his early papers [18, 19] on computable and mu-recursive operators and recursively complete arithmetical operations, and the subsequent ones [20, 21] on universal functions, Professor Skordev had over 30 publications on Recursion Theory during the period subsequent to 1974. Most of those publications were actually on Algebraic Recursion Theory, including the monographs [23, 24]. Likewise, it was for his research in Algebraic Recursion Theory that Professor Skordev got his Doctor of Sciences Degree and was awarded the *Nikola Obreshkov Prize*, this country's most prestigious award for achievements in the area of mathematics.

Professor Skordev set about his undertaking to generalize and axiomatize Classical Recursion Theory in the early seventies. That happened in the context of particularly interesting developments connected with a number of attempts to expand the scope of Recursion Theory. Probably, the first substantial advancement in that direction were the papers of Kleene [9, 10], affording a presentation of the hyperarithmetic theory via recursion in a second order object embodying quantification over natural numbers. Kleene's generalization was specifically important for not only initiating a new area in Recursion Theory known as Higher Recursion Theory, which was considerably advanced in the sequel, but also for setting a pattern and paving the way for other generalizations, especially those of Platek [16] and Moschovakis [12]. Research on computability over algebraic structures occurred as early as in the sixties, but the appropriate concepts of such computability were devised by Moschovakis [12] and by Friedman [5], the finite algorithmic procedures of the latter accounting for the lightface version. Incidentally, the concepts of prime computability and search computability of Moschovakis had a significant influence on the genesis of Skordev's generalization itself.

It is of interest to clarify the motives behind the various endeavours to generalize Recursion Theory beyond the classical study of effectively computable number-theoretic functions. For instance, recursion on infinite ordinals originated in Takeuti's papers [25, 26] with the necessity of introducing and studying such recursion, arising most naturally out of several areas of Mathematical Logic: Proof Theory, Model Theory and Set Theory. Such recursion was needed in order to deal with concrete problems such as 'effectivity' of proofs and 'arithmetical' undefinability in a generalized sense, as well as to achieve a more precise understanding of set structure, based on which to find solution to some problems already formulated in Set Theory.

Apart of particular problems originating in other areas, the study of effective computability in a more general context was put on the agenda also by certain general principles ensuing from Recursion Theory itself. These comprised the common aims of a mathematical generalization: to design abstract structures that are not only new and support a rich in content theory, but which also clarify Classical Recursion Theory and would possibly prove useful in application. More than that, it was hoped that if successful, such developments would eventually provide an axiomatical foundation of Recursion Theory.

The effort of some of the most brilliant logicians of the sixties and the seventies led to successful generalizations of Classical Recursion Theory in several directions, in the sense that suitable notions of effective computability were identified, providing the means for desired applications in the areas for which the relevant generalizations had been intended. The resulting Generalized Recursion Theory, initially regarded as technically forbidding but for a small community of devoted experts, later got much better and streamlined presentations. The progress in axiomatizing Recursion Theory, however, was less than satisfactory, at least until the invention of Skordev combinatory spaces.

Skordev's ideas of generalizing and axiomatizing Recursion Theory evolved around 1974 by way of extracting certain algorithmic properties of multiple-valued functions which turned out to permit axiomatical treatment. Professor Skordev successfully materialized his ideas by a combination of mathematical intuition and a refined technique based on an excellent command of the apparatus of Classical Recursion Theory and related domains of Logic. In the process, however, he not only achieved the aims he had set, but went far beyond his original goals, taking advantage of the rich opportunities offerred by the very approach invented by him. Actually, within few years Skordev laid the foundations and outlined the scope of a general theory notable for its deepness and elegance combined with an unusually wide scope of application. If the place of Skordev's theory in mathematics is to be described in few words, one might say that from a philosophical viewpoint Skordev's theory captured the nature of effective computability very much in the same way as Group Theory related to the concept of symmetry.

The hard core of Skordev's axiomatic approach was based on the algebraic structure of combinatory space. The principal characteristics of those spaces comprised: first, dealing with more general mathematical objects, members of a partially ordered semigroup rather than just functions or functionals; and second, choosing few basic or initial operations and setting forth their fundamental properties by means of a small number of elegant algebraically-styled axioms including a mu-induction principle. The basic operations of a combinatory space corresponded both intuitively and in a direct way to certain constructions to be found in structural programming or to certain patterns of combining computational devices, namely composition, branching or if-then-else statement, loop or while-do statement. Their axioms were first order axioms and also a first order mu-induction axiom sufficed for the bulk of the theory.

It is instructive to notice that in essence the basic operations of combinatory spaces occurred independently in other works, mainly in Computer Science, e.g. in the functional programming structures of Backus [1] and the schemes of Böhm and Jacopini [2], where, however, their mathematical potential had not been profitably exploited due to a number of reasons. The method of mu-induction, too, could be found in Computer Science; indeed the mu-induction axiom of combinatory space was a particular instance of Scott's mu-induction rule. A comparison shows that, due to the right choice of basic operations and initial elements, mu-induction in combinatory spaces was a powerful technical device, while the general Scott's rule was not, precisely because the system of Scott [17] lacked such suitable basic elements and operations.

Owing to the combination of aptly chosen basic operations and the mu-induction technique, a fairly non-trivial results were obtained in the general theory of combinatory spaces. Typical of that theory are assertions such as the Normal Form Theorem, the Enumeration Theorem and the First and Second Recursion Theorems, abstract Rice and Rogers Theorems. Needless to say, representation of the ordinary partial recursive functions was available too, hence the Classical Recursion Theory was not just a particular instance (i.e. model of the general theory) but at the same time was always imbedded as a minimum component. This was very much the case of Kleene-recursiveness in finite type objects, which was both a particular instance of relative recursiveness within a suitable combinatory space, and was also represented (and thus imbedded) in hierarchies of spaces and in a certain kind of more sophisticated spaces studied in Ivanov [6, 8].

On the other hand, the appropriate choice — or one might say design — of the basic operations and their axiomatically captured properties resulted in a surprising variety of models or particular spaces with essentially different semantics of various order. Apart of the standard case of single-valued and multiple-valued functions forming first order spaces, and monotonous functionals and second order relations forming second order spaces, these included also spaces of first and higher order related to certain concepts of everywhere-definedness and complexity of data processing, or comprising functions with finite types arguments, ordinal functions, probabilistic functions, fuzzy relations and the like. This abundance of spaces made it possible, first, to generalize via Skordev's approach already existing notions of effective computability, thereby paving the way for ample applications of the general theory. And second, it allowed to introduce notions of effective computability in areas which had not supported such notions before.

The approach initiated by Skordev provided a good illustration to another aspect of generalization by contributing to better understanding of Classical Recursion Theory and Generalized Recursion Theory. Certain phenomena which in Classical Recursion Theory were muted by 'too much arithmetic', i.e. by the availability of uninherently strong tools, had been known to emerge even in Generalized Recursion Theory. Such was, for instance, the distinction between lightface and boldface versions of the theory; also the understanding that Classical Recursion Theory traditionally employed operations which fitted better in arithmetic, but belong less naturally in Recursion Theory. Indeed, unsuccessful attempts to make use of minimization (or least number) operator in Generalized Recursion Theory had shown that operation to be inadequate for the purposes of prime or search computability or, as a matter of fact, recursion in higher types or recursion on ordinals. In contrast, the iteration operation of Skordev that superseded the least number operator was always suitable, because it was defined by its properties needed for the theory.

Of course, this universality of the axioms of combinatory space had most interesting semantical implications for its operations, resulting in semantic multiplicity even within a single higher order space. That applies particularly to multiplication and, as a consequence, to iteration operation. The semigroup multiplication would usually be a sort of composition, executed however in an opposite order, respectively in first and higher order spaces. The first order semantics of iteration was more or less of a loop nature, while in higher order spaces iteration at the higher level was nothing else but the least fixed point operator over the preceding level. Thus in the context of Algebraic Recursion Theory one could ascertain a sort of identity between seemingly completely different operations: the least fixed point operator was a particular instance of iteration which in turn was a particular instance of the least fixed point operator. Another similar phenomenon in the axiomatic theory was explicated by Skordev's pairing operation, which drew the lightface-boldface division line in the theory. Its first order semantics dealt with coding of pairs of data other than natural numbers, while its higher order semantics, as shown in Ivanov [8], took care of lambda abstraction.

An important aspect of any mathematical theory are not just its statements but their proofs as well. Here we see one of the unmistakable symptoms of a non-trivial generalization in the fact that quite a few of the proofs in Skordev's general theory were new rather than just modified proofs extracted from particular instances. More often than not those proofs tended to be streamlined and elegant on account of avoiding the temptation to solve problems 'by force' due to availability of excessive tools. At the same time, Skordev's axiomatic theory established common proofs and direct links between theorems belonging to different theories which otherwise seemed to be analogous, but actually proved to be particular instances of one and the same abstract proposition of Algebraic Recursion Theory; the situation earlier discussed for operations applies here to statements. For example, the First Recursion Theorem of Skordev generalized both the Kleene First Recursion Theorem and the Moschovakis Induction Completeness Theorem.

One of the popular and quite natural approaches to generalizing Recursion Theory was by way of employing inductive definability as a foundation, an idea stemming from Moschovakis [14] and supported by Feferman [3], too. The interesting try of Moschovakis [15] was further aimed at elevating the theory of inductive definability to a more abstract axiomatic level comparable with that of Skordev's setting for Recursion Theory. From the point of view of Recursion Theory however, Skordev's approach had the advantage of being not transplanted but intrinsic to that theory. Moreover, his approach made it possible for the inductive definability itself to be dealt with as a particular instance of relative recursiveness in a suitable combinatory space, i.e. within Recursion Theory, thus showing that Recursion Theory was just as fundamental as Inductive Definability Theory.

Returning to the strive for building axiomatic foundations of Recursion Theory, the attempts prior to Skordev's one might be regarded as partially successful, as far as their results and acceptance by the logician community were concerned. It was true that considerable effort had been allotted to the detailed elaboration of certain axiomatic approaches to Recursion Theory; typical example of that were the so-called computation theories of Moschovakis [13] studied extensively by Fenstad [4]. It turned out eventually that it was possible to embrace a number of notions of Generalized Recursion Theory in the framework of the computation theories and to reaffirm once again the relevant results from particular theories, leaving however the feeling of a persisting necessity to readapt the general setting, i.e. lack of true uniform general approach. Combined with the domination of modified proofs, that hinted at a certain creative potential deficiency.

Needless to say, there are still many open problems in Algebraic Recursion Theory. One of the major challenges at this stage appears to be the necessity to identify a reasonable concept of 'finite' in Algebraic Recursion Theory, needed, e.g., to deepen the study of abstract degrees initiated by Ivanov [7]. The importance of such a step in any generalization of Recursion Theory was stressed by Kreisel [11].

With its undoubted quality of good mathematics the approach of Skordev inspired natural interest among a number of other logicians as well as computer scientists. That resulted in dozens of publications, M.Sc. and Ph.D. theses by N. Georgieva, J. Zashev, O. Ignatov, L. Ivanov, R. Lukanova, S. Nikolova, E. Pazova, V. Petrov, A. Radenski, I. Soskov, M. Tabakov and others. Most interesting are the works of Zashev [27–29] in a related new area, Recursion Theory on partially ordered combinatory algebras and further generalizations at categorial level. During the last two decades Professor Skordev worked out a new portion of Recursion Theory which, with the contribution of his followers, evolved to form an original school in the Theory of Effective Computability. Apart of that, ideas and methods originating in Skordev's approach were applied to other areas of Recursion Theory and to Non-Classical Logic by A. Dichev, I. Soskov, A. Soskova, D. Vakarelov, G. Gargov, S. Passy, T. Tinchev and V. Goranko. As a matter of fact, a good deal of Bulgarian logicians have had a more than passing interest in this subject matter.

In conclusion, as a witness of these developments during the last twenty years or so, in which I was honoured to participate, I would like to take this opportunity to most cordially congratulate Professor Skordev as my teacher, on the occasion of his anniversary, and wish him best health and further twenty years of tireless and fruitful work.

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A SCHEMATIC PROOF OF STRONG NORMALIZATION FOR THE SYSTEMS OF THE λ-CUBE*

MILENA STEFANOVA

This paper describes a set-theoretical argument for proving *Strong Normalization* (SN) for the systems of the so-called λ-cube. The argument is relatively simple and, moreover, flexible. It can be adapted to extensions of the systems considered, such as additional sorts, inductive types or sub-types.

Keywords: typed lambda calculus, normalization, inductive types
1991/95 Math. Subject Classification: 03B15, 03B40

1. INTRODUCTION

In the recent years a lot of attention has been paid to the property of Strong Normalization for second- and higher-order dependent type systems. The number of the existing SN-proofs can be informally divided into two groups:

- *'syntactically-oriented' proofs*—proofs which are based on mixed syntactical and semantical methods ([6, 5, 3, 14, 15]), and
- *'semantically-oriented' proofs*—pure semantical proofs ([1, 7, 8, 9, 16]).

Most of these proofs make use of the idea of interpreting all typable terms as elements of sets of strongly-normalizing terms. Further, one can prove that a typable term belongs to the interpretation of its type and thus it is strongly-normalizing. However, semantically-oriented proofs make use of fully-compositional

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models, while syntactically-oriented ones are based on models which disregard some of the dependencies in a typable term. This has several consequences for both the kinds SN-proofs.

In general, the syntactically-oriented proofs are relatively simple but lacking flexibility and modularity. It is not easy and sometimes impossible to adapt them to various extensions such as additional type-constructions, more universes or sorts or sub-types. Any extension of the systems requires reconsideration and significant changes in such a SN-proof (see, for example, [14, 15, 13]). Furthermore, the relative simplicity of such a syntactically-oriented proof is usually lost after adapting it to a richer system.

Semantically-oriented proofs are based on operational or denotational semantics of the system under consideration (see [7, 1, 8]). These proofs seem to be more flexible than the syntactically-oriented ones in the sense that they can be easily adapted to various extensions of the system in question. Furthermore, they suggest generic methods for normalization proofs of PTSs (see [8, 9, 1, 16]).

However, in order to obtain compositional interpretations, most of them introduce very complicated structures, which are difficult to be mapped intuitively to the corresponding type system. Most of them use a realizability or categorical semantics (see [1, 8]) instead of a naive set-theoretic semantics as in the syntactically-oriented proofs.

The SN-proof considered in this paper combines advantages of syntactically- and semantically-oriented proofs: simplicity, flexibility and genericity. It can be classified as semantically-oriented. It is based on a naive set-theoretical semantics and as so is similar to the syntactically-oriented proofs. The principal difference with them is that type-dependencies are not disregarded in the interpretations, i.e., the interpretations are fully-compositional. This is achieved by defining simultaneously the interpretations of types and their elements.

The benefits one gets from this proof are in general the same as those in [1, 8, 9] — extendibility to more powerful systems. However, it is still simpler to interpret new type-constructors and reductions in the present set-theoretical setting. The flexibility of the proof presented is shown by extending it to systems with inductive types. We treat the case of *Natural Numbers* in the last section.

2. BARENDREGT'S CUBE

In this section precise definitions of the pure type systems in Barendregt's cube are given (see also [2]).

Definition 2.1 (PTS-definition). A *system of Barendregt's cube* λS is a triple $\lambda S = \langle \mathcal{T}, \mathcal{R}, \mathbf{R} \rangle$ such that:

- $\mathcal{T}$ is a set of *pseudoterms* defined by the abstract syntax

$$\mathcal{T} := Var \mid \{*, \Box\} \mid \mathcal{T}\mathcal{T} \mid \lambda Var{:}\mathcal{T}.\mathcal{T} \mid \Pi Var{:}\mathcal{T}.\mathcal{T},$$

where $Var = Var^* \bigcup Var^\square$ and Var^*, and $Var^\square$ are infinite enumerable disjoint sets of *object* and *constructor variables*, respectively. The object variables will be denoted by the small Latin letters x, y, z (with or without subscripts) and the constructor variables — by the small Greek letters α, β, γ. When we do not want to make a distinction between object and constructor variables, we will use the small Latin letters u, v, w.

The notions of β-reduction and β-conversion are defined on $\mathcal{T}$ by the contraction rule

$$(\lambda v{:}T_1.T_2)T_3 \to_\beta T_2[T_3/v];$$

• $\mathcal{R}$ is the set of *rules* of the system λS and consists of ordered pairs (s_1, s_2), such that $s_1, s_2 \in \{*, \square\}$ and $(*, *) \in \mathcal{R}$;

• **R** is the set of *derivation rules* of λS specified bellow.

(*axiom*) $\quad \vdash * : \square,$

(*var*) $$\frac{T \vdash s :}{v{:}T \vdash v : T}, \qquad s \in \{*, \square\}\ ,\ v \in Var^* \setminus FV(\Gamma),$$

(*weak*) $$\frac{T \vdash s : \quad M \vdash U :}{v{:}T \vdash M : U}, \qquad s \in \{*, \square\}\ ,\ v \in Var^* \setminus FV(\Gamma),$$

($\prod$) $$\frac{T \vdash s_1 : \quad v{:}T \vdash U : s_2}{\Pi v{:}T.U \vdash s_2 :}, \qquad (s_1, s_2) \in \mathcal{R}\ ,\ v \in Var^{s_1},$$

(λ) $$\frac{v{:}T \vdash M : U \quad \Pi v{:}T.U \vdash s :}{\lambda v{:}T.M \vdash \Pi v{:}T.U :}, \qquad s \in \{*, \square\},$$

(*app*) $$\frac{M \vdash \Pi v{:}T.U : \quad N \vdash T :}{MN \vdash U[N/v] :},$$

(*conv*) $$\frac{M \vdash T : \quad U \vdash s :}{M \vdash U :} \quad T =_\beta U, \qquad s \in \{*, \square\}.$$

The eight systems of the λ-cube are listed below (see Table 1) according to the sets of their rules. The set of (*typable*) *terms* of the system λS is defined by

$$\texttt{Terms} := \{T \in \mathcal{T} \mid \exists \Gamma, U\ (T \vdash U : \text{ or } U \vdash T :)\ \}.$$

It is convenient to divide the typable terms into subsets (see [2, 4]) in the following way:

$$\begin{aligned}
\texttt{Kind}(\lambda S) &:= \{A \in \mathcal{T} \mid \exists \Gamma\ (A \vdash \square :)\ \},\\
\texttt{Constr}(\lambda S) &:= \{C \in \mathcal{T} \mid \exists \Gamma, A\ (C \vdash A :) : \square\ \},\\
\texttt{Type}(\lambda S) &:= \{\sigma \in \mathcal{T} \mid \exists \Gamma\ (\sigma \vdash * :)\ \},\\
\texttt{Obj}(\lambda S) &:= \{t \in \mathcal{T} \mid \exists \Gamma, \sigma\ (t \vdash \sigma :) : *\ \}.
\end{aligned}$$

Table 1. The systems of the λ-cube

System	$(*,*)$	$(*,\Box)$	$(\Box,*)$	$(\Box,\Box)$
$\lambda\rightarrow$	×	-	-	-
λP	×	×	-	-
$\lambda 2$	×	-	×	-
$\lambda P2$	×	×	×	-
$\lambda\underline{\omega}$	×	-	-	×
$\lambda P\underline{\omega}$	×	×	-	×
$\lambda\omega$	×	-	×	×
λC	×	×	×	×

We will skip the subscript S in the above notations when it is clear which is the system under consideration.

3. INFORMAL OUTLINE OF THE PROOF

Let λS be a system of the λ-cube. Classification of typable terms of λS into objects, types, constructors and kinds determines a hierarchical structure which will be called *type hierarchy* in the sequel (see Fig. 1(a)). The type hierarchy has a fine structure — it contains two sub-hierarchies: the one of types and the other of kinds (see Fig. 1(b)).

Intuitively, every type is the set of objects of this type, and every kind is the set of constructors typable with it. All of these four levels are connected by **Type** $\subseteq$ **Constr**.

The typable terms of the system λS are interpreted in levels according to their level in the type hierarchy. In fact, the type hierarchy is mapped into a set-theoretical hierarchy, which will be called *λS-hierarchy*. The carrier, or the bottom level of the λS hierarchy is simply the set $\mathcal{T}$ of pseudo-terms.

Each system λS of the λ-cube is determined by its PTS-specification and its derivation rules. There are two sorts in each of the systems of the λ-cube: one of types ($*$) and another of kinds ($\Box$). Suppose that these sorts are interpreted by the set-universes $\mathbf{U}^*_S$ and $\mathbf{U}^\Box_S$. The conditions which $\mathbf{U}^*_S$ and $\mathbf{U}^\Box_S$ should satisfy are determined by the rest of the specification of λS, i.e., by its axioms and PTS-rules.

There is one axiom for each λS of the λ-cube, namely, $* : \Box$. This corresponds to the requirement $\mathbf{U}^*_S \in \mathbf{U}^\Box_S$. Further, suppose that (s_1, s_2) is a rule of λS. That means that one of the derivation rules of λS is

$$\frac{\Gamma \vdash T : s_1 \quad \Gamma, v{:}T \vdash U : s_2}{\Gamma \vdash \Pi v{:}T.U : s_2}.$$

This rule says informally that the sort s_2 is closed under dependent-product terms. The corresponding "meaning" in the model of this derivation rule would be that

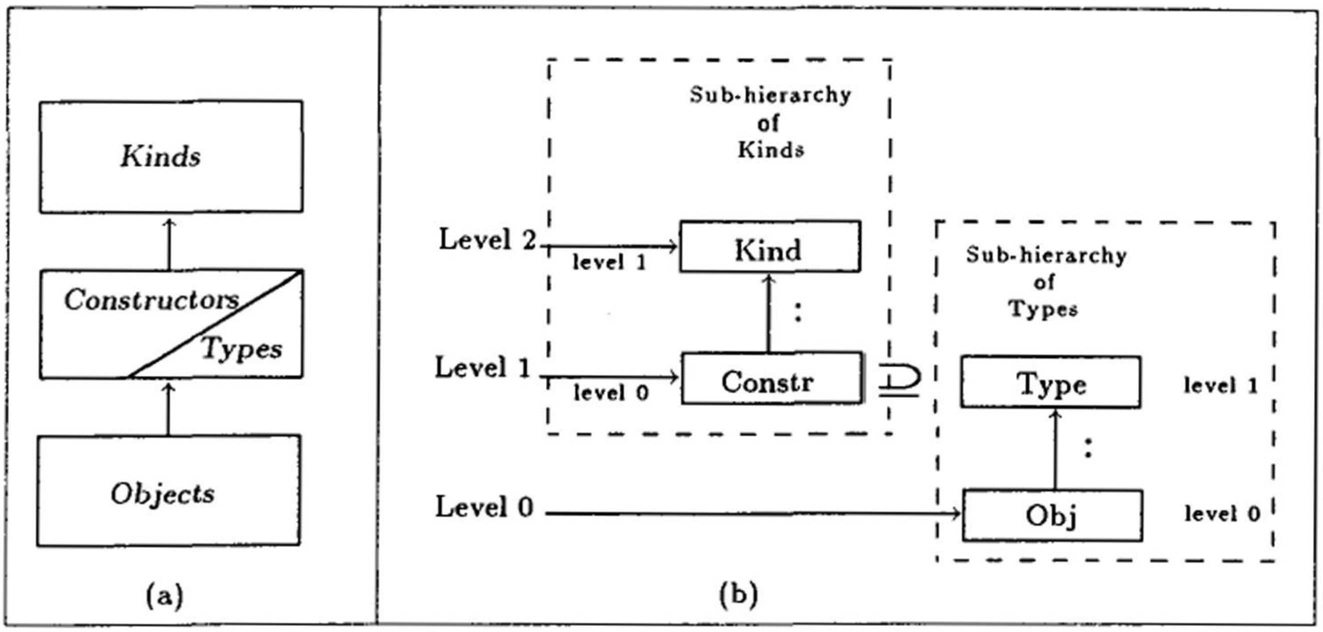

Fig. 1. The type-hierarchy

the universe $\mathbf{U}_S^{s_2}$ is closed under some suitable operation $\Pi_{s_2}^{s_1}$. More precisely, $\Pi_{s_2}^{s_1}$ takes as arguments a set $X \in \mathbf{U}_S^{s_1}$ and a family $\{Y_x\}_{x\in X}$ of sets of $\mathbf{U}_S^{s_2}$ and returns a result again in $\mathbf{U}_S^{s_2}$. In order to improve readability, we will denote the application $\Pi_{s_2}^{s_1}(X, \{Y_x\}_{x\in X})$ by

$$\Pi_{s_2}^{s_1} x \in X.Y_x.$$

Now, suppose we have found the collections $\mathbf{U}_S^*$ and $\mathbf{U}_S^\Box$ and the operations $\Pi_{s_2}^{s_1}$. The type-hierarchy of λS is mapped into a set-theoretical hierarchy (λS-hierarchy) through the interpretation functions $[\![\ ]\!]^2$, $[\![\ ]\!]^1$ and $[\![\ ]\!]^0$ (see Fig. 2). Note that it is not allowed to construct elements of a lower level of the λS-hierarchy by means of elements on higher levels (the crossed arrows in Fig. 2). The typing relation : between legal terms is mapped into the relation $\in$ on sets, so that if X is an element on level i of the λS-hierarchy, then there is an element Y on level $i+1$ such that $X \in Y$.

The typable terms of λS are interpreted as follows:

• Every kind A, $(\Gamma \vdash A : \Box)$ is mapped by the interpretation function $[\![\ ]\!]^2$ to an element of $\mathbf{U}_S^\Box$. Intuitively, dependent kinds are interpreted with the help of the operations $\Pi_\Box^*$ and $\Pi_\Box^\Box$, if respectively the rules $(*, \Box)$ and $(\Box, \Box)$ are present in the specification of λS;

• Every constructor C, $(\Gamma \vdash C : A : \Box)$ is mapped by the function $[\![\ ]\!]^1$ to an element of the collection

$$\bigcup \mathbf{U}_S^\Box = \{X \mid \exists Z(X \in Z \ \wedge \ Z \in \mathbf{U}_S^\Box)\}$$

in such a way that $[\![C]\!]^1 \in [\![A]\!]^2$. In particular, every type σ, $(\Gamma \vdash \sigma : *)$ is interpreted as an element of the universe $\mathbf{U}_S^*$. Impredicative types (i.e., types formed by the rule $(\Box, *)$) are interpreted with the help of the operation $\Pi_*^\Box$. Pure product types (rule $(*, *)$) are interpreted by using Π_*^*. Constructors formed by λ-abstrac-

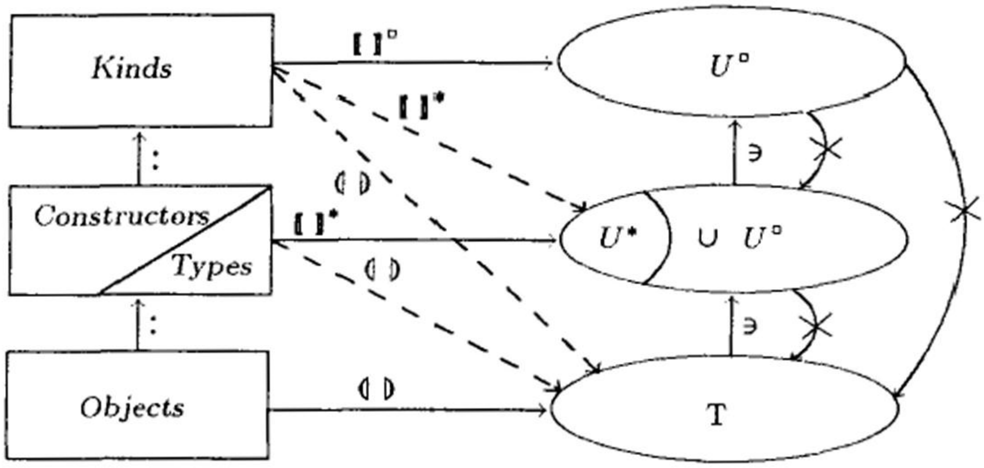

Fig. 2. λS-hierarchy

tion are interpreted as set-theoretical functions and applications of constructors to terms — as set-theoretical applications of functions to their arguments;

• Finally, every objects t, $(\Gamma \vdash t : \sigma : *)$ is mapped to a pseudo-term in a trivial way by using abstraction and application operations on pseudo-terms. For the interpretations of objects we have $\llbracket t \rrbracket^0 \in \llbracket \sigma \rrbracket^1$.

The interpretations $\llbracket\ \rrbracket^0$, $\llbracket\ \rrbracket^1$ and $\llbracket\ \rrbracket^2$ are compositional, i.e., the interpretation of a term is built up from the interpretations of its sub-terms by means of proper operations. For that reason all constructors are mapped into the set of pseudo-terms, in order to be able to interpret objects of the form $\lambda x{:}\sigma.t$ as $\lambda x{:}\llbracket \sigma \rrbracket^0.\llbracket t \rrbracket^0$. This implies that all kinds should be mapped into $\mathbf{U}^*_S$ in order to prove proper inclusion properties for the new interpretations of constructors. To summarize (see also Fig. 2):

— Every kind A is mapped into $\mathbf{U}^*_S$ by the function $\llbracket\ \rrbracket^1$ and into $\mathcal{T}$ by the function $\llbracket\ \rrbracket^0$.

— Every constructor C $(\Gamma \vdash C : A : \Box)$ is mapped into $\mathcal{T}$ by the function $\llbracket\ \rrbracket^0$ in a way that $\llbracket C \rrbracket^0 \in \llbracket A \rrbracket^1$.

Note that for the systems $\lambda \rightarrow$ and λP it is not necessary to interpret kinds as pseudo-terms, but we will do it in order to obtain more uniform treatment for all the systems of the cube. However, these two cases can be treated separately.

A final remark is that the interpretation functions $\llbracket\ \rrbracket^2$ and $\llbracket\ \rrbracket^1$ are constructed simultaneously on the structure of the typable terms. Due to this, it is possible to keep type dependencies in the interpretations.

4. THE FORMALIZATION

Any system λS of the λ-cube is interpreted into ZF-set theory. The typable terms are interpreted as sets and the typing relation ":" as the inclusion relation

$\in$ between sets. In particular, every object is mapped into a pseudo-term[1], every constructor — into a set-theoretical function, every type — into a set of pseudo-terms, and every kind — into a set of set-theoretical functions.

Note that the existence of this model does not contradict the result in [11], which simply says that in polymorphic λ-calculus one cannot interpret all abstraction-terms (i.e., terms of the form $\lambda v{:}T_1.T_2$) as set-theoretical functions and all application-terms (i.e., terms of the form T_1T_2) as function-applications. We interpret only the abstraction and application terms which are at the predicative level of λS as set-theoretical functions and function applications. The terms which are at the impredicative level are interpreted as λ-abstractions and applications of pseudo-terms.

4.1. PRELIMINARIES

As it has been mentioned before, the set $\mathcal{T}$ of pseudo-terms will be identified with the set ω. Thus an additional equality to the usual set-theoretical equality on ω will be used in order to represent β-equality. It will be denoted ambiguously by $=_\beta$.

Definition 4.1. Let a and b be sets. We say that a is *v-equal* to b (notation $a =_v b$) iff a and b are both pseudo-terms or are both sets and:

(i) $a =_\beta b$ in the case $a, b \in \mathcal{T}$;

(ii) $a = b$, otherwise.

Note that if $t_1 =_\beta t_2$ and $t_1 \in a$, it is not necessarily $t_2 \in a$. We extend the equality $=_v$ on sequences of elements of U in the following way.

Definition 4.2. Let α, γ be sequences of elements of U. Then $\alpha =_v \gamma$ iff $|\alpha| = |\gamma|$ and $\alpha(i) =_v \gamma(i)$ for all $i = 1, \ldots, |\alpha|$.

The set-theoretical functions which will be used in the model of λS form a restricted class of the functions in set theory. They are defined below.

Definition 4.3. Let a and b be sets such that $a, b \notin \mathcal{T}$. The set F is a *v-function* from a to b (notation $F : a \xrightarrow{v} b$) iff F consists of ordered pairs $\langle x, y\rangle$ such that

$$\forall x \in a\ \exists y_x \in b\ (\langle x, y_x\rangle \in F)$$

and

$$\forall x_1, x_2, y_1, y_2\ (x_1 =_v x_2 \wedge \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in F \Rightarrow y_1 =_v y_2).$$

[1] Note that the set of pseudo-terms can be identified with the set ω of standard sets representing natural numbers, so every pseudo-term t can be thought as a numeral $\underline{n}_t$ which is uniquely assigned to it. For convenience we will use in the sequel the set of pseudo-terms instead of the set ω of their images into ZF set theory.

Remarks 4.4.

• Let $F : a \xrightarrow{v} b$. The *v-function application* is defined in the usual way. That is, if $\langle x, y\rangle \in F$, then $\mathsf{App}_v(F, x) =_v y$. For simplicity the v-function application will be written as $F(x)$, since the usual function-application will not be used in the sequel. Thus, if $F(x)$ is defined, that means that F is a v-function, i.e., it respects β-equality.

• Let $F : a \xrightarrow{v} b$ and $F(x) =_v y_x$ for $x \in a$. This F will be denoted by $\lambda_v x{\in}a.y_x$.

Lemma 4.5. *The v-functions $F_i : a_i \xrightarrow{v} b_i$, $i = 1, 2$, are v-equal, e.g. $F_1 =_v F_2$ iff $a_1 =_v a_2$ and for all $x \in a_1$, $F_1(x) =_v F_2(x)$.*

Definition 4.6. Let a be a set such that $a \notin \mathcal{T}$, and let $\{b_x\}_{x\in a}$ be a family of sets such that $b_x \notin \mathcal{T}$ for all $x \in a$:

(i) The set-theoretical v-dependent product is defined as

$$\Pi_v x{\in}a.b_x := \{F : a \xrightarrow{v} \bigcup_{x\in a} b_x \mid \forall x \in a\ (F(x) \in b_x)\}.$$

(ii) The dependent sum is defined as

$$\Sigma_v x{\in}a.b_x := \{\langle m, n\rangle \mid m \in a, n \in b_m\}.$$

Note that if $F \in \Pi_v x{\in}a.b_x$ and $x_1 =_v x_2$, then $F(x_1) =_v F(x_2)$.

Lemma 4.7. *Let $a, a' \notin \mathcal{T}$ and let $\{b_x\}_{x\in a}$, $\{b'_x\}_{x\in a'}$ be families of sets such that $b_x, b'_x \notin \mathcal{T}$. Then*

$$\Pi_v x{\in}a.b_x =_\beta \Pi_v x{\in}a'.b'_x \iff a =_\beta a' \& \forall x \in a\ (b_x =_\beta b'_x).$$

The hierarchy of sets into which the typable terms (kinds, constructors, types and objects) will be mapped is specified as follows:

Definition 4.8. For every ordinal number $\alpha \in \mathtt{Or}$ the sets $\mathtt{V}_\alpha(\mathcal{T})$ are defined in the following way:

(i) $\mathtt{V}_0(\mathcal{T}) = \mathcal{T}$;

(ii) $\mathtt{V}_{\alpha+1}(\mathcal{T}) = \mathtt{V}_\alpha(\mathcal{T}) \bigcup \mathcal{P}(\mathtt{V}_\alpha(\mathcal{T}))$;

(iii) $\mathtt{V}_\alpha(\mathcal{T}) = \bigcup_{\beta<\alpha} \mathtt{V}_\beta(\mathcal{T})$ if α is a limit ordinal.

Definition 4.9. Let $\circ$ be an operation which takes as arguments a set and a family of sets, indexed by this set, and gives as a result a set:

(i) The set $\mathcal{A} \subseteq \mathtt{V}_\alpha(\mathcal{T})$ is *α-closed* under the operation $\circ$ if for any set $a \in \mathcal{A}$ and any family $\{b_x\}_{x\in a}$ of sets from $\mathcal{A}$, the set $\circ(a, \{b_x\}_{x\in a})$ belongs to $\mathcal{A}$;

(ii) The set $\mathcal{A} \subseteq \mathtt{V}_\alpha(\mathcal{T})$ is *weakly-α-closed* under the operation $\circ$ if there exists an ordinal β such that $\beta < (\alpha - 1)$ and moreover, for all ordinals γ such that $\beta \leq \gamma < \alpha$ and for any family of sets $\{b_x\}_{x \in a}$, for which $b_x \in \mathcal{A} \cap \mathtt{V}_\gamma(\mathcal{T})$, the set $\circ(a, \{b_x\}_{x \in a})$ belongs to $\mathcal{A}$.

4.2. THE UNIVERSE $\mathbf{U}_S^*$ AND $\mathbf{U}_S^\square$

The interpretation $\mathbf{U}_S^\square$ of the predicative universe $\square$ is chosen to be the set

$$\mathtt{V}_\omega(\mathcal{T}) \setminus \mathtt{V}_1(\mathcal{T}).$$

The next lemma shows that it is weakly-ω-closed under the set-theoretical dependent product Π_v (defined in Definition 4.6).

Lemma 4.10. *The universe $\mathbf{U}_S^\square$ is weakly-ω-closed under the operation Π_v.*

Proof. We have to find an ordinal $\beta < \omega$ such that for all $n \geq \beta$ it holds that

$$\Pi_v x{\in}a.b_x \in \mathbf{U}_S^\square$$

if $a \in \mathbf{U}_S^\square$ and $\{b_x\}_{x \in a}$ is a family of sets such that $b_x \in \mathbf{U}_S^\square \cap \mathtt{V}_n(\mathcal{T})$ for any $x \in a$. Note that

$$\mathbf{U}_S^\square \bigcup \mathtt{V}_n(\mathcal{T}) =_v \mathtt{V}_n(\mathcal{T}) \setminus \mathtt{V}_1(\mathcal{T}).$$

Now, let us choose $\beta = 2$. Let $a \in \mathbf{U}_S^\square$ and let $n \geq 2$. From the definition of $\mathbf{U}_S^\square$ it follows that there exists a natural number $m \geq 2$ such that $a \in \mathtt{V}_m(\mathcal{T})$.

The elements of the set $\Pi_v x{\in}a.b_x$ are v-functions and thus sets of pairs of the form (x, y), where $x \in a$ and $y \in b_x$. By definition, a pair (x, y) is a set $\{x, \{x, y\}\}$. Thus, if $x \in a \in \mathtt{V}_m(\mathcal{T})$ and $y \in b_x \in \mathtt{V}_n(\mathcal{T})$, then $(x, y) \in \mathtt{V}_{\max(m-1,n-1)+2}(\mathcal{T})$. Consequently,

$$\Pi_v x{\in}a.b_x \in \mathtt{V}_{\max(m-1,n-1)+4}(\mathcal{T}),$$

and hence $\Pi_v x{\in}a.b_x \in \mathbf{U}_S^\square$ since obviously

$$\Pi_v x{\in}a.b_x \notin \mathtt{V}_1(\mathcal{T}).$$ ■

It is convenient to specify the interpretation $\mathbf{U}_S^*$ of the impredicative universe $*$ to be the collection SAT_β of β-saturated sets. SAT_β is closed under arbitrary non-empty intersections and under an operation of dependent product defined on the set $\mathcal{T}$ of pseudo-terms.

Let $\mathsf{SN}_\beta \subset \mathcal{T}$ be the set of pseudo-terms which are strongly normalizing under β-reduction.

Definition 4.11. The set $\mathcal{B}_\beta$ of *β-base terms* is defined as the smallest set satisfying the following conditions:

(i.) $Var^* \bigcup Var^\square \subset \mathcal{B}_\beta$;

(ii.) If $M \in \mathcal{B}_\beta$ and $N \in \mathsf{SN}_\beta$, then $MN \in \mathcal{B}_\beta$.

Definition 4.12. The *β-key-reduction* is the relation $\xrightarrow{k}_\beta$ defined by the contraction schemes for β-reduction and the following compatibility condition:

$$M_1 \xrightarrow{k}_\beta M_2 \Longrightarrow M_1 N \xrightarrow{k}_\beta M_2 N.$$

Lemma 4.13. *If the proper sub-terms of a term M are β-strongly normalizing,* $M \xrightarrow{k}_\beta N$ *and* $N \in \mathsf{SN}_\beta$*, then* $M \in \mathsf{SN}_\beta$.

Definition 4.14. The set $X \subset \mathcal{T}$ of pseudo-terms is called *β-saturated* if the following conditions hold:

(i.) $X \subseteq \mathsf{SN}_\beta$;

(ii.) $\mathcal{B}_\beta \subset X$;

(iii.) If $M \xrightarrow{k}_\beta N$, $N \in X$ and the proper subterms of M are β-strongly normalizing, then $M \in X$.

The collection of all β-saturated sets will be denoted by SAT_β. Thus one chooses $\mathbf{U}^* \equiv \mathsf{SAT}_\beta$.

Definition 4.15. The operation Π^*_* of dependent product on $\mathcal{T}$ takes as arguments a set $X \subseteq \mathcal{T}$ and a function $F : X \to \mathcal{P}(\mathcal{T})$ and is defined as follows:

$$\Pi^*_* m \in X.F(m) := \{t \in \mathcal{T} \mid \forall q \in X \ (tq \in F(q))\ \}.$$

The operation $\Pi^\square_*$ is defined as intersection of sets. Namely,

$$\Pi^*_* x \in X.Y_x = \bigcap_{x \in X} Y_x.$$

Note that $X \neq \emptyset$ for any $X \in \mathbf{U}^\square_S$. The next lemma shows that the universe $\mathbf{U}^*_S$, e.g. SAT_β, satisfies the necessary closureness properties.

Lemma 4.16. *The set* SAT_β *is closed under* Π^*_* *and under arbitrary non-empty intersections.*

4.3. THE INTERPRETATIONS

In this subsection the interpretations $\llbracket\ \rrbracket^2$, $\llbracket\ \rrbracket^1$ and $\llbracket\ \rrbracket^0$ are defined (see Fig. 2). For that purpose we need two valuations

$$\xi : Var^\square \to \bigcup \mathbf{U}^\square_S \quad \text{and} \quad \rho : Var^* \bigcup Var^\square_* \to \mathcal{T}$$

to interpret all constructor variables at the middle level of the λS-hierarchy (see Fig. 2) and all constructor and object variables at the level of atoms. The interpretation $\llbracket\ \rrbracket^0_{\xi,\rho}$ is obtained simply by applying the substitution ρ on its argument. Thus it does not depend on the assignment ξ and for this reason it will be written as $\llbracket\ \rrbracket^0_\rho$.

The other two interpretations are constructed simultaneously by induction on the structure of typable terms of the system λS.

Definition 4.17. Let $\rho : Var^* \bigcup Var^{\Box} \to \mathcal{T}$ be a valuation of constructor and object variables. The *atom-interpretation*,

$$[\![\ ]\!]^0_\rho : \{\Box\} \bigcup \mathtt{Kind} \bigcup \mathtt{Constr} \bigcup \mathtt{Obj} \to \mathcal{T},$$

is defined as $[\![T]\!]^0_\rho = \rho(T)$, where $\rho(T)$ is the term obtained from T by applying the substitution ρ to T.

Definition 4.18. Let $\mathcal{R}_{\lambda S}$ be the set of PTS-rules of the system λS. Let $\rho : Var^* \bigcup Var^{\Box} \to \mathcal{T}$ and $\xi : Var^{\Box} \to \bigcup \mathbf{U}^{\Box}_S$ be valuations. The *constructor-interpretation* of constructor and kinds,

$$[\![\ ]\!]^1_{\xi,\rho} : \{\Box\} \bigcup \mathtt{Kind} \bigcup \mathtt{Constr} \to \bigcup \mathbf{U}^{\Box}_S,$$

and the *kind-interpretation* of kinds,

$$[\![\ ]\!]^2_{\xi,\rho} : \{\Box\} \bigcup \mathtt{Kind} \to \mathbf{U}^{\Box}_S,$$

are defined simultaneously by induction on the structure of the typable terms as follows:

Sorts of λS:

$$[\![*]\!]^2_{\xi,\rho} = [\![\Box]\!]^2_{\xi,\rho} = \mathsf{SAT}_\beta,$$

$$[\![*]\!]^1_{\xi,\rho} = [\![\Box]\!]^1_{\xi,\rho} = \mathsf{SN}_\beta.$$

Kinds of λS:

- $(\Box, \Box) \in \mathcal{R}_{\lambda S}$ ($A, B \in \mathtt{Kind}(\lambda S)$, $\alpha \in Var^{\Box}$).

$$[\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho} \simeq \Pi_v a \in [\![A]\!]^2_{\xi,\rho}.\Pi_v m \in [\![A]\!]^1_{\xi,\rho}.[\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=m]},$$

$$[\![\Pi\alpha{:}A.B]\!]^1_{\xi,\rho} \simeq \bigcap_{a\in[\![A]\!]^2_{\xi,\rho}} \Pi^*_* m \in [\![A]\!]^1_{\xi,\rho}.[\![B]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}.$$

- $(*, \Box) \in \mathcal{R}_{\lambda S}$ ($A \in \mathtt{Kind}(\lambda S)$, $\sigma \in \mathtt{Type}(\lambda S)$, $x \in Var^*$).

$$[\![\Pi x{:}\sigma.A]\!]^2_{\xi,\rho} \simeq \Pi_v m \in [\![\sigma]\!]^1_{\xi,\rho}.[\![A]\!]^2_{\xi,\rho[x:=m]},$$

$$[\![\Pi x{:}\sigma.A]\!]^1_{\xi,\rho} \simeq \Pi^*_* m \in [\![\sigma]\!]^1_{\xi,\rho}.[\![A]\!]^1_{\xi,\rho[x:=m]}.$$

Constructors of λS:

- Constructor-variables ($\alpha \in Var^{\Box}$).

$$[\![\alpha]\!]^1_{\xi,\rho} = \xi(\alpha).$$

- $(\square,\square) \in \mathcal{R}_{\lambda S}$ ($P, Q \in$ `Constr`(λS), $A \in$ `Kind`(λS), $\alpha \in Var^{\square}$).

$$\begin{aligned} [\![PQ]\!]^1_{\xi,\rho} &\simeq [\![P]\!]^1_{\xi,\rho}([\![Q]\!]^1_{\xi,\rho}, [\![Q]\!]^0_{\rho}), \\ [\![\lambda\alpha{:}A.P]\!]^1_{\xi,\rho} &\simeq \lambda_v a \in [\![A]\!]^2_{\xi,\rho}.\lambda_v m \in [\![A]\!]^1_{\xi,\rho}.[\![P]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}. \end{aligned}$$

- $(*,\square) \in \mathcal{R}_{\lambda S}$ ($P \in$ `Constr`(λS), $t \in$ `Obj`(λS), $\sigma \in$ `Type`(λS), $x \in Var^{*}$).

$$\begin{aligned} [\![Pt]\!]^1_{\xi,\rho} &\simeq [\![P]\!]^1_{\xi,\rho}([\![t]\!]^0_{\rho}), \\ [\![\lambda x{:}\sigma.P]\!]^1_{\xi,\rho} &\simeq \lambda_v m \in [\![\sigma]\!]^1_{\xi,\rho}.[\![P]\!]^1_{\xi,\rho[x:=m]}. \end{aligned}$$

- $(\square,*) \in \mathcal{R}_{\lambda S}$ ($A \in$ `Kind`(λS), $\sigma \in$ `Type`(λS), $\alpha \in Var^{\square}$).

$$[\![\Pi\alpha{:}A.\sigma]\!]^1_{\xi,\rho} \simeq \bigcap_{a \in [\![A]\!]^2_{\xi,\rho}} \Pi^*_* m \in [\![A]\!]^1_{\xi,\rho}.[\![\sigma]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}.$$

- $(*,*) \in \mathcal{R}_{\lambda S}$ ($\sigma, \tau \in$ `Type`(λS), $x \in Var^{*}$).

$$[\![\Pi x{:}\sigma.\tau]\!]^1_{\xi,\rho} \simeq \Pi^*_* m \in [\![\sigma]\!]^1_{\xi,\rho}.[\![\tau]\!]^1_{\xi,\rho[x:=m]}.$$

Remark 4.19. The equality $\simeq$ is the usual Kleene equality as the interpretations $[\![\ ]\!]^2_{\xi,\rho}$ and $[\![\ ]\!]^1_{\xi,\rho}$ may not be always defined.

The next lemma says that the atom-interpretations of β-equal terms of λS are also β-equal.

Lemma 4.20. *If $t_1, t_2 \in$ `Term`(S) and $t_1 =_\beta t_2$, then $[\![t_1]\!]^0_\rho =_\beta [\![t_2]\!]^0_\rho$.*

For the interpretations $[\![\ ]\!]^k_{\xi,\rho}$, $k = 1, 2$, the substitution property, which is stated in the next lemma, holds.

Lemma 4.21 (Substitution). *If $C \in$ `Constr`(λS), $t \in$ `Obj`(λS), $M, M[C/\alpha]$, $M[t/x] \in$ `Kind`$(\lambda S) \cap$ `Constr`(λS), then*

$$\begin{aligned} [\![M[C/\alpha]]\!]^k_{\xi,\rho} &\simeq [\![M]\!]^k_{\xi[\alpha:=[\![C]\!]^1_{\xi,\rho}],\rho[\alpha:=[\![C]\!]^0_\rho]}, \\ [\![M[t/x]]\!]^k_{\xi,\rho} &\simeq [\![M]\!]^k_{\xi,\rho[x:=[\![t]\!]^0_\rho]} \end{aligned}$$

for $k = 1, 2$.

Lemma 4.22. *Let $M_1, M_2 \in$ `Constr`$(\lambda S) \cup$ `Kind`(λS). If $M_1 \to_\beta M_2$ and $[\![M_1]\!]^k_{\xi,\rho}$ is defined, then $[\![M_1]\!]^k_{\xi,\rho} =_v [\![M_1]\!]^k_{\xi,\rho}$ for $k = 1, 2$.*

Proof. Let $M_1 \to_\beta M_2$. The following cases are treated:

- Let $(\lambda\alpha{:}A.C)Q \to_\beta C[\alpha := Q]$ for $A \in$ `Kind`(λS) and $C, Q, (\lambda\alpha{:}A.C)Q \in$ `Constr`(λS). Assume that the interpretation $[\![(\lambda\alpha{:}A.C)Q]\!]^1_{\xi,\rho}$ is defined, i.e., it is equal to $[\![C]\!]^1_{\xi[\alpha:=[\![Q]\!]^1_{\xi,\rho}],\rho[\alpha:=[\![Q]\!]^0_\rho]}$ (see Definition 4.18). Thus, from Lemma 4.21, it follows that the constructor-interpretation of $C[\alpha := Q]$ is defined and moreover, it is equal to the constructor-interpretation of $(\lambda\alpha{:}A.C)Q$.

• Let $Ct_1 \to_\beta Ct_2$, where $C \in \texttt{Constr}(\lambda S)$ and $t_1, t_2 \in \texttt{Obj}(\lambda S)$, and let $[\![Ct_1]\!]^1_{\xi,\rho}$ be well-defined. That means that $[\![C]\!]^1_{\xi,\rho}$ is a v-function (see Remarks 4.4) and since $[\![t_1]\!]^0_\rho =_\beta [\![t_2]\!]^0_\rho$, it follows that $[\![Ct_1]\!]^1_{\xi,\rho} =_\beta [\![Ct_2]\!]^1_{\xi,\rho}$.

The rest of the cases are trivial and their proof is similar. We have proved that if $M_1 \to_\beta M_2$ and $[\![M_1]\!]^k_{\xi,\rho}$ is defined for $k = 1, 2$, then the interpretation $[\![M_2]\!]^k_{\xi,\rho}$ is also defined and v-equal to $[\![M_1]\!]^k_{\xi,\rho}$. ■

Definition 4.23. The object interpretations ρ_1 and ρ_2 are *compatible* under the β-equality if for all $v \in Var$, $\rho_1(v) =_\beta \rho_2(v)$.

The proofs of the next two lemmas are trivial by induction on the structure of typable terms.

Lemma 4.24. *If $M \in \texttt{Term}(\lambda S)$ and ρ_1 and ρ_2 are compatible object-valuations, then*

$$[\![M]\!]^0_{\rho_1} =_\beta [\![M]\!]^0_{\rho_2}.$$

Lemma 4.25. *Let $M \in \texttt{Constr}(\lambda S) \cup \texttt{Kind}(\lambda S)$ and let ρ_1 and ρ_2 be compatible object-interpretations. If the interpretations $[\![M]\!]^k_{\xi,\rho_1}$ and $[\![M]\!]^k_{\xi,\rho_2}$ are defined ($k =$ 1, 2), then*

$$[\![M]\!]^k_{\xi,\rho_1} =_v [\![M]\!]^k_{\xi,\rho_1}.$$

We have mentioned earlier that the interpretations of the typable terms should satisfy some inclusion properties (Section 3). For that purpose, we introduce the notion of satisfaction of a context Γ. In such a way we restrict the possible valuations, so that the interpretations $[\![\ ]\!]^1_{\xi,\rho}$ and $[\![\ ]\!]^2_{\xi,\rho}$ are defined.

Definition 4.26. The valuations

$$\xi : Var^{\square} \to \bigcup \mathbf{U}^{\square}_S \quad \text{and} \quad \rho : Var^{*} \bigcup Var^{\square} \to T$$

satisfy the context Γ (notation $\xi, \rho \models \Gamma$) iff:

(i) for every constructor variable α and kind A, such that $(\alpha : A) \in \Gamma$,

$$\xi(\alpha) \in [\![A]\!]^2_{\xi,\rho} \quad \text{and} \quad \rho(\alpha) \in [\![A]\!]^1_{\xi,\rho}, \quad \text{and}$$

(ii) for every object variable x and type σ, such that $(x : \sigma) \in \Gamma$,

$$\rho(x) \in [\![\sigma]\!]^1_{\xi,\rho}.$$

Definition 4.27. The (legal) context Γ *models that the (typable) term M has a type T* (notation $\Gamma \models M : T$) iff:

(i) if $M \in \texttt{Obj}(\lambda S)$, then for all $\xi, \rho \models \Gamma$,

$$[\![M]\!]^0_\rho \in [\![T]\!]^1_{\xi,\rho};$$

(ii) if $M \in \texttt{Constr}(\lambda S)$, then for all $\xi, \rho \models \Gamma$,

$$[\![M]\!]^0_\rho \in [\![T]\!]^1_{\xi,\rho} \text{ and } [\![M]\!]^1_{\xi,\rho} \in [\![T]\!]^2_{\xi,\rho};$$

(iii) if $M \in \texttt{Kind}(\lambda S)$, then there exists a natural number $n \geq 2$, such that for all $\xi, \rho \models \Gamma$,

$$[\![M]\!]^2_{\xi,\rho} \in \texttt{V}_n(T), \ [\![M]\!]^1_{\xi,\rho} \in \mathsf{SAT}_\beta \text{ and } [\![M]\!]^0_\rho \in \mathsf{SN}_\beta.$$

Theorem 4.28 (Soundness). *If* $\Gamma \vdash M : T$, *then* $\Gamma \models M : T$.

Proof. The proof of 1–3 is done by induction on derivations. The following cases are treated:

• The (λ) rule. The case when the bound variable is a constructor variable and the term formed by the λ-rule is a constructor (i.e., $(\Box, \Box) \in \mathcal{R}_{\lambda S}$) is considered. The proof for all other cases of the λ-rule is done in a similar way.

$$\frac{\Gamma, \alpha{:}A \vdash P : B \quad \Gamma \vdash \Pi\alpha{:}A.B : \Box}{\Gamma \vdash \lambda\alpha{:}A.P : \Pi\alpha{:}A.B}$$

Let $\xi, \rho \models \Gamma$. We have to prove that

$$[\![\lambda\alpha{:}A.P]\!]^1_{\xi,\rho} \in [\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho} \text{ and } [\![\lambda\alpha{:}A.P]\!]^0_\rho \in [\![\Pi\alpha{:}A.B]\!]^1_{\xi,\rho}.$$

From the induction hypothesis $[\![A]\!]^2_{\xi,\rho}$ is an element of $\mathbf{U}^\Box_S$, $[\![A]\!]^1_{\xi,\rho}$ is an element of SAT_β, and for all $a \in [\![A]\!]^2_{\xi,\rho}$ and $m \in [\![A]\!]^1_{\xi,\rho}$

$$[\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=m]} \in \mathbf{U}^\Box_S.$$

Furthermore, for $a \in [\![A]\!]^2_{\xi,\rho}$ and $m \in [\![A]\!]^1_{\xi,\rho}$

$$[\![P]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]} \in [\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=m]}. \tag{1}$$

From Lemma 4.25 it follows

$$m_1 =_\beta m_2 \Longrightarrow [\![P]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m_1]} =_v [\![P]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m_2]}$$

for all $a \in [\![A]\!]^2_{\xi,\rho}$ and $m_1, m_2 \in [\![A]\!]^1_{\xi,\rho}$, and hence the function

$$\lambda_v a{\in}[\![A]\!]^2_{\xi,\rho}.\lambda_v m{\in}[\![A]\!]^1_{\xi,\rho}.[\![P]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}$$

is indeed a v-function. Thus, from (1) and by the definition of $[\![\lambda\alpha{:}A.P]\!]^1_{\xi,\rho}$ it follows

$$[\![\lambda\alpha{:}A.P]\!]^1_{\xi,\rho} \in [\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho}.$$

To prove $[\![\lambda\alpha{:}A.P]\!]^0_\rho \in [\![\Pi\alpha{:}A.B]\!]^1_{\xi,\rho}$, we have to prove that for any $a \in [\![A]\!]^2_{\xi,\rho}$ and $m \in [\![A]\!]^1_{\xi,\rho}$

$$[\![\lambda\alpha{:}A.P]\!]^0_\rho m \in [\![B]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}.$$

The term $[\![\lambda\alpha{:}A.P]\!]^0_\rho m$ key-reduces to the term $[\![P]\!]^0_{\rho[\alpha:=m]}$, which by the induction hypothesis belongs to the saturated set $[\![B]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}$ and hence the term $[\![\lambda\alpha{:}A.P]\!]^0_\rho m$ itself belongs to $[\![B]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=m]}$.

• The (*app*) rule. Again only the case when the applied terms are constructors is considered (i.e., $(\square,\square) \in \mathcal{R}_{\lambda S}$). The proof of the other cases is done in the same or even simpler way.

$$\frac{\Gamma \vdash P : \Pi\alpha{:}A.B : \square \quad \Gamma \vdash Q : A : \square}{\Gamma \vdash PQ : B[Q/\alpha]}$$

Let $\xi,\rho \models \Gamma$. From the induction hypothesis it follows

$$[\![P]\!]^1_{\xi,\rho} \in [\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho}, \quad [\![Q]\!]^1_{\xi,\rho} \in [\![A]\!]^2_{\xi,\rho} \quad \text{and} \quad [\![Q]\!]^0_\rho \in [\![A]\!]^1_{\xi,\rho}.$$

Thus,

$$[\![PQ]\!]^1_{\xi,\rho} \in [\![B]\!]^2_{\xi[\alpha:=[\![Q]\!]^1_{\xi,\rho}],\rho[\alpha:=[\![Q]\!]^0_\rho]}.$$

From the Substitution Lemma (see Lemma 4.21) it follows

$$[\![PQ]\!]^1_{\xi,\rho} \in [\![B[\alpha := Q]]\!]^2_{\xi,\rho}.$$

• The ($\prod$) rule. The case of $(\square,\square) \in \mathcal{R}_{\lambda S}$ is considered again. The proof in the case $(*,\square)$ is similar and the proofs in the cases $(\square,*)$ or $(*,*)$ follow directly form the closure properties of SAT_β (see Lemma 4.16). Let now the last rule in the derivation of $\Gamma \vdash M : T$ be

$$\frac{\Gamma \vdash A : \square \quad \Gamma, \alpha{:}A \vdash B : \square}{\Gamma \vdash \Pi\alpha{:}A.B : \square}$$

Let $\xi,\rho \models \Gamma$. From the induction hypothesis it follows that there exist $m,n \geq 2$ such that

$$[\![A]\!]^2_{\xi,\rho} \in \mathtt{V}_m(\mathcal{T}), \quad [\![A]\!]^1_{\xi,\rho} \in \mathsf{SAT}_\beta, \quad \text{and} \quad [\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=m]} \in \mathtt{V}_n(\mathcal{T})$$

for any $a \in [\![A]\!]^2_{\xi,\rho}$ and $b \in [\![A]\!]^1_{\xi,\rho}$. Thus $[\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho}$ is defined and equal to the set

$$\Pi_v a{\in}[\![A]\!]^2_{\xi,\rho}.\Pi_v b{\in}[\![A]\!]^1_{\xi,\rho}.[\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=b]}. \tag{2}$$

The elements of this set are v-functions which consist of triples of the form (a,b,c) with $a \in [\![A]\!]^2_{\xi,\rho}$, $b \in [\![A]\!]^1_{\xi,\rho}$ and $c \in [\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=b]}$. Thus for any such triple (a,b,c) it follows $a \in \mathtt{V}_{m-1}(\mathcal{T})$, $b \in \mathtt{V}_0(\mathcal{T})$ and $c \in \mathtt{V}_{n-1}(\mathcal{T})$. Note that by definition

$$(a,b,c) \equiv \{a,\{a,\{b,\{b,c\}\}\}\}.$$

Thus $(a,b,c) \in \mathtt{V}_{\max(n+1,m-1)+2}(\mathcal{T})$ and hence

$$\Pi_v a{\in}[\![A]\!]^2_{\xi,\rho}.\Pi_v b{\in}[\![A]\!]^1_{\xi,\rho}.[\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=b]} \in \mathtt{V}_{\max(n-1,m-1)+4}(\mathcal{T}).$$

Consequently,

$$[\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho} \in \mathtt{V}_{\max(n-1,m-1)+3}(\mathcal{T}).$$

• The $(conv)$ rule.

$$\frac{\Gamma \vdash M : T \quad \Gamma \vdash U : s}{\Gamma \vdash M : U} \quad T =_\beta U.$$

Let, for example, U be a kind, i.e., $s \equiv \Box$. From the induction hypothesis it follows

$$[\![M]\!]^k_{\xi,\rho} \in [\![T]\!]^{k+1}_{\xi,\rho}$$

for $k = 0, 1$. Moreover, the interpretations $[\![T]\!]^{k+1}_{\xi,\rho}$ and $[\![U]\!]^{k+1}_{\xi,\rho}$ are defined for $k = 0, 1$. Since the property of confluence holds for β-conversion on the set of pseudo-terms, it follows that there exists a pseudo-term V such that $T \twoheadrightarrow_\beta V$ and $U \twoheadrightarrow_\beta V$. Thus, from Lemma 4.22 it follows

$$[\![T]\!]^{k+1}_{\xi,\rho} =_v [\![V]\!]^{k+1}_{\xi,\rho} \text{ and } [\![U]\!]^{k+1}_{\xi,\rho} =_v [\![V]\!]^{k+1}_{\xi,\rho}$$

for $k = 1, 2$, and hence $[\![T]\!]^{k+1}_{\xi,\rho} =_v [\![U]\!]^{k+1}_{\xi,\rho}$. Consequently,

$$[\![M]\!]^k_{\xi,\rho} \in [\![T]\!]^{k+1}_{\xi,\rho}.$$

The theorem is proved. ■

5. STRONG NORMALIZATION

The property of Strong Normalization for the system λS of the λ-cube is obtained as a corollary of the Soundness Theorem.

Theorem 5.1 (Strong Normalization for λS). *For every context Γ of λS and for every terms M and T, such that $\Gamma \vdash M : T$, it follows that $M \in \mathsf{SN}_\beta$.*

Proof. We define a maximum element for every kind-interpretation of kinds in the following way:

$$\begin{aligned} max(\mathsf{SAT}_\beta) &= \mathsf{SN}_\beta, \\ max([\![\Pi\alpha{:}A.B]\!]^2_{\xi,\rho}) &= \lambda_v a {\in} [\![A]\!]^2_{\xi,\rho}.\lambda_v m {\in} [\![A]\!]^1_{\xi,\rho}.max([\![B]\!]^2_{\xi[\alpha:=a],\rho[\alpha:=m]}), \\ max([\![\Pi x{:}\sigma.B]\!]^2_{\xi,\rho}) &= \lambda_v m {\in} [\![\sigma]\!]^1_{\xi,\rho}.max([\![B]\!]^2_{\xi,\rho[x:=m]}). \end{aligned}$$

Let $\rho(v) = v$ for every variable v, and $\xi(\alpha) = max([\![A]\!]^2_{\xi,\rho})$ for every $(\alpha{:}A) \in \Gamma$. (This is possible due to the linearity of the legal contexts.) Obviously, the so-chosen valuations satisfy Γ. From the Soundness Theorem it follows that $[\![M]\!]^0_\rho \in [\![T]\!]^1_{\xi,\rho} \subseteq \mathsf{SN}_\beta$, and hence $M \in \mathsf{SN}_\beta$. ■

6. INDUCTIVE TYPES

In the following a method for extending the present SN-proof to systems with *inductive types* is presented. For simplicity we consider only the system λC. It is the most general system of the λ-cube and all non-trivial cases are captured.

In order to use a typing system for practical applications, there should be a certain mechanism for defining data types (e.g., inductive types) in it. The study of inductive types, however, happens to be a rather difficult task as well for defining a general inductive scheme as for studying the metatheory of type systems with inductive types. A general scheme for defining inductive types is presented, for example, in [10]. Here we will use a particular example, e.g., the type Nat of *Natural Numbers*, to show the flexibility of the present SN-proof. The system obtained from λC by adding the derivation and reduction rules for Nat will be denoted by $\lambda C + \mathsf{Nat}$.

The main problems for studying the metatheory of λC with inductive types arise when there are inductive types at the impredicative level of λC (i.e., which are of type $*$) and it is possible to define a type (respectively, predicate) over the elements of some inductive type. That means that for different elements of this inductive type the elimination scheme yields different types. Thus, the elements of an inductive type are distinguishable and one can prove inequalities like $0 \neq 1$. Therefore, the well-known syntactically-oriented proofs of Strong Normalization for λC and similar systems ([5, 6]), which exploit the idea of unifying all inhabitants of a type, are not directly adaptable for systems with inductive types. In such proofs many new technical complications must be added in order to adapt them to a system with inductive types (see [14, 15, 13]). Further, it is very likely that it is not possible to extend such syntactically-oriented proofs to normalization proofs of systems with mixed inductive types and kinds. Since the dependencies between constructors and objects are not disregarded in the interpretations presented here, we do not face the above problems. The present SN-proof is extended in a straightforward way to systems with other type-constructors. In the present section such an extension is shown for the system $\lambda C + \mathsf{Nat}$.

The additional rules of $\lambda C + \mathsf{Nat}$ are listed in Table 2.

The rule (elim, $*$) is called *small elimination* and the rule (elim, $\Box$) — *large elimination*.

There are two additional reduction rules for computing the values of recursive functions over Nat. This sort of reduction is called *ι-reduction*. The contraction rules for the ι-reduction are defined as follows:

$$\begin{array}{lcl} \mathsf{Rec}(P[x]; f_0, f_s[x,v])(0) & \to_\iota & f_0, \\ \mathsf{Rec}(P[x]; f_0, f_s[x,v])(sn) & \to_\iota & f_s[x := n, v := \mathsf{Rec}(P[x]; f_0, f_s[x,v])(n)]. \end{array}$$

The proof is extended as follows. First, the notion of v-equality is modified in order to comprise ι-reduction as well. This modification is obvious (see Definition 4.1). Further, the notion of saturated set should be adapted to $\beta\iota$-reduction. This is done below. Let $\mathsf{SN}_{\beta\iota} \subseteq \mathcal{T}$ be the set of pseudo-terms which are strongly normalizing under $\beta\iota$-reduction.

Table 2. Rules for the type Nat

(form)	$\vdash \mathsf{Nat} : *$	
(intro1)	$\vdash 0 : \mathsf{Nat}$	
(intro2)	$\dfrac{\Gamma \vdash n : \mathsf{Nat}}{\Gamma \vdash \mathsf{s}(n) : \mathsf{Nat}}$	
(elim, s)	$\dfrac{\Gamma \vdash n : \mathsf{Nat} \quad \Gamma, x{:}\mathsf{Nat} \vdash P : s \quad \Gamma \vdash f_0 : P[x := 0] \quad \Gamma, x{:}\mathsf{Nat}, v{:}P \vdash f_s : P[x := \mathsf{s}(x)]}{\Gamma \vdash \mathtt{Rec}(P[x]; f_0, f_s[x,v])(n) : P[x := n]}$	$s \in \{*, \Box\}$

Definition 6.1. The set of *$\beta\iota$-base terms* $\mathcal{B}_{\beta\iota}$ is defined as the smallest set satisfying the following conditions:

(i) $Var \subseteq \mathcal{B}_{\beta\iota}$;

(ii) if $M \in \mathcal{B}_{\beta\iota}$ and $N \in \mathsf{SN}_{\beta\iota}$, then $MN \in \mathcal{B}_{\beta\iota}$;

(iii) if $M \in \mathcal{B}_{\beta\iota}$ and $P, f_0 f_s, t \in \mathsf{SN}_{\beta\iota}$, then $\mathtt{Rec}(P[x]; f_0, f_s[x,v])(M) \in \mathcal{B}_{\beta\iota}$.

Definition 6.2. One step *$\beta\iota$-key-reduction* is defined by the contraction scheme

$$\begin{array}{rcl} (\lambda v{:}T_1.T_2)T_3 & \xrightarrow{k}_{\beta\iota} & T_2[v := T_3], \\ \mathtt{Rec}(P[x]; f_0, f_s[x,v])(0) & \xrightarrow{k}_{\beta\iota} & f_0, \\ \mathtt{Rec}(P[x]; f_0, f_s[x,v])(sn) & \xrightarrow{k}_{\beta\iota} & f_s[x := n, v := \mathtt{Rec}(P[x]; f_0, f_s[x,v)(n)] \end{array}$$

and by the compatibility extensions

$$\begin{array}{lcl} T_1 \xrightarrow{k}_{\beta\iota} T_2 & \Longrightarrow & T_1 M \xrightarrow{k}_{\beta\iota} T_2 M, \\ T_1 \xrightarrow{k}_{\beta\iota} T_2 & \Longrightarrow & \mathtt{Rec}(P[x]; f_0, f_s[x,v])(T_1) \xrightarrow{k}_{\beta\iota} \mathtt{Rec}(P[x]; f_0, f_s[x,v])(T_2). \end{array}$$

Fact 6.3. If the proper sub-terms of a term T_1 are $\beta\iota$-strongly-normalizing, $T_1 \xrightarrow{k}_{\beta\iota} T_2$ and $T_2 \in \mathsf{SN}_{\beta\iota}$, then $T_1 \in \mathsf{SN}_{\beta\iota}$.

The next definition describes the collection $\mathsf{SAT}_{\beta\iota}$ of $\beta\iota$-saturated sets. Note that only the subscripts differ from those in Definition 4.14.

Definition 6.4. The set X of pseudo-terms is *$\beta\iota$-saturated* if:

(i) $X \subseteq \mathsf{SN}_{\beta\iota}$;

(ii) $\mathcal{B}_{\beta\iota} \subseteq X$;

(iii) if $T_1 \xrightarrow{k}_{\beta\iota} T_2$, $T_2 \in X$ and the proper sub-terms of T_1 are $\beta\iota$-strongly-normalizing, then $T_1 \in X$.

It is easy to check that the closureness properties listed in Lemma 4.16 are also valid for the collection $\mathsf{SAT}_{\beta\iota}$.

The interpretations which are needed to be added to those in Definitions 4.18 and 4.17 are specified as follows:

Atom-interpretations:

$$\begin{aligned}
&[\![\mathsf{Nat}]\!]^0_\rho := \mathsf{Nat},\\
&[\![\mathsf{s}(n)]\!]^0_\rho := \mathsf{s}([\![n]\!]^0_\rho),\\
&[\![\mathtt{Rec}(P[x]; f_0, f_s[x,v])(n)]\!]^0_\rho :=\\
&\qquad\qquad \mathtt{Rec}([\![P]\!]^0_{\rho[x:=x]}[x]; [\![f_0]\!]^0_\rho, [\![f_s]\!]^0_{\rho[x:=x,v:=v]}[x,v])([\![n]\!]^0_\rho).
\end{aligned}$$

Constructor-interpretations:

- The constructor-interpretation of the type Nat is defined to be the smallest $\beta\iota$-saturated set which contains 0 and is closed under s, that is

$$[\![\mathsf{Nat}]\!]^1_{\xi,\rho} := \mu X \in \mathsf{SAT}_{\beta\iota}(0 \in X \& (x \in X \Longrightarrow \mathsf{s}(x) \in X)). \tag{3}$$

- The constructor-interpretation of terms of the form

$$\mathtt{Rec}(A[x]; f_0, f_s[x,\alpha])(n),$$

obtained by applying the rule for large elimination over Nat, is defined below. Its definition uses recursion over the set $[\![\mathsf{Nat}]\!]^1_{\xi,\rho}$. First some auxiliary functions are defined. Let in the following

$$G_0 \equiv [\![f_0]\!]^1_{\xi,\rho}. \tag{4}$$

Let also $g(n)$ and $G_s(n,a)$ be (dependent) set-theoretical functions defined by the equations ($n \in [\![\mathsf{Nat}]\!]^1_{\xi,\rho}$, $a \in [\![A]\!]^2_{\xi,\rho[x:=n]}$ and $z \notin FV(A, f_0, f_s, x, \alpha)$)

$$g(n) := [\![\mathtt{Rec}(A[x]; f_0, f_s[x,\alpha])(z)]\!]^0_{\rho[z:=n]}, \tag{5}$$

$$G_2(n,a) = [\![f_s]\!]^1_{\xi[\alpha:=a],\rho[\alpha:=g(n),x:=n]}. \tag{6}$$

Finally, let $G(n)$ be a set-theoretical function with domain $[\![\mathsf{Nat}]\!]^1_{\xi,\rho}$, defined by

$$G(n) = max([\![A]\!]^2_{\xi,\rho[x:=n]}). \tag{7}$$

Now we define a function $F(n)$ by recursion on $n \in [\![\mathsf{Nat}]\!]^1_{\xi,\rho}$:

$$\begin{array}{lcll}
F(0) & = & G_0, & \\
F(\mathsf{s}(n)) & = & G_s(n, F(n)), & \\
F(b) & = & G(b) & \text{if } b \in \mathcal{B}_{\beta\iota}, \\
F(M) & = & F(M') & \text{if } M \xrightarrow{k}_{\beta\iota} M'.
\end{array} \tag{8}$$

Then we can define

$$\llbracket \mathsf{Rec}(P[x]; f_0, f_s[x,v])(n) \rrbracket^1_{\xi,\rho} \simeq F(\llbracket n \rrbracket^0_\rho). \tag{9}$$

The Lemmas 4.20–4.22, 4.24, 4.25 and their proof are extended in an obvious way. The proof of the Soundness Theorem 4.28 is also extended in a trivial way for the new rules. The case when the last rule in the derivation of a judgment $\Gamma \vdash M : T$ is the rule for large elimination over Nat, will be treated here:

$$\frac{\Gamma \vdash t : \mathsf{Nat} \quad \Gamma, x{:}\mathsf{Nat} \vdash A : \Box \qquad \Gamma \vdash f_0 : A[x := 0] \quad \Gamma, x{:}\mathsf{Nat}, v{:}A \vdash f_s : A[x := \mathsf{s}(x)]}{\Gamma \vdash \mathsf{Rec}(A[x]; f_0, f_s[x,\alpha])(t) : A[x := t]}$$

We have to prove that if $\xi, \rho \models \Gamma$, then

$$\llbracket \mathsf{Rec}(A[x]; f_0, f_s[x,v])(t) \rrbracket^1_{\xi,\rho} \in \llbracket A[x := t] \rrbracket^2_{\xi,\rho}$$

and

$$\llbracket \mathsf{Rec}(A[x]; f_0, f_s[x,\alpha])(t) \rrbracket^0_\rho \in \llbracket A[x := t] \rrbracket^1_{\xi,\rho}.$$

From the induction hypothesis the following inclusions follow:

(i) $\llbracket t \rrbracket^0_\rho \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$;

(ii) $\forall n \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$:

$$\begin{aligned} \llbracket A \rrbracket^2_{\xi,\rho[x:=n]} &\in \mathbf{U}^\Box, \\ \llbracket A \rrbracket^1_{\xi,\rho[x:=n]} &\in \mathsf{SAT}_{\beta\iota}, \\ \llbracket A \rrbracket^0_{\rho[x:=n]} &\in \mathsf{SN}_{\beta\iota}; \end{aligned}$$

(iii)
$$\begin{aligned} \llbracket f_0 \rrbracket^1_{\xi,\rho} &\in \llbracket A \rrbracket^2_{\xi,\rho[x:=0]}, \\ \llbracket f_0 \rrbracket^0_\rho &\in \llbracket A \rrbracket^1_{\xi,\rho[x:=0]}; \end{aligned}$$

(iv) $\forall n \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$, $\forall a \in \llbracket A \rrbracket^2_{\xi,\rho[x:=n]}$, $\forall l \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$:

$$\begin{aligned} \llbracket f_s \rrbracket^1_{\xi[\alpha:=a],\rho[\alpha:=l,x:=n]} &\in \llbracket A \rrbracket^2_{\xi,\rho[x:=\mathsf{s}(n)]}, \\ \llbracket f_s \rrbracket^0_{\rho[\alpha:=l,x:=n]} &\in \llbracket A \rrbracket^1_{\xi,\rho[x:=\mathsf{s}(n)]}. \end{aligned}$$

Note that G_0 and the function $G(n)$ are well-defined (see (4), (iii), (7) and (ii)). We will prove also that the function $G_s(n,a)$ is well-defined. First, we shall prove that $g(n) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$ (see (5)) by induction on $n \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$:

1. Let $n \equiv 0$.

$$g(0) = \mathsf{Rec}(\llbracket A \rrbracket^0_{\rho[x:=x]}[x]; \llbracket f_0 \rrbracket^0_\rho, \llbracket f_s \rrbracket^0_{\rho[x:=x,\alpha:=\alpha]})(0),$$

and hence $g(0) \xrightarrow{k}_{\beta\iota} \llbracket f_0 \rrbracket^0_\rho$. Thus, from (iii), (ii) and Definition 6.4 it follows

$$g(0) \in \llbracket A \rrbracket^1_{\rho[x:=0]}.$$

2. Let $n \equiv \mathbf{s}(m)$ for some $m \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$ and let us assume $g(m) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=m]}$.

$$g(\mathbf{s}(m)) = \mathtt{Rec}(\llbracket A \rrbracket^0_{\rho[x:=x]}[x]; \llbracket f_0 \rrbracket^0_\rho, \llbracket f_s \rrbracket^0_{\rho[x:=x,\alpha:=\alpha]})(\mathbf{s}(\llbracket m \rrbracket^0_\rho)).$$

Thus $g(\mathbf{s}(m)) \xrightarrow{k}_{\beta\iota} \llbracket f_s \rrbracket^0_{\rho[\alpha:=g(m),x:=m]}$ and hence from (iv), (ii) and Definition 6.4 we obtain

$$g(\mathbf{s}(m)) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=\mathbf{s}(m)]}.$$

3. Let $n \in \mathcal{B}_{\beta\iota}$. Then also $g(n) \in \mathcal{B}_{\beta\iota}$ (see Definition 6.1 and (5)) and hence $g(n) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$ since $\llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$ is a $\beta\iota$-saturated set.

4. Finally, let $n \xrightarrow{k}_{\beta\iota} n'$ and let us assume $g(n') \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n']}$. Note that $g(n) \xrightarrow{k}_{\beta\iota} g(n')$ (see Definition 6.2) and hence $g(n) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n']}$ since $\llbracket A \rrbracket^1_{\xi,\rho[x:=n']}$ is a saturated set. Further, from Lemma 4.25 it follows

$$\llbracket A \rrbracket^1_{\xi,\rho[x:=n']} = \llbracket A \rrbracket^1_{\xi,\rho[x:=n]},$$

and hence $g(n) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$.

Thus we have proved that $g(n) \in \llbracket A \rrbracket^1_{\xi,\rho[x:=n]}$. This implies that, first, the function $G_s(n,a)$ (see (6)) is well-defined, and second,

$$\llbracket \mathtt{Rec}(A[x]; f_0, f_s[x,\alpha])(t) \rrbracket^0_\rho \in \llbracket A[x := t] \rrbracket^1_{\xi,\rho} \tag{10}$$

since $\llbracket \mathtt{Rec}(A[x]; f_0, f_s[x,\alpha])(t) \rrbracket^0_\rho = g(\llbracket t \rrbracket^0_\rho)$.

It follows now that the function $F(n)$ (see (8)) is well-defined, because G_0 and the functions G and G_s are well-defined. Now we shall prove that $F(n) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=n]}$ by induction on $n \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$:

1. Let $n \equiv 0$. By definition

$$F(0) = \llbracket f_0 \rrbracket^1_{\xi,\rho}.$$

Thus, from (iii) it follows $F(0) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=0]}$.

2. Let $n \equiv \mathbf{s}(m)$ for some $m \in \llbracket \mathsf{Nat} \rrbracket^1_{\xi,\rho}$ and let us assume

$$F(m) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=m]}.$$

By definition (see (8)) $F(\mathbf{s}(m)) = \llbracket f_s \rrbracket^1_{\xi[\alpha:=F(m)],\rho[\alpha:=g(m),x:=m]}$. Thus, from (iv) it follows $F(\mathbf{s}(m)) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=\mathbf{s}(m)]}$.

3. Let $n \in \mathcal{B}_{\beta\iota}$. In this case $F(n) = max(\llbracket A \rrbracket^2_{\xi,\rho[x:=n]})$ and hence $F(n) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=n]}$.

4. Let now $n \xrightarrow{k}_{\beta\iota} n'$ and let $F(n') \in \llbracket A \rrbracket^2_{\xi,\rho[x:=n']}$. By definition $F(n) = F(n')$ and thus from Lemma 4.25 it follows

$$F(n) \in \llbracket A \rrbracket^2_{\xi,\rho[x:=n]}.$$

Thus we have proved that $F(n) \in [\![A]\!]^2_{\xi,\rho[x:=n]}$ and hence

$$[\![\mathbf{Rec}(A[x]; f_0, f_s[x,v])(t)]\!]^1_{\xi,\rho} \in [\![A[x := t]]\!]^2_{\xi,\rho}. \tag{11}$$

The proof of the Soundness Theorem in the case of the rule for large elimination over Nat follows directly from (10) and (11).

The proof of Strong Normalization for $\lambda C + \mathsf{Nat}$ follows in a trivial way from the Soundness Theorem (see Theorem 5.1).

7. DISCUSSION

We have presented a simple semantical proof of Strong Normalization for the systems of the λ-cube. We have shown that the property of Strong Normalization can be derived directly from a simple denotational semantics of the system considered. Further, the flexibility of this semantical proof has been illustrated by extending the system $\lambda C + \mathsf{Nat}$.

We have not addressed the following questions, which deserve some attention:

• *Generalized Inductive Definitions.* The proof presented here is extendible in a straightforward way to a proof of Strong Normalization of systems with generalized inductive definitions. Such definitions are a convenient tool for defining various inductive types, such as lists of a type σ, sigma types, finite sets, etc. A proof of Strong Normalization of λC enriched with generalized inductive definitions is presented in [12].

• *Inductive kinds.* In some systems there is a clear distinction between the level of formulas ($*$) and the level of domains ($\Box$). In such systems one prefers to define data-types rather as kinds than as types. An interesting issue, which seems to have not been considered yet in the literature, is the metatheory of a system in which inductive definitions are allowed at the both levels $*$ and $\Box$. In such systems one can define inductive predicates (for example $=_A : A \to A \to *$) at the level of formulas and inductive data types (for example $Nat : \Box$) at the level of domains. The proof described here is adapted to systems with inductive kinds in [12].

• *Generic strong normalization argument.* The proof presented above suggests a generic method for proving Strong Normalization for PTSs. The genericity lies in the fact that the properties of interpretations $\mathbf{U}^*$ and $\mathbf{U}^\Box$ of the universes $*$ and $\Box$ are derived directly from the PTS-presentation of the systems in the λ-cube. For example, the axiom $* : \Box$ is interpreted by $\mathbf{U}^* \in \mathbf{U}^\Box$, and the PTS-rules — by requiring adequate closure properties on $\mathbf{U}^*$ and $\mathbf{U}^\Box$.

We outline how one can generalize the method to a subclass PTSs.

A PTS $\mathcal{S} = \langle S, \mathcal{A}, \mathcal{R}\rangle$ is specified by three sets: S of sorts, $\mathcal{A}$ of axioms, and $\mathcal{R}$ of rules (see [2] or [4] for a detailed presentation of PTS-s). The set S of sorts is simply a set of fixed constants s_i. Every axiom has the form $s_i : s_j$, and every rule — (s_i, s_j, s_k). The PTS-rules say what kind of dependent products can be constructed inside the system $\mathcal{S}$. For example, if $(s_i, s_j, s_k) \in \mathcal{R}$, then the following ($\prod$)-rule is allowed in the system $\mathcal{S}$:

$$\frac{A \vdash s_i : \quad v{:}A \vdash B : s_j}{\Pi v{:}A.B \vdash s_k :}.$$

A relation $<$ is defined on the sets of sorts to be the smallest relation satisfying the following conditions:

(i) $s_i : s_j \Longrightarrow s_i < s_j$;

(ii) $s_i < s_j$ and $s_j < s_k \Longrightarrow s_i < s_k$.

Below we sketch out the properties each interpretation $\mathbf{U}^s$ of a sort s should possess:

— for any axiom $(s_i : s_j) \in \mathcal{A}$ it holds that $\mathbf{U}^{s_i} \in \mathbf{U}^{s_j}$;

— for any rule $(s_i, s_j, s_k) \in \mathcal{R}$, such that $s_i \leq s_k$, one can define an operation $\Pi^{s_i}_{s_j}$ for which holds

$$\forall X \in \mathbf{U}^{s_i}\ \forall \{Y_x\}_{x \in X} \in \mathbf{U}^{s_j}\ \ \Pi^{s_i}_{s_j} x \in X.Y_x\ \in \mathbf{U}^{s_k};$$

— for any rule $(s_i, s_j, s_k) \in \mathcal{R}$, such that $s_i > s_k$, it follows that $\mathbf{U}^{s_k}$ is closed under arbitrary non-empty intersections;

— for each sort s, $\emptyset \notin \mathbf{U}^s$.

It is interesting to see for which PTSs the universes U_s exist. For example, it is clear that for PTSs, for which the relation $<$ is not a strict order, i.e. $s < s$ for some sort s, such universes can not be found. Further, one needs to study more precisely the dependencies in the PTS considered, in order to specify as precise as possible the operations $\Pi^{s_i}_{s_j}$.

• *Models.* The Strong Normalization proof presented here is based on specific models of the systems of the λ-cube. In [12] an abstract notion of a model of λC will be presented. This abstract model construction generalizes the ideas presented here.

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FIRST ORDER AXIOMATIZABILITY OF RECURSION THEORY IN CARTESIAN LINEAR COMBINATORY ALGEBRAS*

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A modification of recursion theorem in Cartesian linear combinatory algebras is proved which yields first order formalizability of theory of the last algebras. Some other improvements of this theory are demonstrated.

Keywords: axiomatic recursion theory, algebraic recursion theory, combinatory algebras

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1. Cartesian linear combinatory algebras (shortly CLCA) were introduced in [1]; the principal objective was to provide a theoretical example to be compared with other partially ordered algebras used for abstract axiomatical treatment of the fundamentals of recursion theory. In the present note we are going to give an improved exposition of principal results of [1], which is based on replacement of the concept of iterative CLCA with that of *strictly* iterative one.

Let $\mathcal{F} = \langle |\mathcal{F}|, \leq, \mathtt{App}, O, A, C, K, C', D' \rangle$ be a Cartesian linear combinatory algebra in the sense of [1]; App is the application operation and we write as usual $\varphi\psi$ for $\mathtt{App}(\varphi, \psi)$ and adopt the other traditional notational conventions for application (association to left, etc.). By definition this means that $|\mathcal{F}|$ is a set partially ordered by $\leq$, App is a binary operation in $|\mathcal{F}|$ increasing on both arguments, O is the least element of $|\mathcal{F}|$ with respect to $\leq$, and A, C, K, C', D' are elements of $|\mathcal{F}|$ such that the following equalities hold for all $\varphi, \psi, \chi \in |\mathcal{F}|$:

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$$D'OO = O; \quad A\varphi\psi\chi = \varphi(\psi\chi); \quad C\varphi\psi = \psi\varphi; \quad K\varphi\psi = \varphi;$$

and

$$C'\varphi(D'\psi\chi) = \varphi\psi\chi.$$

We shall write $\mathcal{F}$ for $|\mathcal{F}|$ below, and we shall use some other notations and terminology from [1]. Especially, for any set $\mathcal{C}$ of operations in $\mathcal{F}$ (which may include elements of $\mathcal{F}$ considered as operations of zero arguments) an element of $\mathcal{F}$ or an operation in $\mathcal{F}$ will be called $\mathcal{C}$-expressible iff it can be defined by an explicit expression containing application and operations from $\mathcal{C}$.

A set $\mathcal{A}$ will be called an *admissible iteration domain* (of first, second, etc. kind, respectively) iff it has one of the following four forms:

i) $\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi\varphi \leq \psi\}$;

ii) $\mathcal{A} = \{(\xi, \vartheta) \in \mathcal{F}^2 \mid \xi \leq \vartheta \;\&\; D'O\vartheta \leq \vartheta \;\&\; \vartheta \leq \psi\}$;

iii) $\mathcal{A} = \{(\xi, \vartheta, \eta) \in \mathcal{F}^3 \mid \xi \leq \vartheta \;\&\; \vartheta \leq \chi\eta\vartheta \;\&\; D'\eta\varphi' \leq \psi'\}$;

iv) $\mathcal{A} = \{\xi \in \mathcal{F} \mid D'\xi\xi \leq \psi\}$,

where φ, ψ, χ, φ', ψ' are elements of $\mathcal{F}$ such that $D'O\varphi' \leq \psi'$.

A CLCA $\mathcal{F}$ will be called *strictly iterative* iff for every $\varphi \in \mathcal{F}$ the inequality $\varphi\xi \leq \xi$ has the least solution $\mathbb{I}(\varphi) \in \mathcal{F}$ with respect to ξ such that the following three conditions are fulfilled:

I_1) For every admissible iteration domain $\mathcal{A}$ of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$ we have $\mathbb{I}(\varphi) \in \mathcal{A}$;

I_2) For every admissible iteration domain $\mathcal{A}$ of second kind and every $\alpha \in \mathcal{F}$ such that $(\varphi\xi, \alpha\vartheta) \in \mathcal{A}$ for all $(\xi, \vartheta) \in \mathcal{A}$ there is $\vartheta' \in \mathcal{F}$ such that $(\mathbb{I}(\varphi), \vartheta') \in \mathcal{A}$;

I_3) For every admissible iteration domain $\mathcal{A}$ of third kind and every $\mathcal{F}$-expressible mapping $\Gamma : \mathcal{F} \to \mathcal{F}$, and every $\alpha \in \mathcal{F}$, if $(\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$ for all $(\xi, \vartheta, \eta) \in \mathcal{A}$, then there are $\vartheta', \eta' \in \mathcal{F}$ such that $(\mathbb{I}(\varphi), \vartheta', \eta') \in \mathcal{A}$.

The element $\mathbb{I}(\varphi)$ will be called *iteration* of φ; it is the least fixed point of the mapping $\xi \mapsto \varphi\xi$.

This notion of strict iterativity is clearly first order formalizable, while the previous notion of iterativity of a CLCA in the sense of [1] is not. It seems, however, that a formalization of I_3) would require infinitely many (first order) axioms, because it involves arbitrary $\mathcal{F}$-expressible mappings Γ. This is not really the case, since we may safely restrict condition I_3) to mappings Γ of the form $\Gamma(\xi) = \varphi\xi\xi$ (for fixed $\varphi \in \mathcal{F}$) only, as it will be explained below in Remark 1.

The next Proposition 1 is an analog of the usual criteria of iterativity in algebraic recursion theory; it shows that in typical cases CLCA will be strictly iterative.

Proposition 1. *Let $\mathcal{F}$ be a CLCA and let k be a cardinal number such that $\sup_{i<l} \varphi_i$ exists for all increasing (transfinite) sequences $\varphi_i \in \mathcal{F}$ and all ordinal numbers $l \leq k$. Suppose at least one of the following two conditions holds:*

1) $k = \omega$ and $\sup_{i<k} \varphi\varphi_i = \varphi \sup_{i<k} \varphi_i$ for all increasing sequences φ_i in $\mathcal{F}$ and all $\varphi \in \mathcal{F}$;

2) card$\mathcal{F} < k$ *and the following equalities hold for all increasing transfinite sequences* φ_i *in* $\mathcal{F}$, *all* $l \leq k$ *and all* $\psi \in \mathcal{F}$:

2a) $\sup\limits_{i<l}(\varphi_i\psi) = (\sup\limits_{i<l}\varphi_i)\psi$;

2b) $\sup\limits_{i<l} D'\varphi_i\psi = D'(\sup\limits_{i<l}\varphi_i)\psi$;

2c) $\sup\limits_{i<l} D'\psi\varphi_i = D'\psi(\sup\limits_{i<l}\varphi_i)$.

Then $\mathcal{F}$ *is strictly iterative.*

Proof. In the case of condition 1) we define by induction $\varphi_0 = O$ and $\varphi_{n+1} = \varphi\varphi_n$, where φ is a fixed element of $\mathcal{F}$. The usual argument shows that the sequence φ_n increases and $\mathbb{I}(\varphi) = \sup\limits_{n<\omega}\varphi_n$ is the least solution of $\varphi\xi \leq \xi$ with respect to ξ in $\mathcal{F}$. If $\mathcal{A}$ is an admissible iteration domain of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$, then by induction on n we have $\varphi_n \in \mathcal{A}$. Indeed, $O \in \mathcal{A}$ since $D'OO = O$ and $O\beta = O$ for all $\beta \in \mathcal{F}$, because $O\beta \leq KO\beta = O$, and the induction step is obvious. Using the supposition that increasing suprema commute with application, we get $\mathbb{I}(\varphi) = \sup\limits_{n<\omega}\varphi_n \in \mathcal{A}$. (Note that condition 1) implies $\sup(\varphi_n\psi) = \sup(C\psi\varphi_n) = C\psi\sup\varphi_n = (\sup\varphi_n)\psi$.) To show that condition I_2) holds, consider an admissible iteration domain $\mathcal{A}$ of second kind and an element $\alpha \in \mathcal{F}$ such that

$$(\xi, \vartheta) \in \mathcal{A} \;\Rightarrow\; (\varphi\xi, \alpha\vartheta) \in \mathcal{A}$$

for all $\xi, \vartheta \in \mathcal{F}$. Define inductively $\alpha_0 = O$ and $\alpha_{n+1} = \alpha\alpha_n$. The sequence α_n increases and $\alpha_\omega = \sup\alpha_n$ exists in $\mathcal{F}$. Since obviously $(O, O) \in \mathcal{A}$, we have by induction on n that $(\varphi_n, \alpha_n) \in \mathcal{A}$, whence, using condition 1), we obtain $(\mathbb{I}(\varphi), \alpha_\omega) \in \mathcal{A}$. In similar way we see that condition I_3) holds: for an admissible iteration domain $\mathcal{A}$ of third kind and an element $\alpha \in \mathcal{F}$ and a mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ such that

$$(\xi, \vartheta, \eta) \in \mathcal{A} \;\Rightarrow\; (\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$$

we define α_n as before and γ_n as $\Gamma^n(O)$ and prove by induction on n that $(\varphi_n, \alpha_n, \gamma_n) \in \mathcal{A}$, whence $(\mathbb{I}(\varphi), \alpha_\omega, \sup\gamma_n) \in \mathcal{A}$.

In the case of condition 2) the usual Platek argument holds. We define by transfinite recursion a sequence $\varphi_i \in \mathcal{F}$ $(i < k)$, and prove simultaneously that $\varphi_i = \sup\limits_{j<i}\varphi\varphi_j$, $\varphi_i \leq \varphi\varphi_i$, and $(\varphi_j)_{j<i}$ increases for all $i < k$. Then $\mathbb{I}(\varphi) = \varphi_m$, where m is the least ordinal number for which $\varphi_m = \varphi_{m+1}$, is the least solution of $\varphi\xi \leq \xi$ with respect to ξ in $\mathcal{F}$. To show that condition I_1) holds, we prove by induction on i that $\varphi_i \in \mathcal{A}$ for all $i < k$ and every admissible iteration domain $\mathcal{A}$ of first or fourth kind such that $\varphi\mathcal{A} \subseteq \mathcal{A}$, using 2a)–2c). Then $\mathbb{I}(\varphi) = \varphi_m \in \mathcal{A}$. For condition I_2), given an admissible iteration domain $\mathcal{A}$ of second kind and an element $\alpha \in \mathcal{F}$ such that $(\xi, \vartheta) \in \mathcal{A}$ implies $(\varphi\xi, \alpha\vartheta) \in \mathcal{A}$ for all $\xi, \vartheta \in \mathcal{F}$, we construct in a similar way a transfinite increasing sequence $\alpha_i \in \mathcal{F}$ such that $\alpha_i = \sup\limits_{j<i}\alpha\alpha_j$ for all $i < k$, whence, using induction on i, we see that $(\varphi_i, \alpha_i) \in \mathcal{A}$ for all $i < k$ and therefore $(\mathbb{I}(\varphi), \alpha_m) \in \mathcal{A}$. Finally, given an admissible iteration domain $\mathcal{A}$ of third kind, an element $\alpha \in \mathcal{F}$ and an expressible mapping $\Gamma : \mathcal{F} \to \mathcal{F}$

such that $(\xi, \vartheta, \eta) \in \mathcal{A}$ implies $(\varphi\xi, \alpha\vartheta, \Gamma(\eta)) \in \mathcal{A}$ for all $\xi, \eta, \vartheta \in \mathcal{F}$, we define transfinite increasing sequences α_i as before and $\gamma_i = \sup_{j<i} \Gamma(\gamma_j)$ (using monotonicity of expressible mappings Γ), and prove by induction on i that $(\varphi_i, \alpha_i, \gamma_i) \in \mathcal{A}$ for all $i < k$, whence $(\mathbb{I}(\varphi), \alpha_m, \gamma_m) \in \mathcal{A}$.

Proposition 1 is applicable to all the examples of CLCA in [1] and shows that these CLCA are strictly iterative; particularly, the algebras $\mathcal{F}$ in the examples 1 and 2 in [1] satisfy condition 1) in the last proposition, and the algebras $\mathcal{F}$ in examples 3 and 4 in [1] satisfy condition 2) of the same proposition.

Let us note in this connection that the Proposition 5.2 in [1], treating the same question of general iterativity criteria, is incorrect. A correct version of this proposition would be that a CLCA is iterative if it satisfies the conditions of the above Proposition 1 in such a way that 1) in the last proposition holds. However, this correct version does not imply the iterativity of the CLCA $\mathcal{F}$ in the examples 3 and 4 in [1], and the last CLCA are indeed non-iterative. Thus the notion of strict iterativity provides also the necessary improvement to comprise these examples as well.

Theorem 1. *Let $\mathcal{F}$ be a strictly iterative CLCA and let $\mathcal{C} \subseteq \mathcal{F}$. Then for every $\mathcal{C}$-expressible unary operation $\Gamma : \mathcal{F} \to \mathcal{F}$ the least fixed point of Γ exists and is $\mathcal{C} \cup \{A,\ C,\ K,\ C',\ D',\ \mathbb{I}\}$-expressible.*

Proof. The proof begins as that of Theorem 5.3 in [1]; using a short notation $\varphi^n(\xi)$ for $\varphi(\varphi(\ldots\varphi(\varphi\xi)\ldots))$ (where we have n occurrences of φ and $\varphi, \xi \in \mathcal{F}$ are arbitrary) and the basic equalities for the constants A, C, C', D' in the definition of CLCA, we find an $\mathcal{C} \cup \{A,\ C,\ C',\ D'\}$-expressible element $c_\Gamma \in \mathcal{F}$ such that for all $\varphi, \psi, \vartheta \in \mathcal{F}$

$$c_\Gamma\vartheta((D'\varphi)^k(\psi)) = D'\Gamma(\varphi)(\vartheta\psi),$$

where k is the number of occurrences of the variable (for) ξ in the explicit expression defining $\Gamma(\xi)$. Next we define $\gamma = \mathbb{I}(c_\Gamma)$ and $\nabla(\varphi) = \mathbb{I}(D'\varphi)$ and prove that for all $\varphi \in \mathcal{F}$

$$\nabla(\Gamma(\varphi)) = \gamma\nabla(\varphi). \tag{1}$$

This is done by making use of strict iterativity of $\mathcal{F}$, especially condition I_1). Namely, the set

$$\mathcal{A}_0 = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq \nabla(\Gamma(\varphi))\}$$

is an admissible iteration domain of first kind. If $\xi \in \mathcal{A}_0$, then

$$c_\Gamma\xi\nabla(\varphi) = c_\Gamma\xi((D'\varphi)^k(\nabla(\varphi))) = D'\Gamma(\varphi)(\xi\nabla(\varphi)) \leq D'\Gamma(\varphi)\nabla(\Gamma(\varphi)) = \nabla(\Gamma(\varphi)),$$

since $\nabla(\varphi) = D'\varphi\nabla(\varphi) = (D'\varphi)^k(\nabla(\varphi))$, because $\nabla(\varphi)$ is the least fixed point of the mapping $\xi \mapsto D'\varphi\xi$. Thus $c_\Gamma\mathcal{A}_0 \subseteq \mathcal{A}_0$ and by condition I_1) $\gamma = \mathbb{I}(c_\Gamma) \in \mathcal{A}_0$, i.e.

$$\gamma\nabla(\varphi) \leq \nabla(\Gamma(\varphi)).$$

The reverse inequality follows from

$$D'\Gamma(\varphi)(\gamma\nabla(\varphi)) = c_\Gamma\gamma((D'\varphi)^k(\nabla(\varphi))) = c_\Gamma\gamma\nabla(\varphi) = \gamma\nabla(\varphi)$$

and proves (1). Then for an arbitrary $\xi \in \mathcal{F}$ such that $\Gamma(\xi) \leq \xi$ we have

$$\gamma\nabla(\xi) = \nabla(\Gamma(\xi)) \leq \nabla(\xi),$$

whence $\mathbb{I}(\gamma) \leq \nabla(\xi)$ and

$$L\mathbb{I}(\gamma) \leq L\nabla(\xi) = L(D'\xi\nabla(\xi)) = \xi$$

(where $L = C'K$ and therefore $L(D'\varphi\psi) = \varphi$ for all $\varphi, \psi \in \mathcal{F}$).

Therefore it remains to show that $\Gamma(L\mu) \leq L\mu$, where we are writing shortly μ for $\mathbb{I}(\gamma)$. For this we show first that

$$D'O\mu \leq \mu. \tag{2}$$

Indeed, consider the set

$$\mathcal{A} = \{(\xi, \vartheta) \in \mathcal{F}^2 \mid \xi \leq \vartheta \;\&\; D'O\vartheta \leq \vartheta \;\&\; \vartheta \leq \mu\}.$$

It is an admissible iteration domain of second kind. To apply condition $\mathbf{I}_2$) suppose $(\xi, \vartheta) \in \mathcal{A}$, i.e. $\xi \leq \vartheta$, $D'O\vartheta \leq \vartheta$ and $\vartheta \leq \mu$. Then we have $\gamma\xi \leq \gamma\vartheta$ and

$$\gamma\vartheta \leq \gamma\mu = \gamma\mathbb{I}(\gamma) \leq \mathbb{I}(\gamma) = \mu.$$

Moreover, by induction on n we see that for all natural n

$$(D'O)^n(\vartheta) \leq \vartheta,$$

and using the definition of c_Γ we have

$$D'O(\gamma\vartheta) \leq D'\Gamma(O)(\gamma\vartheta) = c_\Gamma\gamma((D'O)^k(\vartheta)) \leq c_\Gamma\gamma\vartheta = \gamma\vartheta.$$

So we see that $(\gamma\xi, \gamma\vartheta) \in \mathcal{A}$. Then by condition I_2) $(\mu, \vartheta) \in \mathcal{A}$ for some $\vartheta \in \mathcal{F}$, whence we obtain (2). From (2) it follows that the set

$$\mathcal{B} = \{(\xi, \vartheta, \eta) \in \mathcal{F} \mid \xi \leq \vartheta \;\&\; \vartheta \leq D'\eta\vartheta \;\&\; D'\eta\mu \leq \mu\}$$

is an admissible iteration domain of third kind. To apply condition I_3), suppose $(\xi, \vartheta, \eta) \in \mathcal{B}$, i.e. $\xi \leq \vartheta$, $\vartheta \leq D'\eta\vartheta$ and $D'\eta\mu \leq \mu$. Then $\gamma\xi \leq \gamma\vartheta$ and by induction on n we have

$$\vartheta \leq (D'\eta)^n(\vartheta)$$

and

$$(D'\eta)^n(\mu) \leq \mu$$

for all natural n, whence

$$\gamma\vartheta \leq \gamma((D'\eta)^k(\vartheta)) = c_\Gamma\gamma((D'\eta)^k(\vartheta)) = D'\Gamma(\eta)(\gamma\vartheta)$$

and

$$D'\Gamma(\eta)\mu = D'\Gamma(\eta)(\gamma\mu) = c_\Gamma\gamma((D'\eta)^k(\mu)) \leq c_\Gamma\gamma\mu = \gamma\mu = \mu.$$

Therefore $(\gamma\xi, \gamma\vartheta, \Gamma(\eta)) \in \mathcal{B}$, and by condition I_3) $(\mu, \vartheta, \eta) \in \mathcal{B}$ for some $\vartheta, \eta \in \mathcal{F}$. Thus we have $\mu \leq \vartheta$, $\vartheta \leq D'\eta\vartheta$ and $D'\eta\mu \leq \mu$, whence $L\mu \leq L\vartheta \leq L(D'\eta\vartheta) = \eta$ and

$$D'(L\mu)\mu \leq D'\eta\mu \leq \mu.$$

By definition of the operation ∇ this inequality shows that $\nabla(L\mu) \leq \mu$, whence by (1)

$$\nabla(\Gamma(L\mu)) = \gamma\nabla(L\mu) \leq \gamma\mu = \mu,$$

and

$$\Gamma(L\mu) = L(D'\Gamma(L\mu)\nabla(\Gamma(L\mu))) = L\nabla(\Gamma(L\mu)) \leq L\mu.$$

2. Let $\mathcal{F}$ be a strictly iterative CLCA and define ∇ as in Section 1. Then we have

Theorem 2. *There is an* $\{A, C, C', D', \mathbb{I}\}$*-expressible element* $\delta \in \mathcal{F}$ *such that for all* $\varphi \in \mathcal{F}$ *we have*

$$\delta\nabla(\varphi) = D'\nabla(\varphi)\nabla(\varphi).$$

Proof. Using the basic equalities in the definition of a CLCA, we define two elements $D_1, D_2 \in \mathcal{F}$ such that

$$D_1\xi\eta(D'\vartheta_0\vartheta_1) = D'(D'\xi\vartheta_0)(D'\eta\vartheta_1)$$

for all $\xi, \eta, \vartheta_0, \vartheta_1 \in \mathcal{F}$, and

$$D_2\vartheta(D'\xi(D'\eta\zeta)) = D_1\xi\eta(\vartheta\zeta)$$

for all $\xi, \eta, \vartheta, \zeta \in \mathcal{F}$; and let $\delta = \mathbb{I}(D_2)$. To prove the inequality

$$D'\nabla(\varphi)\nabla(\varphi) \leq \delta\nabla(\varphi), \tag{3}$$

consider the set

$$\mathcal{A} = \{\xi \in \mathcal{F} \mid D'\xi\xi \leq \delta\nabla(\varphi)\}$$

which is an admissible iteration domain of fourth kind. We shall show that $D'\varphi\mathcal{A} \subseteq \mathcal{A}$. Suppose $\xi \in \mathcal{A}$. Then

$$\begin{aligned} D'(D'\varphi\xi)(D'\varphi\xi) &= D_1\varphi\varphi(D'\xi\xi) \leq D_1\varphi\varphi(\delta\nabla(\varphi)) = D_2\delta(D'\varphi(D'\varphi\nabla(\varphi))) \\ &= D_2\delta\nabla(\varphi) = \delta\nabla(\varphi), \end{aligned}$$

whence $D'\varphi\xi \in \mathcal{A}$. By condition I_1) $\nabla(\varphi) = \mathbb{I}(D'\varphi) \in \mathcal{A}$, which proves (3). To prove the reverse inequality, consider the admissible iteration domain $\mathcal{B}$ of first kind defined by

$$\mathcal{B} = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq D'\nabla(\varphi)\nabla(\varphi)\}.$$

Then for $\xi \in \mathcal{B}$ we have

$$\begin{aligned} D_2\xi\nabla(\varphi) &= D_2\xi(D'\varphi(D'\varphi\nabla(\varphi))) = D_1\varphi\varphi(\xi\nabla(\varphi)) \leq D_1\varphi\varphi(D'\nabla(\varphi)\nabla(\varphi)) \\ &= D'(D'\varphi\nabla(\varphi))(D'\varphi\nabla(\varphi)) = D'\nabla(\varphi)\nabla(\varphi), \end{aligned}$$

which by definition of $\mathcal{B}$ means that $D_2\xi \in \mathcal{B}$; thus we have $D_2\mathcal{B} \subseteq \mathcal{B}$ and $\delta = \mathbb{I}(D_2) \in \mathcal{B}$.

Corollary 1. *There is* $\{A, C, C', D', \mathbb{I}\}$*-expressible* $\kappa \in \mathcal{F}$ *such that for all* $\varphi \in \mathcal{F}$ *we have*

$$\kappa\nabla(\varphi) = \nabla^2(\varphi) = \nabla(\nabla(\varphi)).$$

Proof. Define $D_3 \in \mathcal{F}$ so that the equality

$$D_3\vartheta(D'\eta\zeta) = D'\eta(\vartheta\zeta)$$

holds for all $\vartheta, \eta, \zeta \in \mathcal{F}$. Next define $\delta_1 \in \mathcal{F}$ to satisfy

$$\delta_1\vartheta\xi = D_3\vartheta(\delta\xi)$$

for all $\vartheta, \xi \in \mathcal{F}$, and define $\kappa = \mathbb{I}(\delta_1)$. Then

$$\kappa\nabla(\varphi) = \delta_1\kappa\nabla(\varphi) = D_3\kappa(\delta\nabla(\varphi)) = D_3\kappa(D'\nabla(\varphi)\nabla(\varphi)) = D'\nabla(\varphi)(\kappa\nabla(\varphi)),$$

whence

$$\nabla^2(\varphi) \leq \kappa\nabla(\varphi).$$

To prove the reverse inequality, consider the admissible iteration domain of first kind $\mathcal{A}_1 = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq \nabla^2(\varphi)\}$. If $\xi \in \mathcal{A}_1$, then

$$\delta_1\xi\nabla(\varphi) = D_3\xi(\delta\nabla(\varphi)) = D_3\xi(D'\nabla(\varphi)\nabla(\varphi)) = D'\nabla(\varphi)(\xi\nabla(\varphi))$$
$$\leq D'\nabla(\varphi)\nabla^2(\varphi) = \nabla^2(\varphi),$$

which shows that $\delta_1\xi \in \mathcal{A}_1$. Thus $\delta_1\mathcal{A}_1 \subseteq \mathcal{A}_1$ and $\kappa = \mathbb{I}(\delta_1) \in \mathcal{A}_1$.

The next theorem is Lemma 5.5 in [1], stated for strictly iterative CLCA instead of iterative ones.

Theorem 3. *There are $\{A, C, C', K, \mathbb{I}\}$-expressible $\iota \in \mathcal{F}$ and $\{A, C, C', D', \mathbb{I}\}$-expressible $\mu \in \mathcal{F}$ such that for all $\varphi, \psi \in \mathcal{F}$ the following two equalities hold:*
(a) $\iota\nabla(\varphi) = \mathbb{I}(\varphi)$;
(b) $\mu\nabla(\varphi)\nabla(\psi) = \nabla(\varphi\psi)$.

Proof. There is an $\{A, C, C'\}$-expressible element $e \in \mathcal{F}$ such that for all ξ, η, ζ, ζ' in $\mathcal{F}$ we have

$$e\xi\eta(D'\zeta\zeta') = \xi\zeta(\eta\zeta').$$

We shall show that for all $\varphi, \chi \in \mathcal{F}$

$$\mathbb{I}(e\chi)\nabla(\varphi) = \mathbb{I}(\chi\varphi). \tag{4}$$

Indeed,

$$\chi\varphi(\mathbb{I}(e\chi)\nabla(\varphi)) = e\chi\mathbb{I}(e\chi)(D'\varphi\nabla(\varphi)) = \mathbb{I}(e\chi)\nabla(\varphi),$$

whence $\mathbb{I}(\chi\varphi) \leq \mathbb{I}(e\chi)\nabla(\varphi)$. To prove the reverse inequality, consider the admissible iteration domain $\mathcal{A}$ of first kind, defined by

$$\mathcal{A} = \{\xi \in \mathcal{F} \mid \xi\nabla(\varphi) \leq \mathbb{I}(\chi\varphi)\}.$$

If $\xi \in \mathcal{A}$, then

$$e\chi\xi\nabla(\varphi) = e\chi\xi(D'\varphi\nabla(\varphi)) = \chi\varphi(\xi\nabla(\varphi)) \leq \chi\varphi\mathbb{I}(\chi\varphi) = \mathbb{I}(\chi\varphi),$$

i.e. $e\chi\xi \in \mathcal{A}$. Since $\mathcal{F}$ is supposed strictly iterative, this implies $\mathbb{I}(e\chi) \in \mathcal{A}$, which means that $\mathbb{I}(e\chi)\nabla(\varphi) \leq \mathbb{I}(\chi\varphi)$ and proves (4). For $\iota = \mathbb{I}(eI)$, where $I = A(CA)K$, this gives the equality (a) of the theorem. To define μ, consider an $\{A, C, C', D', \mathbb{I}\}$-expressible element $b \in \mathcal{F}$ such that for all ξ, η, ζ, ζ' in $\mathcal{F}$ we have

$$b\xi\eta(D'\zeta\zeta') = D'(\xi\zeta)(\eta\zeta').$$

The equalities

$$D'(\varphi\psi)(\mathbb{I}(b\varphi)\nabla(\psi)) = b\varphi\mathbb{I}(b\varphi)(D'\psi\nabla(\psi)) = \mathbb{I}(b\varphi)\nabla(\psi)$$

show that $\nabla(\varphi\psi) \leq \mathbb{I}(b\varphi)\nabla(\psi)$, and the reverse equality follows from the inclusion $b\varphi\mathcal{B} \subseteq \mathcal{B}$ for the admissible iteration domain $\mathcal{B}$ of first kind defined by

$$\mathcal{B} = \{\xi \in \mathcal{F} \mid \xi\nabla(\psi) \leq \nabla(\varphi\psi)\}.$$

Indeed, for $\xi \in \mathcal{B}$ we have

$$b\varphi\xi\nabla(\psi) = b\varphi\xi(D'\psi\nabla(\psi)) = D'(\varphi\psi)(\xi\nabla(\psi)) \leq D'(\varphi\psi)\nabla(\varphi\psi) = \nabla(\varphi\psi),$$

i.e. $b\varphi\xi \in \mathcal{B}$. Therefore $\mathbb{I}(b\varphi)\nabla(\psi) = \nabla(\varphi\psi)$, and defining $\mu = \mathbb{I}(eb)$, we obtain from (4) the equality (b) of the theorem.

Corollary 2 (First normal form theorem). *There is an element $\lambda \in \mathcal{F}$, recursive in $\{A, C, K, C', D'\}$, such that for every recursive in $\mathcal{C} \subseteq \mathcal{F}$ mapping $\Gamma : \mathcal{F} \to \mathcal{F}$ there is an $\mathcal{C} \cup \{A, C, C', D'\}$-expressible mapping $\Delta : \mathcal{F} \to \mathcal{F}$ such that $\Gamma(\xi) = \lambda\mathbb{I}(\Delta(\xi))$ for all $\xi \in \mathcal{F}$.*

Proof. By Proposition 1.2 in [1] and the proof of Theorem 1 we have $\Gamma(\xi) = L(L\mathbb{I}(\mathbb{I}(c_{\Gamma'})))$ for suitable $\mathcal{C} \cup \{\xi\}$-expressible mapping $\Gamma' : \mathcal{F} \to \mathcal{F}$. It is clear by the definition of $c_{\Gamma'}$ in the proof of Theorem 1 that $c_{\Gamma'} = \Delta'(\xi)$ for certain $\mathcal{C}$-expressible mapping $\Delta' : \mathcal{F} \to \mathcal{F}$ and all ξ in $\mathcal{F}$. Then by Corollary 1 and Theorem 3 we have

$$\begin{aligned}\Gamma(\xi) &= L(L(\iota\nabla(\iota\nabla(\Delta'(\xi))))) = L(L(\iota(\mu\nabla(\iota)\nabla^2(\Delta'(\xi)))))\\ &= ALL(A\iota(\mu\nabla(\iota))(\kappa\nabla(\Delta'(\xi)))) = A(ALL)(A\iota(\mu\nabla(\iota)))(\kappa\nabla(\Delta'(\xi)))\\ &= A(A(ALL)(A\iota(\mu\nabla(\iota))))\kappa\mathbb{I}(D'\Delta'(\xi)),\end{aligned}$$

and we can take $D'\Delta'(\xi)$ for $\Delta(\xi)$ and $A(A(ALL)(A\iota(\mu\nabla(\iota))))\kappa$ for λ.

Corollary 3. *The algebra $\mathcal{F}$ is a combinatory algebra with respect to the application operation* App, *defined by* $\texttt{App}(\varphi, \psi) = \varphi\nabla(\psi)$, *and with recursive in $\{A, C, K, C', D'\}$ combinators.*

Proof. This follows from Propositions 1 and 2 in [2] and [3], since ∇ is a 'DW-producing' operator (a storage operation would be a better terminology) in terms of [2]. By definition, the last means that there are five constants I^*, M^*, Q^*, P^*, D^* in $\mathcal{F}$ such that the following five equalities hold for all $\varphi, \psi \in \mathcal{F}$:

$$I^*\nabla(\varphi) = \varphi; \tag{5}$$

$$M^*\nabla(\varphi)\nabla(\psi) = \nabla(\varphi\psi); \tag{6}$$

$$Q^*\nabla(\varphi) = \nabla^2(\varphi); \tag{7}$$

$$P^*\nabla(\varphi)\psi = \psi; \tag{8}$$

$$D^*\nabla(\varphi) = D\varphi\varphi, \tag{9}$$

where D is an $\{A, C\}$-expressible element of $\mathcal{F}$ such that

$$D\varphi\psi\chi = \chi\varphi\psi$$

for all φ, ψ, χ in $\mathcal{F}$. We may find such elements I^*, M^*, Q^*, P^*, D^*, as follows. Define $I^* = L$; $M^* = \mu$ (the element defined in Theorem 3); $Q^* = \kappa$ (defined by Corollary 1); $P^* = A(AR)D'$, where R is an $\{A, C, K, C'\}$-expressible element of $\mathcal{F}$ such that $R(D'\xi\eta) = \eta$ for all $\xi, \eta \in \mathcal{F}$; and define D^* by the condition that

$$D^*(D'\xi\eta) = C'D(D'\xi(L\eta))$$

for all $\xi, \eta \in \mathcal{F}$. Then the equalities (5)–(7) are immediate and for the last two ones we have

$$P^*\nabla(\varphi)\psi = AR(D'\nabla(\varphi))\psi = R(D'\nabla(\varphi)\psi) = \psi$$

and

$$D^*\nabla(\varphi) = D^*(D'\varphi\nabla(\varphi)) = C'D(D'\varphi(L\nabla(\varphi))) = C'D(D'\varphi\varphi) = D\varphi\varphi.$$

The equalities (5)–(9) form with the basic equalities for the constants A and C a combinatory type-free variant of axioms for a 'decomposed' application operation (in the sense of the decomposition of the application first observed by Girard for his coherence spaces semantics of the typed lambda calculus and used by him for the development of linear logic). The fact that they imply the usual combinatory axioms for the operation **App** can be easily verified by a direct calculation, as follows. Define

$$K^* = A(AI^*)P^*,$$

and define S^* as a $\{A, C, I^*, M^*, Q^*, D^*\}$-expressible element such that

$$S^*\xi\eta\zeta = C(S_0\xi\eta)(D^*\zeta)$$

for all $\xi, \eta, \zeta \in \mathfrak{F}$, where S_0 is an $\{A, C, I^*, M^*, Q^*\}$-expressible element such that for all $\xi, \eta, \zeta, \vartheta \in \mathfrak{F}$ we have

$$S_0\xi\eta\zeta\vartheta = I^*\xi\zeta(M^*\eta(Q^*\vartheta)).$$

Then for all $\varphi, \psi, \zeta \in \mathfrak{F}$ we have

$$\begin{aligned}\mathtt{App}(\mathtt{App}(K^*, \varphi), \psi) &= K^*\nabla(\varphi)\nabla(\psi) = AI^*(P^*\nabla(\varphi))\nabla(\psi)\\ &= I^*(P^*\nabla(\varphi)\nabla(\psi)) = I^*\nabla(\psi) = \psi;\end{aligned}$$

and

$$\begin{aligned}\mathtt{App}(\mathtt{App}(\varphi, \zeta), \mathtt{App}(\psi, \zeta)) &= \varphi\nabla(\zeta)\nabla(\psi\nabla(\zeta))\\ &= I^*\nabla(\varphi)\nabla(\zeta)(M^*\nabla(\psi)(Q^*\nabla(\zeta))) = S_0\nabla(\varphi)\nabla(\psi)\nabla(\zeta)\nabla(\zeta)\\ &= D\nabla(\zeta)\nabla(\zeta)(S_0\nabla(\varphi)\nabla(\psi)) = C(S_0\nabla(\varphi)\nabla(\psi))(D\nabla(\zeta)\nabla(\zeta))\\ &= C(S_0\nabla(\varphi)\nabla(\psi))(D^*\nabla(\zeta)) = S^*\nabla(\varphi)\nabla(\psi)\nabla(\zeta)\\ &= \mathtt{App}(\mathtt{App}(\mathtt{App}(S^*, \varphi), \psi), \zeta).\end{aligned}$$

Corollary 4 (Second normal form theorem). *For every recursive in $\mathfrak{C} \subseteq \mathfrak{F}$ mapping $\Phi : \mathfrak{F} \to \mathfrak{F}$ there is recursive in $\mathfrak{C} \cup \{A, C, K, C', D'\}$ element $\varphi \in \mathfrak{F}$ such that $\Phi(\xi) = \varphi\nabla(\xi) = \varphi\mathbb{I}(D'\xi)$ for all $\xi \in \mathfrak{F}$.*

Proof. It follows easily from (5) that the original application in $\mathfrak{F}$ is explicitly expressible through application operation **App** from the last corollary. (Indeed, the element $a = A(CI^*)(AAI^*)$ satisfies the equality $a\xi\eta = I^*\xi(I^*\eta)$ for all $\xi, \eta \in \mathfrak{F}$, whence

$$\varphi\psi = I^*\nabla(\varphi)(I^*\nabla(\psi)) = a\nabla(\varphi)\nabla(\psi) = \mathtt{App}(\mathtt{App}(a, \varphi), \psi)$$

for all $\varphi, \psi \in \mathfrak{F}$.) Then Corollary 3 implies that every $\mathfrak{C}$-expressible mapping $\Gamma : \mathfrak{F} \to \mathfrak{F}$ is representable in the form $\Gamma(\xi) = \gamma\nabla(\xi)$ for certain $\gamma \in \mathfrak{F}$ recursive in $\mathfrak{C} \cup \{A, C, K, C', D'\}$. Thence by Corollary 2

$$\Phi(\xi) = \lambda\mathbb{I}(\gamma\nabla(\xi)) = \lambda(\iota\nabla(\gamma\nabla(\xi)))$$

for a similar γ, and by (6) and (7) we get an element $\varphi \in \mathfrak{F}$ satisfying the conditions of Corollary 4.

Remark 1. The element γ in the last proof of Corollary 4 is actually $\mathfrak{C} \cup \{A, C, K, C', D', \mathbb{I}\}$-expressible and this is seen without using Theorem 1. Then the inequality $\Gamma(\xi) \leq \xi$ being equivalent to $\gamma\nabla(\xi) \leq \xi$, any inequality of this kind may be reduced to a system of inequalities of the form

$$\gamma\eta \leq \xi, \quad D'\xi\eta \leq \eta, \tag{10}$$

since the first member ξ_0 of the least solution (ξ_0, η_0) of the last system with respect to ξ, η is the least solution of $\gamma\nabla(\xi) \leq \xi$ and therefore of $\Gamma(\xi) \leq \xi$. On the other hand, it is easy to see that if ζ_0 is the least solution of the inequality

$$D'(\gamma(R\zeta))\zeta \leq \zeta,$$

where $R = C'K'$ and $K'\xi'\eta' = \eta'$ for all $\xi', \eta' \in \mathcal{F}$, then $\xi_0 = L\zeta_0$ and $\eta_0 = R\zeta_0$ is the least solution of (10). Thus, the inequality $\Gamma(\xi) \leq \xi$ may be reduced to an inequality of the form $\varphi\zeta\zeta \leq \zeta$ for certain $\mathcal{C} \cup \{A, C, K, C', D', \mathbb{I}\}$-expressible $\varphi \in \mathcal{F}$. Hence, it would be enough to prove Theorem 1 for mappings of the form $\Gamma(\xi) = \varphi\xi\xi$, for which we need condition I_3) for such mappings only.

Remark 2. Admissible iteration domains of fourth kind were used in the proof of Theorem 2 only. We may exclude them from axioms by the restricting condition I_1) to such domains of first kind. Still all results above remain valid if we replace the storage operation ∇ with its square $\nabla^2(\varphi) = \nabla(\nabla(\varphi))$. This can be shown by using the obvious analogue of the normal form theorem from [1] (for strictly iterative CLCA instead of iterative ones), which is seen to hold (even after such excluding of admissible iteration domains of fourth kind) in the same way as in [1].

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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A CATEGORICAL FRAMEWORK FOR CODE EVALUATION METHOD*

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In the middle of the seventies Skordev proposed to consider in general the so-called fixpoint complete partially ordered algebras, introduced in [3]. The code evaluation method is an universal method for establishing a fixpoint completeness of such algebras. Its principal result — the code evaluation theorem (or the coding theorem, as it was called before) — implies easily all basic results of algebraic recursion theory. In the present work we give a categorical analysis of code evaluation proofs for operative spaces. Thus we obtain an algebraic formulation of the fundamentals of recursion theory which can be considered as an abstract recursion theory of higher level — by one level higher, compared with the usual theory of operative spaces [1]; and it may be otherwise considered as a generalization of the last theory in a new categorical direction, in which the role of multiplication in partially ordered semigroups is played by some kind of weak tensor product in partially ordered (weak) premonoidal categories.

Keywords: least fixed point of an endofunctor, operative space

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1. CATEGORICAL PRELIMINARIES

Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. By $\mathcal{C}_F$ we denote the category of *F-algebras in* $\mathcal{C}$; objects of $\mathcal{C}_F$ are the arrows $\varphi : F(X) \to X$ in $\mathcal{C}$, and arrows between two objects $\varphi : F(X) \to X$ and $\psi : F(Y) \to Y$ of $\mathcal{C}$ are the arrows $f : X \to Y$ in $\mathcal{C}$ such that $f \circ \varphi = \psi \circ F(f)$. The least fixed point of

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F is an initial object $\underline{m} : F(M) \to M$ in $\mathcal{C}_F$. For every object $\varphi : F(X) \to X$ of $\mathcal{C}_F$ there is an unique $h : M \to X$ in $\mathcal{C}$ such that $h \circ \underline{m} = \varphi \circ F(h)$. The last equality can be considered as an abstract definition of the *evaluator* h by primitive recursive iteration. To treat the more general case with primitive recursion, we use the following concept: Let Δ and F' be endofunctors in $\mathcal{C}$, and let $\vartheta : \Delta \circ F \dot{\to} F' \circ \Delta$ be a natural transformation. Then we say that a *parameterized evaluation* holds for the least fixed point $\underline{m} : F(M) \to M$ of F with respect to Δ, F' and ϑ iff for every arrow $f : F'(X) \to X$ in $\mathcal{C}$ there is an unique $\mathcal{C}$-arrow $\xi : \Delta(M) \to X$ such that $\xi \circ \Delta(\underline{m}) = f \circ F'(\xi) \circ \vartheta$:

$$\begin{array}{ccccc}
\Delta(F(M)) & & \xrightarrow{\quad\Delta(\underline{m})\quad} & & \Delta(M) \\
\Big\downarrow{\scriptstyle\vartheta} & & & & \Big\downarrow{\scriptstyle\xi} \\
F'(\Delta(M)) & \xrightarrow{F'(\xi)} & F'(X) & \xrightarrow{f} & X
\end{array}$$

Theorem 1.1. *Suppose $\mathcal{C}$ has an initial object O and co-limits of all ω-sequences $X_0 \to X_1 \to \cdots$, and the functors F and Δ commute with those co-limits and $\Delta(O) \cong O$. Then the least fixed point $\underline{m} : F(M) \to M$ of F exists and a parameterized evaluation holds for it with respect to Δ, F' and ϑ, where F' and ϑ are arbitrary.*

Proof. The least fixed point $\underline{m} : F(M) \to M$ is obtained from a limiting cone $\bar\vartheta_n : F^n(O) \to M$ of the sequence $O \to F(O) \to F^2(O) \to \cdots$ of arrows $\vartheta_n = F^n(\vartheta_0) : F^n(O) \to F^{n+1}(O)$ in a well-known way, namely: since F preserves these co-limits, then $F(\bar\vartheta_n) : F^{n+1}(O) \to F(M)$ is a limiting cone for the sequence $F(O) \to F^2(O) \to \cdots$, whence there is an unique $\underline{m} : F(M) \to M$ such that $\bar\vartheta_{n+1} = \underline{m} \circ F(\bar\vartheta_n)$ for all natural n; this $\underline{m}$ is the least fixed point of F.

Now let $f : F'(X) \to X$ be an $\mathcal{C}$-arrow. Since Δ commutes with co-limits of ω-sequences in $\mathcal{C}$, we have a limiting cone

$$\Delta(\bar\vartheta_n) : \Delta(F^n(O)) \to \Delta(M) \tag{1}$$

for the sequence

$$\Delta(\vartheta_n) : \Delta(F^n(O)) \to \Delta(F^{n+1}(O)).$$

Define a sequence of arrows $\xi_n : \Delta(F^n(O)) \to X$ by induction on n: ξ_0 is determined uniquely, since $\Delta(O)$ is an initial object in $\mathcal{C}$, and $\xi_{n+1} = f \circ F'(\xi_n) \circ \vartheta$. Then by induction on n we have

$$\xi_n = \xi_{n+1} \circ \Delta(\vartheta_n). \tag{2}$$

Indeed, for $n = 0$ this is trivial, since $\Delta(O)$ is an initial object in $\mathcal{C}$, and for the induction step:

$$\begin{aligned}
\xi_{n+2} \circ \Delta(\vartheta_{n+1}) &= f \circ F'(\xi_{n+1}) \circ \vartheta \circ \Delta(\vartheta_{n+1}) = f \circ F'(\xi_{n+1}) \circ F'(\Delta(\vartheta_n)) \circ \vartheta \\
&= f \circ F'(\xi_{n+1} \circ \Delta(\vartheta_n)) \circ \vartheta = f \circ F'(\xi_n) \circ \vartheta = \xi_{n+1}.
\end{aligned}$$

From the limiting cone (1) we obtain an unique arrow $\xi : \Delta(M) \to X$ such that $\xi_n = \xi \circ \Delta(\bar\vartheta_n)$ for all n. Next we show that

$$\xi \circ \Delta(\underline{m}) = f \circ F'(\xi) \circ \vartheta \tag{3}$$

by proving that for all n

$$\xi_n = f \circ F'(\xi) \circ \vartheta \circ \Delta(\underline{\mathrm{m}}^{-1}) \circ \Delta(\bar{\vartheta}_n).$$

For $n = 0$ the last equality is trivial, and for $n > 0$ we have

$$\begin{aligned} f \circ F'(\xi) \circ \vartheta \circ \Delta(\underline{\mathrm{m}}^{-1} \circ \bar{\vartheta}_n) &= f \circ F'(\xi) \circ \vartheta \circ \Delta(F(\bar{\vartheta}_{n-1})) \\ &= f \circ F'(\xi) \circ F'(\Delta(\bar{\vartheta}_{n-1}))) \circ \vartheta = f \circ F'(\xi_{n-1}) \circ \vartheta = \xi_n. \end{aligned}$$

Conversely, if $\xi : \Delta(M) \to X$ satisfies (3), then for all n

$$\xi_n = \xi \circ \Delta(\bar{\vartheta}_n), \tag{4}$$

whence it follows that the arrow ξ, satisfying (3), is unique. For $n = 0$ the equality (4) is obvious, and for the other cases we use induction:

$$\begin{aligned} \xi \circ \Delta(\bar{\vartheta}_{n+1}) = \xi \circ \Delta(\underline{\mathrm{m}} \circ F(\bar{\vartheta}_n)) &= f \circ F'(\xi) \circ \vartheta \circ \Delta(F(\bar{\vartheta}_n)) \qquad \text{(by (3))} \\ &= f \circ F'(\xi) \circ F'(\Delta(\bar{\vartheta}_n)) \circ \vartheta = f \circ F'(\xi_n) \circ \vartheta = \xi_{n+1}. \end{aligned}$$

The term 'parameterized evaluation' is motivated by the following example: $\mathcal{C}$ is the category of sets, F is a 'polinomial' $F(X) = \sum_{j=0}^{m} A_j \times X^j$, $\Delta(X) = Y \times X$ and $F'(X) = Y \times F(X)$ for a fixed set Y of 'parameters'. An F-algebra $f : F(X) \to X$ in $\mathcal{C}$ is then an universal algebra with a set (corresponding to A_j) of j-ary operations for all $j \leq m$. For the least fixed point $\underline{\mathrm{m}} : F(M) \to M$ of F, M is the set of terms freely generated by those operations. The equality (3) then may be interpreted as

$$\xi(y, a(t_0, \ldots, t_{j-1})) = f(y, a(\xi(y, t_0), \ldots, \xi(y, t_{j-1}))),$$

where $y \in Y$ is a parameter, a is a j-ary operation from basic ones, and $t_0, \ldots, t_{j-1}$ are terms from M; it is a definition of ξ by some kind of parameterized recursion.

A *partially ordered category* is a category $\mathcal{C}$ with partial order in every hom-set such that a composition of arrows is increasing on both arguments. We denote the partial order with the usual symbol $\leq$, i.e. $f \leq g$ for two arrows in a partially ordered category $\mathcal{C}$ means that f and g have the same domain and co-domain and f precedes g in the sense of the partial order in the corresponding hom-set. The universal example of partially ordered category is the category of posets and increasing mappings, shortly referred to as 'category of posets'. The partial order in the last category is defined in an obvious way: $f \leq g$ means that $f(x) \leq g(x)$ for all x in the domain of f and g.

The notion of increasing functor $F : \mathcal{C} \to \mathcal{D}$ between two partially ordered categories $\mathcal{C}$ and $\mathcal{D}$ is also obvious: F is increasing iff $f \leq g$ implies $F(f) \leq F(g)$ for every pair of arrows f, g in $\mathcal{C}$.

Let $\mathcal{C}$ be a partially ordered category and let $F : \mathcal{C} \to \mathcal{C}$ be an increasing endofunctor in $\mathcal{C}$. Then we shall call an F-algebra $\underline{\mathrm{m}} : F(M) \to M$ in $\mathcal{C}$ a *proper least fixed point* of F iff for every F-algebra $f : F(X) \to X$ in $\mathcal{C}$ there is an arrow $\tilde{f} : M \to X$ in $\mathcal{C}$ such that $\tilde{f} \circ \underline{\mathrm{m}} = f \circ F(\tilde{f})$ and the following two conditions hold moreover:

a) for every $\mathcal{C}$-arrow $\varphi : M \to X$, such that $f \circ F(\varphi) \leq \varphi \circ \underline{\mathrm{m}}$, we have $\tilde{f} \leq \varphi$;

b) for every $\mathcal{C}$-arrow $\varphi : M \to X$, such that $\varphi \circ \underline{\mathrm{m}} \leq f \circ F(\varphi)$, we have $\varphi \leq \tilde{f}$.

Obviously, a proper least fixed point of F is also such one in the usual sense, and $\tilde{f}$ is the corresponding evaluator for any F-algebra f. We shall desist from discussing general criteria of existence of proper least fixed points, restricting ourselves with the remark that this is a natural concept. Typically, the usual least fixed points of increasing endofunctors are proper. For instance: consider the 'polinomial' $F(X) = \sum_{j=0}^{m} A_j \times X^j$ in the category $\mathcal{C}$ of partially ordered sets and increasing mappings. The category $\mathcal{C}$ is partially ordered in an obvious way: $f \leq g$ means that $f(x) \leq g(x)$ for every x in the domain of f and g. The least fixed point $\underline{m} : F(M) \to M$ exists and the object M is the set of all terms generated by j-ary operations corresponding to the elements of A_j (for all $j \leq m$) with the trivial partial order coinciding with the equality. This least fixed point is proper one and conditions a) and b) express in abstract way the possibility of proving inequalities by induction on complexity of terms (for instance in a) we prove $\tilde{f}(t) \leq \varphi(t)$ by induction on the complexity of the term $t \in M$).

2. NORMAL EVALUATION IN STRUCTURED RING-CATEGORIES

A structured ring-category (shortly, SRC) is by definition a 5-tuple $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$, where: $\mathcal{C}$ is a category with finite co-products and co-limits of ω-sequences $X_0 \to X_1 \to \cdots$; $\odot$ is a bi-endofunctor $\mathcal{C}^2 \to \mathcal{C}$ in $\mathcal{C}$ such that for any fixed object Y of $\mathcal{C}$ the functor $Y \odot \_$ preserves those co-products and co-limits; $\bar{a} : X \odot (Y \odot Z) \to (X \odot Y) \odot Z$ is a natural transformation (not necessarily isomorphism) satisfying Mac Lane pentagonal $\bar{a} \circ \bar{a} = (\bar{a} \odot 1) \circ \bar{a} \circ (1 \odot \bar{a})$; R is an endofunctor in $\mathcal{C}$ and $\vartheta_R : X \odot R(Y) \to R(X \odot Y)$ is a natural in $X, Y \in \mathcal{C}$ transformation satisfying *R-coherence*: $R(\bar{a}) \circ \vartheta_R \circ (1 \odot \vartheta_R) = \vartheta_R \circ \bar{a}$, i.e. the following commutative diagram:

$$\begin{array}{ccccc} X \odot (Y \odot R(Z)) & \xrightarrow{(1 \odot \vartheta_R)} & X \odot R(Y \odot Z) & \xrightarrow{\vartheta_R} & R(X \odot (Y \odot Z)) \\ \downarrow \bar{a} & & & & \downarrow R(\bar{a}) \\ (X \odot Y) \odot R(Z) & & \xrightarrow{\qquad\qquad \vartheta_R \qquad\qquad} & & R((X \odot Y) \odot Z) \end{array}$$

We fix a SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ and we shall write shortly $\mathcal{C}$ for the last 5-tuple. To ensure existence of some least fixed points and applicability of Theorem 1.1 in some cases below, we shall suppose that the endofunctors R, $F_1(X) = X \odot X$ and $F_2(X) = X \odot B$ in $\mathcal{C}$ commute with co-limits of ω-sequences $X_0 \to X_1 \to \cdots$ for any fixed object B of $\mathcal{C}$. Binary co-products in $\mathcal{C}$ will be denoted by $+$, and we write I_0 and I_1 for the canonical monics $X_i \to X_0 + X_1$ of the co-product $X_0 + X_1$ $(i = 0, 1)$. Thus I_0 and I_1 are natural in X_0, X_1 transformations, and we shall use short notations for their compositions, for instance: $I_{01} = I_0 \circ I_1$, $I_{101} = I_1 \circ I_0 \circ I_1$, etc. We also write $[f_0, f_1]$ for the unique arrow $X_0 + X_1 \to Y$ such that $[f_0, f_1] \circ I_i = f_i : X_i \to Y$ $(i = 0, 1)$. Since the functor $Y \odot \_$ preserves binary co-products, there is an isomorphism

$$\delta_{\odot} : Y \odot (X_0 + X_1) \to (Y \odot X_0) + (Y \odot X_1),$$

natural in Y, X_0, X_1, such that $\delta_{\odot}^{-1} = [1 \odot I_0, 1 \odot I_1]$. This means that $\delta_{\odot} \circ (1 \odot I_i) = I_i$ for both $i = 0, 1$, and for every pair of arrows $\varphi, \psi : Y \odot (X_0 + X_1) \to A$ in $\mathcal{C}$, such that $\varphi \circ (1 \odot I_i) = \psi \circ (1 \odot I_i)$ for both $i = 0, 1$, we have $\varphi = \psi$. A proof of an equality $\varphi = \psi$ of this kind based on the last principle will be called below a 'proof of $\varphi = \psi$ by considering cases'.

Algebraic structures concerning the present paper are represented in this context by *standard $\mathcal{C}$-algebras*, i.e. arrows $a : (X \odot X) + R(X) \to X$ in $\mathcal{C}$, satisfying the following two equalities:

$$a_0 \circ (1 \odot a_0) = a_0 \circ (a_0 \odot 1) \circ \bar{a}, \tag{5}$$

$$a_0 \circ (1 \odot a_1) = a_1 \circ R(a_0) \circ \vartheta_R, \tag{6}$$

where $a_i = a \circ I_i$, $i = 0, 1$. The equality (5) means that $a_0 : X \circ X \to X$ is a 'premonoid' in $\mathcal{C}$ (note that $\mathcal{C}$ is not supposed to be premonoidal category with respect to $\odot$, since the associativity transformation $\bar{a}$ may not be an isomorphism). Equality (6) corresponds to the equality $(\varphi, \psi)\chi = (\varphi\chi, \psi\chi)$ in operative spaces in notations of Ivanov [1]. Thus operative spaces are standard $\mathcal{C}$-algebras in the SRC of sets, i.e. the SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$, where $\mathcal{C}$ is the category of sets, $\odot$ is the usual Cartesian product, $R(X) = X \times X$, $\bar{a}$ is the usual associativity isomorphism, and $\vartheta_R : X \times (Y \times Y) \to (X \times Y) \times (X \times Y)$ is the natural transformation defined by

$$\vartheta_R((x, (y, y'))) = ((x, y), (x, y')) \qquad (x \in X,\ y, y' \in Y).$$

The forgetful functor $P : \mathbf{SA}(\mathcal{C}) \to \mathcal{C}$ from the category $\mathbf{SA}(\mathcal{C})$ of standard $\mathcal{C}$-algebras to $\mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathbf{SA}(\mathcal{C})$, which in the case of the SRC of sets assigns to each set X the free standard $\mathcal{C}$-algebra $L(X)$ generated by X. In the present section we shall give an explicit construction of this adjunction in terms of the least fixed points, and this construction is essential for the categorical axiomatization of code evaluation which we give in the next two sections 3 and 4. The reader will probably notice the analogy between sections 2 and 3.

Now consider two bi-endofunctors S and $\dot{S}$ in the fixed SRC $\mathcal{C}$, defined by

$$S(B, X) = B + ((X \odot X) + R(X))$$

and

$$\dot{S}(B, X) = B + ((X \odot B) + R(X))$$

for objects B, X of $\mathcal{C}$ and arrows as well. We shall fix the object B of $\mathcal{C}$ and write shortly $S(X)$ and $\dot{S}(X)$ for $S(B, X)$ and $\dot{S}(B, X)$, respectively (for arrows $S(f)$ means $S(B, f) = S(1_B, f)$ and, similarly, for $\dot{S}$). We have a functor $N : \mathcal{C}_S \to \mathcal{C}_{\dot{S}}$ from the category of S-algebras in $\mathcal{C}$ to that one of $\dot{S}$-algebras, defined by

$$N(f) = [f_0, [f_{10} \circ (1 \odot f_0), f_{11}]]$$

for objects $f : S(X) \to X$ of $\mathcal{C}_S$, i.e. S-algebras in $\mathcal{C}$, and $N(\varphi) = \varphi$ for arrows φ in $\mathcal{C}_S$, where $f_0 = f \circ I_0$, $f_{10} = f \circ I_{10}$, and $f_{11} = f \circ I_{11}$, as we shall write shortly below for any suitable arrow f in $\mathcal{C}$. N is indeed a functor, since for an arrow $\varphi : f \to g$ between two algebras $f : S(X) \to X$ and $g : (Y) \to Y$ in $\mathcal{C}_S$, i.e. an arrow $\varphi : X \to Y$ in $\mathcal{C}$ such that $\varphi \circ f = g \circ S(\varphi)$, we have

$$\begin{aligned}
\varphi \circ N(f) &= [\varphi \circ f_0, [\varphi \circ f_{10} \circ (1 \odot f_0), \varphi \circ f_{11}]] \\
&= [g \circ S(\varphi) \circ I_0, [g \circ S(\varphi) \circ I_{10} \circ (1 \odot f_0), g \circ S(\varphi) \circ I_{11}]] \\
&= [g_0, [g_{10} \circ (\varphi \odot \varphi) \circ (1 \odot f_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (\varphi \odot g \circ S(\varphi) \circ I_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (\varphi \odot g_0), g_{11} \circ R(\varphi)]] \\
&= [g_0, [g_{10} \circ (1 \odot g_0), g_{11}]] \circ (1_B + ((\varphi \odot 1_B) + R(\varphi))) \\
&= [g_0, [g_{10} \circ (1 \odot g_0), g_{11}]] \circ \dot{S}(\varphi) = N(g) \circ \dot{S}(\varphi),
\end{aligned}$$

i.e. $\varphi : N(f) \to N(g)$ in $\mathcal{C}_{\dot{S}}$.

Let $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$ be the least fixed points of S and $\dot{S}$, respectively, in $\mathcal{C}$. (Actually, T and $\dot{T}$ are endofunctors in $\mathcal{C}$ and $\tau(B) : S(B, T(B)) \to T(B)$ and $\dot{\tau}(B) : \dot{S}(B, \dot{T}(B)) \to \dot{T}(B)$ are natural in B isomorphisms.) In the SRC of sets T is the set of terms generated from elements of B by means of two binary operations — $\tau_{10} = \tau \circ I_{10}$ and $\tau_{11} = \tau \circ I_{11}$, and $\dot{T}$ is the set of *normal* terms.

Denote by $\mathcal{D}$ the full subcategory of $\mathcal{C}_S$, consisting of those S-algebras $f : S(X) \to X$ for which $f_1 = f \circ I_1 : (X \odot X) + R(X) \to X$ is a standard $\mathcal{C}$-algebra. We are looking for an algebra $f : S(X) \to X$ in $\mathcal{D}$ such that $N(f) = \dot{\tau}$.

To find such an algebra, consider the natural in $X, Y \in \mathcal{C}$ transformation

$$\dot{\vartheta} : Y \odot \dot{S}(X) \to \dot{S}(Y) + \dot{S}(Y \odot X),$$

defined by

$$\dot{\vartheta} = (I_{10} + I_1 \circ (\bar{a} + \vartheta_R) \circ \delta_\odot) \circ \delta_\odot.$$

This definition is equivalent to the following three equalities:

$$\dot{\vartheta} \circ (1 \odot I_0) = I_{010}, \tag{7}$$

$$\dot{\vartheta} \circ (1 \odot I_{10}) = I_{110} \circ \bar{a}, \tag{7'}$$

$$\dot{\vartheta} \circ (1 \odot I_{11}) = I_{111} \circ \vartheta_R. \tag{7''}$$

Proposition 2.1. *Every algebra $f : S(X) \to X$ from $\mathcal{D}$ satisfies the equality*

$$f_{10} \circ (1 \odot N(f)) = N(f) \circ [1, \dot{S}(f_{10})] \circ \dot{\vartheta}, \tag{8}$$

$$\begin{array}{ccccc}
X \odot \dot{S}(X) & & \xrightarrow{\quad 1 \odot N(f) \quad} & & X \odot X \\
\downarrow \dot{\vartheta} & & & & \downarrow f_{10} \\
\dot{S}(X) + \dot{S}(X \odot X) & \xrightarrow{[1, \dot{S}(f_{10})]} & \dot{S}(X) & \xrightarrow{N(f)} & X
\end{array}$$

Proof. By considering cases. Denoting by φ and ψ the left- and right-hand sides of (8), respectively, we shall conclude $\varphi = \psi$ by showing that $\varphi \circ (1 \odot I_0) = \psi \circ (1 \odot I_0)$, $\varphi \circ (1 \odot I_{10}) = \psi \circ (1 \odot I_{10})$ and $\varphi \circ (1 \odot I_{11}) = \psi \circ (1 \odot I_{11})$. Consider, for instance, the second of the last three equalities, leaving the other ones to the reader:

$$\varphi \circ (1 \odot I_{10}) = f_{10} \circ (1 \odot N(f) \circ I_{10}) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)),$$

and

$$\begin{aligned}\psi \circ (1 \odot I_{10}) &= N(f) \circ [1, \dot{S}(f_{10})] \circ I_{110} \circ \bar{a} = N(f) \circ \dot{S}(f_{10}) \circ I_{10} \circ \bar{a} \\ &= N(f) \circ I_{10} \circ (f_{10} \odot 1) \circ \bar{a} = f_{10} \circ (1 \odot f_0) \circ (f_{10} \odot 1) \circ \bar{a} \\ &= f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} = f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)),\end{aligned}$$

the last equality being the equality (5) for the algebra $f \in \mathcal{D}$. In the case with $(1 \odot I_{11})$ the equality (6) is used in a similar way.

Now suppose an algebra $f : S(X) \to X$ from $\mathcal{D}$ satisfies $N(f) = \dot{\tau}$. Then, obviously, $X = \dot{T}$, and composing the equality $N(f) = \dot{\tau}$ from right by I_0 and I_{11}, we obtain $f_0 = \dot{\tau}_0 = \dot{\tau} \circ I_0$ and $f_{11} = \dot{\tau}_{11} = \dot{\tau} \circ I_{11}$, whence f should be of the form $[\dot{\tau}_0, [\mu, \dot{\tau}_{11}]]$ for some arrow $\mu : \dot{T} \odot \dot{T} \to \dot{T}$ in $\mathcal{C}$. Then the equality $N(f) = \dot{\tau}$ is equivalent to $\mu \circ (1 \odot \dot{\tau}_0) = \dot{\tau}_{10} = \dot{\tau} \circ I_{10}$. If $f \in \mathcal{D}$, then by (8) we obtain

$$\mu \circ (1 \odot \dot{\tau}) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta}, \tag{9}$$

$$\begin{array}{ccccc} \dot{T} \odot \dot{S}(\dot{T}) & \xrightarrow{\;1 \odot \dot{\tau}\;} & \dot{T} \odot \dot{T} & \xrightarrow{\;\mu\;} & \dot{T} \\ \big\downarrow \dot{\vartheta} & & & & \big\uparrow \dot{\tau} \\ \dot{S}(\dot{T}) + \dot{S}(\dot{T} \odot \dot{T}) & & \xrightarrow{\quad [1, \dot{S}(\mu)] \quad} & & \dot{S}(\dot{T}) \end{array}$$

The last equality determines μ uniquely by the principle of the parameterized evaluation, i.e. by Theorem 1.1 (with the functors $\dot{S}$ for F, $\dot{T} \odot X$ for $\Delta(X)$ and $\dot{S}(\dot{T}) + F(X)$ for $F'(X)$). This suggests to *define* μ by (9). Then the arrow $f = [\dot{\tau}_0, [\mu, \dot{\tau}_{11}]]$ satisfies $N(f) = \dot{\tau}$, because a composition of (9) from right by $1 \odot I_0$ yields

$$\mu \circ (1 \odot \dot{\tau}_0) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_0) = \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} = \dot{\tau}_{10}.$$

The arrow f will be denoted below by τ^N, i.e.

$$\tau^N = [\dot{\tau}_0, [\mu, \dot{\tau}_{11}]],$$

where μ satisfies (9). Thus we see that $\tau^N : S(\dot{T}) \to \dot{T}$ is the unique S-algebra f satisfying (8), such that $N(f) = \dot{\tau}$.

Proposition 2.2. *For every S-algebra $f : S(X) \to X$ in $\mathcal{C}$ satisfying* (8) *there is an unique arrow $h : \tau^N \to f$ in $\mathcal{C}_S$, i.e. an unique $h : \dot{T} \to X$ in $\mathcal{C}$, such that*

$$h \circ \tau^N = f \circ S(h), \tag{10}$$

and the arrow h is the unique one $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$, i.e.

$$h \circ \dot{\tau} = N(f) \circ \dot{S}(h). \tag{11}$$

Proof. If $h : \tau^N \to f$ is an arrow in $\mathcal{C}_S$, then $h = N(h) : \dot{\tau} \to N(f)$ is such one in $\mathcal{C}_{\dot{S}}$, but the arrow $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$ is unique, since $\dot{\tau}$ is the least fixed point of $\dot{S}$. Therefore the arrow $h : \dot{T} \to X$, satisfying (10), can be only the unique arrow $h : \dot{\tau} \to N(f)$ in $\mathcal{C}_{\dot{S}}$. To show that the last arrow satisfies (10), we consider the cases:

$h \circ \tau^N \circ I_0 = h \circ \dot{\tau}_0 = N(f) \circ \dot{S}(h) \circ I_0 = N(f) \circ I_0 = f_0 = f \circ I_0 = f \circ S(h) \circ I_0;$

$h \circ \tau^N \circ I_{11} = h \circ \dot{\tau}_{11} = N(f) \circ \dot{S}(h) \circ I_{11} = N(f) \circ I_{11} \circ R(h) = f_{11} \circ R(h) = f \circ S(h) \circ I_{11};$

but $h \circ \tau^N \circ I_{10} = h \circ \mu$ and $f \circ S(h) \circ I_{10} = f_{10} \circ (h \odot h)$. Therefore it remains to show that

$$h \circ \mu = f_{10} \circ (h \odot h). \tag{12}$$

We shall do this by the principle of the parameterized evaluation. For that define $\varphi_0 = h \circ \mu$, $\varphi_1 = f_{10} \circ (h \odot h)$, and

$$\eta = N(f) \circ [\dot{S}(h), 1] : \dot{S}(\dot{T}) + \dot{S}(X) \to X.$$

By the principle of the parameterized evaluation (Theorem 1.1) there is an unique $\mathcal{C}$-arrow $\varphi : \dot{T} \odot \dot{T} \to X$ such that

$$\varphi \circ (1 \odot \dot{\tau}) = \eta \circ (1 + \dot{S}(\varphi)) \circ \dot{\vartheta}. \tag{13}$$

We shall show that both φ_0 and φ_1 satisfy (13) with respect to φ, whence it will follow (12) and the proof will be completed. For φ_0 this can be done without using (8):

$$\begin{aligned}\varphi_0 \circ (1 \odot \dot{\tau}) &= h \circ \mu \circ (1 \odot \dot{\tau}) = h \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} = N(f) \circ \dot{S}(h) \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \\ &= N(f) \circ [\dot{S}(h), \dot{S}(\varphi_0)] \circ \dot{\vartheta} = N(f) \circ [\dot{S}(h), 1] \circ (1 + \dot{S}(\varphi_0)) \circ \dot{\vartheta} \\ &= \eta \circ (1 + \dot{S}(\varphi_0)) \circ \dot{\vartheta}.\end{aligned}$$

For φ_1 the equality (8) is used:

$$\begin{aligned}\varphi_1 \circ (1 \odot \dot{\tau}) &= f_{10} \circ (h \odot h \circ \dot{\tau}) = f_{10} \circ (h \odot N(f) \circ \dot{S}(h)) \\ &= f_{10} \circ (1 \odot N(f)) \circ (h \odot \dot{S}(h)) = N(f) \circ [1, \dot{S}(f_{10})] \circ \dot{\vartheta} \circ (h \odot \dot{S}(h)) \\ &= N(f) \circ [1, \dot{S}(f_{10})] \circ (\dot{S}(h) + \dot{S}(h \odot h)) \circ \dot{\vartheta} \\ &= N(f) \circ [\dot{S}(h), \dot{S}(\varphi_1)] \circ \dot{\vartheta} = \eta \circ (1 + \dot{S}(\varphi_1)) \circ \dot{\vartheta}.\end{aligned}$$

Since $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$ are least fixed points, there are unique arrows $\dot{\nu} : T \to \dot{T}$ and $\iota : \dot{T} \to T$ such that

$$\dot{\nu} \circ \tau = \tau^N \circ S(\dot{\nu}) \tag{14}$$

and

$$\iota \circ \dot{\tau} = N(\tau) \circ \dot{S}(\iota), \tag{15}$$

respectively.

Proposition 2.3. *$\dot{\nu}$ is a retraction with inverse ι, i.e.*

$$\dot{\nu} \circ \iota = 1 = 1_{\dot{T}}.$$

Proof. It is enough to show that $\dot{\nu} \circ \iota \circ \dot{\tau} = \dot{\tau} \circ \dot{S}(\dot{\nu} \circ \iota)$, since the arrow $1 : \dot{\tau} \to \dot{\tau}$ in $\mathcal{C}_{\dot{T}}$ is unique. But

$$\begin{aligned}\dot{\nu} \circ \iota \circ \dot{\tau} &= \dot{\nu} \circ N(\tau) \circ \dot{S}(\iota) = \dot{\nu} \circ [\tau_0, [\tau_{10} \circ (1 \odot \tau_0), \tau_{11}]] \circ \dot{S}(\iota) \\ &= [\dot{\nu} \circ \tau_0, [\dot{\nu} \circ \tau_{10} \circ (1 \odot \tau_0), \dot{\nu} \circ \tau_{11}]] \circ \dot{S}(\iota)\end{aligned}$$

$$= [\tau^N \circ S(\dot\nu) \circ I_0, [\tau^N \circ S(\dot\nu) \circ I_{10} \circ (1 \odot \tau_0), \tau^N \circ S(\dot\nu) \circ I_{11}]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\mu \circ (\dot\nu \odot \dot\nu) \circ (1 \odot \tau_0), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\mu \circ (\dot\nu \odot \dot\tau_0), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\dot\tau_{10} \circ (\dot\nu \odot 1), \dot\tau_{11} \circ R(\dot\nu)]] \circ \dot S(\iota)$$
$$= [\dot\tau_0, [\dot\tau_{10}, \dot\tau_{11}]] \circ (1 + ((\dot\nu \odot 1) + R(\dot\nu))) \circ \dot S(\iota)$$
$$= \dot\tau \circ \dot S(\dot\nu) \circ \dot S(\iota) = \dot\tau \circ \dot S(\dot\nu \circ \iota).$$

Denote by ν the morphism $\iota \circ \dot\nu : T \to T$. This is the 'normalizing' morphism, in the SRC of sets ν assigns to each term its normal form. For any S-algebra $f : S(X) \to (X)$ in $\mathcal{C}$ denote by $\tilde f$ the evaluator of f with respect to τ, i.e. the unique arrow $\tilde f : T \to X$ such that $\tilde f \circ \tau = f \circ S(\tilde f)$. (In the case of SRC of sets $\tilde f$ assigns to each term in T its value in the algebra X.)

Corollary 2.1. *For any S-algebra $f : S(X) \to X$ in $\mathcal{C}$ the following conditions are equivalent*:

(a) $\tilde f \circ \nu = \tilde f$;

(b) *there is a morphism* $h : \tau^N \to f$ *in* $\mathcal{C}_S$;

(b$'$) *there is an unique morphism* $h : \tau^N \to f$ *in* $\mathcal{C}_S$;

(c) *there is a morphism* $h : \dot T \to X$ *in* $\mathcal{C}$ *such that* $h \circ \dot\nu = \tilde f$;

(c$'$) *there is an unique morphism* $h : \dot T \to X$ *in* $\mathcal{C}$ *such that* $h \circ \dot\nu = \tilde f$;

and when they hold, the unique arrows h in (b$'$) *and* (c$'$) *are the same as the evaluator of $N(f)$ with respect to $\dot\tau$ or the unique morphism $h : \dot\tau \to N(f)$ in $\mathcal{C}_{\dot S}$.*

Proof. (a) $\Rightarrow$ (b) Let $\tilde f \circ \nu = \tilde f$ and $h = \tilde f \circ \iota$. Then

$$\begin{aligned} h \circ \tau^N &= \tilde f \circ \iota \circ \tau^N \circ S(\dot\nu \circ \iota) && \text{(by Proposition 2.3)}\\ &= \tilde f \circ \iota \circ \dot\nu \circ \tau \circ S(\iota) && \text{(by (14))}\\ &= \tilde f \circ \nu \circ \tau \circ S(\iota) = \tilde f \circ \tau \circ S(\iota) = f \circ S(\tilde f) \circ S(\iota) = f \circ S(h). \end{aligned}$$

(b) $\Rightarrow$ (c) & (b$'$) Let $h : \dot T \to X$ and $h \circ \tau^N = f \circ S(h)$. Then

$$h \circ \dot\nu \circ \tau = h \circ \tau^N \circ S(\dot\nu) = f \circ S(h) \circ S(\dot\nu) = f \circ S(h \circ \dot\nu),$$

and by the uniqueness of the evaluator $\tilde f$ we have $\tilde f = h \circ \dot\nu$. Thence, also $\tilde f \circ \iota = h \circ \dot\nu \circ \iota = h$ and therefore the morphism $h : \tau^N \to f$ in $\mathcal{C}_S$ is unique.

(c) $\Rightarrow$ (a) Let $h : \dot T \to X$ and $h \circ \dot\nu = \tilde f$. Then

$$\tilde f \circ \nu = h \circ \dot\nu \circ \nu = h \circ \dot\nu \circ \iota \circ \dot\nu = h \circ \dot\nu = \tilde f.$$

(c) $\Rightarrow$ (c$'$) Because $h \circ \dot\nu = \tilde f$ implies $h = h \circ \dot\nu \circ \iota = \tilde f \circ \iota$.

The implications (b$'$) $\Rightarrow$ (b) and (c$'$) $\Rightarrow$ (c) are trivial. The equivalence of (a)–(c$'$) is proved. If they hold, then for the morphism $h : \tau^N \to f$ in (b$'$) we have $h = N(h) : \dot\tau = N(\tau^N) \to N(f)$ in $\mathcal{C}_{\dot S}$. From the proof of (b) $\Rightarrow$ (c) it is clear also that the morphism satisfying (b$'$) coincides with the unique morphism in (c$'$).

The morphism h from this corollary, when (a)–(c$'$) hold, will be called below a *normal evaluator* of f. The next proposition is partially a reverse one to Proposition 2.1.

Proposition 2.4. *If the S-algebra $f : S(X) \to X$ satisfies* (8) *and the arrow* $1_X \odot (1_X \odot \tilde{f})$ *is an epic, then* $f \in \mathcal{D}$.

Proof. We have to prove that

$$f_{10} \circ (1 \odot f_{10}) = f_{10} \circ (f_{10} \odot 1) \circ \bar{a} \tag{16}$$

and

$$f_{10} \circ (1 \odot f_{11}) = f_{11} \circ R(f_{10}) \circ \vartheta_R. \tag{17}$$

The equality (17) follows easily from (8) by a composition from right with $1 \odot I_{11}$. Similarly, a composition with $1 \odot I_{10}$ yields the equality

$$f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) = f_{10} \circ (f_{10} \odot f_0) \circ \bar{a}, \tag{18}$$

which is weaker than (16). To prove the last one, we shall show that

$$f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) = f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a}. \tag{19}$$

From (19) the equality (16) will follow immediately by canceling from right the arrow $1 \odot (1 \odot \tilde{f} \circ \iota)$. This can be done because $1 \odot (1 \odot \tilde{f})$ is right-cancelable by suppositions of Proposition 2.4, and $\tilde{f} = \tilde{f} \circ \nu = \tilde{f} \circ \iota \circ \dot{\nu}$, whence $1 \odot (1 \odot \tilde{f} \circ \iota)$ is also right cancelable. Therefore it remains to prove (19). For that we shall use the principle of the parameterized evaluation. Consider the $\mathcal{C}$-arrow

$$\psi = [f_{10} \circ (1 \odot N(f)), N(f)] : (X \odot \dot{S}(X)) + \dot{S}(X) \to X$$

and the natural in $X, Y \in \mathcal{C}$ transformation

$$\vartheta_1 : X \odot (X \odot \dot{S}(Y)) \to (X \odot \dot{S}(X)) + \dot{S}(X \odot (X \odot Y))$$

defined by

$$\vartheta_1 = ((1 \odot I_{10}) + I_1 \circ (\bar{a} \circ (1 \odot \bar{a}) + \vartheta_R \circ (1 \odot \vartheta_R))) \circ (1 + \delta_{\odot}) \circ \delta_{\odot} \circ (1 \odot (1 + \delta_{\odot}) \circ \delta_{\odot}),$$

which is equivalent to the following three equalities:

$$\vartheta_1 \circ (1 \odot (1 \odot I_0)) = I_0 \circ (1 \odot I_{10}); \tag{20}$$
$$\vartheta_1 \circ (1 \odot (1 \odot I_{10})) = I_{110} \circ \bar{a} \circ (1 \odot \bar{a}); \tag{20'}$$
$$\vartheta_1 \circ (1 \odot (1 \odot I_{11})) = I_{111} \circ \vartheta_R \circ (1 \odot \vartheta_R). \tag{20''}$$

By Theorem 1.1, applied to functors $F(Y) = \dot{S}(Y)$, $\Delta(Y) = X \odot (X \odot Y)$ and $F'(Y) = (X \odot \dot{S}(X)) + F(Y)$, there is an unique $\mathcal{C}$-arrow $\varphi : X \odot (X \odot \dot{T}) \to X$ such that

$$\varphi \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi)) \circ \vartheta_1. \tag{21}$$

Therefore, to complete the proof, it is enough to show that both sides of (19) satisfy (21) with respect to φ. Denote the left- and right-hand sides of (19) by φ_0 and φ_1, respectively. To prove

$$\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi_0)) \circ \vartheta_1, \tag{22}$$

we use the following form of the principle of considering cases: composed from right by $1 \odot (1 \odot I_0)$, $1 \odot (1 \odot I_{10})$ and $1 \odot (1 \odot I_{11})$, the two sides of (22) become equal, hence we shall conclude (22). We shall show this for $1 \odot (1 \odot I_{10})$, leaving the other two cases for the reader (they are similar or simpler). First, notice that by Proposition 2.2 and Corollary 2.1 we have $\tilde{f} = h \circ \dot{\nu}$, where h is the normal evaluator of f, and therefore $\tilde{f} \circ \iota = h$ (by Proposition 2.3), whence

$$\tilde{f} \circ \iota \circ \dot{\tau} = N(f) \circ \dot{S}(\tilde{f} \circ \iota), \tag{23}$$

and

$$\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota \circ \dot{\tau})) = f_{10} \circ (1 \odot f_{10} \circ (1 \odot N(f) \circ \dot{S}(\tilde{f} \circ \iota))).$$

Then

$$\begin{aligned}\varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})) &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot N(f) \circ \dot{S}(\tilde{f} \circ \iota) \circ I_{10})) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0))).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\psi \circ (1 + \dot{S}(\varphi_0)) \circ \vartheta_1 \circ (1 \odot (1 \odot I_{10})) = \psi \circ (1 + \dot{S}(\varphi_0)) \circ I_{110} \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= \psi \circ I_1 \circ \dot{S}(\varphi_0) \circ I_{10} \circ \bar{a} \circ (1 \odot \bar{a}) = N(f) \circ I_{10} \circ (\varphi_0 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (\varphi_0 \odot f_0) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (f_{10} \circ (1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) \odot f_0) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (f_{10} \odot f_0) \circ ((1 \odot f_{10} \circ (1 \odot \tilde{f} \circ \iota)) \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} \circ (1 \odot (f_{10} \circ (1 \odot \tilde{f} \circ \iota) \odot 1)) \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \circ (1 \odot \tilde{f} \circ \iota) \odot 1)) \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \odot 1)) \circ (1 \odot ((1 \odot \tilde{f} \circ \iota) \odot 1) \circ \bar{a}) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (1 \odot (f_{10} \odot 1)) \circ (1 \odot \bar{a} \circ (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &= f_{10} \circ (1 \odot f_{10} \circ (f_{10} \odot f_0)) \circ (1 \odot \bar{a}) \circ (1 \odot (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0))) \circ (1 \odot (1 \odot (\tilde{f} \circ \iota \odot 1))) \\ &= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0))) = \varphi_0 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})).\end{aligned}$$

To prove

$$\varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) = \psi \circ (1 + \dot{S}(\varphi_1)) \circ \vartheta_1, \tag{24}$$

consider the cases as in the proof of (22). Again, we shall consider the case with $1 \odot (1 \odot I_{10})$ only, leaving the other ones to the reader (note that in the case with $1 \odot (1 \odot I_{11})$ the R-coherence is used in the same way in which the Mac Lane pentagonal diagram for $\bar{a}$ is used in the case with $1 \odot (1 \odot I_{10})$). We have, using (23) as before,

$$\begin{aligned}\varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})) &= f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a} \circ (1 \odot (1 \odot \dot{\tau}_{10})) \\ &= f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota \circ \dot{\tau} \circ I_{10}) \circ \bar{a} = f_{10} \circ (f_{10} \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0)) \circ \bar{a}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\psi \circ (1 + \dot{S}(\varphi_1)) \circ \vartheta_1 \circ (1 \odot (1 \odot I_{10})) &= \psi \circ (1 + \dot{S}(\varphi_1)) \circ I_{110} \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= N(f) \circ \dot{S}(\varphi_1) \circ I_{10} \circ \bar{a} \circ (1 \odot \bar{a}) = N(f) \circ I_{10} \circ (\varphi_1 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (\varphi_1 \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \circ \bar{a} \odot 1) \circ \bar{a} \circ (1 \odot \bar{a}) \\ &= f_{10} \circ (1 \odot f_0) \circ (f_{10} \circ (f_{10} \odot \tilde{f} \circ \iota) \odot 1) \circ \bar{a} \circ \bar{a}\end{aligned}$$

$$= f_{10} \circ (f_{10} \odot f_0) \circ \bar{a} \circ (f_{10} \odot (\tilde{f} \circ \iota \odot 1)) \circ \bar{a}$$
$$= f_{10} \circ (1 \odot f_{10} \circ (1 \odot f_0)) \circ (f_{10} \odot (\tilde{f} \circ \iota \odot 1)) \circ \bar{a}$$
$$= f_{10} \circ (f_{10} \odot f_{10} \circ (\tilde{f} \circ \iota \odot f_0)) \circ \bar{a} = \varphi_1 \circ (1 \odot (1 \odot \dot{\tau})) \circ (1 \odot (1 \odot I_{10})).$$

Corollary 2.2. *The algebra $\tau^N : S(\dot{T}) \to \dot{T}$ belongs to $\mathcal{D}$, and therefore it is an initial object of $\mathcal{D}$.*

Proof. Since the evaluator $\dot{\nu} : T \to \dot{T}$ of τ^N is right-invertible by Proposition 2.3, the arrow $1 \odot (1 \odot \dot{\nu})$ is an epic. Then by Proposition 2.4 the algebra τ^N belongs to $\mathcal{D}$ and by Proposition 2.2 it is an initial object of $\mathcal{D}$.

3. MINIMAL EVALUATION IN PARTIALLY ORDERED SRC

A *partially ordered* SRC is a SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ such that $\mathcal{C}$ is in the same time a partially ordered category and all involved functors (i.e. $\odot$, R and $+$) are increasing with respect to the partial order in $\mathcal{C}$ on every argument. In the present section we shall fix a partially ordered SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ satisfying all conditions from the previous section, and we shall suppose moreover that the least fixed point

$$\tau(B) : S(T(B)) \to T(B)$$

is a proper one with respect to the partial order in $\mathcal{C}$. The bi-endofunctors S and $\dot{S}$, defined as in the previous section, are increasing. Consider the bi-endofunctor $S^+(B,X) = \dot{S}(B,X) + X$ which also is increasing. As in the previous section, we shall write shortly $S(X)$, $\dot{S}(X)$ and $S^+(X)$ for $S(B,X)$, $\dot{S}(B,X)$ and $S^+(B,X)$, respectively. We have the functor $\dot{N} : \mathcal{C}_{\dot{S}} \to \mathcal{C}_{S^+}$ defined by $\dot{N}(f) = [f,1]$ for objects $f : \dot{S}(X) \to X$ of $\mathcal{C}_{\dot{S}}$ and $\dot{N}(\varphi) = \varphi$ for morphisms φ. The composition $\dot{N} \circ N : \mathcal{C}_S \to \mathcal{C}_{S^+}$ preserves morphisms and we shall write shortly f^+ for the value $\dot{N}(N(f))$ of $\dot{N} \circ N$ for objects, i.e.

$$f^+ = [[f_0, [f_{10} \circ (1 \odot f_0), f_{11}], 1].$$

Now, for the fixed object B of $\mathcal{C}$ we shall suppose that $B = B_0 + B_1$, where B_0 and B_1 are two fixed objects of $\mathcal{C}$. As usual, we denote the canonical monics $B_0 \to B$ and $B_1 \to B$ by I_0 and I_1, respectively. Intuitively, the object B_0 will be considered as the object of 'parameters', and B_1 — as the object of 'variables', which is just the case for the SRC of sets. We use the short notations $S_0(X)$, $\dot{S}_0(X)$ and $S_0^+(X)$ for $S(B_0,X)$, $S(\dot{B}_0,X)$ and $S^+(B_0,X)$, respectively. (Thus we define endofunctors S_0, $\dot{S}_0$ and S_0^+ in $\mathcal{C}$, for instance, $S_0(f) = S(B_0,f) = S(1_{B_0},f)$ for an arrow f in $\mathcal{C}$, etc.) We have a functor $P : \mathcal{C}_S \to \mathcal{C}_{S_0}$ defined by $P(f) = f \circ S(I_0,1)$ for objects f, and $P(\varphi) = \varphi$ for arrows φ of $\mathcal{C}_S$, where I_0 is here the canonical monic $I_0 : B_0 \to B$ of the co-product $B_0 + B_1$. Intuitively, the functor P simply ignores interpretation of variables. We have also another functor $Q : \mathcal{B} \to \mathcal{C}_S$ which is in some sense inverse to P. Here $\mathcal{B}$ is the category, defined as follows: objects of $\mathcal{B}$ are pairs (x,f), where $x : B_1 \to X$ and $f : S_0(X) \to X$ are $\mathcal{C}$-arrows with the same co-domain X, and morphisms $\varphi : (x,f) \to (y,g)$ in $\mathcal{B}$, where $y : B_1 \to Y$ and

$g : S_0(Y) \to Y$, are the $\mathcal{C}$-arrows $\varphi : X \to Y$ which are simultaneously morphisms in the comma category $(B_1 \downarrow \mathcal{C})$ and in $\mathcal{C}_{S_0}$, i.e. $y = \varphi \circ x$ and $\varphi \circ f = g \circ S_0(\varphi)$. For objects $(x, f) \in \mathcal{B}$ the functor Q is defined by

$$Q(x, f) = [[f_0, x], f_1] : S(X) \to X,$$

and for arrows φ in $\mathcal{B}$ the functor Q is defined trivially: $Q(\varphi) = \varphi$. The reader can easily check that Q is indeed a functor and $Q(f_{01}, P(f)) = f$ for any object $f : S(X) \to X$ of $\mathcal{C}_S$, where, as usual, $f_{01} = f_0 \circ I_1 : B_1 \to X$ and also $P(Q(x, f)) = f$ for all $(x, f) \in \mathcal{B}$.

Next we define a natural in $X, Y \in \mathcal{C}$ transformation

$$\vartheta^+ = \vartheta_S^+ : X \odot S^+(Y) \to S^+(X) + S^+(X \odot Y),$$

similar to the transformation $\dot{\vartheta}$ in the previous section, namely,

$$\vartheta^+ = [(I_0 + I_0) \circ \dot{\vartheta}, I_{11}] \circ \delta_{\odot},$$

which is equivalent to the pair of equalities

$$\vartheta^+ \circ (1 \odot I_0) = (I_0 + I_0) \circ \dot{\vartheta}$$

and

$$\vartheta^+ \circ (1 \odot I_1) = I_{11}.$$

The transformation $\vartheta_{S_0}^+$, defined in the same way for the functor S_0 instead of S, will be denoted shortly by ϑ_0^+.

Proposition 3.1. *For any algebra $f : S(X) \to X$ in $\mathcal{C}_S$ the equalities* (8) *and*

$$f_{10} \circ (1 \odot f^+) = f^+ \circ [1, S^+(f_{10})] \circ \vartheta^+ \tag{25}$$

are equivalent, and therefore (25) *holds for every object f of $\mathcal{D}$.*

Proof. An easy consequence of definitions.

Now consider a morphism $\sigma : B_1 \to \dot{T} = \dot{T}(B)$. In the case of SRC of posets and the trivial order (coinciding with the equality) in B_1, σ assigns to each variable $v \in B_1$ a normal term $\sigma(v) \in \dot{T}$ which may contain any variable from B_1. Thus σ determines a system of inequalities $\{\sigma(v) \leq v \mid v \in B_1\}$. A solution of the last system in an S_0-algebra $f : S_0(X) \to X$ in this SRC is a function $x : B_1 \to X$ such that the evaluator $h : \dot{T} \to X$ of the algebra $N(Q(x, f)) : \dot{S}(X) \to X$ with respect to the least fixed point $\dot{\tau}$ satisfies the inequality

$$h \circ \sigma \leq h \circ \dot{\tau}_{01}, \tag{26}$$

where $\dot{\tau}_{01} = \dot{\tau} \circ I_0 \circ I_1 : B_1 \to \dot{T}$ is the mapping, which assigns to each variable in B_1 the same one considered as a normal term from $\dot{T}$. When $Q(x, f) \in \mathcal{D}$, i.e. f_1 is a standard algebra in the SRC of posets, the mapping $h : \dot{T} \to X$ is a morphism $h : \tau^N \to Q(x, f)$ in $\mathcal{C}_S$ (by Propositions 2.1 and 2.2), and therefore $P(h) = h : P(\tau^N) \to f$ is such one in $\mathcal{C}_{S_0}$, i.e.

$$h \circ \tau^N \circ S(I_0, 1) = f \circ S_0(h). \tag{27}$$

And the solution x can be restored from h, namely, $x = h \circ \dot{\tau}_{01}$, which follows from the equality $h \circ \dot{\tau} = N(Q(x, f)) \circ \dot{S}(h)$ by a composition from right with I_{01}.

In the general case, when we have an arbitrary partially ordered SRC $\mathcal{C}$ and an S_0-algebra $f : S_0(X) \to X$ in it, such that $f_1 \in \mathbf{SA}(\mathcal{C})$, the above consideration suggests to treat an arbitrary arrow $\sigma : B_1 \to \dot{T}$ as a system of inequalities and morphisms $h : P(\tau^N) \to f$ of S_0-algebras in $\mathcal{C}$, satisfying (26) as ***solutions*** of the system σ. The next proposition will give us a more convenient form of (26).

First we define an arrow $\alpha : \dot{T} \to S_0^+(\dot{T})$, called ***analyzer*** of the system σ, by the following equality:

$$\alpha = [(I_0 + \sigma),\ [(I_{10} + \mu \circ (1 \odot \sigma)) \circ \delta_\odot,\ I_{011}]] \circ \dot{\tau}^{-1}.$$

An equivalent and perhaps more clear form of this equality is the following definition 'by cases':

$$\alpha \circ \dot{\tau}_{00} = I_{00}; \tag{28.1}$$

$$\alpha \circ \dot{\tau}_{01} = I_1 \circ \sigma; \tag{28.2}$$

$$\alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = I_{010}; \tag{28.3}$$

$$\alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) = I_1 \circ \mu \circ (1 \odot \sigma); \tag{28.4}$$

$$\alpha \circ \dot{\tau}_{11} = I_{011}. \tag{28.5}$$

Proposition 3.2. *For any S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ and any morphism $h : P(\tau^N) \to f$ of such algebras, i.e. any $\mathcal{C}$-arrow $h : \dot{T} \to X$ for which* (27) *holds, we have the equivalence*

$$h \circ \sigma \le h \circ \dot{\tau}_{01} \ \Leftrightarrow\ f^+ \circ S_0^+(h) \circ \alpha \le h.$$

Proof. The reverse direction ($\Leftarrow$) of the last equivalence is easy to be proved and does not use (27):

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ I_1 \circ \sigma = f^+ \circ I_1 \circ h \circ \sigma = h \circ \sigma. \tag{29}$$

To prove that $h \circ \sigma \le h \circ \dot{\tau}_{01}$ implies $f^+ \circ S_0^+(h) \circ \alpha \le h$, suppose $h \circ \sigma \le h \circ \dot{\tau}_{01}$. Since $\dot{\tau}$ is an isomorphism, it is enough to prove the inequality

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau} \le h \circ \dot{\tau}. \tag{30}$$

We shall do this by considering cases as in the definition of α. We have, using (28.1) and (27),

$$\begin{aligned} f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ I_0 &= f^+ \circ S_0^+(h) \circ I_{00} = f^+ \circ I_0 \circ \dot{S}_0(h) \circ I_0 \\ &= N(f) \circ I_0 = f_0 = f \circ S_0(h) \circ I_0 = h \circ \tau^N \circ S(I_0, 1) \circ I_0 \\ &= h \circ \tau^N \circ I_{00} = h \circ \dot{\tau}_0 \circ I_0. \end{aligned} \tag{31}$$

On the other hand, by (29) and the supposition $h \circ \sigma \le h \circ \dot{\tau}_{01}$ we have

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ I_1 \le h \circ \dot{\tau}_0 \circ I_1,$$

whence (using the supposition that the functor $+$ is increasing on both arguments) we conclude that

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \le h \circ \dot{\tau}_0.$$

In this way it would be enough to show that

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{1i} \leq h \circ \dot{\tau}_{1i}$$

for both $i = 0, 1$. The case with $i = 1$ is easier:

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{11} = f^+ \circ S_0^+(h) \circ I_{011} = N(f) \circ \dot{S}_0(h) \circ I_{11} = N(f) \circ I_{11} \circ R(h)$$
$$= f_{11} \circ R(h) = f \circ S_0(h) \circ I_{11} = h \circ \tau^N \circ S(I_0, 1) \circ I_{11} = h \circ \dot{\tau}_{11}.$$

For the case $i = 0$ we again consider cases, as follows:

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = f^+ \circ S_0^+(h) \circ I_{010} = N(f) \circ \dot{S}_0(h) \circ I_{10}$$
$$= N(f) \circ I_{10} \circ (h \odot 1) = f_{10} \circ (1 \odot f_0) \circ (h \odot 1),$$

and using the chain of equalities (31) and the equality (27), we have

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = f_{10} \circ (h \odot h \circ \dot{\tau}_{00}) = f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot{\tau}_{00})$$
$$= f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot{\tau}_{00}) = h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \dot{\tau}_{00})$$
$$= h \circ \mu \circ (1 \odot \dot{\tau}_0) \circ (1 \odot I_0) = h \circ \dot{\tau}_{10} \circ (1 \odot I_0);$$

on the other hand (using twice (27) and the supposition $h \circ \sigma \leq h \circ \dot{\tau}_{01}$),

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) = f^+ \circ S_0^+(h) \circ I_1 \circ \mu \circ (1 \odot \sigma)$$
$$= f^+ \circ I_1 \circ h \circ \mu \circ (1 \odot \sigma) = h \circ \mu \circ (1 \odot \sigma) = h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \sigma)$$
$$= f \circ S_0(h) \circ I_{10} \circ (1 \odot \sigma) = f_{10} \circ (h \odot h \circ \sigma)$$
$$\leq f_{10} \circ (h \odot h \circ \dot{\tau}_{01}) = f \circ S_0(h) \circ I_{10} \circ (1 \odot \dot{\tau}_{01})$$
$$= h \circ \tau^N \circ S(I_0, 1) \circ I_{10} \circ (1 \odot \dot{\tau}_{01}) = h \circ \mu \circ (1 \odot \dot{\tau}_{01})$$
$$= h \circ \mu \circ (1 \odot \dot{\tau}_0) \circ (1 \odot I_1) = h \circ \dot{\tau}_{10} \circ (1 \odot I_1).$$

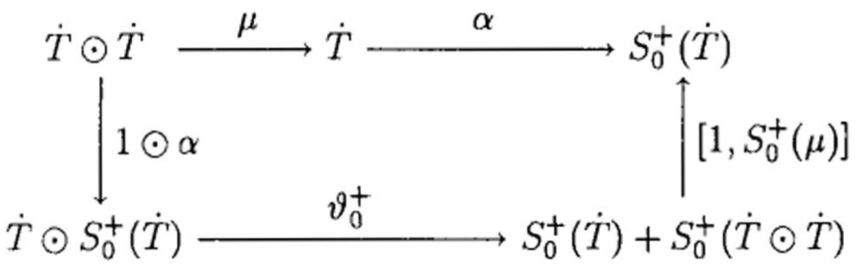
From the last two chains of equalities and inequalities we conclude

$$f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{10} \leq h \circ \dot{\tau}_{10},$$

using again the fact that the composition $\circ$ and the co-product functor $+$ are increasing with respect to $\leq$.

Proposition 3.3. *We have the equality*

$$\alpha \circ \mu = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha). \tag{32}$$



Remark. Note the analogy between (32) and the equality (9) written in the form $\dot{\tau}^{-1} \circ \mu = [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot \dot{\tau}^{-1})$.

Proof. Since $\dot{\tau}$ is an isomorphism, it is enough to show

$$\alpha \circ \mu \circ (1 \odot \dot{\tau}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}),$$

which according to (9) is equivalent to

$$\alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}).$$

To prove the last equality, denote its left- and right-hand sides with φ and ψ, respectively, and consider cases as in the definition of α. Indeed, using definitions of $\dot{\vartheta}$, ϑ_0^+ and α, we have

$$\begin{aligned}\varphi \circ (1 \odot I_{00}) &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{00}) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} \circ (1 \odot I_0) \\ &= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) = I_{010},\end{aligned}$$

and

$$\begin{aligned}\psi \circ (1 \odot I_{00}) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{00}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{00}) \\ &= [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_0) = [I_0, S_0^+(\mu) \circ I_0] \circ I_{010} \\ &= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{010} = I_{010} = \varphi \circ (1 \odot I_{00}),\end{aligned}$$

where $\dot{\vartheta}_0$ is the natural transformation $\dot{\vartheta}$ for the functor S_0 instead of S. In a similar way,

$$\begin{aligned}\psi \circ (1 \odot I_{01}) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{01}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_1 \circ \sigma) \\ &= [1, S_0^+(\mu)] \circ I_{11} \circ (1 \odot \sigma) = S_0^+(\mu) \circ I_1 \circ (1 \odot \sigma) = I_1 \circ \mu \circ (1 \odot \sigma) \\ &= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{010} \circ (1 \odot I_1) \\ &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{01}) = \varphi \circ (1 \odot I_{01}),\end{aligned}$$

whence by considering cases we conclude

$$\varphi \circ (1 \odot I_0) = \psi \circ (1 \odot I_0).$$

Next we have

$$\begin{aligned}\psi \circ (1 \odot I_{10} \circ (1 \odot I_0)) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0)) \\ &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{010}) = [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_{10}) \\ &= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ \dot{\vartheta}_0 \circ (1 \odot I_{10}) = [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{110} \circ \bar{a} \\ &= I_0 \circ \dot{S}_0(\mu) \circ I_{10} \circ \bar{a} = I_{010} \circ (\mu \odot 1) \circ \bar{a},\end{aligned}$$

and

$$\begin{aligned}\varphi \circ (1 \odot I_{10} \circ (1 \odot I_0)) &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{10} \circ (1 \odot I_0)) \\ &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{110} \circ \bar{a} \circ (1 \odot (1 \odot I_0)) = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{10} \circ \bar{a} \circ (1 \odot (1 \odot I_0)) \\ &= \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot I_0)) = \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ (1 \odot I_0) \circ \bar{a} \\ &= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_0) \circ (\mu \odot 1) \circ \bar{a} = I_{010} \circ (\mu \odot 1) \circ \bar{a} = \psi \circ (1 \odot I_{10} \circ (1 \odot I_0)).\end{aligned}$$

Furthermore,

$$\begin{aligned}\psi \circ (1 \odot I_{10} \circ (1 \odot I_1)) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1)) \\ &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_1 \circ \mu \circ (1 \odot \sigma)) = [1, S_0^+(\mu)] \circ I_{11} \circ (1 \odot \mu \circ (1 \odot \sigma)) \\ &= S_0^+(\mu) \circ I_1 \circ (1 \odot \mu \circ (1 \odot \sigma)) = I_1 \circ \mu \circ (1 \odot \mu \circ (1 \odot \sigma)),\end{aligned}$$

and using also Corollary 2.2,

$$\varphi \circ (1 \odot I_{10} \circ (1 \odot I_1)) = \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ \dot{\vartheta} \circ (1 \odot I_{10} \circ (1 \odot I_1))$$
$$= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{110} \circ \bar{a} \circ (1 \odot (1 \odot I_1)) = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{10} \circ \bar{a} \circ (1 \odot (1 \odot I_1))$$
$$= \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot I_1)) = \alpha \circ \dot{\tau}_{10} \circ (\mu \odot 1) \circ (1 \odot I_1) \circ \bar{a}$$
$$= \alpha \circ \dot{\tau}_{10} \circ (1 \odot I_1) \circ (\mu \odot 1) \circ \bar{a} = I_1 \circ \mu \circ (1 \odot \sigma) \circ (\mu \odot 1) \circ \bar{a}$$
$$= I_1 \circ \mu \circ (\mu \odot 1) \circ \bar{a} \circ (1 \odot (1 \odot \sigma)) = I_1 \circ \mu \circ (1 \odot \mu \circ (1 \odot \sigma))$$
$$= \psi \circ (1 \odot I_{10} \circ (1 \odot I_1)),$$

whence we obtain

$$\varphi \circ (1 \odot I_{10}) = \psi \circ (1 \odot I_{10}).$$

Finally,

$$\begin{aligned} \psi \circ (1 \odot I_{11}) &= [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha \circ \dot{\tau}_{11}) = [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot I_{011}) \\ &= [1, S_0^+(\mu)] \circ (I_0 + I_0) \circ \dot{\vartheta}_0 \circ (1 \odot I_{11}) \\ &= [I_0, I_0 \circ \dot{S}_0(\mu)] \circ I_{111} \circ \vartheta_R = I_0 \circ \dot{S}_0(\mu) \circ I_{11} \circ \vartheta_R \\ &= I_{011} \circ R(\mu) \circ \vartheta_R \end{aligned}$$

and

$$\begin{aligned} \varphi \circ (1 \odot I_{11}) &= \alpha \circ \dot{\tau} \circ [1, \dot{S}(\mu)] \circ I_{111} \circ \vartheta_R = \alpha \circ \dot{\tau} \circ \dot{S}(\mu) \circ I_{11} \circ \vartheta_R \\ &= \alpha \circ \dot{\tau}_{11} \circ R(\mu) \circ \vartheta_R = I_{011} \circ R(\mu) \circ \vartheta_R = \psi \circ (1 \odot I_{11}). \end{aligned}$$

Definition. Given an S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ and an $\mathcal{C}$-arrow $\alpha : \dot{T} \to S_0^+(\dot{T})$, another $\mathcal{C}$-arrow $h : \dot{T} \to X$ will be called an *α-minimal evaluator* of f iff h is the least solution of the inequality

$$f^+ \circ S_0^+(\eta) \circ \alpha \leq \eta \tag{33}$$

with respect to η in $\mathcal{C}(\dot{T}, X)$ and for all $\chi \in \mathcal{C}(\dot{T}, X)$ and $\psi \in \mathcal{C}(\dot{T} \odot \dot{T}, X)$ satisfies an additional condition, written symbolically as follows:

$$\forall \eta \in \mathcal{C}(\dot{T}, X)(\varphi \circ (1 \odot \eta) \leq \psi \Rightarrow \varphi \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) \leq \psi) \Rightarrow \varphi \circ (1 \odot h) \leq \psi, \quad (*)$$

where $\varphi = f_{10} \circ (\chi \odot 1) : \dot{T} \odot X \to X$.

Lemma 3.1. *Let $\sigma : B_1 \to \dot{T}$ be a system with analyzer $\alpha : \dot{T} \to S_0^+(\dot{T})$, let $f : S_0(X) \to X$ be an S_0-algebra in $\mathcal{C}$ such that $f_1 \in \mathbf{SA}(\mathcal{C})$, and let $h : \dot{T} \to X$ be an α-minimal evaluator of f. Then for every $\mathcal{C}$-arrow $\chi : \dot{T} \to X$ the arrow $f_{10} \circ (\chi \odot h) : \dot{T} \odot \dot{T} \to X$ is the least solution of the inequality*

$$\chi' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \leq \zeta \tag{34}$$

with respect to ζ in $\mathcal{C}(\dot{T} \odot \dot{T}, X)$, where $\chi' = f^+ \circ [S_0^+(\chi), 1] : S_0^+(\dot{T}) + S_0^+(X) \to X$.

Remark. Note the analogy of (34) with (13).

Proof. The arrow h being a solution of (33), we have

$$\begin{aligned} f_{10} \circ (\chi \odot h) &\geq f_{10} \circ (\chi \odot f^+ \circ S_0^+(h) \circ \alpha) \\ &= f_{10} \circ (1 \odot f^+) \circ (\chi \odot S_0^+(h)) \circ (1 \odot \alpha) \end{aligned}$$

$$= f^+ \circ [1, S_0^+(f_{10})] \circ \vartheta_0^+ \circ (\chi \odot S_0^+(h)) \circ (1 \odot \alpha) \quad \text{(by Proposition 3.1)}$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ (S_0^+(\chi) + S_0^+(\chi \odot h)) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$\text{(since } \vartheta_0^+ \text{ is natural)}$$
$$= f^+ \circ [S_0^+(\chi), S_0^+(f_{10} \circ (\chi \odot h))] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= \chi' \circ (1 + S_0^+(f_{10} \circ (\chi \odot h))) \circ \vartheta_0^+ \circ (1 \odot \alpha) \quad \text{(by definition of } \chi'),$$

i.e. $\zeta = f_{10} \circ (\chi \odot h)$ satisfies (34). For an arbitrary solution ζ of (34) in $\mathcal{C}(\dot{T} \odot \dot{T}, X)$ we shall show that $f_{10} \circ (\chi \odot h) \le \zeta$, using the additional condition (*) in the definition of α-minimal evaluator. For an arbitrary $\eta \in \mathcal{C}(\dot{T}, X)$ suppose $f_{10} \circ (\chi \odot 1) \circ (1 \odot \eta) \le \zeta$, i.e. $f_{10} \circ (\chi \odot \eta) \le \zeta$. Then

$$f_{10} \circ (\chi \odot 1) \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) = f_{10} \circ (1 \odot f^+) \circ (\chi \odot S_0^+(\eta)) \circ (1 \odot \alpha)$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ \vartheta_0^+ \circ (\chi \odot S_0^+(\eta)) \circ (1 \odot \alpha)$$
$$= f^+ \circ [1, S_0^+(f_{10})] \circ (S_0^+(\chi) + S_0^+(\chi \odot \eta)) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= f^+ \circ [S_0^+(\chi), S_0^+(f_{10} \circ (\chi \odot \eta))] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= \chi' \circ (1 + S_0^+(f_{10} \circ (\chi \odot \eta))) \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$\le \chi' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \le \zeta \quad \text{(by (34))}.$$

This proves the hypothesis in (*) with $\psi = \zeta$, whence $\varphi \circ (1 \odot h) \le \zeta$, i.e.

$$f_{10} \circ (\chi \odot h) = f_{10} \circ (\chi \odot 1) \circ (1 \odot h) = \varphi \circ (1 \odot h) \le \zeta.$$

Theorem 3.1. *Let $\sigma : B_1 \to \dot{T}$ be a system with analyzer $\alpha : \dot{T} \to S_0^+(\dot{T})$, let $f : S_0(X) \to X$ be an S_0-algebra in $\mathcal{C}$ such that $f_1 \in \mathbf{SA}(\mathcal{C})$, and let $h : \dot{T} \to X$ be an α-minimal evaluator of f. Then $h : P(\tau^N) \to f$ is a morphism in $\mathcal{C}_{S_0}$, i.e. (27) holds.*

Proof. By Lemma 3.1 $f_{10} \circ (h \odot h)$ is the least solution of

$$h' \circ (1 + S_0^+(\zeta)) \circ \vartheta_0^+ \circ (1 \odot \alpha) \le \zeta \tag{35}$$

with respect to $\zeta \in \mathcal{C}(\dot{T} \odot \dot{T}, X)$, where $h' = f^+ \circ [S_0^+(h), 1]$. But the arrow $h \circ \mu : \dot{T} \odot \dot{T} \to X$ satisfies (35) because

$$h' \circ (1 + S_0^+(h \circ \mu)) \circ \vartheta_0^+ \circ (1 \odot \alpha) = f^+ \circ [S_0^+(h), S_0^+(h \circ \mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha)$$
$$= f^+ \circ S_0^+(h) \circ [1, S_0^+(\mu)] \circ \vartheta_0^+ \circ (1 \odot \alpha) = f^+ \circ S_0^+(h) \circ \alpha \circ \mu \quad \text{(by (32))}$$
$$\le h \circ \mu.$$

Therefore

$$f_{10} \circ (h \odot h) \le h \circ \mu, \tag{36}$$

which is the same as

$$f \circ S_0(h) \circ I_{10} \le h \circ \tau^N \circ S(I_0, 1) \circ I_{10}.$$

On the other hand,

$$f \circ S_0(h) \circ I_0 = f_0 = N(f) \circ \dot{S}_0(h) \circ I_0 = f^+ \circ S_0^+(h) \circ I_{00} = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{00}$$
$$\le h \circ \dot{\tau}_{00} = h \circ \tau^N \circ I_{00} = h \circ \tau^N \circ S(I_0, 1) \circ I_0$$

and

$$f \circ S_0(h) \circ I_{11} = f_{11} \circ R(h) = N(f) \circ \dot{S}_0(h) \circ I_{11} = f^+ \circ S_0^+(h) \circ I_{011}$$
$$= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{11} \leq h \circ \dot{\tau}_{11} = h \circ \tau^N \circ S(I_0, 1) \circ I_{11},$$

whence by considering cases

$$f \circ S_0(h) \leq h \circ \tau^N \circ S(I_0, 1). \tag{37}$$

To prove the reverse inequality, consider the arrows

$$x = h \circ \dot{\tau}_{01} : B_1 \to X$$

and

$$f_h = Q(x, f) = [[f_0, x], f_1] : S(X) \to X.$$

We shall prove that

$$f_h \circ S(h) \leq h \circ \tau^N. \tag{38}$$

Indeed, since h is the least solution of (33) with respect to η, we have $f^+ \circ S_0^+(h) \circ \alpha = h$, whence

$$x = h \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01} = f^+ \circ S_0^+(h) \circ I_1 \circ \sigma = h \circ \sigma$$

and

$$\begin{aligned} f_h \circ S(h) \circ I_0 &= f_h \circ I_0 = [f_0, x] = [f_0, h \circ \sigma] \\ &= [f^+ \circ S_0^+(h) \circ I_{00}, f^+ \circ S_0^+(h) \circ I_1 \circ \sigma] \\ &= [f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{00}, f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{01}] \\ &= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 \circ [I_0, I_1] \\ &= f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_0 = h \circ \dot{\tau}_0 = h \circ \tau^N \circ I_0. \end{aligned}$$

Again, as before,

$$f_h \circ S(h) \circ I_{11} = f_{11} \circ R(h) = f^+ \circ S_0^+(h) \circ \alpha \circ \dot{\tau}_{11} = h \circ \dot{\tau}_{11} = h \circ \tau^N \circ I_{11}$$

and the inequality

$$f_h \circ S(h) \circ I_{10} \leq h \circ \tau^N \circ I_{10}$$

is the same as (36). Thus (38) is proved by considering cases. A composition of the last one with $S(\dot{\nu})$ yields

$$f_h \circ S(h \circ \dot{\nu}) \leq h \circ \tau^N \circ S(\dot{\nu}) = h \circ \dot{\nu} \circ \tau$$

(we use (14)), whence follows the inequality

$$\tilde{f}_h \leq h \circ \dot{\nu} \tag{39}$$

for the evaluator $\tilde{f}_h : T \to X$ of the algebra $f_h : S(X) \to X$ with respect to the least fixed point $\tau : S(T) \to T$, using the supposition that the least fixed point τ is proper one. We shall prove the reverse of (39) by showing that

$$h \leq \tilde{f}_h \circ \iota. \tag{40}$$

Indeed, the algebra f_h belongs to $\mathfrak{D}$ and by Propositions 2.1 and 2.2 the normal e-valuator h_1 of f_h exists, and by Corollary 2.1 (c′) $h_1 \circ \dot{\nu} = \tilde{f}_h$, whence by Proposition 2.3 $h_1 = \tilde{f}_h \circ \iota$. So $\tilde{f}_h \circ \iota$ is a morphism $\tau^N \to f_h$ in $\mathfrak{C}_S$, i.e.

$$\tilde{f}_h \circ \iota \circ \tau^N = f_h \circ S(\tilde{f}_h \circ \iota),$$

and composing the last equality from right with $S(I_0, 1)$, we obtain

$$\tilde{f}_h \circ \iota \circ \tau^N \circ S(I_0, 1) = f \circ S_0(\tilde{f}_h \circ \iota).$$

Moreover, using the inequality $\tilde{f}_h \circ \iota \leq h$ which follows from (39), we have

$$\begin{aligned}\tilde{f}_h \circ \iota \circ \sigma \leq h \circ \sigma = x = [f_0, x] \circ I_1 = f_h \circ I_{01} = f_h \circ S(\tilde{f}_h) \circ I_{01} \\ = \tilde{f}_h \circ \tau \circ I_{01} = \tilde{f}_h \circ N(\tau) \circ \dot{S}(\iota) \circ I_{01} = \tilde{f}_h \circ \iota \circ \dot{\tau}_{01}.\end{aligned}$$

Thence by Proposition 3.2 it follows

$$f^+ \circ S_0^+(\tilde{f}_h \circ \iota) \circ \alpha \leq \tilde{f}_h \circ \iota,$$

and since h is the least solution of (33), this implies (40). Using Corollary 2.1(a), from (40) we obtain the reverse inequality of (39):

$$h \circ \dot{\nu} \leq \tilde{f}_h \circ \iota \circ \dot{\nu} = \tilde{f}_h \circ \nu = \tilde{f}_h.$$

Thus we get the equality $h \circ \dot{\nu} = \tilde{f}_h$, whence it follows

$$h \circ \tau^N \circ S(\dot{\nu}) = h \circ \dot{\nu} \circ \tau = f_h \circ S(h \circ \dot{\nu}),$$

and composing this from right with $S(I_0, \iota)$, we obtain

$$h \circ \tau^N \circ S(I_0, 1) = f \circ S_0(h).$$

4. CODING FORMALIZM AND CODE FACTORIZATION OF THE MINIMAL EVALUATOR

The code evaluation method in algebraic recursion theory uses coding to obtain certain simple standard expression for the *minimal evaluator* of a system of inequalities. In the context of the previous Section 3 the last evaluator may be defined as the evaluator with respect to the least fixed point $\tau : S(T) \to T$ of the algebra $Q(x, f)$, where $f : S_0(X) \to X$ is an S_0-algebra in the SRC $\mathcal{C}$ and $x : B_1 \to X$ is the least solution of a 'system' $\sigma : B_1 \to \dot{T}$ in the algebra f. In the present section we propose a conceptual mechanism for treatment of coding on categorical level in the context of Section 3. We give also an interpretation for the case of SRC of posets, which shows how usual coding theorem in operative spaces (in the sense of [1]) can be obtained as a special case.

We assume suppositions and notations of Section 3, especially, we shall have fixed a partially ordered SRC $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$. In the special case with $\mathcal{C}$ — the category of posets, $\odot$ and $\bar{a}$ — the usual product $\times$ in $\mathcal{C}$ and the natural isomorphism of associativity of $\times$, respectively, $R(X)$ — the Cartesian square $X \times X$, and ϑ_R — the natural transformation defined as in the SRC of sets in Section 2, we shall call the 5-tuple $\langle \mathcal{C}, \odot, \bar{a}, R, \vartheta_R \rangle$ the 'SRC of posets'. It is a partially ordered SRC with respect to the obvious order mentioned in Section 1. We fix also an S_0-algebra $f : S_0(X) \to X$.

Consider a set Ξ of natural in $Y \in \mathcal{C}$ transformations $\xi : Y \to X \odot Y$. We shall say for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ that the last one is *linearized* by a natural in $Y \in \mathcal{C}$ transformation $\lambda : X \odot F(Y) \to F(X \odot Y)$ with respect to Ξ iff for all $\xi \in \Xi$ we have

$$F(\xi) = \lambda \circ \xi,$$

which is a short one for the equality

$$F(\xi(Y)) = \lambda(Y) \circ \xi(F(Y)),$$

expressing the commutativity of the diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{\ \xi\ } & X \odot F(Y) \\ & {}_{F(\xi)}\searrow & \downarrow{\scriptstyle \lambda} \\ & & F(X \odot Y) \end{array}$$

for all $Y \in \mathcal{C}$.

For example, in the SRC of posets the endofunctor R is linearized by the natural transformation ϑ_R with respect to the set of all natural transformations $\xi : Y \to X \times Y$ of the form $\xi(y) = (x, y)$ for fixed $x \in X$.

In the general case, the set Ξ is partially ordered in a natural way:

$$\xi \leq \xi' \iff \forall Y \in \mathcal{C}(\xi(Y) \leq \xi'(Y)).$$

Given a $\mathcal{C}$-morphism $g : X \to X$ such that $(g \odot 1) \circ \xi \in \Xi$ for all $\xi \in \Xi$, we may consider the inequality

$$(g \odot 1) \circ \xi \leq \xi \tag{41}$$

with an unknown $\xi \in \Xi$.

Definition 4.1. An element $\omega \in \Xi$ will be called Φ-pseudominimal solution of (41) for a subset $\Phi \subseteq \mathcal{C}(X \odot \dot{T}, X)$ iff ω is a solution of (41), i.e. $(g \odot 1) \circ \omega(Y) \leq \omega(Y)$ for all $Y \in \mathcal{C}$, and the following two conditions hold for all $\mathcal{C}$-arrows $\chi : \dot{T} \to X$ and $\psi : \dot{T} \odot \dot{T} \to X$ and all $\varphi \in \Phi$:

$$\forall \xi \in \Xi(\varphi \circ \xi \leq \chi \;\Rightarrow\; \varphi \circ (g \odot 1) \circ \xi \leq \chi) \;\Rightarrow\; \varphi \circ \omega \leq \chi, \tag{$*'$}$$

$$\forall \xi \in \Xi(\varphi' \circ (1 \odot \xi) \leq \psi \;\Rightarrow\; \varphi' \circ (1 \odot (g \odot 1) \circ \xi) \leq \psi) \;\Rightarrow\; \varphi' \circ (1 \odot \omega) \leq \psi, \tag{$*''$}$$

where $\varphi' = f_{10} \circ (\chi \odot \varphi)$.

A remark on notations. As usual, we do not write the arguments in a natural transformation; e.g., writing $\varphi \circ \xi$, we mean this instance of ξ which makes it composable with φ; in this way ξ in ($*'$) and ($*''$) is $\xi(\dot{T})$ and ω is $\omega(\dot{T})$.

Definition 4.2. Let Ξ be the set of natural transformations $\xi : Y \to X \odot Y$ as above, and let $g : X \to X$ be a $\mathcal{C}$-arrow such that $(g \odot 1) \circ \xi \in \Xi$ for all $\xi \in \Xi$. Let also $\sigma : B_1 \to \dot{T}$ be a 'system', i.e. an $\mathcal{C}$-arrow with analizer $\alpha : \dot{T} \to S_0^+(\dot{T})$. Then by *coding* for the system σ in the algebra $f : S_0(X) \to X$ with respect to Ξ and g we mean a pair $\langle \kappa, \lambda_0^+ \rangle$ consisting of an $\mathcal{C}$-arrow $\kappa : X \odot \dot{T} \to X$ and a natural transformation

$$\lambda_0^+ : X \odot S_0^+(Y) \to S_0^+(X \odot Y),$$

which linearizes S_0^+ with respect to Ξ, such that the following equality holds for the *coding morphism* κ:

$$\kappa \circ (g \odot 1) = f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha). \tag{42}$$

$$\begin{array}{ccccc}
X \odot \dot{T} & \xrightarrow{\quad g \odot 1 \quad} & X \odot \dot{T} & \xrightarrow{\quad \kappa \quad} & X \\
\big\downarrow 1 \odot \alpha & & & & \big\uparrow f^+ \\
X \odot S_0^+(\dot{T}) & \xrightarrow{\quad \lambda_0^+ \quad} & S_0^+(X \odot \dot{T}) & \xrightarrow{\quad S_0^+(\kappa) \quad} & S_0^+(X)
\end{array}$$

Theorem 4.1. *Let $\langle \kappa, \lambda_0^+ \rangle$ be a coding for $\sigma : B_1 \to \dot{T}$ in $f : S_0(X) \to X$ with respect to Ξ, g be as in the previous definition, and let $\omega \in \Xi$ be a $\{\kappa\}$-pseudominimal solution of (41). Then the morphism $\kappa \circ \omega : \dot{T} \to X$ is an α-minimal evaluator of f, where α is the analizer of σ.*

Proof. The proof is rather straightforward:

$$\begin{aligned}
f^+ \circ S_0^+(\kappa \circ \omega) \circ \alpha &= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \omega \circ \alpha && \text{(because } \lambda_0^+ \text{ linearizes } S_0^+\text{)}\\
&= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \omega && \text{(since } \omega \text{ is natural)}\\
&= \kappa \circ (g \odot 1) \circ \omega && \text{(by (42))}\\
&\le \kappa \circ \omega && \text{(since } \omega \text{ is a solution of (41)),}
\end{aligned}$$

i.e. $\kappa \circ \omega$ is a solution of (33). To show that it is the least solution of (33), suppose $\kappa \circ \xi \le \eta$ for an arbitrary solution $\eta : \dot{T} \to X$ of (33) in $\mathcal{C}(\dot{T}, X)$ and an arbitrary $\xi \in \Xi$. Then

$$\begin{aligned}
\kappa \circ (g \odot 1) \circ \xi &= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \xi && \text{(by (42))}\\
&= f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \xi \circ \alpha && \text{(since } \xi \text{ is natural)}\\
&= f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha && \text{(because } \lambda_0^+ \text{ linearizes } S_0^+\text{)}\\
&\le f^+ \circ S_0^+(\eta) \circ \alpha \le \eta.
\end{aligned}$$

This proves the hypothesis in $(*')$ for $\varphi = \kappa$ and $\chi = \eta$, and since ω is a $\{\kappa\}$-pseudominimal solution of (41), we obtain $\kappa \circ \omega \le \eta$. Therefore $\kappa \circ \omega$ is the least solution of (33). To check the condition $(*)$, take arbitrary $\mathcal{C}$-arrows $\chi : \dot{T} \to X$ and $\psi : \dot{T} \odot \dot{T} \to X$, let $\varphi = f_{10} \circ (\chi \odot 1)$ and suppose also that for every $\mathcal{C}$-arrow $\eta : \dot{T} \to X$

$$\varphi \circ (1 \odot \eta) \le \psi \;\Rightarrow\; \varphi \circ (1 \odot f^+ \circ S_0^+(\eta) \circ \alpha) \le \psi. \tag{43}$$

We have then to prove that

$$\varphi \circ (1 \odot \kappa \circ \omega) \le \psi.$$

For an arbitrary $\xi \in \Xi$ suppose $\varphi \circ (1 \odot \kappa) \circ (1 \odot \xi) \le \psi$. Then by (43)

$$\varphi \circ (1 \odot f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha) \le \psi,$$

and therefore

$$\begin{aligned}
\varphi \circ (1 \odot \kappa) \circ (1 \odot (g \odot 1) \circ \xi) &= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ (1 \odot \alpha) \circ \xi)\\
&= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa) \circ \lambda_0^+ \circ \xi \circ \alpha)\\
&= \varphi \circ (1 \odot f^+ \circ S_0^+(\kappa \circ \xi) \circ \alpha) \le \psi,
\end{aligned}$$

which proves the hypothesis in $(*'')$ for $\varphi' = f_{10} \circ (\chi \odot \kappa) = \varphi \circ (1 \odot \kappa)$, and since ω is a $\{\kappa\}$-pseudominimal solution of (41), we conclude that $\varphi \circ (1 \odot \kappa) \circ (1 \odot \omega) \leq \psi$, i.e. $\varphi \circ (1 \odot \kappa \circ \omega) \leq \psi$. Therefore $\kappa \circ \omega$ is an α-minimal evaluator of f.

Now consider the special case with the SRC $\langle \mathcal{C}, \times, \bar{a}, R, \vartheta_R \rangle$ of posets. Take an *operative space* X in the sense of Ivanov [1]. This is, up to notational variations, a partially ordered algebra X with two binary operations — *multiplication* (denoted in the usual way: xy is the result of applying this operation on $x, y \in X$) and *pairing* (notation: $[x, y]$ for the result of applying this operation on $x, y \in X$) and three constants $\boldsymbol{e}$, $\boldsymbol{i}_0$, $\boldsymbol{i}_1$ such that the multiplication is associative with the unit $\boldsymbol{e}$ and the following three equalities hold for all $x, y, y' \in X$:

$$x[y, y'] = [xy, xy']; \qquad [x, y]\boldsymbol{i}_0 = x; \qquad [x, y]\boldsymbol{i}_1 = y.$$

Consider also a set $B_0 \in \mathcal{C}$ with the trivial partial order (coinciding with equality) and a mapping $f_0 : B_0 \to X$. The set B_0 is supposed to contain three different elements regarded as symbols for the constants $\boldsymbol{e}$, $\boldsymbol{i}_0$, $\boldsymbol{i}_1$, and f_0 is supposed to map those symbols on those constants, respectively. The other elements of B_0 are treated as parameters. The mapping f_0 and the space X determine an S_0-algebra $f : S_0(X) \to X$ in $\mathcal{C}$ such that f_{10} and f_{11} are multiplication and pairing in X, respectively. As in Section 3, we consider also a set B_1 with the trivial partial order, the elements of which are treated as variables. The sum (i.e. the disjoint union) $B = B_0 + B_1$ has also the trivial partial order, and such is the order in the objects T and $\dot{T}$ of the least fixed points $\tau : S(T) \to T$ and $\dot{\tau} : \dot{S}(\dot{T}) \to \dot{T}$, respectively, the elements of which are all terms and normal terms (in the sense, for instance, of [5]), respectively.

Take for Ξ the set of all natural in $Y \in \mathcal{C}$ transformations $\xi : Y \to X \times Y$ defined for all $Y \in \mathcal{C}$ and all $y \in Y$ by

$$\xi(y) = (x, y), \tag{44}$$

where $x \in X$. In the category $\mathcal{C}$ of posets Ξ and X are isomorphic — the obvious isomorphism assigns to each $\xi \in \Xi$ the unique $x \in X$ for which (44) holds for every $y \in Y$ and all $Y \in \mathcal{C}$. This isomorphism transforms the inequality (41) into the inequality $g(x) \leq x$ with one unknown $x \in X$ for every $\mathcal{C}$-arrow (i.e. an increasing mapping) $g : X \to X$. The notion of the Φ-pseudominimal solution of (41) is transformed as follows.

An element $\omega \in \Xi$ is a Φ-pseudominimal solution of (41) iff every subset $J \subseteq X$ of one of the following two forms:

$$\begin{cases} \{x \in X \mid \varphi(x, t) \leq \chi(t) \text{ for all } t \in \dot{T}\}, \\ \{x \in X \mid \chi(s)\varphi(x, t) \leq \psi(s, t) \text{ for all } t, s \in \dot{T}\}, \end{cases} \tag{45}$$

where $\varphi \in \Phi$, $\chi : \dot{T} \to X$ and $\psi : \dot{T} \times \dot{T} \to X$ are arbitrary $\mathcal{C}$-mappings, which is invariant with respect to g, i.e. $g(J) \subseteq J$, contains the element $w \in X$ corresponding to ω (i.e. $\omega(y) = (w, y)$ for all $y \in Y$ and $Y \in \mathcal{C}$).

Take for Φ the set of all mappings $\varphi : X \times \dot{T} \to X$ of the form $\varphi(x, t) = xk(t)$, where $k : \dot{T} \to X$ is an arbitrary function (since the order in $\dot{T}$ is trivial, all such functions belong to $\mathcal{C}$); and take for g the mapping $g : X \to X$ defined by

$g(x) = [e, x]r$, where $r \in X$. We shall call an element $w \in X$ an *iteration* of $r \in X$ iff $g(w) = w$ and $w \in J$ for every set of one of the forms (45) such that $g(J) \subseteq J$. The supposition that every $r \in X$ has an iteration is a possible version of the notion of iterativity for the operative space X. (This version differs from the version of Ivanov in [1] and is close to the version in [5]. However, it is a natural version — the examples of iterative spaces in [1] are typically iterative in this sense also.) Therefore, supposing the space X iterative in this sense, we have that for every mapping g of the form $g(x) = [e, x]r$ there is a Φ-pseudominimal solution $\omega \in X$ of (41).

Next define the natural in $Y \in \mathcal{C}$ transformation

$$\lambda_0^+ : X \odot S_0^+(Y) \to S_0^+(X \odot Y)$$

by

$$\lambda_0^+ = ((\pi + (\bar{a} + \vartheta_R) \circ \delta_\odot) \circ \delta_\odot + 1) \circ \delta_\odot,$$

where π is the projection $X \times B_0 \to B_0$. A direct checking shows that λ_0^+ linearizes S_0^+ with respect to Ξ. For the coding morphism $\kappa : X \times \dot{T} \to X$ we have to ask that $\kappa \in \Phi$, i.e. $\kappa = f_{10} \circ (1 \times k)$ for a suitable $k : \dot{T} \to X$, and that the coding equality (42) is satisfied. In terms of elements, the last equality is equivalent to the following five ones:

$g(x)k(b) = f_0(b)$	for all $b \in B_0$,
$g(x)k(v) = xk(\sigma(v))$	for all $v \in B_1$,
$g(x)k(tb) = xk(t)f_0(b)$	for all $t \in \dot{T}$ and $b \in B_0$,
$g(x)k(tv) = xk(\mu(t, \sigma(v)))$	for all $t \in \dot{T}$ and $v \in B_1$,
$g(x)k([t, s]) = x[k(t), k(s)]$	for all $t, s \in \dot{T}$.

Here we use short notations for terms in $\dot{T}$: b for $\dot{\tau}_{00}(b)$, v for $\dot{\tau}_{01}(v)$, tb for $\dot{\tau}_{10}(t, \dot{\tau}_{00}(b))$, tv for $\dot{\tau}_{10}(t, \dot{\tau}_{01}(v))$, and $[t, s]$ for $\dot{\tau}_{11}(t, s)$. The mapping σ represents a system of inequalities: $\sigma(v) \leq v$ ($v \in B_1$). The last five equalities follow easily from the following ones:

$rk(b) = i_0 f_0(b)$	for all $b \in B_0$,
$rk(v) = i_1 k(\sigma(v))$	for all $v \in B_1$,
$rk(tb) = i_1 k(t) f_0(b)$	for all $t \in \dot{T}$ and $b \in B_0$,
$rk(tv) = i_1 k(\mu(t, \sigma(v)))$	for all $t \in \dot{T}$ and $v \in B_1$,
$rk([t, s]) = i_1 [k(t), k(s)]$	for all $t, s \in \dot{T}$,

and when the last ones are fulfilled, we say that k and r provide a coding for the system σ with respect to f_0 (compare with the notion of coding in [5, 6]); they can be satisfied comparatively straightforwardly, using a representation of primitive recursive functions and a weak form of axioms for the translation operation (see [5]). This construction of coding combined with the code evaluation theorem implies easily all basic facts of algebraic recursion theory in operative spaces. The last theorem states that if k and r provide a coding for a system σ and w is iteration of r, then the mapping $x : B_1 \to X$ defined by $x(v) = wk(v)$ is the least solution of the system σ; and it follows from Theorems 4.1 and 3.1. Indeed, by Theorem 4.1 $\kappa \circ \omega = f_{10} \circ (1 \times k) \circ \omega$ is an α-minimal evaluator of f, where $\omega(y) = (w, y)$ for all $y \in Y$ and all $Y \in \mathcal{C}$. Thus, for $t \in \dot{T}$ we have $(\kappa \circ \omega)(t) = f_{10}(w, k(t)) = wk(t)$. By Theorem 3.1 $\kappa \circ \omega$ is a $\mathcal{C}_{S_0}$-morphism, whence by Proposition 3.2 it is the least such morphism $h : P(\tau^N) \to f$ satisfying the inequality

$$h(\sigma(v)) \leq h(v) = h(\dot{\tau}_{01}(v))$$

for all $v \in B_1$. Thence it follows that the mapping $x : B_1 \to X$ defined by $x(v) = (\kappa \circ \omega)(v) = wk(v)$ (i.e. the 'restriction' of h on B_1) is the least solution of the system represented by σ, since every mapping $x : B_1 \to X$ can be uniquely extended to a $\mathcal{C}_{S_0}$-morphism $h : P(\tau^N) \to f$, the mapping $h : \dot{T} \to X$ assigning to each term $t \in \dot{T}$ its value under the evaluation provided by x.

In this sense the theory of operative spaces is a special case of the results of Sections 3 and 4. The natural categorical generality for the last theory being thus reached, various other special cases may be expected to be of interest. Especially, we shall mention one of them, which is connected with an attempt by Petrov and Skordev [4] to generalize Skordev's theory of combinatory spaces for some kind of category-like partial ordered structures in which the role of multiplication is played by a composition of arrows. This special case is obtained by applying the theory of Sections 3 and 4 to an SRC $\langle \mathcal{C}, \odot, \tilde{a}, R, \vartheta_R \rangle$ in which $\mathcal{C}$ is a suitable subcategory of the category of directed graphs and $\odot$ is the product $\times_O$ over a fixed set O of objects in the terminology and notations of [2]. It may be optimistically said that in this way a theory of that kind, which was aimed at by Petrov and Skordev in [4], may be reached in full (in [4] only a part of the desirable results has been reached, especially, the corresponding analogue of the recursion theorem has not been obtained). We are leaving this topic for the possible further publications.

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ADMISSIBILITY IN Σ_n^0-ENUMERATIONS*

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In the paper we introduce the notion of Σ_n^0 *partial enumeration* of an abstract structure $\mathfrak{A}$. Given a $k \leq n$, we obtain a characterization of the subsets of $\mathfrak{A}$ possessing Σ_k^0 pullbacks in all Σ_n^0 partial enumerations of $\mathfrak{A}$.

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1. INTRODUCTION

Let $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$ be a countable abstract structure, where each R_i is an a_i-ary predicate on A.

A total mapping f of the set of the natural numbers N onto A is called a *total enumeration* of $\mathfrak{A}$. Every total enumeration f of $\mathfrak{A}$ determines a unique structure $\mathfrak{B}_f = (N; R_1^f, R_2^f, \ldots, R_l^f)$ of the same relational type as $\mathfrak{A}$, where

$$R_i^f(x_1, \ldots, x_{a_i}) \iff R_i(f(x_1), \ldots, f(x_{a_i})).$$

Let $\alpha < \omega_1^{CK}$. A subset M of A^a is said to be Σ_α^0-*admissible* in $\mathfrak{A}$ if for every total enumeration f of $\mathfrak{A}$ the pullback $f^{-1}(M)$ of M is Σ_α^0 in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

The notion of Σ_1^0-admissibility with respect to injective total enumerations was introduced in 1964 by Lacombe [3] under the name $\forall$-admissibility. Several modifications and generalizations of this notion have appeared since 1964. Among them

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we would like to mention the Σ_1^0-admissibility in partial enumerations introduced in [5] and the relatively intrinsically Σ_α^0 sets introduced in [1] and [2], which are defined by means of Σ_α^0-admissibility with respect to injective total enumerations.

In [5] the author made the observation that the sets on an abstract structure which are Σ_1^0-admissible with respect to partial enumerations with relatively recursively enumerable (r.e.) domains coincide with the sets which are Σ_1^0-admissible with respect to total enumerations.

In the present paper we are going to study further the interplay between admissibility in total and partial enumerations. For we introduce the notion of Σ_n^0-admissibility in partial enumerations with relatively Σ_n^0 domains, and more generally, for $k \leq n$, Σ_k^0-admissibility with respect to partial enumerations with relatively Σ_n^0 domains. A normal form of the admissible sets is obtained. It turns out that for $k < n$ the admissible sets coincide with those which are Σ_k^0-admissible in all partial enumerations and are described by means of quantifier free recursive Σ_k^0 formulas. If $k = n$, then our notion of admissibility leads to a class of sets, described by means of a simple kind of recursive Σ_n^0 formulas on the abstract structure, in which the quantifiers ranging over the domain of the structure are existential and appear only on the last level.

The arguments use the machinery of the so-called regular enumerations, which seems to have a wide range of other applications.

2. PRELIMINARIES

Consider again the countable structure $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$, which from now on we shall suppose fixed.

2.1. Definition. An *enumeration* of $\mathfrak{A}$ is an ordered pair $\langle f, \mathfrak{B}_f \rangle$, where f is a partial surjective mapping of N onto A with an infinite domain, $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \ldots, \sigma_l)$ is a structure of the same relational type as $\mathfrak{A}$, and the following condition holds for every $i \in [1, l]$ and all $x_1, \ldots, x_{a_i} \in \operatorname{dom}(f)$:

$$\sigma_i(x_1, \ldots, x_{a_i}) \iff R_i(f(x_1), \ldots, f(x_{a_i})).$$

2.2. Definition. Let $n \geq 1$. The enumeration $\langle f, \mathfrak{B}_f \rangle$ is called Σ_n^0 if the domain of f is Σ_n^0 in the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$.

2.3. Definition. Let $k \geq 1$. A subset M of A^a is Σ_k^0-*admissible* in $\langle f, \mathfrak{B}_f \rangle$ if there exists a Σ_k^0 in $D(\mathfrak{B}_f)$ subset W of N^a such that for all $x_1, \ldots, x_a \in \operatorname{dom}(f)$

$$(x_1, \ldots, x_a) \in W \iff (f(x_1), \ldots, f(x_a)) \in M.$$

As stated in the introduction, our goal is to obtain an explicit characterization of the sets which are Σ_k^0-admissible in all Σ_n^0 enumerations, $k \leq n$. For we consider two kinds of recursive Σ_k^0 formulas in the language $\mathcal{L}_{\omega_1 \omega}$ of the structure $\mathfrak{A}$, which we call "quantifier-free" and "existential", respectively.

The Σ_k^0, the Π_k^0 and the Δ_{k+1}^0 quantifier-free formulas are defined simultaneously with their indices by induction on k. We shall suppose that a coding of the formulas in $\mathcal{L}$ is fixed. Given an index v, by Φ^v we shall denote the formula having index v. For every formula Φ, by $\Phi(X_1, \dots, X_a)$ we shall denote that the free variables in Φ are among $X_1, \dots, X_a$.

As usual, by $W_0, \dots, W_e, \dots$ we shall denote the standard enumeration of the r.e. sets of natural numbers.

2.4. Definition.

(i) The logical constant $\mathbb{T}$ and all atomic formulas in $\mathcal{L}$ are Σ_0^0 quantifier-free formulas.

The logical constant $\mathbb{F}$ and all negated atomic formulas in $\mathcal{L}$ are Π_0^0 quantifier-free formulas.

The Δ_1^0 quantifier-free formulas are finite conjunctions of Σ_0^0 and Π_0^0 quantifier-free formulas.

The indices of the Σ_0^0, Π_0^0 and Δ_1^0 quantifier-free formulas are their respective codes as formulas in $\mathcal{L}$.

(ii) If every element of W_e is index of some Δ_{k+1}^0 quantifier-free formula with variables among $X_1, \dots, X_a$, then

$$\bigvee_{v \in W_e} \Phi^v(X_1, \dots, X_a)$$

is a Σ_{k+1}^0 quantifier-free formula with index $\langle 0, k+1, e \rangle$.

If Φ is a Σ_{k+1}^0 quantifier-free formula, then $\neg\Phi$ is a Π_{k+1}^0 quantifier-free formula. For every index $\langle 0, k+1, e \rangle$ of Φ, the triple $\langle 1, k+1, e \rangle$ is an index of $\neg\Phi$.

If $\Phi_1, \dots, \Phi_b$ are Σ_r^0 or Π_r^0, $r \leq k+1$, then $\chi = \Phi_1 \& \dots \& \Phi_b$ is a Δ_{k+2}^0 quantifier-free formula. If $v_1, \dots, v_b$ are indices of $\Phi_1, \dots, \Phi_b$, respectively, then $\langle 2, v_1, \dots, v_b \rangle$ is an index of χ.

2.5. Definition. A Σ_k^0 *existential formula*, $k \geq 1$, is a formula of the form

$$\bigvee_{v \in V} \exists Y_1 \cdots \exists Y_{q_v} \Phi^v(Y_1, \dots, Y_{q_v}, X_1, \dots, X_a),$$

where V is an r.e. set of indices of Δ_k^0 formulas.

Let $M \subseteq A^a$ and $\Phi(X_1, \dots, X_a, Z_1, \dots, Z_b)$ be a Σ_k^0 quantifier-free or existential formula.

2.6. Definition. The set M is *definable* by Φ on $\mathfrak{A}$ if for some $t_1, \dots, t_b \in A$

$$(\forall s_1, \dots, s_a \in A)((s_1, \dots, s_a) \in M \iff \mathfrak{A} \models \Phi(s_1, \dots, s_a, t_1, \dots, t_b)).$$

In the rest of the paper we are going to prove the next two theorems.

2.7. Theorem. *Let $M \subseteq A^a$ and $1 \leq k < n$. The set M is Σ_k^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ_k^0 quantifier-free formula on $\mathfrak{A}$.*

2.8. Theorem. *The set M is Σ_n^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ if and only if M is definable by some Σ_n^0 existential formula on $\mathfrak{A}$.*

3. GENERIC ENUMERATIONS

The proofs of Theorem 2.7 and Theorem 2.8 use a forcing construction. In this section we shall describe the fundamentals of this construction.

3.1. Satisfaction relation. To simplify the notations we shall consider only the subsets of the domain of the structure $\mathfrak{A}$. All results can be easily proved for subsets of A^a, $a > 1$.

Let $\langle f, \mathfrak{B}_f \rangle$ be a partial enumeration of the structure $\mathfrak{A} = (A; R_1, R_2, \ldots, R_l)$. And suppose that $\mathfrak{B}_f = (N; \sigma_1, \sigma_2, \ldots, \sigma_l)$. We shall identify the diagram $D(\mathfrak{B}_f)$ of $\mathfrak{B}_f$ with the set consisting of the codes of the atomic and the negated atomic formulas which are true on $\mathfrak{B}_f$. In other words, we shall assume that

$$D(\mathfrak{B}_f) = \{\langle i, x_1, \ldots, x_{a_i}, \varepsilon \rangle : \sigma_i(x_1, \ldots, x_{a_i}) = \varepsilon,\ i \in [1, l]\}.$$

If $u \in N$, then define

$$f \models u \Longleftrightarrow u \in D(\mathfrak{B}_f).$$

If E is a finite subset of N, then

$$f \models E \Longleftrightarrow f \models u \text{ for each } u \in E.$$

Assume also fixed an effective coding of all finite sets of natural numbers. By E_v we shall denote the finite set with the code v.

Let us fix for every $n \geq 1$ and each $e \in N$ a unary predicate letter F_e^n. We adopt the notation $\neg^i F_e^n(x) = F_e^n(x)$ if $i = 0$ and $\neg^i F_e^n(x) = \neg F_e^n(x)$ if $i = 1$. We shall assume that the code of $\neg^i F_e^n(x)$ is $\langle i, n, e, x \rangle$.

For each $x \in N$ and every predicate letter F_e^n the satisfaction relation $f \models \neg^i F_e^n(x)$ is defined by induction on n. Given a finite set E of natural numbers and $n \geq 1$, by $f \models_n E$ we shall denote that every element u of E is of the form $\langle i, n, e, x \rangle$ and $f \models \neg^i F_e^n(x)$.

3.2. Definition.

(i) $f \models F_e^1(x) \iff \exists v(\langle v, x \rangle \in W_e \;\&\; f \models E_v)$;

$f \models \neg F_e^1(x) \iff f \not\models F_e^1(x)$.

(ii) $f \models F_e^{n+1}(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; f \models_n E_v)$;
$f \models \neg F_e^{n+1}(x) \iff f \not\models F_e^{n+1}(x)$.

3.3. Proposition.

(1) *The sets* $\{x : f \models F_e^n(x)\}$ *coincide with the* Σ_n^0 *in* $D(\mathfrak{B}_f)$ *sets.*

(2) *The sets* $\{x : f \models \neg F_e^n(x)\}$ *coincide with the* Π_n^0 *in* $D(\mathfrak{B}_f)$ *sets.*

Proof. The proof is by induction on n.

For $n = 1$ note that from the definition of "$\models$" we have

$$f \models F_e^1(x) \iff x \in \Gamma_e(D(\mathfrak{B}_f)),$$

where Γ_e is the e-th enumeration operator, see [4]

Since $N \setminus D(\mathfrak{B}_f)$ is enumeration reducible to $D(\mathfrak{B}_f)$, the r.e. in $D(\mathfrak{B}_f)$ sets coincide with the sets which are enumeration reducible to $D(\mathfrak{B}_f)$.

The step from n to $n+1$ follows easily by the Strong hierarchy theorem, see [4].

3.4. Corollary. *A set* $M \subseteq A$ *is* Σ_n^0*-admissible in* $\langle f, \mathfrak{B}_f\rangle$ *iff there exists an* $e \in N$ *such that for all* $x \in \mathrm{dom}(f)$

$$f \models F_e^n(x) \iff f(x) \in M.$$

3.5. Finite parts and forcing. The conditions of the forcing are finite mappings of N into A with some additional properties which we call *finite parts*. We use δ, τ, ρ to denote finite parts.

Let $[0, q]$ be an initial segment of N.

3.6. Definition. A *finite part* δ on $[0, q]$ is an ordered triple $\langle \alpha_\delta, H_\delta, D_\delta\rangle$ with the following properties:

(1) α_δ is a partial mapping of $[0, q]$ into A;

(2) $H_\delta \subseteq [0, q]$;

(3) $\mathrm{dom}(\alpha_\delta) \cup H_\delta = [0, q]$ and $\mathrm{dom}(\alpha_\delta) \cap H_\delta = \emptyset$;

(4) D_δ is the diagram of a finite structure of the same relational type as $\mathfrak{A}$ and domain $[0, q]$, and such that if $x_1, \ldots, x_{a_i} \in \mathrm{dom}(\alpha_\delta)$, then

$$\langle i, x_1, \ldots, x_{a_i}, \varepsilon\rangle \in D_\delta \iff R_i(\alpha_\delta(x_1), \ldots, \alpha_\delta(x_{a_i})) = \varepsilon.$$

Let Δ be the set of all finite parts.

3.7. Definition. Given finite parts δ and τ, let

$$\delta \subseteq \tau \iff \alpha_\delta \subseteq \alpha_\tau \;\&\; H_\delta \subseteq H_\tau \;\&\; D_\delta \subseteq D_\tau.$$

If $\langle f, \mathfrak{B}_f\rangle$ is an enumeration, then let

$$\delta \subseteq \langle f, \mathfrak{B}_f\rangle \iff \alpha_\delta \subseteq f \;\&\; H_\delta \cap \mathrm{dom}(f) = \emptyset \;\&\; D_\delta \subseteq D(\mathfrak{B}_f).$$

Let $\delta \in \Delta$.
If $u \in N$, then $\delta \Vdash u$ iff $u \in D_\delta$.
If $E = \{u_1, \ldots, u_r\}$ is a finite subset of N, then let

$$\delta \Vdash E \iff \delta \Vdash u_1 \ \& \ \ldots \ \& \ \delta \Vdash u_r.$$

Now we are ready to define the forcing relation $\delta \Vdash F_e^n(x)$ for all $e, x \in N$ by induction on $n \geq 1$. As before we shall denote by $\delta \Vdash_n E$ that every element u of the finite set E is in the form $\langle i, n, e, x\rangle$ and $\delta \Vdash \neg^i F_e^n(x)$.

3.8. Definition.

(i) $\delta \Vdash F_e^1(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ \delta \Vdash E_v)$;

$\delta \Vdash \neg F_e^1(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \nVdash F_e^1(x))$.

(ii) $\delta \Vdash F_e^{n+1}(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ \delta \Vdash_n E_v)$;

$\delta \Vdash \neg F_e^{n+1}(x) \iff \forall \rho(\rho \supseteq \delta \Longrightarrow \rho \nVdash F_e^{n+1}(x))$.

From the above definition follows immediately the monotonicity of the forcing, i.e. if $\delta \Vdash F_e^n(x)$ and $\delta \subseteq \tau$, then $\tau \Vdash F_e^n(x)$.

3.9. Definition. Let $Y \subseteq \Delta$. The enumeration $\langle f, \mathfrak{B}_f\rangle$ *meets* Y if for some $\delta \in Y$, $\delta \subseteq f$.

3.10. Definition. A subset $Y \subseteq \Delta$ is *dense in the enumeration* $\langle f, \mathfrak{B}_f\rangle$ if

$$(\forall \delta \subseteq f)(\exists \tau \in Y)(\delta \subseteq \tau).$$

3.11. Definition. Let $\mathfrak{F}$ be a family of subsets of Δ. An enumeration $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}$-*generic* if whenever $Y \in \mathfrak{F}$ and Y is dense in $\langle f, \mathfrak{B}_f\rangle$, then $\langle f, \mathfrak{B}_f\rangle$ meets Y.

As usual, we have that for every countable family $\mathfrak{F}$ of subsets of Δ and every $\delta \in \Delta$ there exists an $\mathfrak{F}$-generic enumeration $\langle f, \mathfrak{B}_f\rangle$ such that $f \supseteq \delta$.

Let $\mathfrak{F}_0 = \{\emptyset\}$. For $n \geq 1$ set $Y_{e,x}^n = \{\tau : \tau \Vdash F_e^n(x)\}$ and let $\mathfrak{F}_n = (\bigcup_{e,x} Y_{e,x}^n) \cup \mathfrak{F}_{n-1}$.

The following Truth lemma can be proved by induction on n:

3.12. Lemma. *Let $\langle f, \mathfrak{B}_f\rangle$ be an enumeration, $n \geq 0$. Then for all $e, x \in N$:*

(1) *If $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_n$-generic, then*

$$f \models F_e^{n+1}(x) \iff (\exists \delta \subseteq f)(\delta \Vdash F_e^{n+1}(x)).$$

(2) *If $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_{n+1}$-generic, then*

$$f \models \neg F_e^{n+1}(x) \iff (\exists \delta \subseteq f)(\delta \Vdash \neg F_e^{n+1}(x)).$$

3.13. Definition. Let $\delta \subseteq \tau$. Then τ/δ is the finite part $\langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)), D_\tau\rangle$.

By $\delta \preceq \tau$ we shall denote that $\mathrm{dom}(\alpha_\delta) = \mathrm{dom}(\alpha_\tau)$ and $\delta \subseteq \tau$.

3.14. Lemma.

(1) *If* $\delta \subseteq \tau$, *then* $\delta \preceq \tau/\delta$;
(2) *If* $\delta \subseteq \tau_1 \subseteq \tau_2$, *then* $\tau_1/\delta \preceq \tau_2/\delta$;
(3) *If* $\delta \subseteq \tau$ *and* $\tau/\delta \preceq \rho$, *then there exists a finite part* ρ' *such that* $\tau \preceq \rho'$ *and* $\rho'/\delta = \rho$.

Proof. (3) Let $\delta \subseteq \tau$ and $\tau/\delta \preceq \rho$. Then $\tau/\delta = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)), D_\tau\rangle$. $\tau/\delta \preceq \rho$ implies $\rho = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)) \cup H', D_\rho\rangle$, where $D_\tau \subseteq D_\rho$ and $H' \cap (\mathrm{dom}(\alpha_\tau) \cup H_\tau) = \emptyset$.

Let $\rho' = \langle \alpha_\tau, H_\tau \cup H', D_\rho\rangle$. Then $\tau \preceq \rho'$ and $\rho'/\delta = \langle \alpha_\delta, H_\tau \cup (\mathrm{dom}(\alpha_\tau) \setminus \mathrm{dom}(\alpha_\delta)) \cup H', D_\rho\rangle = \rho$.

3.15. Stared forcing. We define a stared forcing relation $\delta \Vdash^* F_e^n(x)$ for all $n \geq 1$, $e, x \in N$ by means of the following inductive definition:

3.16. Definition.

(i) $\delta \Vdash^* F_e^1(x) \iff \delta \Vdash F_e^1(x)$;
$\delta \Vdash^* \neg F_e^1(x) \iff \forall \rho(\rho \succeq \delta \Longrightarrow \rho \not\Vdash^* F_e^1(x))$.

(ii) $\delta \Vdash^* F_e^{n+1}(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ \delta \Vdash_n^* E_v)$;
$\delta \Vdash^* \neg F_e^{n+1}(x) \iff \forall \rho(\rho \succeq \delta \Longrightarrow \rho \not\Vdash^* F_e^{n+1}(x))$.

Here $\delta \Vdash_n^* E_v$ means, as before, that every element of E_v is in the form $\langle i, n, e, x\rangle$ and $\delta \Vdash^* \neg^i F_e^n(x)$.

From the definition above it follows immediately that the stared forcing is monotone with respect to "$\preceq$", i.e. $\delta \Vdash^* F_e^n(x) \ \&\ \delta \preceq \tau \Longrightarrow \tau \Vdash^* F_e^n(x)$.

3.17. Lemma. *Let* $\delta \subseteq \tau$. *Then for all* $e, x \in N$, $n \geq 1$,

(1) $\tau \Vdash F_e^n(x) \iff \tau/\delta \Vdash^* F_e^n(x)$;

(2) $\tau \Vdash \neg F_e^n(x) \iff \tau/\delta \Vdash^* \neg F_e^n(x)$.

Proof. The proof is by induction on n.

Since $D_\tau = D_{\tau/\delta}$, (1) holds for $n = 1$.

Suppose now that (1) is true for some $n \geq 1$.

(2) ($\Rightarrow$). Let $\tau \Vdash \neg F_e^n(x)$. Assume that $\tau/\delta \not\Vdash^* \neg F_e^n(x)$. Then there is a finite part $\rho \succeq \tau/\delta$ such that $\rho \Vdash^* F_e^n(x)$. By Lemma 3.14 there exists a finite part ρ' such that $\rho' \succeq \tau$ and $\rho'/\delta = \rho$. Then $\rho'/\delta \Vdash^* F_e^n(x)$ and by induction $\rho' \Vdash F_e^n(x)$. Clearly, $\rho' \supseteq \tau$. A contradiction.

(2) ($\Leftarrow$). Let $\tau/\delta \Vdash^* \neg F_e^n(x)$. Assume that $\tau \nVdash \neg F_e^n(x)$. Then there exists $\rho \supseteq \tau$ such that $\rho \Vdash F_e^n(x)$. By induction $\rho/\delta \Vdash^* F_e^n(x)$. By Lemma 3.14 $\rho/\delta \succeq \tau/\delta$. A contradiction.

Now, using the respective definitions, we get immediately that

$$\tau \Vdash F_e^{n+1}(x) \iff \tau \Vdash^* F_e^{n+1}(x).$$

3.18. Lemma. *Let δ be a finite part, $n \geq 1$, $e, x \in N$. Then*

(1) $\delta \Vdash F_e^n(x) \iff \delta \Vdash^* F_e^n(x)$;

(2) $(\exists \tau \supseteq \delta)(\tau \Vdash F_e^n(x)) \iff (\exists \rho \succeq \delta)(\rho \Vdash^* F_e^n(x))$.

Since $\delta/\delta = \delta$, (1) follows from the previous lemma. By the same argument $\delta \Vdash \neg F_e^n(x) \iff \delta \Vdash^* \neg F_e^n(x)$. From here (2) follows by contraposition.

4. REGULAR ENUMERATIONS

Given a finite part δ defined on $[0, q]$, we shall call q the length of δ and denote it by $|\delta|$. If $p \leq q$, then by $\delta \lceil p$ we shall denote the restriction of δ on $[0, p]$, i.e. $\delta \lceil p = \langle \alpha_\delta \lceil [0, p], H_\delta \lceil [0, p], D_\delta \lceil [0, p] \rangle$. Clearly, $\delta \lceil p$ is a finite part and $\delta \lceil p \subseteq \delta$.

Given finite parts τ_1 and τ_2, say that τ_1 is *shorter* than τ_2 if:

(a) $|\tau_1| < |\tau_2|$ or

(b) $|\tau_1| = |\tau_2|$ and the code of the finite set D_{τ_1} is less than the code of D_{τ_2}.

Notice that "being shorter than" is a recursive relation and for every finite part δ it is a well ordering on the set $\{\tau | \delta \preceq \tau\}$.

Let $\mathcal{F}_n^*$ be the sequence $\{X_0^n, X_1^n, \ldots, X_i^n, \ldots\}$ of sets of finite parts, where $X_i^0 = \emptyset$ and $X_i^n = \{\tau : \tau \Vdash^* F_{(i)_0}^n((i)_1)\}$ for $n \geq 1$.

The finite part τ *decides* X_i^n if $\tau \in X_i^n$ or $(\forall \rho \succeq \tau)(\rho \notin X_i^n)$. Clearly, for every δ and i there exists a $\tau \succeq \delta$ such that τ decides X_i^n. By Lemma 3.18, if τ decides X_i^n and $\tau \subseteq \rho$, then ρ also decides X_i^n.

Let

$$\mu_n(i, \delta) = \begin{cases} \delta & \text{if } (\forall \tau \succeq \delta)(\tau \notin X_i^n), \\ (\textit{the shortest } \tau)(\delta \preceq \tau \ \& \ \tau \in X_i^n) & \text{otherwise.} \end{cases}$$

Clearly, $\mu_n(i, \delta)$ decides X_i^n. Notice also that the length of $\mu_n(i, \delta)$ depends only on the length $|\delta|$ of δ and on its diagram D_δ. Moreover, there exists a recursive in $\emptyset^{(n)}$ function λ_n such that

$$\forall i \forall \delta (\lambda_n(i, |\delta|, D_\delta) = |\mu_n(i, \delta)|).$$

4.1. Definition. Let δ be a finite part on $[0, q]$. Then δ is *n-regular* if $0 \in \text{dom}(\alpha_\delta)$, and if $q_0 < q_1 < \ldots < q_r$ are the elements of $\text{dom}(\alpha_\delta)$, then:

(a) $(\forall i < r)(\delta \lceil (q_{i+1} - 1) = \mu_n(i, \delta \lceil q_i))$;

(b) $\delta = \mu_n(r, \delta \lceil q_r)$.

We shall denote the number r from the above definition by $\|\delta\|$.

4.2. Lemma. *Let δ be an n-regular finite part, where $\mathrm{dom}(\alpha_\delta) = \{q_0 < q_1 < \ldots < q_r\}$. Then for each $i < r$, $\delta\lceil(q_{i+1} - 1)$ is n-regular.*

4.3. Definition. An enumeration $\langle f, \mathfrak{B}_f\rangle$ of $\mathfrak{A}$ is called *n-regular* if for each finite part $\delta \subseteq f$ there exists an n-regular finite part τ such that $\delta \subseteq \tau \subseteq f$.

4.4. Lemma. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then for each natural number r there exists an n-regular finite part $\delta \subseteq f$ such that $\|\delta\| = r$.*

Proof. Given an r, consider the first $r+1$ elements $q_0 < q_1 < \ldots < q_r$ of $\mathrm{dom}(f)$. Let δ be the shortest n-regular finite part such that $\{q_0, \ldots, q_r\} \subseteq \mathrm{dom}(\alpha_\delta)$ and $\delta \subseteq f$. Assume that $\|\delta\| > r$. Then there exists an element q_{r+1} of $\mathrm{dom}(\alpha_\delta)$ such that $q_r < q_{r+1}$. By Lemma 4.2 $\delta\lceil(q_{r+1} - 1)$ is n-regular. Clearly, $\delta\lceil(q_{r+1} - 1)$ is shorter than δ and $\{q_0, \ldots, q_r\} \subseteq \mathrm{dom}(\alpha_{\delta\lceil(q_{r+1}-1)})$. The last contradicts the choice of δ.

Recall the family $\mathfrak{F}_n$. Notice that by Lemma 3.18 $\mathfrak{F}_n = \mathfrak{F}_n^*$.

4.5. Proposition. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then $\langle f, \mathfrak{B}_f\rangle$ is $\mathfrak{F}_n$-generic.*

Proof. Skipping the trivial case $n = 0$, suppose that $n \geq 1$. We shall show that $\langle f, \mathfrak{B}_f\rangle$ is generic with respect to the family $\mathfrak{F}_n^*$. Suppose that X_i^n is dense in $\langle f, \mathfrak{B}_f\rangle$. We have to prove that $\langle f, \mathfrak{B}_f\rangle$ meets X_i^n, i.e. there is a $\delta \subseteq f$ such that $\delta \in X_i^n$. By the previous lemma there exists an n-regular $\delta \subseteq f$ such that $\|\delta\| = i$. Clearly, δ decides X_i^n. Assume that $\delta \notin X_i^n$. Then $\delta \Vdash^* \neg F^n_{(i)_0}((i)_1)$ and hence, by Lemma 3.17, $\delta \Vdash \neg F^n_{(i)_0}((i)_1)$. The last contradicts the density of X_i^n.

4.6. Proposition. *Let $\langle f, \mathfrak{B}_f\rangle$ be an n-regular enumeration of $\mathfrak{A}$. Then $\mathrm{dom}(f)$ is Δ^0_{n+1} relative to $D(\mathfrak{B}_f)$.*

Proof. We have the following recursive in $D(\mathfrak{B}_f) \oplus \emptyset^{(n)}$ procedure, which lists the elements of $\mathrm{dom}(f)$ in an increasing order.

We start by printing out 0. Suppose that the first $r + 1$ elements $q_0, \ldots, q_r$ of $\mathrm{dom}(f)$ are listed. Consider the finite part $\delta_r \subseteq f$ on $[0, q_r]$. Using the oracle $D(\mathfrak{B}_f)$, we can obtain the diagram D_{δ_r}. Let q_{r+1} be the first element of $\mathrm{dom}(f)$ greater than q_r. Clearly, there exists an n-regular finite part τ such that $\delta_r \subseteq \tau$ and $q_{r+1} \in \mathrm{dom}(\alpha_\tau)$. By Definition 4.1 $q_{r+1} = \lambda_n(r, q_r, D_{\delta_r}) + 1$.

5. THE NORMAL FORM THEOREMS

In this section we shall obtain a normal form of the Σ^0_k-admissible in all Σ^0_n enumerations of $\mathfrak{A}$ sets for $k \leq n$. We start with the case $k = n$.

Let δ be a finite part, $x = |\delta| + 1$ and $s \in A$. By $\delta * s$ we shall denote the finite part $\langle \alpha', H_\delta, D\rangle$, where $\mathrm{dom}(\alpha') = \mathrm{dom}(\alpha_\delta) \cup \{x\}$, $\alpha_\delta \subseteq \alpha'$, $\alpha'(x) \simeq s$, and D is the appropriate extension of the diagram D_δ.

5.1. Theorem. *Let $M \subseteq A$, $n \geq 1$, and M be a Σ_n^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$ set. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F_e^n(x)). \tag{5.1}$$

Proof. Assume the opposite. We shall construct an $(n-1)$-regular enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ such that M is not admissible in it.

The construction of $\langle f, \mathfrak{B}_f \rangle$ will be carried out by steps. On each step j we shall define an $(n-1)$-regular finite part δ_j, so that $\delta_j \subseteq \delta_{j+1}$, and take $f = \bigcup \alpha_{\delta_j}$ and $\mathfrak{B}_f$ to be the structure with diagram $\bigcup D_{\delta_j}$.

On the even steps we shall ensure that f is onto A. On the odd steps we shall ensure that M is not admissible in $\langle f, \mathfrak{B}_f \rangle$.

Let $t_0, t_1, \ldots, t_i, \ldots$ be a fixed enumeration of the elements of A.

Let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 2e + 1$. Let $x = |\delta_{2e}| + 1$. By the assumption there exists an $s \in A$ such that

$$\neg\big[s \in M \iff (\exists \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular } \ \& \ \tau \Vdash^* F_e^n(x))\big].$$

We have two possibilities:

Case (i). $s \in M$ and $(\forall \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular} \Rightarrow \tau \nVdash^* F_e^n(x))$. In this case let δ_{2e+1} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e} * s$;

Case (ii). $s \notin M$ and $(\exists \tau \supseteq \delta_{2e} * s)(\tau \text{ is } (n-1)\text{-regular and } \tau \Vdash^* F_e^n(x))$. In this case let δ_{2e+1} be the shortest such τ.

Step $j = 2e+2$. Let t be the first $t_i \in A$ such that $t \notin \text{range}(\alpha_{\delta_{2e+1}})$. Let δ_{2e+2} be the shortest $(n-1)$-regular finite part τ such that $\tau \supseteq \delta_{2e+1} * t$.

Clearly, the enumeration $\langle f, \mathfrak{B}_f \rangle$ is $(n-1)$-regular and hence $\text{dom}(f)$ is Σ_n^0 relative to $D(\mathfrak{B}_f)$ and $\langle f, \mathfrak{B}_f \rangle$ is $\mathfrak{F}_{n-1}$-generic.

Towards a contradiction assume that M is Σ_n^0-admissible in $\langle f, \mathfrak{B}_f \rangle$. Then there exists an $e \in N$ such that for all $x \in \text{dom}(f)$

$$f(x) \in M \iff f \models F_e^n(x).$$

Consider the stage $j = 2e + 1$ of the construction. Let $x = |\delta_{2e}| + 1$. Using the Truth lemma (Lemma 3.12), we get that

$$f(x) \in M \iff (\exists \tau)(\delta_{2e+1} \subseteq \tau \subseteq f \ \& \ \tau \Vdash F_e^n(x)).$$

On the other hand, according to our construction this is not the case. So, M is not Σ_n^0-admissible in $\langle f, \mathfrak{B}_f \rangle$.

5.2. Theorem. *Let $k < n$, $M \subseteq A$ and let M be Σ_k^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$. Then there exists a finite part δ and a natural number e such that for each $s \in A$ if $x = |\delta| + 1$, then*

$$s \in M \iff (\exists \tau \succeq \delta * s)(\tau \Vdash^* F_e^k(x)). \tag{5.2}$$

Proof. Assume the contrary. We shall construct an enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ with the following properties:

(1) $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{n-1}$-generic;

(2) dom(f) is Σ^0_n relative to $D(\mathfrak{B}_f)$;

(3) the set M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$.

The construction of the enumeration $\langle f, \mathfrak{B}_f \rangle$ is very similar to that used in the proof of the previous theorem. Again it will be carried out by steps. On steps $j = 3e + 1$ we shall satisfy that $\langle f, \mathfrak{B}_f \rangle$ is an $\mathcal{F}_{n-1}$-generic enumeration. On steps $j = 3e + 2$ — that M is not Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$. And on steps $j = 3e + 3$ we shall ensure that f is a mapping onto A.

Let $t_0, t_1, \ldots, t_i, \ldots$ be a fixed enumeration of the elements of A and let δ_0 be the shortest $(n-1)$-regular finite part such that $\alpha_{\delta_0}(0) = t_0$.

Step $j = 3e + 1$. Let $\delta_{3e+1} = \mu_{n-1}(e, \delta_{3e})$.

Step $j = 3e + 2$. Let $x = |\delta_{3e+1}| + 1$. According to the assumption there exists an $s \in A$ such that

$$\neg\big[s \in M \iff (\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))\big].$$

We have two possibilities:

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \nVdash^* F^k_e(x))$.

Put $\delta_{3e+2} = \delta_{3e+1} * s$;

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))$.

In this case let δ_{3e+2} be the shortest such τ.

Step $j = 3e + 3$. Find the first $t \in A$ such that $t \notin \text{range}(\alpha_{\delta_{3e+2}})$. Let $\delta_{3e+3} = \delta_{3e+2} * t$.

The enumeration $\langle f, \mathfrak{B}_f \rangle$ is constructed as in Theorem 5.1, i.e. $f = \bigcup \alpha_{\delta_j}$ and $D(\mathfrak{B}_f) = \bigcup D_{\delta_j}$.

Arguments very similar to those used in the previous section show that $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{n-1}$-generic and dom(f) is Δ^0_n in $D(\mathfrak{B}_f)$.

Assume that M is Σ^0_k-admissible in $\langle f, \mathfrak{B}_f \rangle$. Then there is an $e \in N$ such that for all $x \in \text{dom}(f)$

$$f \models F^k_e(x) \iff f(x) \in M.$$

Consider the stage $j = 3e+2$ of our construction and let $x = |\delta_{3e+1}|+1$. There exists an $s \in A$ such that:

Case (i). $s \in M$ and $(\forall \tau \succeq \delta_{3e+1} * s)(\tau \nVdash^* F^k_e(x))$.

Since $\delta_{3e+2} \subseteq f$, $f(x) \in M$. Then $f \models F^k_e(x)$. Clearly, $\langle f, \mathfrak{B}_f \rangle$ is $\mathcal{F}_{k-1}$-generic. By Lemma 3.12 and Lemma 3.18 there exists a finite part τ such that $\delta_{3e+1} * s \preceq \tau \;\&\; \tau \Vdash^* F^k_e(x)$. A contradiction;

Case (ii). $s \notin M$ and $(\exists \tau \succeq \delta_{3e+1} * s)(\tau \Vdash^* F^k_e(x))$. Since $\delta_{3e+2} \subseteq f$, $f(x) = s$. Using again Lemma 3.12 and Lemma 3.18, we get $f \models F^k_e(x)$. A contradiction.

If a subset M of A is definable by a Σ_k^0 quantifier-free formula on $\mathfrak{A}$, then it is clear that M is Σ_k^0-admissible in all enumerations of $\mathfrak{A}$. It is easy to verify also that if a set M is definable by a Σ_n^0 existential formula on $\mathfrak{A}$, then M is Σ_n^0-admissible in all Σ_n^0 enumerations of $\mathfrak{A}$.

The proofs of both theorems in the non-trivial directions make use of the respective normal form theorems.

Suppose that the first order language $\mathfrak{L}$ consists of the predicate letters $\{P_1, \dots, P_l\}$ and let var be a recursive one to one mapping of the natural numbers onto the set of all variables.

6.1. Lemma. *Let K, H, D be finite sets and $K = \{z_1, \dots, z_r\}$. Let $Z_1 = \mathrm{var}(z_1), \dots, Z_r = \mathrm{var}(z_r)$. There exists a uniform effective way to define a Δ_1^0 quantifier-free formula $\Pi_{K,H,D}(Z_1, \dots, Z_r)$ such that for all $t_1, \dots, t_r \in A$*

$$\mathfrak{A} \models \Pi_{K,H,D}(Z_1/t_1, \dots, Z_r/t_r) \iff \exists\delta(\mathrm{dom}(\alpha_\delta) = K \ \&\ H_\delta = H \ \&\ D_\delta = D \ \&\ \alpha_\delta(z_i) \simeq t_i).$$

Proof. If $K \cap H \neq \emptyset$ or $K \cup H$ is not an initial segment $[0, q]$ or D is not a diagram of a finite structure of the language $\mathfrak{L}$ with domain $K \cup H$, then set $\Pi_{K,H,D} = \mathbb{F}$. Otherwise, let $\{u_1, \dots, u_v\}$ be all elements of D such that if $u_j = \langle i, x_1, \dots, x_{a_i}, \varepsilon\rangle, i \in [1, l]$, then $\{x_1, \dots, x_{a_i}\} \subseteq K$. For every such u_j let $L_j = \neg^\varepsilon P_i(\mathrm{var}(x_1), \dots, \mathrm{var}(x_{a_i}))$ and define $\Pi_{K,H,D} = L_1 \& \dots \& L_v$.

6.2. Corollary. *There exists a uniform effective way, given finite sets K, H, D and E, to define a Δ_1^0 quantifier-free formula $\Pi_{K,H,D,E}$ with free variables among $\{\mathrm{var}(z) : z \in K\}$ such that if $K = \{z_1, \dots, z_r\}$ and $\mathrm{var}(z_i) = Z_i$, then for all $t_1, \dots, t_r \in A$*

$$\mathfrak{A} \models \Pi_{K,H,D,E}(Z_1/t_1, \dots, Z_r/t_r) \iff \exists\delta(\mathrm{dom}(\alpha_\delta) = K \ \&\ H_\delta = H \ \&\ D_\delta = D \ \&(\forall i \in [1, r])(\alpha_\delta(z_i) \simeq t_i) \ \&\ \delta \Vdash^* E)).$$

Proof. Set $\Pi_{K,H,D,E} = \mathbb{F}$ if $E \not\subseteq D$ and let $\Pi_{K,H,D,E} = \Pi_{K,H,D}$ otherwise.

6.3. Lemma. *Let $k \geq 0$, $\delta = \langle \alpha_\delta, H_\delta, D_\delta\rangle$ be a finite part, $\mathrm{dom}(\alpha_\delta) = \{z_1, \dots, z_r\}$ and $\alpha_\delta(z_1) \simeq t_1, \dots, \alpha_\delta(z_r) \simeq t_r$. Suppose that $\mathrm{var}(z_i) = Z_i$. Then there exists a uniform in $\mathrm{dom}(\alpha_\delta), H_\delta, D_\delta$ effective way, given natural numbers e, x and finite set E of natural numbers, to define:*

(1) *A Δ_{k+1}^0 quantifier-free formula $\Gamma^k_{\mathrm{dom}(\alpha_\delta), H_\delta, D_\delta, E}(Z_1, \dots, Z_r)$ such that*

$$\mathfrak{A} \models \Gamma^k_{\mathrm{dom}(\alpha_\delta), H_\delta, D_\delta, E}(Z_1/t_1, \dots, Z_r/t_r) \iff \delta \Vdash_k^* E;$$

(2) *A* Σ^0_{k+1} *quantifier-free formula* $\Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff \delta \Vdash^* F^{k+1}_e(x);$$

(3) *A* Σ^0_{k+1} *quantifier-free formula* $\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff (\exists\tau \succeq \delta)(\tau \Vdash^* F^{k+1}_e(x));$$

(4) *A* Π^0_{k+1} *quantifier-free formula* $\Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1,\ldots,Z_r)$ *such that*

$$\mathfrak{A} \models \Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}(Z_1/t_1,\ldots,Z_r/t_r) \iff \delta \Vdash^* \neg F^{k+1}_e(x).$$

Proof. Induction on k. Using Corollary 6.2, we shall suppose that (1) is true for k and proceed to prove (2), (3) and (4). After that we shall show the validity of (1) for $k+1$. Let $R_{e,x} = \{v : \langle v, x\rangle \in W_e\}$. Following the definition of the stared forcing, we get

$$\Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{v\in R_{e,x}} \Gamma^{k}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E_v},$$

$$\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \bigvee_{H\supseteq H_\delta,D\supseteq D_\delta} \Pi_{\mathrm{dom}(\alpha_\delta),H,D} \,\&\, \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H,D,e,x},$$

$$\Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x} = \neg\Psi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}.$$

So it remains to construct $\Gamma = \Gamma^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,E}$. Set $\Gamma = \mathbb{F}$ if not all elements u of E are of the form $\langle i, k+1, e, x\rangle$, $i \in \{0,1\}$. Otherwise, for every element $u = \langle i, k+1, e, x\rangle$ of E let $L^u = \Theta^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 0$, and let $L^u = \Phi^{k+1}_{\mathrm{dom}(\alpha_\delta),H_\delta,D_\delta,e,x}$ if $i = 1$. Put $\Gamma = \bigwedge_{u\in E} L^u$.

As a corollary we obtain the proof of Theorem 2.7. Indeed, suppose that $M \subseteq A$, $1 \leq k < n$, and M be Σ^0_k-admissible in all Σ^0_n enumerations. Using Theorem 5.2, we obtain that there exist δ and e such that if $x = |\delta| + 1$, then for all $s \in A$

$$s \in M \iff (\exists\tau \succeq \delta * s)(\tau \Vdash^* F^k_e(x)).$$

Let $\mathrm{dom}(\alpha_\delta) = \{z_1,\ldots,z_r\}$, $\mathrm{var}(z_i) = Z_i$, $\mathrm{var}(x) = X$. Denote by K the finite set $\mathrm{dom}(\alpha_\delta) \cup \{x\}$. Put $\Psi = \Psi^k_{K,H_\delta,D_\delta,e,x}$. Clearly, the variables of Ψ are among $\{Z_1,\ldots,Z_r,X\}$. Let $\alpha_\delta(z_i) \simeq t_i$. Notice that $\alpha_{\delta*s}(x) \simeq s$ for all $s \in A$. Then

$$s \in M \iff \mathfrak{A} \models \Psi(Z_1/t_1,\ldots,Z_r/t_r,X/s).$$

Using Lemma 6.3 and the definition of the regular finite parts, one can easily prove the following

6.4. Lemma. *For every $n \geq 0$ there exists a uniform effective way to construct, given finite sets $K = \{z_1,\ldots,z_r\}$, H and D, a finite disjunction $\Omega^n_{K,H,D}$ of*

Δ^0_{n+1} *quantifier-free formulas with variables among* var$(z_1), \ldots,$ var(z_r) *such that if* var$(z_i) = Z_i$ *and* $t_1, \ldots, t_r$ *are elements of* A, *then*

$$\mathfrak{A} \models \Omega^n_{K,H,D}(Z_1/t_1, \ldots, Z_r/t_r) \iff \exists\delta(\delta \text{ is n-regular } \& \operatorname{dom}(\alpha_\delta) = K \;\&\; H_\delta = H \;\&\; D_\delta = D \;\&\; (\forall i \in [1,r])(\alpha_\delta(z_i) \simeq t_i)).$$

Now we are ready to prove Theorem 2.8. Let $n \geq 1$, $M \subseteq A$. Suppose that M is Σ^0_n admissible in all Σ^0_n enumerations. By Theorem 5.1 there exist δ and e such that if $x = |\delta| + 1$, then for all $s \in A$

$$s \in M \iff (\exists\tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \; \tau \Vdash^* F^n_e(x)).$$

Let $\operatorname{dom}(\alpha_\delta) = \{z_1, \ldots, z_r\}$ and $\alpha_\delta(z_i) = t_i$. Let var$(z_i) = Z_i$ and var$(x) = X$. Given any formula Φ and finite set $K = \{y_1 < \ldots < y_q\}$, by $\exists(y \in K)\Phi$ we shall denote the formula $\exists$var$(y_1)\ldots\exists$var$(y_q)\Phi$. Let $K_\delta = \operatorname{dom}(\alpha_\delta) \cup \{x\}$. Define

$$\Phi(Z_1, \ldots, Z_r, X) = \bigvee_{K \supseteq K_\delta, H \supseteq H_\delta, D \supseteq D_\delta} \exists(y \in K \setminus K_\delta)(\Omega^{n-1}_{K,H,D} \& \; \Theta^n_{K,H,D,e,x}).$$

Clearly, Φ is a Σ^0_n existential formula and

$$\mathfrak{A} \models \Phi(Z_1/t_1, \ldots, Z_r/t_r, X/s) \iff (\exists\tau \supseteq \delta * s)(\tau \text{ is } (n-1)\text{-regular } \& \; \tau \Vdash^* F^n_e(x)).$$

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Π_1^0-POSITIVE INDUCTIVE DEFINABILITY ON ABSTRACT STRUCTURES*

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Inductive definability by means of Π_1^0-positive formulas is studied in the paper. An explicit characterization of the Π_1^0-positive inductive sets on an arbitrary abstract structure with equality is presented. A relationship between these sets and the sets of all points of $\forall$-definedness of non-deterministic programs is established.

Keywords: inductive definability, non-deterministic computability, points of $\forall$-definedness, prime computability

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1. INTRODUCTION

It is well-known that on the first order structure of arithmetic every inductive set can be defined inductively also by means of very simple formula — Π_1^0, and positive with respect to its set variable ([2, 6]). When study induction on abstract structures, it is reasonable to consider the so-called acceptable structures. Even in this case, however, the above mentioned result of Kleene and Spector is no longer valid. In other words, there are acceptable structures for which the class of the Π_1^0-positive inductive sets is strictly included in the class of all inductive sets (cf. for example [1]). So, a question arises to find a characterization for this type of inductive definability.

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Let us mention that a similar problem is considered in [1], where the sets, which have Σ_2^0 and simpler (excepting Π_1^0 positive) inductive definitions, are characterized by means of prime and search computability in Kleene's quantifier $\mathbb{E}$. Here we describe Π_1^0 positive inductive definability in terms of some particular characteristic of the non-deterministic programs — the so-called sets of points of $\forall$-definedness ([3, 5]). It turns out that a set is Π_1^0-positive inductive iff it is the set of all points of $\forall$-definedness of some non-deterministic program.

From here it is easily obtained that every Π_1^0-positive inductive set on acceptable structure $\mathfrak{A}$ can be represented as $\{\bar{s} \mid \forall\alpha\exists n R(\alpha(\bar{n}), n, \bar{s})\}$ and vice versa. Here the predicate R is prime computable over $\mathfrak{A}$ and the second order variable α ranges over the set of all infinite sequences with elements in $|\mathfrak{A}|$.

In view of further applications, we shall consider here some special acceptable structures, namely least acceptable extensions [4]. It will be transparent, however, how to modify the proofs for an arbitrary acceptable structure with a sufficiently simple coding scheme.

2. PRELIMINARIES

Given an arbitrary total structure $\mathfrak{A}_0 = (B; f_1, \dots, f_a; R_1, \dots, R_b)$ (the case f_i is 0-ary is admitted as well), we define its least acceptable extension $\mathfrak{A}$ in the following way. Take an object $\mathbb{O} \notin B$ and fix some pairing operation $\Pi : C \times C \to C$ ($C \supseteq B \cup \{\mathbb{O}\}$) such that no element of $B \cup \{\mathbb{O}\}$ is an ordered pair. Let B^* be the smallest set containing $B \cup \{\mathbb{O}\}$ and closed under Π. Denote by $\langle\ \rangle$ the restriction of Π on B^*. We extend the initial functions and predicates of $\mathfrak{A}$ on B^*, setting $f_i(s_1, \dots, s_n) = \mathbb{O}$ if $(s_1, \dots, s_n) \notin B^n$, and $R_j(s_1, \dots, s_m) =$ "falsity" if $(s_1, \dots, s_m) \notin B^m$.

Now put $\mathfrak{A} = (B^*; \mathbb{O}, \langle\ \rangle, f_1, \dots, f_a; B, R_1, \dots, R_b)$. From now on we shall suppose that the equality relation is among the basic predicates of $\mathfrak{A}$. Throughout the paper we shall assume this structure fixed.

Let $\varphi(x_1, \dots, x_k, X)$ be a formula in the first order language $\mathfrak{L}_{\mathfrak{A}}$ of $\mathfrak{A}$ with k object variables $x_1, \dots, x_k$ and one k-ary relational variable X which occurs in φ only positively. Then φ determines the following mapping $\Gamma_\varphi : (B^*)^k \to (B^*)^k$:

$$\Gamma_\varphi(A) = \{(s_1, \dots, s_k) \mid \varphi(s_1, \dots, s_k, A)\}.$$

Define I_φ^ξ by transfinite induction on ξ as follows:

$$I_\varphi^\xi = \Gamma_\varphi\Big(\bigcup_{\eta<\xi} I_\varphi^\eta\Big).$$

Then the set $I_\varphi = \bigcup_\xi I_\varphi^\xi$ is the least fixed point of Γ_φ (see for example [4, Ch. 1A]).

For every $\bar{s} \in I_\varphi$ set

$$|\bar{s}| = \min\{\xi \mid \bar{s} \in I_\varphi^\xi\}.$$

It will be convenient to consider that $|\bar{s}| = |B^*|^+$ for $\bar{s} \notin I_\varphi$, where $|B^*|^+$ is the least cardinal, greater than the cardinal number of B^*.

A set $A \subseteq (B^*)^k$ is called *inductively definable* (*by* φ *on* $\mathfrak{A}$) if $A = \{(s_1, \dots, s_k) \mid (s_1, \dots, s_k, t_1, \dots, t_n) \in I_\varphi)\}$ for some fixed $t_1, \dots, t_n$, $n \geq 0$, built up from the basic functions and constants of $\mathfrak{A}$.

Remark. The last requirement imposed on $t_1, \dots, t_n$ is a slight deviation from the usual definition in [4, Ch. 1D], where these parameters are supposed arbitrary. In our case perhaps more appropriate would be to say that A is absolutely inductively definable.

We shall say that the set A is Σ_k^0 (Π_k^0) *positive inductive* (*on* $\mathfrak{A}$) iff it is inductively definable by some Σ_k^0 (Π_k^0) X-positive formula $\varphi(\bar{x}, X)$.

In this paper we shall consider non-deterministic programs, in which the non-determinism is understood as possibility of choosing arbitrary elements of $|\mathfrak{A}|$. These programs are built up from the following three types of (eventually labeled) operators: assignment operator $x_i := \tau(x_{j_1}, \dots, x_{j_m})$, conditional operator **if** $R(x_{j_1}, \dots, x_{j_n})$ **then go to** q (τ and R being a term and a quantifier-free formula in $\mathfrak{L}_\mathfrak{A}$, respectively), and choice operator $x_i :=$ **arbitrary**(B^*).

Semantics of the assignment and conditional operators is the usual one. The execution of the choice operator assigns to the variable x_i an arbitrary element of B^*. The choice of this element is arbitrary: it does not depend on the input, on the current configuration, etc.

Now let P be such non-deterministic program. Along with the usual input-output relation R_P in this case we can speak also about the so-called *set of points of* $\forall$*-definedness* of P, to be denoted by D_P. An input $\bar{s}$ belongs to D_P iff all possible executions of P, starting from this input, are finite.

The main part of the exposition is based on a certain syntactical description of these sets of points of $\forall$-definedness. It is easily obtained from a more general uniform characterization of all possible pairs (R_P, D_P) which we are going to formulate below.

Let us call χ elementary if it is atomic or a negation of an atomic formula. A clause is an expression of the form $\Pi \Rightarrow \tau$, where τ is a term and Π is a finite conjunction of elementary formulas in the language $\mathfrak{L}_\mathfrak{A}$. A sequence of clauses $\left\{\Pi^{(n)} \Rightarrow \tau^{(n)}\right\}_n$ is regarded primitive recursive if the function, which assigns to each n the code of $\Pi^{(n)} \Rightarrow \tau^{(n)}$, is primitive recursive.

Throughout the paper $\alpha = \{\alpha(n)\}_{n=1}^\infty$ will denote an infinite sequence with elements from B^*. As usual, $\bar{\alpha}(n)$ will stand for $\langle\!\langle \alpha(1), \dots, \alpha(n) \rangle\!\rangle$, where $\langle\!\langle\ \rangle\!\rangle$ is some effective coding of all finite sequences from B^*.

Proposition 2.1 (Normal form theorem). *Let P be a non-deterministic program with k input and one output variables. Then there exists a primitive recursive sequence of clauses $\{\Pi^{(n)} \Rightarrow \tau^{(n)}\}_n$, each with variables among $x_0, x_1, \dots, x_k$, such that*

$$(s_1, \dots, s_k) \in D_P \Leftrightarrow \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s_1, \dots, s_k),$$

$$(s_1, \dots, s_k, t) \in R_P \Leftrightarrow \exists\alpha\exists n\big(\Pi^{(n)}(\bar{\alpha}(n), s_1, \dots, s_k) \,\&\, \tau^{(n)}(\bar{\alpha}(n), s_1, \dots, s_k) = t \\ \&\, \forall m_{m<n} \neg\Pi^{(m)}(\bar{\alpha}(m), s_1, \dots, s_k)\big)$$

for every $s_1, \ldots, s_k, t$ in B^*.

Conversely, for every such sequence $\{\Pi^{(n)} \Rightarrow \tau^{(n)}\}_n$ *there exists a program* P *such that* D_P *and* R_P *satisfy the above equivalence.*

The proof of this proposition is rather technical to be presented here. We are going to make some comments instead. Every $\alpha : N \to B^*$ can be thought of as being a sequence of successive values of the choice operator (which may be assumed unique). So, every particular (finite or infinite) execution of P is uniquely determined by the input $\bar{s}$ and some choice sequence α. In addition, this execution is carried out in elementary steps in some canonical way. More precisely, given an input $\bar{s}$ and a sequence α, the values $\Pi^{(1)}(\bar{\alpha}(1), \bar{s})$, $\Pi^{(2)}(\bar{\alpha}(2), \bar{s})$, ... are computed in turn until the first n with $\Pi^{(n)}(\bar{\alpha}(n), \bar{s}) =$ true is reached. Then an output $\tau^{(n)}(\bar{\alpha}(n), \bar{s})$ is returned.

From this point of view it is clear that an input $\bar{s}$ belongs to D_P iff for every α there exists n such that $\Pi^{(n)}(\bar{\alpha}(n), s)$ holds. Further we shall be interested in sets of the type D_P rather than of D_P and R_P as a pair. For this purpose it will be enough (and even more appropriate) to consider non-deterministic programs without output variable. In this case a particular execution is regarded finite if the output operator **stop** is reached during the computation.

So, as a consequence of Proposition 2.1, for any non-deterministic program P (with or without output variable) we have

Proposition 2.2. *Let* P *has* k *input variables. Then there exists a primitive recursive sequence* $\{\Pi^{(n)}\}_n$ *with variables among* $x_0, x_1, \ldots, x_k$ *such that*

$$(s_1, \ldots, s_k) \in D_P \;\Leftrightarrow\; \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s_1, \ldots, s_k),$$

whenever $s_1, \ldots, s_k \in |\mathfrak{A}|$.

3. INDUCTIVE DEFINABILITY OF THE SETS OF ALL POINTS OF ∀-DEFINEDNESS

We begin with some preliminary definitions. Set

$$\text{Nat} = \{\mathbb{O},\ \langle\mathbb{O},\mathbb{O}\rangle,\ \langle\langle\mathbb{O},\mathbb{O}\rangle,\mathbb{O}\rangle,\ \langle\langle\langle\mathbb{O},\mathbb{O}\rangle,\mathbb{O}\rangle,\mathbb{O}\rangle,\ \ldots\}.$$

We shall identify the natural numbers $0, 1, 2, \ldots$ with elements of Nat (as listed above). For every $n \in \text{Nat}$, $n+1$ will stand for $\langle n, 0\rangle$. Let L and R be the left and right decoding functions for the mapping $\langle\,\rangle$. We shall assume that $L(0) = R(0) = 0$ and $L(s) = R(s) = 1$ for $s \in B$.

A coding $\langle\!\langle\,\rangle\!\rangle$ of all finite sequences from B^* is defined inductively by the equalities

$$\langle\!\langle\,\rangle\!\rangle = 0,\ \langle\!\langle s_1\rangle\!\rangle = \langle 0, s_1\rangle,\ \langle\!\langle s_1, \ldots, s_{n+1}\rangle\!\rangle = \langle\langle\!\langle s_1, \ldots, s_n\rangle\!\rangle, s_{n+1}\rangle.$$

Set also $\text{Seq} = \{s \mid s = \langle\!\langle s_1, \ldots, s_n\rangle\!\rangle$ for some $s_1, \ldots, s_n,\ n \geq 0\}$. The function lh (length) is defined in the usual way:

$$\text{lh}(s) = \begin{cases} n & \text{if } s = \langle\!\langle s_1, \ldots, s_n\rangle\!\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, denote by $(s)_i$ the decoding function corresponding to $\langle\!\langle\ \rangle\!\rangle$:

$$(s)_i = \begin{cases} s_i & \text{if } s = \langle\!\langle s_1, \ldots, s_n \rangle\!\rangle \text{ and } 1 \le i \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $t = \langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$. As customary, $t * q$ will stand for $\langle t, q \rangle = \langle\!\langle t_1, \ldots, t_n, q \rangle\!\rangle$. For any sequence α we shall write $\alpha \succ t$ to denote that $\alpha(i) = t_i$ for every $i = 1, \ldots, \mathrm{lh}(t)$.

Now let P be an arbitrary non-deterministic program over $\mathfrak{A}$. We shall assume for simplicity that P has one input variable. According to Proposition 2.2, there is a primitive recursive sequence $\left\{\Pi^{(n)}\right\}_{n=0}^{\infty}$ with variables x_0 and x_1 such that for every $s \in B^*$

$$s \in D_P \iff \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s).$$

Let us fix some effective coding $\ulcorner \ldots \urcorner$ of all finite conjunctions of elementary formulas with variables among x_0 and x_1. Let Φ be the universal relation for this class of formulas, defined by the equivalence

$$\Phi(n, t, s) \iff n \text{ is a code of some elementary formula } \chi \text{ and } \chi(t, s) \text{ holds.}$$

Denote by g the primitive recursive function $\lambda n.\ulcorner \Pi^{(n)} \urcorner$. Then we have

$$s \in D_P \iff \forall\alpha\exists n \Phi(g(n), \bar{\alpha}(n), s). \tag{3.1}$$

Now set

$$\varphi^*(n, t, s, X) \iff \mathrm{Nat}(n) \,\&\, \mathrm{Seq}(t) \,\&\, (\Phi(g(n), t, s) \vee \forall q((n+1, t*q, s) \in X))).$$

Let us first check that D_P is a section of $I_{\varphi}*$.

Lemma 3.1. $D_P = \{s \mid (0, 0, s) \in I_{\varphi}*\}$.

Proof. For the inclusion $D_P \supseteq \{s \mid (0, 0, s) \in I_{\varphi^*}\}$ we need the following more general assertion:

$$\mathrm{Seq}(t) \,\&\, \mathrm{lh}(t) = n \,\&\, (n, t, s) \in I_{\varphi^*} \Rightarrow \forall\alpha_{\succ t}\exists m \Phi(g(m), \bar{\alpha}(m), s). \tag{3.2}$$

We are going to prove (3.2) by transfinite induction on $|n, t, s|$. It can be easily seen that

$$|n, t, s| = \begin{cases} 0 & \text{if } \Phi(g(n), t, s), \\ \sup\{|n+1, t*q, s| + 1 \mid q \in B^*\} & \text{otherwise.} \end{cases}$$

If $|n, t, s| = 0$, take $m = n$. We have by assumption

$$\bar{\alpha}(m) = \bar{\alpha}(n) = \langle\!\langle \alpha(1), \ldots, \alpha(n) \rangle\!\rangle = t$$

and therefore $\Phi(g(m), \bar{\alpha}(m), s)$.

Now choose some $(n, t, s) \in I_{\varphi^*}$ with $|n, t, s| > 0$ (and, of course, $\mathrm{Seq}(t)$ and $\mathrm{lh}(t) = n$). Take some $\alpha \succ t$ and set $\alpha(n+1) = q$. We have $|n+1, t*q, s| < |n, t, s|$, as well as $\mathrm{Seq}(t*q)$ and $\mathrm{lh}(t*q) = n+1$. So by induction hypothesis $\Phi(g(m), \bar{\alpha}(m), s)$ for some m, which completes the verification of (3.2).

Now put $t = \langle\!\langle\ \rangle\!\rangle$ ($= 0$) and $n = 0$ in (3.2) and use (3.1) to conclude that $D_P \supseteq \{s \mid (0, 0, s) \in I_{\varphi^*}\}$.

To see that the converse inclusion also holds, take some s such that $(0,0,s) \notin I_{\varphi^*}$. By the definition of φ^* we have $\neg\Phi(g(0),0,s)$ and $(1,\langle\!\langle q_1\rangle\!\rangle,s) \notin I_{\varphi^*}$ for at least one q_1. Assume that for some $n \geq 1$ we have found $q_1,\ldots,q_n$ with $\neg\Phi(g(i),\langle\!\langle q_1,\ldots,q_i\rangle\!\rangle,s)$ for $i=1,\ldots,n-1$ and $(n,\langle\!\langle q_1,\ldots,q_n\rangle\!\rangle,s) \notin I_{\varphi^*}$. From the latter it follows that $\neg\Phi(g(n),\langle\!\langle q_1,\ldots,q_n\rangle\!\rangle,s)$ and $\langle n+1,\langle\!\langle q_1,\ldots,q_{n+1}\rangle\!\rangle,s\rangle \notin I_{\varphi^*}$ for some q_{n+1}.

In this way we construct a sequence $\alpha = q_1,q_2,\ldots$ for which the right-hand side of (3.1) fails. Therefore $s \notin D_P$.

For any list $g_1,\ldots,g_k;Q_1,\ldots,Q_l$ of functions and predicates in B^* denote by $(\mathfrak{A};g_1,\ldots,g_k;Q_1,\ldots,Q_l)$ the extended structure

$$(B^*;\mathbb{O},\langle\ \rangle,f_1,\ldots,f_a,g_1,\ldots,g_k;B,R_1,\ldots,R_b,Q_1,\ldots,Q_l).$$

We have established so far that the set D_P is inductive in the structure $(\mathfrak{A};g;\mathrm{Nat},\mathrm{Seq},\Phi)$. To eliminate the additional function and predicates, we shall need the following refinement of the Transitivity Theorem [4, Th. 1C.3], which follows immediately from the corresponding proof in [4]:

> If $\varphi(\bar{x},X)$ is Σ^0_1 (Π^0_1) formula in $(\mathfrak{A};Q)$, positive with respect to X and Q, and Q is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$, then I_φ is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$. (3.3)

Further we shall apply this fact in the next modified form.

Lemma 3.2 (Transitivity Lemma). *Let $\varphi(\bar{x},X)$ be Σ^0_1 (Π^0_1) formula in $(\mathfrak{A};g_1,\ldots,g_k;Q_1,\ldots,Q_l)$ in which $Q_1,\ldots,Q_l$ and X occur only positively. Suppose also that the graphs $G_{g_1},\ldots,G_{g_k}$ are Σ^0_1 ($\bar{G}_{g_1},\ldots,\bar{G}_{g_k}$ are Π^0_1) positive inductive on $\mathfrak{A}$ and $Q_1,\ldots,Q_l$ are Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$. Then I_φ is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$.*

Proof. Assume first that g_1 has a unique occurrence in φ and let $\tau_1,\ldots,\tau_m$ be the arguments of g_1 in this occurrence. Let ψ be the formula which is obtained from φ when we replace $g_1(\tau_1,\ldots,\tau_m)$ by y. If φ is Σ^0_1, then ψ is Σ^0_1 in $(\mathfrak{A};g_2,\ldots,g_k;Q_1,\ldots,Q_l)$. Set

$$\chi(\bar{x},X) \;\Leftrightarrow\; \exists y\,(G_{g_1}(\tau_1,\ldots,\tau_m,y)\,\&\,\psi)\,.$$

Obviously, $\varphi \Leftrightarrow \chi$. By assumption G_{g_1} is Σ^0_1 positive inductive on $\mathfrak{A}$ and according to (3.3) I_χ (and hence I_φ) is Σ^0_1 positive inductive on $(\mathfrak{A};g_2,\ldots,g_k;Q_1,\ldots,Q_l)$.

When φ is Π^0_1, consider the Π^0_1 formula

$$\chi(\bar{x},X) \;\Leftrightarrow\; \forall y\,(\neg G_{g_1}(\tau_1,\ldots,\tau_m,y)\vee\psi)\,.$$

If g_1 has $i>1$ occurrences in φ, proceed by induction on i, applying the above argument to the innermost g_1. Iterating this procedure, exclude successively $g_2,\ldots,g_k$. For the elimination of $Q_1,\ldots,Q_l$ apply directly (3.3).

Lemma 3.3. *Let $\varphi(\bar{x},X)$ is Σ^0_k (Π^0_k) positive formula and I_φ is the unique fixed point of Γ_φ. Then $\bar{I}_\varphi$ (the complement of I_φ) is Π^0_k (Σ^0_k) positive inductive.*

Proof. We have $\bar{x} \in I_\varphi \;\Leftrightarrow\; \varphi(\bar{x}, I_\varphi)$ and hence

$$\bar{x} \in \bar{I}_\varphi \;\Leftrightarrow\; \neg\varphi(\bar{x}, I_\varphi) \;\Leftrightarrow\; \varphi'(\bar{x}, \bar{I}_\varphi),$$

where $\varphi'(\bar{x}, X)$ is obtained from $\neg\varphi$ after the replacement of every occurrence of X by $\neg X$. Obviously, φ' is equivalent to some $\varphi''(\bar{x}, X)$, which is X-positive and Π^0_k (Σ^0_k) if φ is Σ^0_k (Π^0_k) respectively. From the above equivalence it follows that

$$\bar{x} \in \bar{I}_\varphi \;\Leftrightarrow\; \varphi''(\bar{x}, \bar{I}_\varphi),$$

i.e. $\bar{I}_\varphi$ is a fixed point of $\Gamma_{\varphi''}$. Assuming that A is another fixed point of $\Gamma_{\varphi''}$ we obtain successively

$$\bar{x} \in \bar{A} \;\Leftrightarrow\; \neg(\bar{x} \in A) \;\Leftrightarrow\; \neg\varphi''(\bar{x}, A) \;\Leftrightarrow\; \varphi(\bar{x}, \bar{A}).$$

So $\bar{A}$ is a fixed point of Γ_φ and hence $\bar{A} = I_\varphi$, $A = \bar{I}_\varphi$, i.e. $\bar{I}_\varphi$ is the unique fixed point of $\Gamma_{\varphi''}$.

Say that a set is Δ^0_1 *positive inductive on* $\mathfrak{A}$ if it is both Σ^0_1 and Π^0_1 positive inductive on $\mathfrak{A}$.

For any $s \in B^*$ define $||s||$ (norm of s) as follows: $||s|| = 0$ for $s \in B \cup \{\mathbb{O}\}$, $||\langle s_1, s_2\rangle|| = \max(||s_1||, ||s_2||) + 1$. We shall use systematically an induction on this norm when proving the next lemma.

Lemma 3.4. (i) *Let $f \in \{L, R, \mathrm{lh}, \lambda x, i.(x)_i\}$. Then G_f (the graph of f) is Σ^0_1 positive inductive on $\mathfrak{A}$ and $\bar{G}_f$ is Π^0_1 positive inductive on $\mathfrak{A}$.*

(ii) Nat, $\overline{\mathrm{Nat}}$, Seq *and* $\overline{\mathrm{Seq}}$ *are Δ^0_1 positive inductive on* $\mathfrak{A}$.

Proof. By definition we have

$$L(s) = t \;\Leftrightarrow\; s = t = 0 \vee s \in B \,\&\, t = 1 \vee \exists q(s = \langle t, q\rangle).$$

So the set G_L is explicitly definable by Σ^0_1 formula and, in particular, it is Σ^0_1 positive inductive on $\mathfrak{A}$. Similarly, G_L is Π^0_1 explicitly (and hence inductively) definable on $\mathfrak{A}$. The case $f = R$ is analogous.

Set

$$\varphi(x, X) \;\Leftrightarrow\; x = 0 \vee R(x) = 0 \,\&\, L(x) \in X.$$

Evidently, Nat is a fixed point of Γ_φ. Towards establishing that it is the unique fixed point of Γ_φ, assume that A and A' are some fixed points of Γ_φ. Then

$$x \in A \;\Leftrightarrow\; x = 0 \vee R(x) = 0 \,\&\, L(x) \in A \qquad \text{and}$$
$$x \in A' \;\Leftrightarrow\; x = 0 \vee R(x) = 0 \,\&\, L(x) \in A'. \tag{3.4}$$

Let $s \in$A. We shall use induction on $||s||$ to see that $s \in A'$. Suppose, first, that $||s|| = 0$. The case $s = 0$ is obvious; the other case $s \in B$ is impossible, since then we would have $R(s) = 0$. If $s = \langle s_1, s_2\rangle$, then by (3.4) $R(s) = 0$ and $L(s) = s_1 \in A$. By induction hypothesis $s_1 \in A'$ and applying again (3.4) we conclude that $s \in A'$. So $A \subseteq A'$ and symmetrically, $A' \subseteq A$. Thus Nat is the only fixed point of Γ_φ. Consequently, Nat and (by Lemma 3.3) $\overline{\mathrm{Nat}}$ are Δ^0_0 inductive on $(\mathfrak{A}; L, R)$. Now, having in mind the facts about L and R just established and applying Lemma 3.2, we obtain that Nat and $\overline{\mathrm{Nat}}$ are Δ^0_1 positive inductive on $\mathfrak{A}$.

For the predicate Seq apply the same argument to the formula

$$\psi(x, X) \Leftrightarrow x \notin B \,\&\, (x = 0 \vee L(x) \in X).$$

By virtue of its definition lh satisfies the equivalence

$$\begin{aligned} \mathrm{lh}(s) = n \Leftrightarrow & (\neg \mathrm{Seq}(s) \vee s = 0) \,\&\, n = 0 \\ & \vee \mathrm{Seq}(s) \,\&\, s \neq 0 \,\&\, \mathrm{Nat}(n) \,\&\, n \neq 0 \,\&\, \mathrm{lh}(L(s)) = L(n). \end{aligned}$$

An easy induction on $||s||$ convinces us that lh is the only function with this property. In other words, G_{lh} is the unique fixed point of Γ_χ, where

$$\begin{aligned} \chi(x, y, X) \Leftrightarrow & (\neg \mathrm{Seq}(x) \vee x = 0) \,\&\, y = 0 \\ & \vee \mathrm{Seq}(x) \,\&\, x \neq 0 \,\&\, \mathrm{Nat}(y) \,\&\, y \neq 0 \,\&\, (L(x), L(y)) \in X. \end{aligned}$$

So, by Lemma 3.3 G_{lh} and $\bar{G}_{\mathrm{lh}}$ are Δ_0^0 positive inductive on $(\mathfrak{A}; L, R, \mathrm{Nat}, \mathrm{Seq}, \overline{\mathrm{Seq}})$. To see that G_{lh} is Σ_1^0 positive inductive on $\mathfrak{A}$, apply again the Transitivity Lemma and the previous results; similarly for $\bar{G}_{\mathrm{lh}}$.

Finally, for the function $(x)_i$ it is also immediate that $G_{(x)_i}$ is the unique fixed point of Γ_θ, where

$$\begin{aligned} \theta(x, i, y, X) \Leftrightarrow & (\neg \mathrm{Nat}(i) \vee i = 0 \vee i > \mathrm{lh}(x)) \,\&\, y = 0 \\ & \vee \mathrm{Seq}(x) \,\&\, x \neq 0 \,\&\, (\mathrm{lh}(x) = i \,\&\, y = R(x) \vee (L(x), i, y) \in X). \end{aligned}$$

Here the predicate "$>$" (greater than) over Nat is defined inductively as

$$n > k \Leftrightarrow \mathrm{Nat}(n) \,\&\, \mathrm{Nat}(k) \,\&\, n \neq 0 \,\&\, (k = 0 \vee L(n) > L(k))$$

and therefore is Δ_1^0 positive inductive on $\mathfrak{A}$. To complete the proof, repeat the arguments used above.

Let f is a k-ary function in B^*. Say that f is primitive recursive if the restriction of f over Nat^k is primitive recursive (considered as a function over the natural numbers) and $f(\bar{s}) = 0$ for $\bar{s} \notin \mathrm{Nat}^k$.

Lemma 3.5. *Let f be primitive recursive. Then G_f is Σ_1^0 positive inductive on $\mathfrak{A}$ and $\bar{G}_f$ is Π_1^0 positive inductive on $\mathfrak{A}$.*

Proof. By induction on the definition of f. If f is initial primitive recursive, then it has a Δ_0^0 explicit definition on $\mathfrak{A}$. If f is a superposition, say $f = f_0(f_1, \dots, f_n)$, then we have the representation

$$\begin{aligned} f(s_1, \dots, s_k) = t \Leftrightarrow & \mathrm{Nat}(s_1) \,\&\, \dots \,\&\, \mathrm{Nat}(s_k) \,\&\, \exists q_1 \dots \exists q_k (f_1(s_1, \dots, s_k) = q_1 \,\&\, \dots \\ & \&\, f_n(s_1, .., s_k) = q_n \,\&\, f_0(q_1, .., q_n) = t) \\ & \vee \neg(\mathrm{Nat}(s_1) \,\&\, \dots \,\&\, \mathrm{Nat}(s_k)) \,\&\, t = 0. \end{aligned}$$

Now the result follows easily from the induction hypothesis, Lemma 3.4 and the Transitivity Lemma.

Finally, assume that f is obtained by primitive recursion from some g and h. Then

$$\begin{aligned} f(s_1, .., s_k, q) = t \Leftrightarrow & \mathrm{Nat}(s_1) \,\&\, \dots \,\&\, \mathrm{Nat}(s_k) \,\&\, \mathrm{Nat}(q) \,\&\, (q = 0 \,\&\, t = g(s_1, .., s_k) \\ & \vee q \neq 0 \,\&\, \exists r (f(s_1, \dots, s_k, L(q)) = r \,\&\, h(s_1, \dots, s_k, L(q), r) = t) \\ & \vee \neg(\mathrm{Nat}(s_1) \,\&\, \dots \,\&\, \mathrm{Nat}(s_k) \,\&\, \mathrm{Nat}(q)) \,\&\, t = 0. \end{aligned}$$

A trivial induction on $q \in \text{Nat}$ convinces us that f is the unique function, satisfying this equivalence. To see that f has the desired properties, proceed as in the proof of the previous lemma.

It remains to check that the universal relation Φ from the definition of φ^* is inductive on $\mathfrak{A}$. Of course, Φ depends on the particular coding of the syntactical objects that we have fixed. Below we specify some primitive recursive coding, which allows us to assert that Φ has Δ_1^0 inductive definition on $\mathfrak{A}$.

In order to save space, here we shall assume that the basic functions and predicates of $\mathfrak{A}_0 = (B; f_1, \ldots, f_a; R_1, \ldots, R_b)$ are unary. We shall use also the same letters for the corresponding symbols in $\mathfrak{L}_{\mathfrak{A}}$.

Let p_i be the i-th prime number (starting from $p_0 = 2$). Set also $\langle 0 \rangle_i = 0$; $\langle n \rangle_i = \max\{j \mid p_i^j \text{ divides } n\}$ for $n > 0$.

Now put

$$\ulcorner \mathbb{O} \urcorner = 0, \quad \ulcorner x_0 \urcorner = 1, \quad \ulcorner x_1 \urcorner = 2, \quad \ulcorner \langle \tau_1, \tau_2 \rangle \urcorner = 2^2.3^{\ulcorner \tau_1 \urcorner} 5^{\ulcorner \tau_2 \urcorner},$$
$$\ulcorner f_i(\tau) \urcorner = 2^{2+i}.3^{\ulcorner \tau \urcorner} \text{ for } 1 \leq i \leq a,$$
$$\ulcorner R_i(\tau) \urcorner = 3^i.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \neg R_i(\tau) \urcorner = 3^{b+i}.5^{\ulcorner \tau \urcorner} \text{ for } 1 \leq i \leq b,$$
$$\ulcorner B(\tau) \urcorner = 3^{2b+1}.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \neg B(\tau) \urcorner = 3^{2b+2}.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \psi_1 \,\&\, \psi_2 \urcorner = 3^{2b+3}.5^{\ulcorner \psi_1 \urcorner}.7^{\ulcorner \psi_2 \urcorner}.$$

Obviously, the predicates

$$K(n) \Leftrightarrow n \text{ is a code of a term with variables among } x_0, x_1,$$
$$M(n) \Leftrightarrow n \text{ is a code of some finite conjunction of elementary formulas with variables among } x_0, x_1$$

are primitive recursive.

Let U be the universal for the class of all terms with variables among x_0 and x_1, in other words,

$$U(n,t,s) = q \Leftrightarrow n \text{ is a code of a term } \tau \text{ with variables } x_1, x_2 \text{ and } \tau(t,s) = q.$$

Lemma 3.6. (i) *G_U is Σ_1^0 positive inductive on $\mathfrak{A}$ and $\bar{G}_U$ is Π_1^0 positive inductive on $\mathfrak{A}$.*

(ii) *Φ is Δ_1^0 positive inductive on $\mathfrak{A}$.*

Proof. (i) By definition we have

$$\begin{aligned} U(n,t,s) = q \Leftrightarrow\ & \neg K(n) \,\&\, q = 0 \\ & \vee M(n) \,\&\, (n = 0 \,\&\, q = \mathbb{O} \vee n = 1 \,\&\, q = t \vee n = 2 \,\&\, q = s \\ & \vee \langle n \rangle_0 = 2 \,\&\, \exists q_1 \exists q_2 (U(\langle n \rangle_1, t, s) = q_1 \,\&\, U(\langle n \rangle_2, t, s) = q_2 \,\&\, q = \langle q_1, q_2 \rangle) \\ & \vee \langle n \rangle_0 = 3 \,\&\, \exists q_1 (U(\langle n \rangle_1, t, s) = q_1 \,\&\, f_1(q_1) = q) \vee \ldots \\ & \vee \langle n \rangle_0 = a + 2 \,\&\, \exists q_1 (U(\langle n \rangle_1, t, s) = q_1 \,\&\, f_a(q_1) = q)). \end{aligned}$$

Moreover, U is the unique function which satisfies this equivalence (a simple induction on n). All arithmetic functions which appear in the above formula are primitive recursive. Now proceed again as in the proof of Lemma 3.4, using also the previous lemma.

(ii) To see that Φ is both Σ_1^0 and Π_1^0 positive inductive on $\mathfrak{A}$, use the same argumentation, noticing first that Φ is the unique relation satisfying the equivalence

$$\begin{aligned}\Phi(n,t,s) \Leftrightarrow\ & M(n) \,\&\, (\langle n\rangle_1 = 1 \,\&\, R_1(U(\langle n\rangle_2,t,s)) \vee \ldots \\ & \vee \langle n\rangle_1 = b \,\&\, R_b(U(\langle n\rangle_2,t,s)) \vee \langle n\rangle_1 = b+1 \,\&\, \neg R_1(U(\langle n\rangle_2,t,s)) \vee \ldots \\ & \vee \langle n\rangle_1 = 2b \,\&\, \neg R_b(U(\langle n\rangle_2,t,s)) \vee \langle n\rangle_1 = 2b+1 \,\&\, B(U(\langle n\rangle_2,t,s)) \\ & \vee \langle n\rangle_1 = 2b+2 \,\&\, \neg B(U(\langle n\rangle_2,t,s)) \\ & \vee \langle n\rangle_1 = 2b+3 \,\&\, \Phi(\langle n\rangle_2,t,s) \,\&\, \Phi(\langle n\rangle_3,t,s)).\end{aligned}$$

Now we are in a position to claim

Proposition 3.7. *Every set of points of $\forall$-definedness D_P is Π^0_1 positive inductive on $\mathfrak{A}$.*

Proof. By Lemma 3.1 D_P is a section of I_{φ^*}, where φ^* is Π^0_1 positive formula in some extended structure $(\mathfrak{A}; g; \mathrm{Nat}, \mathrm{Seq}, \Phi)$. Here g is primitive recursive, so by Lemma 3.5 $\bar{G}_g$ is Π^0_1 positive inductive on $\mathfrak{A}$. According to Lemma 3.4 and Lemma 3.6 the predicates Nat, Seq and Φ are Π^0_1 positive inductive on $\mathfrak{A}$. Now apply the Transitivity Lemma to conclude that I_{φ^*} is Π^0_1 positive inductive on $\mathfrak{A}$.

4. PROGRAM CHARACTERIZATION OF THE Π^0_1 POSITIVE INDUCTIVE DEFINITIONS

Let $\varphi(x_1,\ldots,x_k,X)$ be an arbitrary Π^0_1 formula in which the k-ary relational variable X occurs positively. After contracting the quantifiers and converting the matrix into the disjunctive normal form, φ becomes equivalent to a formula of the following type:

$$\begin{aligned}\forall y\Big(\psi(\bar{x},y) \vee \psi_1(\bar{x},y) \,\&\, \left(\tau^{(1)}_{1,1},\ldots,\tau^{(k)}_{1,1}\right) \in X \,\&\, \ldots \,\&\, \left(\tau^{(1)}_{1,n_1},\ldots,\tau^{(k)}_{1,n_1}\right) \in X \vee \ldots \\ \vee \psi_m(\bar{x},y) \,\&\, \left(\tau^{(1)}_{m,1},\ldots,\tau^{(k)}_{m,1}\right) \in X \,\&\, \ldots \,\&\, \left(\tau^{(1)}_{m,n_m},\ldots,\tau^{(k)}_{m,n_m}\right) \in X\Big),\end{aligned}$$

where $\psi, \psi_1, \ldots, \psi_m$ are quantifier-free formulas in which X does not occur.

Further we shall consider the case $k = 1$, $m = 2$ and $n_1 = n_2 = 1$, since it is sufficiently representative. Without essential loss of generality we may omit also ψ_1 and ψ_2 (dropping the formula ψ, however, trivializes the problem). So φ takes the form

$$\forall y(\psi(x,y) \vee \tau(x,y) \in X \vee \mu(x,y) \in X).$$

Now consider the following simple non-deterministic program P, for which we are going to establish that D_P coincides with the fixed point I_φ:

```
P:  input(x); x := ⟪x⟫;
 1: y := arbitrary(B*); z := head(x);
    if ψ(z, y) then stop;
    x := append(tail(x), ⟪τ(z, y), μ(z, y)⟫);
    if x = x then go to 1.
```

Head, *tail* and *append* act as the usual string-transforming operations, here applied to codes of sequences. It is an easy exercise to show that these functions can be computed by means of programs of the type considered here. Of course, in the above program P the operators, involving this functions, should be considered as macros rather than as assignments.

The proof of the equality $D_P = I_\varphi$ will be carried out by two lemmas. For the first one let us denote by D the set of all $t \in B^*$ such that $\mathrm{Seq}(t)$ and every computation of P, starting from the choice operator $y := \mathtt{arbitrary}(B^*)$ with current value of the variable x equal to t, is terminating.

Obviously, $s \in D_P$ iff $\langle\langle s\rangle\rangle \in D$. Further, if $\mathrm{Seq}(t)$, then

$$t \in D \Leftrightarrow \forall q(\psi(head(t), q) \vee append(tail(t), \langle\langle\tau(head(t), q), \mu(head(t), q)\rangle\rangle) \in D). \quad (4.1)$$

Lemma 4.1. *Let $s \in I_\varphi$. Then for every $t = \langle\langle t_1, \ldots, s, \ldots, t_n\rangle\rangle$ is true that $t \in D$.*

Proof. Transfinite induction on $|s|$. Let us notice (having in mind the agreement $|s| = |B^*|^+$ for $s \notin I_\varphi$) that whenever $s \in I_\varphi$

$$|s| = \begin{cases} 0 & \text{if } \forall q \psi(s, q), \\ \sup\limits_{q: \neg\psi(s,q)} (\min(|\tau(s, q)|, |\mu(s, q)|) + 1) & \text{otherwise.} \end{cases} \quad (4.2)$$

For every $t = \langle\langle t_1, \ldots, s, \ldots, t_n\rangle\rangle$ set $\mathrm{pos}(s, t) = \min\{i \mid s = t_i\}$.

Now let $s \in I_\varphi$ and suppose first that $|s| = 0$. Using induction on $\mathrm{pos}(s, t)$, we are going to prove that $t \in D$ for every $t = \langle\langle t_1, \ldots, s, \ldots, t_n\rangle\rangle$.

Case 1: $t = \langle\langle s, t_2, \ldots, t_n\rangle\rangle$. From the assumption $|s| = 0$ it follows that $\forall q \psi(s, q)$, in other words, $\forall q \psi(head(t), q)$ and therefore by (4.1) $t \in D$.

Case 2: $t = \langle\langle t_1, \ldots, t_n\rangle\rangle$ with $s = t_i$ for some $i > 1$. Pick any $q \in B^*$ and set $t' = \langle\langle t_2, \ldots, t_n, \tau(t_1, q), \mu(t_1, q)\rangle\rangle$. Obviously, $\mathrm{pos}(t', s) = i - 1$ and hence $t' \in D$ by induction supposition. Since q is arbitrary, applying again (4.1) we obtain $t \in D$.

Now let $s \in I_\varphi$, $|s| > 0$ and assume that for all s' with $|s'| < |s|$ the lemma is true. We shall use second induction on $\mathrm{pos}(s, t)$ again. Suppose first that $t = \langle\langle s, t_2, \ldots, t_n\rangle\rangle$. In order to establish that the right-hand side of (4.1) holds, take an arbitrary $q \in B^*$. If $\psi(s, q)$, there is nothing to prove; if not, by (4.2) $|s| > \min\{|\tau(s, q)|, |\mu(s, q)|\}$. Suppose for definiteness that $|s| > |\tau(s, q)|$. Then, in particular, $\tau(s, q) \in I_\varphi$ and by induction hypothesis for $s' = \tau(s, q)$, applied to $t' = \langle\langle t_2, \ldots, t_n, \tau(s, q), \mu(s, q)\rangle\rangle$, we obtain $t' \in D$. So by (4.1) $t \in D$. Finally, consider $t = \langle\langle t_1, \ldots, t_n\rangle\rangle$ with $\mathrm{pos}(s, t) = i > 1$. For $t' = \langle\langle t_2, \ldots, t_n, \tau(t_1, q), \mu(t_1, q)\rangle\rangle$, where q is any element of B^*, we have $\mathrm{pos}(s, t') = i - 1$. Therefore $t' \in D$ and hence $t \in D$.

Applying this result to $t = \langle\langle s\rangle\rangle$, we obtain $I_\varphi \subseteq D_P$. The opposite inclusion is given by the next lemma.

Lemma 4.2. $D_P \subseteq I_\varphi$.

Proof. Consider particular execution (finite or infinite) of P with some input s. Let x_n and y_n be the current values of the variables x and y immediately after the n-th running of the choice operator (if the execution has stopped after the m-th run of this operator, we assume that $x_n = x_m$ and $y_n = y_m$ for $n > m$). Set $\alpha(n) = y_n$, $n = 1, 2, \ldots$ Obviously, every x_n is uniquely determined by the input s and α. Let F be such that $F(s, \alpha, n) = x_n$.

Now choose some $s \notin I_\varphi$. In order to prove that $s \notin D_P$, it is enough to find a sequence α such that

$$\neg\psi(head(F(s, \alpha, n)), \alpha(n)) \text{ for every } n = 1, 2, \ldots \tag{4.3}$$

To this end we are going to define recursively two sequences $\{s_n\}_n$ and $\{q_n\}_n$ satisfying the condition

$$\begin{aligned} &s_1 = s; \quad s_{2k} = \tau(s_k, q_k); \quad s_{2k+1} = \mu(s_k, q_k); \\ &\neg\psi(s_k, q_k) \text{ and } s_k \notin I_\varphi \text{ for } k = 1, 2, \ldots \end{aligned} \tag{$*$}$$

We shall see later that (4.3) is true for $\alpha = \{q_n\}_n$.

Indeed, set $s_1 = s$. By assumption $s \notin I_\varphi$ and therefore $\neg\varphi(s, I_\varphi)$. Consequently, there is $q_1 \in B^*$ such that $\neg\psi(s_1, q_1)$, $s_2 = \tau(s_1, q_1) \notin I_\varphi$ and $s_3 = \mu(s_1, q_1) \notin I_\varphi$. Let us assume that for some $n > 1$ we have found $s_1, \ldots, s_{2^n-1}$ and $q_1, \ldots, q_{2^{n-1}-1}$ with the property

$$\begin{aligned} &s_1 = s; \quad s_{2k} = \tau(s_k, q_k); \quad s_{2k+1} = \mu(s_k, q_k); \\ &\neg\psi(s_k, q_k) \text{ for every } k = 1, 2, \ldots, 2^{n-1} - 1; \quad s_1 \notin I_\varphi, \ldots, s_{2^n-1} \notin I_\varphi. \end{aligned} \tag{$*$}_n$$

We shall construct elements $q_{2^{n-1}}, \ldots, q_{2^n-1}$ and $s_{2^n}, \ldots, s_{2^{n+1}-1}$ such that $(*)_{n+1}$ holds for $s_1, \ldots, s_{2^{n+1}-1}$ and $q_1, \ldots, q_{2^n-1}$. Let k be an arbitrary number between 2^{n-1} and $2^n - 1$. From the fact that $s_k \notin I_\varphi$ it follows that for some $q_k \in B^*$, $\neg\psi(s_k, q_k)$, $\tau(s_k, q_k) \notin I_\varphi$ and $\mu(s_k, q_k) \notin I_\varphi$. Set $s_{2k} = \tau(s_k, q_k)$ and $s_{2k+1} = \mu(s_k, q_k)$. Obviously, all the requirements of $(*)_{n+1}$ are satisfied. Therefore $(*)$ is true for the sequences $\{s_n\}_n$ and $\{q_n\}_n$ constructed in this way.

Set finally $\alpha(n) = q_n$, $n = 1, 2, \ldots$ We are going to check that (4.3) holds for this sequence α. Let us first notice that for every n, $F(s, \alpha, n) = \langle\langle s_n, \ldots, s_{2n-1}\rangle\rangle$. Indeed, the case $n = 1$ is obvious. Assuming that this is true for some $n \geq 1$, we shall have $head(F(s, \alpha, n)) = s_n$. By $(*)$ $\neg\psi(s_n, q_n)$ and therefore

$$\begin{aligned} F(s, \alpha, n+1) &= append(tail(F(s, \alpha, n)), \langle\langle \tau(s_n, q_n), \mu(s_n, q_n)\rangle\rangle) \\ &= append(\langle\langle s_{n+1}, \ldots, s_{2n-1}\rangle\rangle, \langle\langle s_{2n}, s_{2n+1}\rangle\rangle) = \langle\langle s_{n+1}, \ldots, s_{2n+1}\rangle\rangle. \end{aligned}$$

So, in particular, $head(F(s, \alpha, n)) = s_n$ for every $n = 1, 2, \ldots$ and using again $(*)$ we conclude that (4.3) is true.

Proposition 4.3. *Let A be a Π^0_1 positive inductive set. Then there exists a non-deterministic program P such that $A = D_P$.*

Proof. If A is I_φ, the result follows directly from the lemmas just verified. Otherwise A is a section of some I_φ, i.e.

$$A = \{(s_1, \ldots, s_k) \mid (s_1, \ldots, s_k, t_1, \ldots, t_m) \in I_\varphi)\}$$

for some fixed $t_1, \ldots, t_m$, built up from the basic constants and functions of $\mathfrak{A}$. Choose some $\tau_1, \ldots, \tau_m$ such that $\tau_{1\mathfrak{A}} = t_1$, $\ldots$, $\tau_{m\mathfrak{A}} = t_m$. Let P_0 be such that

$I_\varphi = D_P$. Denote by $x_1, \ldots, x_{k+m}$ the input variables of P_0. Now consider the following program P with input variables $x_1, \ldots, x_k$:

$$P:\ x_{k+1} := \tau_1;\ \ldots;\ x_{k+m} := \tau_m;\ P_0.$$

Obviously, $D_{P_0} = A$.

5. INDUCTIVE DEFINABILITY BY EXISTENTIALLY RESTRICTED FORMULAS

Let us call φ *existentially restricted* (*e.r.*) iff all existential quantifiers of φ range over the set $\mathrm{Nat} \subseteq B^*$. In this section we establish that these quantifiers do not increase the inductive expressive power, i.e. the least fixed point I_φ of every e.r. X-positive $\varphi(\bar{x}, X)$ is Π^0_1 positive inductive. This result is based on the next lemma, which ascertains the same for the Π^0_2 positive formulas.

Lemma 5.1. *Let $\varphi(\bar{x}, X)$ be existentially restricted Π^0_2 positive formula. Then I_φ is Π^0_1 positive inductive.*

Proof. Our aim is to build a non-deterministic program Q with $D_Q = I_\varphi$ and, applying Proposition 3.7, to conclude that I_φ is Π^0_1 positive inductive.

We shall assume for simplicity that φ has one object variable x, so it is in the following general form:

$$\forall y (\exists z \in \mathrm{Nat})(\psi(x,y,z) \vee \psi_1(x,y,z) \,\&\, \tau_{1,1} \in X \,\&\, \ldots \,\&\, \tau_{1,n_1} \in X \vee \ldots \\ \vee \psi_m(x,y,z) \,\&\, \tau_{m,1} \in X \,\&\, \ldots \,\&\, \tau_{m,n_m} \in X), \quad (5.1)$$

where ψ, ψ_1, $\ldots$, ψ_m are quantifier-free.

For the sake of clarity we shall confine ourselves to the case $m = 1$. When $m = 2$ (a case which is typical of the general case), combine the idea used in the construction of the program Q below with the in-width search in an appropriate binary tree, as carried out in the previous section.

Dropping also ψ_1 in (5.1) (since it is unessential here), we come to the following formula φ:

$$\forall y (\exists z \in \mathrm{Nat})(\psi(x,y,z) \vee \tau(x,y,z) \in X).$$

Now define the program Q as follows:

$Q:$ **input**(x); $t := \langle\!\langle\rangle\!\rangle$; $u := 0$;
  1: $z := x$; $y :=$ **arbitrary**(B^*);
    $t := append(t, \langle\!\langle y \rangle\!\rangle)$; $u := u + 1$; $v := 1$;
  2: **if** $\psi(z, (t)_v, [u]_v)$ **then stop**;
    if $u = v$ **then go to** 1;
    $z := \tau(z, (t)_v, [u]_v)$; $v := v + 1$;
    if $x = x$ **then go to** 2.

By $\lambda n, i.[n]_i$ we have denoted the decoding function for a fixed effective coding $\varkappa$ of all finite sequences of natural numbers (assuming, as customary, that $[n]_i = 0$ if

$i > \mathrm{lh}(n)$). Obviously, $\lambda n, i.[n]_i$, being recursive, can be computed with a program of our type.

Let us mention that the program Q that we propose here is far from being the most efficient one with the property $D_Q = I_\varphi$. Its advantage is the easy way to prove this fact.

Let us consider a particular execution of Q with input s. Suppose that during the computation we have arrived at the operator 2: **if** $\psi(z, (t)_v, [u]_v)$ **then stop** with current values of variables z, t, u and v, respectively q, r, n and i. Obviously, q is uniquely determined by s, r, n, i, i.e. there is a function g_0 such that $q = g_0(s, r, n, i)$. Set $g(s, \alpha, n, i) = g_0(s, \alpha(\bar{n}), n, i)$. Clearly, g satisfies the following equalities:

$$g(s, \alpha, n, 1) = s,$$
$$g(s, \alpha, n, i+1) = \tau(g(s, \alpha, n, i), \alpha(i), [n]_i) \text{ for } 1 < i \leq n.$$

Using this observation, one can easily check that

$$s \in D_Q \quad \text{iff} \quad \forall \alpha \exists n_{>0} (\exists i_{1 \leq i \leq n} \psi(g(s, \alpha, n, i), \alpha(i), [n]_i). \tag{5.2}$$

We shall use this equivalence in proving that the program Q has the desired property $I_\varphi = D_Q$.

Now let us agree until the end of the proof that n (eventually indexed) denotes an element of Nat.

For the first inclusion $I_\varphi \subseteq D_Q$ we shall use induction on $|s|$. A straightforward verification convinces us that for every $s \in I_\varphi$

$$|s| = \begin{cases} 0 & \text{if } \forall q \exists n \psi(s, q, n), \\ \sup\limits_{q : \forall n \neg \psi(s,q,n)} (\min\{|\tau(s, q, n)| + 1 \mid n \in \mathrm{Nat}\}) & \text{otherwise.} \end{cases}$$

Now take some $s \in I_\varphi$. In order to show that $s \in D_Q$, it suffices to see that the right-hand side of (5.2) holds. To do this, pick a sequence α and denote its first element by q. If $|s| = 0$, then there exists $n_q : \psi(s, q, n_q)$. Set $i = 1$ and $n = \varkappa(n_q)$ (or, for example, $n = \varkappa(n_q, 0)$ if $\varkappa(n_q)$ happens to be 0). Then, obviously, $\psi(g(s, \alpha, n, i), \alpha(i), [n]_i)$.

Now suppose that $|s| > 0$. If there is n_q with $\psi(s, q, n_q)$, proceed as above. Otherwise there should exist n_q such that $\tau(s, q, n_q) \in I_\varphi$ and $|\tau(s, q, n_q)| < |s|$. Set $\beta(n) = \alpha(n+1)$, $n = 1, 2, \ldots$ By induction hypothesis $\tau(s, q, n_q) \in D_Q$ and according to (5.2) exist m and $j_{1 \leq j \leq m}$:

$$\psi(g(\tau(s, q, n_q), \beta, m, j), \beta(j), [m]_j). \tag{5.3}$$

Now take $n \geq j + 1$ such that

$$[n]_1 = n_q \quad \text{and} \quad [n]_l = [m]_{l-1} \text{ for } l = 2, \ldots, j+1.$$

An easy induction on $i = 1, \ldots, j$ convinces us that

$$g(s, \alpha, n, i+1) = g(\tau(s, q, n_q), \beta, m, i).$$

In particular, $g(s, \alpha, n, j+1) = g(\tau(s, q, n_q), \beta, m, j)$. From here, using (5.3) and taking $i = j + 1$, we get the desired

$$\psi(g(s, \alpha, n, i), \alpha(i), [n]_i).$$

Towards establishing the converse inclusion $D_Q \subseteq I_\varphi$, suppose that A is an arbitrary fixed point of Γ_φ and take some $s \notin A$. Then $\neg\varphi(s, A)$ and therefore for at least one $q_1 \in B^*$

$$\forall n_1 \neg\psi(s, q_1, n_1) \quad \text{and} \quad \forall n_1 \tau(s, q_1, n_1) \notin A.$$

Analogously, from the latter there exists some $q_2 \in B^*$:

$$\forall n_1 \forall n_2 \neg\psi(\tau(s, q_1, n_1), q_2, n_2) \quad \text{and} \quad \forall n_1 \forall n_2 \tau(\tau(s, q_1, n_1), q_2, n_2) \notin A.$$

Iterating this procedure, we build a sequence $\alpha = q_1, q_2, \ldots$ satisfying for each n and $i \in \{1, \ldots, n\}$

$$\neg\psi(g(s, \alpha, n, i), \alpha(i), [n]_i).$$

From here, applying (5.2), we get $s \notin D_Q$.

Proposition 5.2. *Let $\varphi(\bar{x}, X)$ be an arbitrary existentially restricted positive formula. Then I_φ is Π^0_1 positive inductive.*

Proof. Our idea is to reduce φ to a Π^0_2 positive formula φ^* such that I_φ is a section of I_{φ^*} and to apply the result just obtained.

With no loss of generality we may assume that φ has one object variable x. Now consider first the case when φ is Π^0_4 formula, i.e. it is equivalent to

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} \psi(x, z, t, z', t')$$

with ψ — a quantifier-free. Let $\varphi^*(x, y, X)$ be the formula

$$\forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x, z, t \rangle\!\rangle, 1) \in X \vee y = 1 \,\&\, \hat{\psi}(x, z, t, X)),$$

where $\hat{\psi}$ is constructed from ψ in the following way: first replace simultaneously the variables x, z, t, z', t' by $(x)_1$, $(x)_2$, $(x)_3$, z and t, respectively. Then in the formula thus obtained replace each formula $\tau \in X$ by $(\tau, 0) \in X$. We claim that for every $s \in B^*$

$$s \in I_\varphi \;\Leftrightarrow\; (s, 0) \in I_{\varphi^*}. \tag{5.4}$$

We are going to prove (5.4) for the case when the matrix ψ is in the following simple form:

$$\chi(x, z, t, z', t') \vee \alpha(x, z, t, z', t') \in X,$$

since the verification of the general case is much similar to it.

So the corresponding formula $\varphi^*(x, y, X)$ is the following:

$$\begin{aligned} \forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x, z, t \rangle\!\rangle, 1) \in X \\ \vee\, y = 1 \,\&\, (\chi((x)_1, (x)_2, (x)_3, z, t) \vee (\alpha((x)_1, (x)_2, (x)_3, z, t), 0) \in X). \end{aligned}$$

The set I_{φ^*} is a fixed point of Γ_{φ^*}, therefore for any x

$$\begin{aligned} (x, 0) \in I_{\varphi^*} \;&\Leftrightarrow\; \forall z \exists t_{\in \mathrm{Nat}} ((\langle\!\langle x, z, t \rangle\!\rangle, 1) \in I_{\varphi^*}) \\ &\Leftrightarrow\; \forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(x, z, t, z', t') \vee (\alpha(x, z, t, z', t'), 0) \in I_{\varphi^*}). \end{aligned} \tag{5.5}$$

Towards establishing the equivalence (5.4) suppose that $s \in I_\varphi$. Then $s \in I^\xi_\varphi$ for some ordinal ξ. Using transfinite induction on ξ, we are going to check that $(s, 0) \in I_{\varphi^*}$. Indeed, under definition, $s \in I^\xi_\varphi$ if and only if

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} \Big(\chi(s, z, t, z', t') \vee \alpha(s, z, t, z', t') \in \bigcup_{\eta < \xi} I^\eta_\varphi \Big). \tag{5.6}$$

Taking an ordinal $\eta < \xi$, we get by inductive supposition that

$$\alpha(s, z, t, z', t') \in I^\eta_\varphi \;\Rightarrow\; (\alpha(s, z, t, z', t'), 0) \in I_{\varphi^*}.$$

So, using (5.6), we obtain

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s, z, t, z', t') \vee (\alpha(s, z, t, z', t'), 0) \in I_{\varphi^*}),$$

which according to (5.5) means that $(s, 0) \in I_{\varphi^*}$.

Now, conversely, assuming that $(s, 0) \in I^\xi_{\varphi^*}$ for some ξ, by induction on ξ we prove that $s \in I_\varphi$. We have

$$(s, 0) \in I^\xi_{\varphi^*} \;\Leftrightarrow\; (s, 0) \in \Gamma_{\varphi^*}\Big(\bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big) \;\Leftrightarrow\; \forall z \exists t_{\in \mathrm{Nat}} \Big((\langle\!\langle s, z, t \rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big).$$

Now suppose that $(\langle\!\langle s, z, t \rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}$ for some $\eta < \xi$. Then

$$\forall z' \exists t'_{\in \mathrm{Nat}} \Big(\chi(s, z, t, z', t') \vee \Big(\alpha(s, z, t, z', t'), 0) \in \bigcup_{\mu<\eta} I^\mu_{\varphi^*}\Big)\Big)$$

and by the induction hypothesis for μ

$$\forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s, z, t, z', t') \vee \alpha(s, z, t, z', t') \in I_\varphi).$$

So we obtained

$$(\langle\!\langle s, z, t \rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*} \;\Rightarrow\; \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s, z, t, z', t') \vee \alpha(s, z, t, z', t') \in I_\varphi).$$

From here

$$\forall z \exists t_{\in \mathrm{Nat}} \Big((\langle\!\langle s, z, t \rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big)$$
$$\Rightarrow\; \forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s, z, t, z', t') \vee \alpha(s, z, t, z', t') \in I_\varphi),$$

in other words,

$$(s, 0) \in I^\xi_{\varphi^*} \;\Rightarrow\; s \in I_\varphi$$

and hence $s \in I_\varphi$.

Thereby, the verification of equivalence (5.4) is completed. So I_φ is a section of I_{φ^*}, which is Π^0_2 positive inductive on $(\mathfrak{A}, \lambda x, i.(x)_i)$. From here, I_{φ^*} is Π^0_2 positive inductive on $\mathfrak{A}$ (under Lemma 3.4 and Transitivity Lemma). By Lemma 5.1 I_{φ^*} is Π^0_1 positive inductive on $\mathfrak{A}$ and hence I_φ is Π^0_1 positive inductive on $\mathfrak{A}$.

Now let $\varphi(x, X)$ be an arbitrary e.r. positive formula. We may assume that φ is Π^0_{2k}, $k \geq 2$, i.e. that φ is equivalent to

$$\forall z^1 \exists t^1_{\in \mathrm{Nat}} \ldots \forall z^k \exists t^k_{\in \mathrm{Nat}} \psi(x, z^1, t^1, \ldots, z^k, t^k, X)$$

with ψ — a quantifier-free. Set φ^* to be the following formula:

$$\forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x, z, t \rangle\!\rangle, 1) \in X \vee \ldots \vee y = k-2 \,\&\, (\langle\!\langle x, z, t \rangle\!\rangle, k-1) \in X$$
$$\vee\, y = k-1 \,\&\, \hat{\psi}(x, z, t)).$$

Here the formula $\hat{\psi}(x, z, t)$ is constructed from ψ as it follows: first replace in ψ variables x, z^1 and t^1 by $(x)_1$, $(x)_2$ and $(x)_3$, respectively, and denote the formula thus obtained by $\psi^{(1)}$. To define $\psi^{(2)}$, replace in $\psi^{(1)}$ each occurrence of $(x)_i$ by $((x)_1)_i$ for $i = 1, 2, 3$ and z^2 and t^2 by $(x)_2$ and $(x)_3$, respectively. Repeat this procedure $k-1$ times. Finally, in the formula $\psi^{(k-1)}$ replace z^k and t^k by z and t, respectively, and then replace all formulas of the type $\tau \in X$ by $(\tau, 0) \in X$. The formula, constructed in this way, is $\hat{\psi}$.

Now the equality $I_\varphi = \{s \mid (s,0) \in I_{\varphi^*}\}$ is verified as above. As we have already seen, it immediately implies that I_φ is Π_1^0 positive inductive on $\mathfrak{A}$.

Theorem 5.3. *Let $A \subseteq (B^*)^k$. The following conditions are equivalent:*

(i) *A is the set of all points of $\forall$-definedness for some non-deterministic program P;*

(ii) *A is inductively definable by some Π_1^0 positive formula;*

(iii) *A is inductively definable by some existentially restricted positive formula;*

(iv) *$A = \{\bar{s} \mid \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s})\}$, where the predicate R is prime computable on $\mathfrak{A}$.*

Proof. The equivalence between the first three conditions follows from Proposition 3.7, Proposition 4.3 and Proposition 5.2. The easiest way to complete the proof of the theorem, is to show that (i) and (iv) are equivalent. We shall use the observation that prime computability (PC) is equivalent to computability by means of deterministic programs (see, for example, [5, Ch. 1.3]).

Now, assuming that the non-deterministic program P is determined by the recursive sequence $\{\Pi^{(n)}\}_n$ (in the sense of Proposition 2.2), let us set

$$R(t,n,\bar{s}) \Leftrightarrow \mathrm{Seq}(t) \,\&\, \mathrm{Nat}(n) \,\&\, \mathrm{lh}(t) = n \,\&\, \Pi^{(n)}(t,\bar{s}).$$

Having in mind some basic facts about prime computability, we may assert that R is prime computable on $\mathfrak{A}$. It is clear that $D_P = (\bar{s} \mid \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s})\}$.

Conversely, let $R(t,n,\bar{s})$ be a prime computable predicate and P_0 be some deterministic program that computes it. Denote by y, z, x_1, ..., x_k the input variables of P_0. Now set

$P:$ **input**$(x_1, \ldots, x_k)$; $y := \langle\langle\ \rangle\rangle$; $z := 0$;
 1: $t :=$ **arbitrary**(B^*); $y := y * t$; $z := z + 1$;
 P_0; **if** $x_1 = x_1$ **then go to** 1.

It is immediate by the construction of P that

$$\bar{s} \in D_P \Leftrightarrow \forall\alpha\exists n(P_0 \text{ stops at input } (\bar{\alpha}(n), n, \bar{s})) \Leftrightarrow \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s}).$$

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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AN INEQUALITY OF DUFFIN-SCHAEFFER-SCHUR TYPE*

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It is shown here that the transformed Chebyshev polynomial of the second kind $\overline{U}_n(x) := U_n(x\cos\frac{\pi}{n+1})$ has the greatest uniform norm in $[-1, 1]$ of its k-th derivative ($k = 1,\dots,n$) among all algebraic polynomials of degree not exceeding n, which vanish at ± 1 and whose absolute value is less than or equal to 1 at the points $\{\cos\frac{j\pi}{n} / \cos\frac{\pi}{n+1}\}_{j=1}^{n-1}$.

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1. INTRODUCTION AND STATEMENT OF RESULT

Denote by π_n the set of all real algebraic polynomials of degree at most n. As usual, $T_n(x) = \cos n \arccos x$ denotes the n-th Chebyshev polynomial of the first kind. In what follows, $\|\cdot\|$ will mean the uniform norm in $[-1,1]$, $\|f\| := \sup\limits_{x\in[-1,1]} |f(x)|$.

The classical inequality of I. Schur [15] asserts that the transformed Chebyshev polynomial $\overline{T}_n(x) = T_n\left(x\cos\dfrac{\pi}{2n}\right)$ has the greatest uniform norm of its first derivative on $[-1, 1]$ among all $f \in \pi_n$, which vanish at the boundary points ± 1, and whose uniform norm is less than or equal to 1.

Recently, this result was extended to higher order derivatives by Milev and Nikolov [10] (the special cases $k = 2$ and $k = 3$ have been examined earlier by

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Milev [8, 9]).

Theorem A ([10, Theorem 1.1]). *If $f \in \pi_n$ satisfies*

$$f(-1) = f(1) = 0 \tag{1.1}$$

and

$$\|f\| \le 1, \tag{1.2}$$

then

$$\|f^{(k)}\| \le \|\overline{T}_n^{(k)}\| \tag{1.3}$$

for $k = 1, \ldots, n$. Equality in (1.3) is possible if and only if $f = \pm\overline{T}_n$.

Let $\{y_j^*\}_{j=1}^{n-1}$ be defined by

$$y_j^* = \frac{\cos(j\pi/n)}{\cos(\pi/2n)}.$$

For $k \ge 2$ Milev and Nikolov proved the following extension of Theorem A.

Theorem B ([10, Theorem 1.2]). *Let $f \in \pi_n$ satisfy (1.1) and*

$$|f(y_j^*)| \le 1, \quad j = 1, \ldots, n-1. \tag{1.4}$$

Then the inequality (1.3) holds for $k = 2, \ldots, n$. Moreover, equality is possible if and only if $f = \pm\overline{T}_n$.

Theorem B asserts that the condition (1.2) in Theorem A is unnecessarily restrictive, and that for $k \ge 2$ the inequality (1.3) remains valid if (1.2) is replaced by the weaker requirement $|f(x)| \le |\overline{T}_n(x)|$ at the extremal points of $\overline{T}_n$, i.e., at $\{y_j^*\}_{j=1}^{n-1}$. This is very similar to the extension of the Markov inequality, found by Duffin and Schaeffer [4]. For some related results the reader may consult [1, 2, 5, 11–13, 16].

Regarding Theorem B, the following question arises in a natural way: what would happen if the "comparison points" $\{y_j^*\}_{j=1}^{n-1}$ in (1.4) are replaced by some other points? Answering this question for arbitrary $\{y_j\}_{j=1}^{n-1}$ seems to be a very difficult task.

In this paper we examine completely the case

$$y_j = \frac{\cos(j\pi)}{\cos(\pi/(n+1))}, \quad j = 1, \ldots, n-1.$$

It turns out that in this case the extremizer for all $k \in \{1, \ldots, n\}$ is the transformed Chebyshev polynomial of the second kind $\overline{U}_n$,

$$\overline{U}_n(x) := U_n\left(x \cos\frac{\pi}{n+1}\right).$$

Precisely, we prove the following Duffin-Schaeffer-Schur type inequality:

Theorem 1.1. *Let $f \in \pi_n$ satisfy* (1.1) *and*

$$\left| f\left(\frac{\cos(j\pi/n)}{\cos(\pi/(n+1))}\right)\right| \le 1, \quad j = 1, \dots, n-1. \tag{1.5}$$

Then

$$\|f^{(k)}\| \le \|\overline{U}_n^{(k)}\| \tag{1.6}$$

for all $k \in \{1, \dots, n\}$. Moreover, equality in (1.6) *is possible if and only if $f = \pm\overline{U}_n$.*

The paper is organized as follows. In Section 2 we prove a pointwise inequality (Theorem 2.1), which is the main ingredient of the proof of Theorem 1.1. The necessary auxiliary results are proven in Section 3, with the exception of Lemma 3.5, the proof of which is the content of Section 5. In Section 4 we prove Theorem 1.1.

2. A POINTWISE INEQUALITY

For the sake of convenience we examine the usual Chebyshev polynomial of the second kind $U_n(x) := \dfrac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}$ on the interval $[-\eta, \eta]$, where $\eta := \cos\dfrac{\pi}{n+1}$. For this reason the conditions (1.1) and (1.5) are replaced by

$$f(-\eta) = f(\eta) = 0, \tag{2.1}$$

and

$$\left| f\left(\cos\frac{j\pi}{n}\right)\right| \le 1, \quad j = 1, \dots, n-1. \tag{2.2}$$

Throughout, $\|\cdot\|_\star$ will mean the uniform norm in $[-\eta, \eta]$, i.e.,

$$\|f\|_\star := \sup_{x\in[-\eta,\eta]} |f(x)|.$$

Theorem 1.1 is proved with the help of the pointwise inequality, given by the next theorem.

Theorem 2.1. *Let $f \in \pi_n$ satisfy the conditions* (2.1)–(2.2). *Then for each $k \in \{1, \dots, n\}$ and for every $x \in [-\eta, \eta]$*

$$|f^{(k)}(x)| \le \max\{|U_n^{(k)}(x)|, |Z_{n,k}(x)|\},$$

where

$$Z_{n,k}(x) := \frac{1}{k\eta}\left[\left(x^2 - \frac{n+k}{n}\eta^2\right) T_n^{(k+1)}(x) + kxT_n^{(k)}(x)\right]. \tag{2.3}$$

Proof. Let $x_0 < x_1 < \ldots < x_n$ be the zeros of $\omega(x) := (x^2 - \eta^2)T_n'(x)$, and let $\omega_\nu(x) := \omega(x)/(x - x_\nu)$, $\nu = 0, \ldots, n$. For every polynomial f of degree at most n the Lagrange interpolation formula yields

$$f^{(k)}(x) = \sum_{\nu=0}^{n} \frac{f(x_\nu)}{\omega_\nu(x_\nu)} \omega_\nu^{(k)}(x). \tag{2.4}$$

In particular, for $f \in \pi_n$ satisfying (2.1)–(2.2), (2.4) yields

$$|f^{(k)}(x)| \leq \sum_{\nu=1}^{n-1} \left| \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)} \right|. \tag{2.5}$$

According to a well-known result of V. Markov, if the zeros of two polynomials interlace, then the interlacing property is inherited by the zeros of their derivatives (for a proof see, e.g., [14, Lemma 2.7.1]). In particular, for polynomials of the same degree this result could be interpreted as monotone dependence of the zeros of the derivative of a polynomial on its zeros ([1, p. 39]). Since for $i > j$ the zeros of $\omega_i(x)$ are less than or equal to the corresponding zeros of $\omega_j(x)$, we conclude that the same relation remains valid for the zeros of $\omega_i^{(k)}$ and $\omega_j^{(k)}$. Hence, the j-th zeros of the polynomials $\{\omega_i^{(k)}\}_{i=1}^{n-1}$ are located between the j-th zero of $\omega_n^{(k)}$ and the j-th zero of $\omega_0^{(k)}$. Denote by $\{\beta_i\}_{j=1}^{n-k}$ and $\{\alpha_i\}_{i=2}^{n-k+1}$ the zeros of $\omega_n^{(k)}$ and $\omega_0^{(k)}$, respectively, arranged in increasing order. Set $\alpha_1 := -\eta$, $\beta_{n-k+1} := \eta$, then the above reasoning implies that

$$\text{sign}\,\{\omega_\nu^{(k)}(x)\} \quad \text{is the same for all } \nu \in \{1, \ldots, n-1\} \text{ when } x \in [\alpha_j, \beta_j]. \tag{2.6}$$

We observe that the zeros of ω and U_n interlace, and in addition

$$U_n(x_\nu) = \text{sign}\,\{\omega_\nu(x_\nu)\} = (-1)^{n-\nu}, \quad \nu = 1, \ldots, n-1.$$

Therefore, for $x \in [\alpha_j, \beta_j]$ ($j \in \{1, \ldots, n-k+1\}$) the substitution $f = U_n$ in (2.4) yields

$$|U_n^{(k)}(x)| = \left| \sum_{\nu=1}^{n-1} \frac{\omega_\nu^{(k)}(x)}{|\omega_\nu(x_\nu)|} \right| = \sum_{\nu=1}^{n-1} \left| \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(x_\nu)} \right|. \tag{2.7}$$

Comparison of (2.7) and (2.5) implies that if f satisfies the assumptions of Theorem 2.1, then

$$|f^{(k)}(x) \leq |U_n^{(k)}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k+1} [\alpha_j, \beta_j]. \tag{2.8}$$

Our next goal is to prove that under the same assumptions

$$|f^{(k)}(x)| \leq |Z_{n,k}(x)| \quad \text{for all } x \in \bigcup_{j=1}^{n-k} (\beta_j, \alpha_{j+1}). \tag{2.9}$$

Observing that the j-th zero of U_n is located between the j-th zero of ω_n and the j-th zero of ω_0 (precisely, the first zeros of U_n and ω_n and the last zeros of U_n and ω_0 coincide), we conclude on the basis of V. Markov's result that each interval (β_j, α_{j+1}), $j = 1, \ldots, n-k$, contains exactly one zero of $U_n^{(k)}$, and consequently

$$\operatorname{sign}\{U_n^{(k)}(\alpha_j)\} = \operatorname{sign}\{U_n^{(k)}(\beta_j)\} = (-1)^{n+1-k-j}, \quad j = 1, \ldots, n-k. \tag{2.10}$$

Using the identity (cf. [17, eqs. (4.7.28)])

$$U_n^{(k)}(x) = xU_{n-1}^{(k)}(x) + (n+k)U_{n-1}^{(k-1)}(x), \tag{2.11}$$

it is not difficult to see that

$$Z_{n,k}(x) - U_n^{(k)}(x) = \frac{1}{k\eta}\left(x - \frac{n+k}{n}\eta\right)\omega_n^{(k)}(x), \tag{2.12}$$

$$Z_{n,k}(x) + U_n^{(k)}(x) = \frac{1}{k\eta}\left(x + \frac{n+k}{n}\eta\right)\omega_0^{(k)}(x), \tag{2.13}$$

whence

$$Z_{n,k}(x) = \begin{cases} -U_n^{(k)}(x) & \text{for } x = \alpha_j,\ j = 2, \ldots, n-k+1, \\ U_n^{(k)}(x) & \text{for } x = \beta_j,\ j = 1, \ldots, n-k. \end{cases} \tag{2.14}$$

If $f \in \pi_n$ satisfies the assumptions of Theorem 2.1, then according to the above reasonings $|f^{(k)}| \le |U_n^{(k)}|$ at the zeros of $\omega_0^{(k)}$ and $\omega_n^{(k)}$. The relations (2.14) and (2.10) then imply that each of the polynomials $Z_{n,k} \pm f^{(k)}$ has at least one zero in each of the intervals $[\alpha_j, \beta_j]$, $j = 2, \ldots, n-k$. Moreover, $\operatorname{sign}\{(Z_{n,k} \pm f^{(k)})(\alpha_{n-k+1})\} = -\operatorname{sign}\{U_n^{(k)}(\alpha_{n-k+1})\} = -1$. Since $Z_{n,k} \pm f^{(k)}$ have positive leading coefficients, it follows that each of them has at least one zero located to the right of α_{n-k+1}. Similar arguments show that $Z_{n,k} \pm f^{(k)}$ must have at least one zero located to the left of β_1. Hence, each of the polynomials $Z_{n,k} \pm f^{(k)}$ has maximal possible number of zeros $(n-k+1)$, and all these zeros lie outside the set $\cup_{j=1}^{n-k}(\beta_j, \alpha_{j+1})$. Now the observation that $|f^{(k)}| \le |Z_{n,k}|$ on the boundary of this set completes the proof of (2.9). Theorem 2.1 is proved. ∎

Remark 1. The claims of both Theorem 1.1 and Theorem 2.1 are trivial when $n \le 2$. Following the proof of Theorem 2.1, one observes that if $k = n$, then $|f^{(k)}(x)| \le |U_n^{(k)}(x)|$ on the whole real axis for every function f satisfying (2.1) and (2.2). Thus, for $k = n$

$$\|f^{(k)}\|_* \le \|U_n^{(k)}\|_*. \tag{2.15}$$

Furthermore, if $k = n-1$, then $f^{(k)}$ is a polynomial of degree 1, and therefore $\|f^{(k)}\|_*$ is attained at $x = -\eta$ or at $x = \eta$. According to (2.8), at these points we have $|f^{(k)}| \le |U_n^{(k)}|$, and therefore again (2.15) holds. The statement of Theorem

1.1 then follows for $k = n-1, n$ from (2.15) after a linear transformation (see also [16, Corollary 4]).

For this reason, we may restrict our considerations to the case $n \geq k+2$.

Remark 2. Studying carefully the proof of Theorem 2.1, one can see that for any fixed point $x_0 \in \bigcup_{j=1}^{n-k} (\beta_j, \alpha_{j+1})$ the exact upper bound for $|f^{(k)}(x_0)|$ in (2.5) subject to the constraints (2.1)–(2.2) is attained for a polynomial, which alternates between -1 and 1 at the points $\{x_i\}_{i=1}^{n-1}$ with only one exception (i.e., $|f(x_i)| = 1$, $i = 1, \ldots, n-1$, there is a $\lambda \in \{1, \ldots, n-2\}$ such that $f(x_\lambda)f(x_{\lambda+1}) > 0$ and $f(x_i)f(x_{i+1}) < 0$ for $i \neq \lambda$). Following the notations of Gusev [6], we may call these polynomials as Zolotarev polynomials. The number of the essentially different Zolotarev polynomials is $[(n-1)/2]$, e.g., 1, if $n = 3, 4$; 2, if $n = 5, 6$; 3, if $n = 7, 8$, etc. Hence, for small n one can examine directly all the possible extremal polynomials in Theorem 1.1.

3. AUXILIARY RESULTS

We begin with listing in a lemma some well-known properties of the ultraspherical polynomials $P_n^{(\lambda)}$ $(\lambda > -1/2)$. Recall that $P_n^{(\lambda)}$ is the n-th orthogonal polynomial in $[-1, 1]$ with respect to the weight $w_\lambda(x) = (1-x^2)^{\lambda - 1/2}$ and normalized (for $\lambda \neq 0$) by $P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}$ (in particular, $P_n^{(1)} = U_n$). In the case $\lambda = 0$ the Chebyshev polynomial T_n is orthogonal and satisfies $T_n(1) = 1$.

Lemma 3.1. (i) *For every* $\lambda > -1/2$, $\lambda \neq 0$,

$$\frac{d}{dx}\{P_n^{(\lambda)}(x)\} = 2\lambda P_{n-1}^{(\lambda+1)}(x)$$

(*the case* $\lambda = 0$ *reads as* $T_n'(x) = nP_{n-1}^{(1)}(x)$).

(ii) *For every* $\lambda \geq \mu > -1/2$, $P_m^{(\lambda)}$ *obeys a representation*

$$P_n^{(\lambda)}(x) = \sum_{m=0}^{n} a_{m,n}(\lambda, \mu) P_m^{(\mu)}(x) \quad \textit{with } a_{m,n}(\lambda,\mu) \geq 0,\ m = 0, \ldots, n\,.$$

(iii) *For* $\lambda > 0$ *the absolute values of the local extrema of* $P_n^{(\lambda)}$ *increase as the distance between the points of local extrema and the origin increases.*

(iv) $y = P_n^{(\lambda)}$ *satisfies the differential equation*

$$y'' - (2\lambda + 1)xy' + n(n+2\lambda)y = 0.$$

For easy reference we formulate in a lemma two simple facts from calculus, which will be used frequently in the sequel.

Lemma 3.2. (i) *For any fixed* $\alpha \in (0,2)$ *the sequence* $a_n = n \sin \frac{\alpha\pi}{n}$, $n = 2, 3, \dots,$ *is monotone increasing.*

(ii) *For any fixed* $0 < \alpha < \beta \le 2$ *the sequence* $b_n := \frac{\sin(\beta\pi/n)}{\sin(\alpha\pi/n)}$, $n = 3, 4, \dots,$ *is monotone increasing.*

Lemma 3.3. *For every natural* $n \ge 2$ *and for* $k = 2, \dots, n$ *there holds*

$$\max_{x \in [-\cos(\pi/n), \cos(\pi/n)]} |T_n^{(k)}(x)| = T_n^{(k)}(\eta). \tag{3.1}$$

Proof. It suffices to prove only the case $k = 2$. Indeed, if (3.1) is established for $k = 2$, then it follows that $\|T_m''\|_\star = T_m''(\eta)$ for all $m \le n$. For $k \ge 3$, Lemma 3.1(i)–(ii) yields

$$T_n^{(k)}(x) = \sum_{m=2}^{n-k+2} a_m T_m''(x) \quad \text{with } a_m \ge 0,$$

and consequently,

$$\|T_n^{(k)}\|_\star \le \sum_{m=2}^{n-k+2} a_m \|T_m''\|_\star = \sum_{m=2}^{n-k+2} a_m T_m''(\eta) = T_n^{(k)}(\eta).$$

Thus it remains to prove (3.1) for $k = 2$. The cases $n = 2$, 3 are trivial, therefore we suppose that $n \ge 4$. According to Lemma 3.1(i), (iii), we have to compare $T_n''(\eta)$ with $T_n''(z)$, where $z = \cos\tau$ is the last critical point for T_n'', i.e., the last zero of T_n'''. The explicit representation of T_n yields, with $x = \cos\theta$, $0 < \theta < \pi$,

$$T_n''(x) = n\cos\theta \frac{\sin n\theta}{\sin^3\theta} - n^2 \frac{\cos n\theta}{\sin^2\theta}, \tag{3.2}$$

$$T_n'''(x) = \frac{n}{\sin^5\theta} \left\{ [3 - (n^2+2)\sin^2\theta] \sin n\theta - 3n \sin\theta \cos\theta \cos n\theta \right\}. \tag{3.3}$$

Putting $\theta = 2\pi/n$ and $\theta = 3\pi/(2n)$ in (3.3), we get

$$T_n'''\left(\cos\frac{2\pi}{n}\right) = -3n^2 \frac{\cos(2\pi/n)}{\sin^4(2\pi/n)} \le 0,$$

$$T_n'''\left(\cos\frac{3\pi}{2n}\right) = \frac{n}{\sin^5(3\pi/(2n))} \left[(n^2+2)\sin^2\frac{3\pi}{2n} - 3\right] > 0,$$

hence T_n''' has a zero in the interval $\left[\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$. This zero is readily seen to be unique, and it is the last critical point of T_n''.

The equation (3.2) can be rewritten as

$$T_n''(\cos\theta) = \frac{n}{2\sin^3\theta}\varphi(\theta), \tag{3.4}$$

where $\varphi(\theta) = (n+1)\sin(n-1)\theta - (n-1)\sin(n+1)\theta$. The points in $(0, \pi)$, at which the function φ has local extrema, are $\theta_k = \dfrac{k\pi}{n}$, $k = 1, \ldots, n-1$, and

$$|\varphi(\theta_k)| = 2n \sin \frac{k\pi}{n}. \tag{3.5}$$

Taking into account the inequalities $\cos\dfrac{2\pi}{n} < \cos\tau < \cos\dfrac{3\pi}{2n}$, we obtain from (3.4) and (3.5)

$$|T_n''(\cos\tau)| < n^2 \frac{\sin(2\pi/n)}{\sin^3(3\pi/(2n))}. \tag{3.6}$$

The substitution $\theta = \dfrac{\pi}{n}$ in (3.2) yields

$$\left|T_n''\left(\cos\frac{\pi}{n}\right)\right| = \frac{n^2}{\sin^2(\pi/n)} \tag{3.7}$$

and the lemma will be proved if we succeed to show that the right-hand side of (3.7) is greater than the right-hand side of (3.6), which is equivalent to

$$\left(\frac{\sin(3\pi/(2n))}{\sin(\pi/n)}\right)^3 \geq 2\cos\frac{\pi}{n}. \tag{3.8}$$

According to Lemma 3.2(i), the left-hand side of (3.8) is increasing with respect to n, and for $n \geq 4$ it is greater than 2. This completes the proof of Lemma 3.3. ∎

Remark 3. A more precise examination of equation (3.3) shows that for $n \geq 5$ the last critical point of T_n'' is located in $\left(\cos\dfrac{2\pi}{n}, \cos\dfrac{7\pi}{4n}\right)$ (see the proof of Lemma 5.1 below).

As an immediate consequence from Lemma 3.3 we get

Corollary 3.1. *For all natural $k \leq n$ there hold:*

(a) $||U_n^{(k)}||_* = U_n^{(k)}(\eta)$;

(b) $||T_n^{(k)}||_* = T_n^{(k)}(\eta)$ for $k \geq 2$.

The function $Z_{n,k}(x)$ appearing in Theorem 2.1 can be represented as

$$Z_{n,k}(x) = c_k[u_{n,k}(x) - v_{n,k}(x)], \tag{3.9}$$

where $c_k := 1/(k\eta)$ and

$$u_{n,k}(x) := (x^2-1)T_n^{(k+1)}(x) + kxT_n^{(k)}(x), \tag{3.10}$$

$$v_{n,k}(x) := \left(\frac{n+k}{n}\eta^2 - 1\right)T_n^{(k+1)}(x). \tag{3.11}$$

We quote without proof the following simple lemma:

Lemma 3.4. *The inequality*

$$\frac{n+k}{n}\eta^2 \geq 1$$

holds for every $n \geq 9$ if $k = 1$, for every $n \geq 5$ if $k = 2$, and for every $n, k \geq 3$.

Corollary 3.2. *For n and k as in Lemma 3.4 there holds*

$$\|v_{n,k}\|_\star = v_{n,k}(\eta).$$

The next lemma shows that a similar conclusion holds for the function $u_{n,k}(x)$.

Lemma 3.5. *For all natural $k \geq 3$ there holds*

$$\|u_{n,k}\|_\star = u_{n,k}(\eta). \tag{3.12}$$

The proof of this lemma requires more work, and we put it off to the last section.

4. PROOF OF THEOREM 1.1

According to Remark 1, we may assume that $n \geq k+2$. We exclude also the cases $k = 1$, $3 \leq n \leq 8$, and $k = 2$, $n = 4$, which are verified directly as indicated in Remark 2. For the remaining n and k we shall prove the inequality

$$\|Z_{n,k}\|_\star < \|U_n^{(k)}\|_\star. \tag{4.1}$$

Having established (4.1), we can readily deduce Theorem 1.1 as a corollary of Theorem 2.1. Indeed, if $f \in \pi_n$ satisfies (1.1) and (1.5), then $p(x) := f(x/\eta)$ will satisfy (2.1)–(2.2). It follows then from Theorem 2.1 that for every $x \in [-\eta, \eta]$

$$|p(x)| \leq \max\{|U_n^{(k)}(x)|, |Z_{n,k}(x)|\} \leq \|U_n^{(k)}\|_\star = U_n^{(k)}(\eta). \tag{4.2}$$

Then $f(x) = p(x\eta)$ will satisfy

$$\|f^{(k)}\| = \eta^k \|p^{(k)}\|_\star \leq \eta^k U_n^{(k)}(\eta) = \|\overline{U}_n^{(k)}\|,$$

whence the inequality of Theorem 1.1 follows. To clarify the cases in which equality holds, we observe that equality in (4.2) is possible only if $x = \pm\eta$ and $p = \pm U_n$. This completes the proof of Theorem 1.1.

It remains to prove (4.1). Equation (3.9) and Corollary 3.2 yield

$$\begin{aligned}
\|Z_{n,k}\|_\star &\leq c_k[\|u_{n,k}\|_\star + \|v_{n,k}\|_\star] = c_k[v_{n,k}(\eta) + \|u_{n,k}\|_\star] \\
&= -Z_{n,k}(\eta) + c_k[u_{n,k}(\eta) + \|u_{n,k}\|_\star] \\
&= U_n^{(k)}(\eta) - c_k\frac{k(2n+k)}{n}\eta T_n^{(k)}(\eta) + c_k[u_{n,k}(\eta) + \|u_{n,k}\|_\star]
\end{aligned}$$

(for the last equality we used equation (2.13)). Clearly, the inequality (4.1) will hold if we succeed in showing that

$$u_{n,k}(\eta) + \|u_{n,k}\|_\star < \frac{k(2n+k)}{n}\eta T_n^{(k)}(\eta). \tag{4.3}$$

In the proof of (4.3) we shall distinguish between the cases $k=1$, $k=2$ and $k \geq 3$. Note that

$$T_n(\eta) = -\cos\frac{\pi}{n+1}, \quad T_n'(\eta) = n, \quad T_n''(\eta) = n(n+1)\frac{\cos(\pi/(n+1))}{\sin^2(\pi/(n+1))}. \tag{4.4}$$

Case $k=1$. Lemma 3.1(iv) implies that $u_{n,1}(x) = n^2 T_n(x)$ and the inequality (4.3) in this case reduces to

$$\cos\frac{\pi}{n+1} > \left(\frac{n}{n+1}\right)^2,$$

the verification of which causes no difficulties.

Case $k=2$. Using Lemma 3.1(iv) we obtain $u_{n,2}(x) = (n^2-1)T_n'(x) - xT_n''(x)$. The explicit form of T_n' and Lemma 3.3 yield the estimate

$$\|u_{n,2}\|_\star \leq (n^2-1)\|T_n'\|_\star + \|xT_n''(x)\|_\star \leq \frac{n(n^2-1)}{\sin(\pi/(n+1))} + \eta T_n''(\eta),$$

and consequently,

$$u_{n,2}(\eta) + \|u_{n,2}\|_\star \leq n(n^2-1)\left(1 + \sin^{-1}\frac{\pi}{n+1}\right).$$

Therefore, (4.3) will follow with $k=2$ if

$$n(n-1)\left[\sin\frac{\pi}{n+1} + \sin^2\frac{\pi}{n+1}\right] < 4(n+1)\cos^2\frac{\pi}{n+1}.$$

Putting $A_n := (n+1)\sin\dfrac{\pi}{n+1}$, we rewrite the latter inequality in the following form

$$(n-2)A_n + A_n^2 + (A_n+2)\sin\frac{\pi}{n+1} + 2\sin^2\frac{\pi}{n+1} < 4(n+1). \tag{4.5}$$

According to Lemma 3.2(i), $A_n < A_\infty = \pi$, and we increase the left-hand side of (4.5) to obtain the inequality

$$(n-2)\pi + \pi^2 + (\pi+2)\sin\frac{\pi}{n+1} + 2\sin^2\frac{\pi}{n+1} < 4(n+1),$$

which is easily seen to be true for all $n \geq 4$.

Case $k \geq 3$. According to Lemma 3.5, in this case $\|u_{n,k}\|_\star = u_{n,k}(\eta)$ and the inequality (4.3) becomes

$$(1-\eta^2)T_n^{(k+1)}(\eta) + \frac{k^2}{2n}\eta T_n^{(k)}(\eta) \geq 0,$$

which is obviously true, since η is located to the right from the right-most zero of $T_n^{(k)}$, $k = 1, \ldots, n$. With this (4.3) is proved and therefore Theorem 1.1. ∎

5. PROOF OF LEMMA 3.5

We first observe that the general case is a consequence of the case $k = 3$. Indeed, let

$$\|u_{n,3}\|_* = u_{n,3}(\eta). \tag{5.1}$$

It is readily seen that $u_{n,3}$ is strictly monotone increasing to the right of $x = \eta$. This implies that (5.1) follows also with n replaced by m, $m \le n$. Then for $k \ge 4$ Lemma 3.1(ii) and Corollary 3.1(b) yield

$$T_n^{(k)}(x) = \sum_{m=3}^{n-k+3} b_m T_m'''(x) \quad \text{with non-negative } b_m,\ m = 3, \ldots, n-k+3,$$

$$\begin{aligned}
\|u_{n,k}\|_* &= \|(x^2-1)T_n^{(k+1)}(x) + 3xT_n^{(k)}(x) + (k-3)xT_n^{(k)}(x)\|_* \\
&= \|\sum_{m=3}^{n-k+3} b_m u_{m,3}(x) + (k-3)xT_n^{(k)}(x)\|_* \\
&\le \sum_{m=3}^{n-k+3} b_m \|u_{m,3}\|_* + (k-3)\eta T_n^{(k)}(\eta) \\
&= \sum_{m=3}^{n-k+3} b_m u_{m,3}(\eta) + (k-3)\eta T_n^{(k)}(\eta) = u_{n,k}(\eta).
\end{aligned}$$

The proof of (5.1) goes through several lemmas. For the sake of simplicity we suspend the indices in $u_{n,3}(x)$ and simply write $u(x)$, where

$$u(x) = (x^2-1)T_n^{\mathrm{IV}}(x) + 3xT_n'''(x) = (n^2-4)T_n''(x) - 2xT_n'''(x).$$

It is not difficult to verify that (5.1) is true for $n \le 10$ and we suppose in what follows $n \ge 11$.

We shall need information about the location of the last critical points of $u(x)$ (i.e., the last zero of $u'(x)$), which we denote by ξ. As a first, we observe that the zeros of $u'(x)$ and $T_n^{(4)}(x)$ interlace and a brief examination shows that $\xi \in \left(\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$. Sharper bounds are given in the next lemma.

Lemma 5.1. *For every natural $n > 10$ there holds*

$$\cos\frac{7\pi}{4n} < \xi < \cos\frac{5\pi}{3n}. \tag{5.2}$$

Proof. Putting $x = \cos\theta$, after some straightforward calculations we obtain

$$u'(x) := \frac{-n \sin n\theta}{\sin^7\theta} \left[(5+\sigma_n)t_n^3 - 30t_n\right] \left[g(t_n, \sigma_n) + \cos\theta \cot n\theta\right], \tag{5.3}$$

where $\sigma_n := 4/n^2$, $t_n = t_n(\theta) := n \sin\theta$, and

$$g(t,\sigma) := \frac{(1+2\sigma)t^4 - (15+6\sigma)t^2 + 30}{(5+\sigma)t^3 - 30t}. \tag{5.4}$$

Since $\xi \in \left(\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}\right)$ and $n \geq 11$, we may assume that $t_n(\theta) > 11 \sin\frac{3\pi}{22} > 4.5$ and $0 < \sigma_n < 1/30$, i.e., $(t_n, \sigma_n) \in \mathcal{A}$, where

$$\mathcal{A} := \{(t,\sigma) \mid t > 4.5,\ 0 \leq \sigma < 1/30\}.$$

The function $g(t,\sigma)$ has continuous derivatives in $\mathcal{A}$ and $\frac{\partial g}{\partial \sigma} > 0$ therein. This implies for $(t,\sigma) \in \mathcal{A}$

$$g_1(t) := \frac{t^4 - 15t^2 + 30}{5t^3 - 30t} \leq g(t,\sigma) \leq \frac{32t^4 - 456t^2 + 900}{151t^3 - 900t} := g_2(t), \tag{5.5}$$

where $g_1(t) = g(t,0)$ and $g_2(t) = g(t, 1/30)$. Moreover, $g_1(t)$ and $g_2(t)$ are monotone increasing for $t > 4.5$. Looking at (5.3) and taking into account $\theta \in (3\pi/(2n), 2\pi/n)$, we observe that

$$\text{sign}\{u'(\cos\theta)\} = \text{sign}\{g(t_n, \sigma_n) + \cos\theta \cot n\theta\} := \text{sign}\{h(\theta)\}. \tag{5.6}$$

For $\theta_1 = \frac{7\pi}{4n}$ Lemma 3.2(i) yields $t_n(\theta_1) \leq t_\infty(\theta_1) < 5.498$, hence for $n \geq 11$

$$h(\theta_1) \leq g_2(5.498) - \cos\frac{7\pi}{44} = -0.065 < 0.$$

For $\theta_2 = \frac{5\pi}{3n}$ and $n \geq 11$ Lemma 3.2(i) asserts $t_n(\theta_2) \geq t_{11}(\theta_2) > 5.04$ and therefore

$$h(\theta_2) \geq g_1(5.04) + \cot\frac{5\pi}{3} = 0.024 > 0.$$

Consequently, we obtain $u'\left(\cos\frac{7\pi}{4n}\right) < 0$ and $u'\left(\cos\frac{5\pi}{3n}\right) > 0$. This completes the proof of Lemma 5.1. ∎

Lemma 5.2. *The local maxima of $|u(x)|$ increase as $|x|$ increases.*

For the proof of Lemma 5.2 we apply the following result (see, e.g., [17, (7.31)]):

Lemma 5.3 (Theorem of Sonin-Pòlya). *Let $y(x)$ be a non-trivial solution of the differential equation*

$$(py')' + Py = 0, \tag{5.7}$$

where $p(x)$ and $P(x)$ are continuously differentiable and positive in the interval (a,b), and let the function $p(x)P(x)$ be non-decreasing (non-increasing) on (a,b). Then the relative maxima of $|y|$ in (a,b) form a non-increasing (non-decreasing) sequence.

The application of Lemma 5.3 with $y = u$ is possible because of the next lemma.

Lemma 5.4. *The function $u(x)$ satisfies a differential equation of the type* (5.7) *with*

$$p(x) = \frac{(1-x^2)^{7/2}}{n^2(1-x^2)-6} \tag{5.8}$$

and

$$P(x) = \frac{(1-x^2)^{5/2}\left[(n^2-4)n^2(1-x^2)-10n^2+48\right]}{\left[n^2(1-x^2)-6\right]^2}. \tag{5.9}$$

The proof of Lemma 5.4 is by direct verification, applying Lemma 3.1(iv). For the proof of Lemma 5.2 one only have to check that the functions p and P defined by (5.8) and (5.9) are positive in $\left(-\cos\frac{5\pi}{3n}, \cos\frac{5\pi}{3n}\right)$ and that $(pP)'$ is negative in $\left(0, \cos\frac{5\pi}{3n}\right)$. This is an easy exercise if the inequality $n^2(1-x^2) \geq t_{11}^2\left(\frac{5\pi}{3n}\right) > 25$ is taken into account. ∎

Now we are in a position to prove (5.1). According to Lemma 5.2,

$$\|u\|_* = \max\{|u(\xi)|, u(\eta)\}$$

and it suffices to show that

$$|u(\xi)| \leq u\left(\cos\frac{\pi}{n+1}\right). \tag{5.10}$$

We have

$$u\left(\cos\frac{\pi}{n+1}\right) = \frac{n(n+1)\cos(\pi/(n+1))}{\sin^4(\pi/(n+1))}\left[n(n+2)\sin^2\frac{\pi}{n+1} - 6\right]$$

and for $n \geq 11$ the application of Lemma 3.2(i) yields the estimate

$$u\left(\cos\frac{\pi}{n+1}\right) > 3.457\frac{n(n+1)}{\sin^4(\pi/(n+1))}. \tag{5.11}$$

At the point ξ we have $(n^2-6)T_n^{(3)}(\xi) = 2\xi T_n^{(4)}(\xi)$, and using Lemma 3.1(iv) repeatedly, we obtain

$$u(\xi) = \left[1 + \frac{4\xi^2}{(n^2+4)(1-\xi^2)-10}\right](n^2-4)T_n''(\xi). \tag{5.12}$$

According to Remark 3, $T_n''(x)$ is monotone increasing in $\left(\cos\frac{7\pi}{4n}, \cos\frac{5\pi}{3n}\right)$ and equation (3.2) shows that T_n'' is negative therein. Therefore $|T_n''(\xi)|$ is bounded from above by $\left|T_n''\left(\cos\frac{7\pi}{4n}\right)\right|$. For $n \geq 11$, an upper bound for the first factor in equation (5.12) is given by $1 + 4/(121\sin^2 5\pi/33 - 10)$. Substituting these bounds in (5.12), we obtain

$$|u(\xi)| < 5.788\frac{n(n^2-4)}{\sin^3(7\pi/(4n))}. \tag{5.13}$$

In view of (5.11) and (5.13), (5.10) will hold if

$$\left(\frac{\sin(7\pi/(4n))}{\sin(\pi/(n+1))}\right)^3 \geq 1.6743\frac{n^2-4}{(n+1)^2}\left[(n+1)\sin\frac{\pi}{n+1}\right],$$

or if the following stronger inequality holds:

$$\left(\frac{\sin(7\pi/(4n))}{\sin(\pi/n)}\right)^3 \geq 5.26\frac{n^2-4}{(n+1)^2}.$$

According to Lemma 3.2(ii), the left-hand side of the latter inequality increases monotonically as n increases and for $n \geq 24$ it is greater than 5.26. By verification its validity is seen also for $11 \leq n \leq 23$. This proves (5.10), (5.1) and Lemma 3.5. ∎

Remark 4. It is not difficult to see that Theorem 1.1 remains true even if the polynomials under consideration are allowed to have complex coefficients (the same applies to Theorem B). Indeed, let p be the extremal polynomial from this larger class, and let

$$\sup_f\{\|f^{(k)}\|\} = |p^{(k)}(\tau)| = e^{i\theta}p^{(k)}(\tau), \quad \tau \in [-1, 1]$$

with some real θ. Then the polynomial $g(x) = \operatorname{Re}\{e^{i\theta}p(x)\}$ also belongs to the class under consideration and satisfies $g^{(k)}(\tau) = |p^{(k)}(\tau)|$. Thus we found another extremal polynomial, which, in addition, has real coefficients. Following the proof of Theorem 1.1, we conclude that this is only possible if $\tau = \pm 1$ and $g = \pm\overline{U}_n$.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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FACTORIZATIONS OF THE GROUPS $\Omega_7(q)$*

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The following result is proved:
Let $G = \Omega_7(q)$ and q is odd. Suppose that $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:

(1) $q = 3$ and $A \cong L_4(3)$ or $G_2(3)$, $B \cong Sp_6(2)$ or A_9;
(2) $q \equiv -1 \pmod 4$ and $A \cong G_2(q)$, $B \cong L_4(q)$;
(3) $q \equiv 1 \pmod 4$ and $A \cong G_2(q)$, $B \cong U_4(q)$;
(4) $q = 3^{2n+1} > 3$ and $A \cong {}^2G_2(q)$, $B \cong L_4(q)$;
(5) $q = 3^{2n+1}$ and $A \cong U_3(q)$, $B \cong L_4(q)$;
(6) $q = 3^{2n}$ and $A \cong L_3(q)$, $B \cong U_4(q)$;
(7) $A \cong G_2(q)$, $B \cong PSp_4(q)$.

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1. INTRODUCTION

In [1–3] we determined all the factorizations with two proper simple subgroups of all groups G of Lie type of Lie rank 3 except for $G = \Omega_7(q)$. In the present work we extend this investigation to the simple groups $\Omega_7(q)$ of Lie type (B_3) over the finite fields $GF(q)$. Thus we complete the determination of all factorizations (into the product of two simple groups) of all simple groups of Lie type of Lie rank 3. Here we may assume that q is odd in view of the well-known isomorphism

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$\Omega_7(q) \cong PSp_6(q)$ if q is even (recall that the factorizations of the groups $PSp_6(q)$ have been determined in [3]). We prove the following

Theorem. *Let $G = \Omega_7(q)$ (q is odd) and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 3$ *and* $A \cong L_4(3)$ *or* $G_2(3)$, $B \cong Sp_6(2)$ *or* A_9;
(2) $q \equiv -1 \pmod 4$ *and* $A \cong G_2(q)$, $B \cong L_4(q)$;
(3) $q \equiv 1 \pmod 4$ *and* $A \cong G_2(q)$, $B \cong U_4(q)$;
(4) $q = 3^{2n+1} > 3$ *and* $A \cong {}^2G_2(q)$, $B \cong L_4(q)$;
(5) $q = 3^{2n+1}$ *and* $A \cong U_3(q)$, $B \cong L_4(q)$;
(6) $q = 3^{2n}$ *and* $A \cong L_3(q)$, $B \cong U_4(q)$;
(7) $A \cong G_2(q)$, $B \cong PSp_4(q)$.

The factorizations of $\Omega_7(q)$ into the product of two maximal subgroups have been determined in [7]. We make use of this result here.

We shall freely use the notation and basic information on the finite (simple) classical groups given in [6]. $L_n^\varepsilon(q)$ denotes $L_n(q)$ if $\varepsilon = +$ and $U_n(q)$ if $\varepsilon = -$. Let V be the natural 7-dimensional orthogonal space over $GF(q)$ on which G acts, and let $(\ ,\ )$ be a non-singular symmetric bilinear form on V. There is a basis $\{d, e_i, f_i \mid i = 1,2,3\}$ of V, called a standart basis, such that $(d,d) = 2$, $(d,e_i) = (d,f_i) = (e_i,e_j) = (f_i,f_j) = 0, (e_i,f_j) = \delta_{ij}$ for $i,j = 1,2,3$. Let P_k be the stabilizer in G of a totally singular k-dimensional subspace of V. If W is a non-singular subspace of V of dimension k, we denote the stabilizer G_W of W in G by N_k^ε ($\varepsilon = \pm$), where $W^\perp$ has type O_{7-k} if k is odd, and W has type O_k^ε if k is even. From Propositions 4.1.6 and 4.1.20 in [6] we can obtain the structure of P_k and N_k^ε. In particular, it follows that

$$P_1 \cong [q^6] : ((q-1)/2 \times PSp_4(q)).2, \qquad P_3 \cong [q^6] : \frac{1}{2}GL_3(q),$$

$$N_1^\varepsilon \cong \Omega_6^\varepsilon(q).2 \cong (2,(q-\varepsilon 1)/2).L_4^\varepsilon(q).2, \qquad N_2^\varepsilon \cong ((q-\varepsilon 1)/2 \times PSp_4(q)).[4].$$

From this it follows immediately that N_1^ε contains a subgroup isomorphic to $L_4^\varepsilon(q)$ if and only if $q \equiv -\varepsilon 1 \pmod 4$; also, in P_3 there exists a subgroup isomorphic to $L_3(q)$ only if $q \not\equiv 1 \pmod 3$. Lemma 4.1.12 in [6] gives us a possibility to describe in P_1 the subgroup L isomorphic to $PSp_4(q)$, namely, we may regard L (up to conjugacy in G) as the subgroup of G fixing the vectors e_1, f_1 and stabilizing the subspace $\langle d, e_2, e_3, f_2, f_3\rangle$ of V. In the same way , using again Lemma 4.1.12 in [6], we may take (if $q \not\equiv 1 (\mathrm{mod}\ 3)$) the $L_3(q)$ subgroup of P_3 to be the subgroup K of G fixing the vector d and stabilizing each of the subspaces $\langle e_1, e_2, e_3\rangle$ and $\langle f_1, f_2, f_3\rangle$ on which K induces an $SL_3(q)$ subgroup. Note that each of the groups N_1^ε and N_2^ε also contains a subgroup isomorphic to $PSp_4(q)$.

Let $G = \Omega_7(q)$, where $q = p^m$ and p is an odd prime, and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. The factorizations of $\Omega_7(3)$ are determined in [4]; this gives (1) and (2), (5), (7) (with $q = 3$) in the theorem. Thus we can assume that $q > 3$. The list of maximal factorizations of G is given in [7]. This leads, by order considerations, to the following possibilities:

1) $A \cong U_4(q)$ (in N_1^-), $B \cong G_2(\sqrt{q})$ (in a $G_2(q)$ subgroup of G), m even;

2) $A \cong U_4(q)$ (in N_1^-), $B \cong L_3(q)$ (in P_3), $q \equiv 1 (\mathrm{mod}\ 4)$ and $q \not\equiv 1 (\mathrm{mod}\ 3)$;

3) $A \cong G_2(q)$, $B \cong PSp_4(q)$ or $B \cong L_4^\varepsilon(q)$ (in N_1^ε with $q \equiv -\varepsilon 1 (\mathrm{mod}\ 4)$);

4) $A \cong {}^2G_2(q)$, $B \cong L_4(q)$ (in N_1^+), $q = 3^{2n+1} > 3$;

5) $A \cong L_3^\varepsilon(q)$ (in a $G_2(q)$ subgroup of G), $B \cong L_4^{-\varepsilon}(q)$ (in $N_1^{-\varepsilon}$), $q \not\equiv \varepsilon 1 \pmod 3$ and $q \equiv \varepsilon 1 \pmod 4$.

We consider these possibilities case by case.

Case 1. Here $|A \cap B| = q - 1$. Now let $B_1 \cong G_2(q)$ be a subgroup of G containing B. Then $G = AB_1$ and $|A \cap B_1| = |SU_3(q)|$. Since $(A \cap B_1) \cap B = A \cap B$ has order $q - 1$, it follows (by order considerations) $B_1 = (A \cap B_1)B$. However, the group $B_1 \cong G_2(q)$ possesses no such factorization ([5]), a contradiction.

Case 2. Here we use the following two realizations of the group $G_1 = SO_7(q)$:

(i) $SO_7(q) = \{X \in SL_7(q) \mid X^t H X = H\}$, where

$$H = \left(\begin{array}{c|cc} 2 & \multicolumn{2}{c}{0} \\ \hline \multirow{2}{*}{0} & 0 & E \\ & E & 0 \end{array}\right)$$

is the matrix of the bilinear form (,) in the standart basis $d, e_1, e_2, e_3, f_1, f_2, f_3$;

(ii) $SO_7(q) = \{Y \in SL_7(q) \mid Y^t I Y = I\}$, where

$$I = \left(\begin{array}{c|ccccc|c} 2\lambda & & & 0 & & & 0 \\ \hline & & & & & 1 & \\ & & & & 1 & & \\ 0 & & & 2 & & & 0 \\ & & 1 & & & & \\ & 1 & & & & & \\ \hline 0 & & & 0 & & & -2\lambda \end{array}\right)$$

is the matrix of the form (,) in the basis $e_1 + \lambda f_1$, e_2, e_3, d, f_3, f_2, $e_1 - \lambda f_1$ with λ a non-square in $GF(q)$.

Let $X, Y \in SL_7(q)$ and $Y = T_0^{-1} X T_0$, where

$$T_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $Y^t I Y = I$ if and only if $X^t H X = H$.

Now, from the above description of the $L_3(q)$ subgroup in P_3, with respect to (i), we have

$$B = \left\{ \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & M & 0 \\ \hline 0 & 0 & M^{-t} \end{array} \right) \quad \Big| \quad M \in SL_3(q) \right\} \cong L_3(q).$$

Further, we may take A to be the $U_4(q)$ subgroup in the subgroup P of $SO_7(q)$ which has the following form about (ii):

$$P = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \star \end{array} \right) \in SO_7(q) \right\} \cong SO_6^-(q) \cong 2 \times U_4(q).$$

Moreover, we have $P \cap G = A$ and hence $P \cap B = A \cap B$. A direct calculation shows that

$$P \cap B = T_0^{-1} \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \begin{array}{c|c} 1 & 0 \\ \hline 0 & T \end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c|c} 1 & 0 \\ \hline 0 & T^{-t} \end{array} \end{array} \right) T_0,$$

where $T \in SL_2(q)$. Thus $A \cap B \cong SL_2(q)$ and order considerations imply $G \neq AB$.

Now we proceed to prove that in the remaining cases 3–5 (with suitable q) the factorizations exist.

Case 3. Let us consider the following realization of the group $SO_7(q)$:

(iii) $SO_7(q) = \{Z \in SL_7(q) \mid Z^t J Z = J\}$, where

$$J = \begin{pmatrix} & & & & & & 1 \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & 2 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \end{pmatrix}$$

is the matrix of the bilinear form in the basis $e_1, e_2, e_3, d, f_3, f_2, f_1$ (see (i) above). Now, with respect to (iii), we make use of the well-known 7-dimensional representation of the group $G_2(q)$ over the field $GF(q)$ ([8]).

The root system of type (G_2) is

$$\sum = \{\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm(\xi_1 - \xi_2), \pm(\xi_2 - \xi_3), \pm(\xi_3 - \xi_1)\},$$

where $\xi_1+\xi_2+\xi_3 = 0$. Let E and E_{ij}, $-3 \leq i, j \leq 3$, denote the 7×7 identity matrix and matrix units, respectively. Then the generators $x_r(t)$ $(r \in \sum,\ t \in GF(q))$ of $G_2(q)$ are represented as follows:

$$x_{\xi_i-\xi_j}(t) = E + t(E_{-i-j} - E_{ji}),$$

$$x_{\pm\xi_i}(t) = E + t(\pm 2E_{\mp io} \mp E_{o\pm i} \pm E_{\pm j\mp k} \mp E_{\pm k\mp j}) - t^2 E_{\mp i\pm i},$$

where (i, j, k) is an even permutation of 1, 2, 3. Note that

$$\omega_r = x_r(1)x_{-r}(-1)x_r(1), \quad h_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)\omega_r^{-1}.$$

Any element of $A \cong G_2(q)$ can be written uniquely in the form

$$x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\,\omega\, x_a(s_1)x_b(s_2)$$
$$x_{a+b}(s_3)x_{2a+b}(s_4)x_{3a+b}(s_5)x_{3a+2b}(s_6),$$

where $a = \xi_2$, $b = \xi_1 - \xi_2$, $t_i, s_i \in GF(q)$, $u, v \in GF(q)^*$ and $\omega = 1$, ω_a, ω_b, $\omega_a\omega_b$, $\omega_b\omega_a$, $\omega_a\omega_b\omega_a$, $\omega_b\omega_a\omega_b$, $(\omega_a\omega_b)^2$, $(\omega_b\omega_a)^2$, $(\omega_a\omega_b)^2\omega_a$, $(\omega_b\omega_a)^2\omega_b$ or $(\omega_a\omega_b)^3$.

On the other hand, using the above description of the $PSp_4(q)$ subgroup in P_1, with respect to (iii) we may take $B \cong PSp_4(q)$ to be a subgroup in the following subgroup Q of $SO_7(q)$:

$$Q = \left\{ \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \star & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \in SO_7(q) \right\} \cong SO_5(q) \cong PSp_4(q).2 \quad .$$

Moreover, $Q \cap G = B$ and hence $A \cap Q = A \cap B$. A direct computation shows that $A \cap B$ consists of the following elements of A:

$$x_b(t_2)h_b(v), \quad x_b(t_2)h_b(v)\omega_b x_b(s_2) \qquad (v \in GF(q)^*,\ t_2, s_2 \in GF(q)).$$

Hence $|A \cap B| = q(q^2 - 1)$ (in fact, $A \cap B = \langle x_b(t), x_{-b}(t)\rangle \cong SL_2(q)$). Now order considerations imply $G = AB$. This is the factorization in (7) of the theorem.

Further, let $A \cong G_2(q)$ be the subgroup of G described in the above paragraph and $B \cong L_4(q)$ be the subgroup of G in the subgroup R of $SO_7(q)$ which has the following form with respect to (iii):

$$R = \left\{ \left(\begin{array}{c|c|c} \star & 0 & \star \\ \hline 0 & 1 & 0 \\ \hline \star & 0 & \star \end{array} \right) \in SO_7(q) \right\} \cong SO_6^+(q) \cong 2 \times L_4(q).$$

Here, again $R \cap G = B$ and thus $A \cap R = A \cap B$. Just as above, we can find the common elements of these $G_2(q)$ and $L_4(q)$ subgroups of G; they are as follows:

$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\omega_b x_b(s_2)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)\omega_a\omega_b\omega_a x_{3a+b}(s_5)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_a\omega_b)^2 x_b(s_2)x_{3a+2b}(s_6)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_b\omega_a)^2 x_{3a+b}(s_5)x_{3a+2b}(s_6)$,
$x_b(t_2)x_{3a+b}(t_5)x_{3a+2b}(t_6)h_a(u)h_b(v)(\omega_b\omega_a)^2\omega_b x_b(s_2)x_{3a+b}(s_5)x_{3a+2b}(s_6)$.

Hence $|A\cap B| = q^3(q^3-1)(q^2-1)$ (in fact, $A\cap B = \langle x_{\pm b}(t), x_{\pm(3a+b)}(t), x_{\pm(3a+2b)}(t) \mid t \in GF(q)\rangle \cong SL_3(q)$). Again order considerations imply $G = AB$. This is the factorization in (2) of the theorem.

Now, with respect to (iii), let $A \cong G_2(q)$ be the same subgroup of G described above and $B \cong U_4(q)$ be a subgroup, in realization (ii), of the group P considered in the previous case. Let $Y, Z \in SL_7(q)$ and $Y = K_0^{-1} Z K_0$, where

$$K_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \lambda & 0 & 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

Then $Y^t I Y = I$ if and only if $Z^t J Z = J$; here I and J are the matrices described above.

A direct computation shows that the common elements (in realization (ii)) of the above $G_2(q)$ and $U_4(q)$ subgroups are

$$K_0^{-1}(x_a(t_1)x_b(t_2)x_{a+b}(t_3)x_{2a+b}(t_4)x_{3a+b}(t_5)x_{3a+2b}(t_6)$$
$$h_a(u)h_b(v)\omega x_a(s_1)x_b(s_2)x_{a+b}(s_3)x_{2a+b}(s_4)x_{3a+b}(s_5)x_{3a+2b}(s_6))K_0,$$

where:

$\omega = 1$ and $u = 1$, $t_1 = t_3 = t_4 = t_5 = t_6 = s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0$;

$\omega = \omega_a$ and $s_2 = s_3 = s_4 = s_5 = s_6 = 0$, $s_1 = -u$, $t_1 = -uv^{-1}$, $t_3 = t_1t_2$, $t_4 = t_1t_3$, $t_5 = -t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 + u^{-1}v\lambda$;

$\omega = \omega_b$ and $u = 1$, $t_1 = t_3 = t_4 = t_5 = t_6 = s_1 = s_3 = s_4 = s_5 = s_6 = 0$;

$\omega = \omega_a\omega_b$ and $s_1 = s_4 = s_5 = s_6 = 0$, $s_3 = u$, $t_1 = -uv^{-1}$, $t_3 = t_1t_2$, $t_4 = t_1t_3$, $t_5 = -t_1t_4$, $t_6 = \lambda u^{-1}v - t_3t_4$;

$\omega = \omega_b\omega_a$ and $s_2 = s_3 = s_4 = s_6 = 0$, $s_1 = -u$, $s_5 = -u^{-1}vt_1$, $t_3 = t_1t_2 - u^{-2}v$, $t_4 = t_1t_3$, $t_5 = -t_1^2t_3 - \lambda u^2v^{-1}$, $t_6 = t_1^2t_2t_3 - 2t_1t_3^2$;

$\omega = \omega_a\omega_b\omega_a$ and $s_2 = s_3 = s_6 = 0$, $s_1 = u^{-1}vt_1$, $s_4 = t_1t_3 - t_4$, $s_5 = -u - 2s_1s_4$, $t_3 = t_1t_2 + u^{-2}v\lambda - u^{-2}vs_4^2$, $t_5 = u^2v^{-1} - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = \omega_b\omega_a\omega_b$ and $s_1 = s_4 = s_5 = 0$, $s_3 = u$, $s_6 = u^{-1}vt_1$, $t_3 = -u^{-2}v + t_1t_2$, $t_4 = t_1t_3$, $t_5 = -u^2v^{-1}\lambda - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_a\omega_b)^2$ and $s_1 = s_5 = 0$, $s_6 = u - 2s_3s_4$, $t_1 = -uv^{-1}s_3$, $t_3 = t_1t_2 - u^{-2}vs_4^2 + u^{-2}v\lambda$, $t_4 = t_1t_3 - s_4$, $t_5 = u^2v^{-1} - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_b\omega_a)^2$ and $s_2 = s_3 = 0$, $s_5 = -u - 2s_1s_4$, $t_1 = uv^{-1}(s_4^2 - s_1s_6 - \lambda)$, $t_3 = u^{-2}vs_1 + t_1t_2$, $t_4 = t_1t_3 - s_4$, $t_5 = -u^2v^{-1}s_6 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 + u^{-1}v$;

$\omega = (\omega_a\omega_b)^2\omega_a$ and $s_2 = 0$, $t_1 = uv^{-1}(2s_1s_4 + s_5 - s_1^2s_3)$, $t_3 = u^{-2}v(s_6 + 2s_3s_4) + t_1t_2$, $t_4 = t_1t_3 + s_4 - s_1s_3$, $t_5 = -u^2v^{-1}s_1 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 - u^{-1}vs_3$, $s_4^2 + s_3s_5 - s_1s_6 - 2s_1s_3s_4 = u + \lambda$;

$\omega = (\omega_b\omega_a)^2\omega_b$ and $s_1 = 0$, $s_6 = u - 2s_3s_4 - s_2s_5$, $t_1 = uv^{-1}(s_4^2 + s_3s_5 - \lambda)$, $t_3 = u^{-2}vs_3 + t_1t_2$, $t_4 = t_1t_3 - s_4$, $t_5 = -u^2v^{-1}s_5 - t_1t_4$, $t_6 = u^{-1}v + t_1t_2t_4 - t_1t_3^2 - t_3t_4$;

$\omega = (\omega_a\omega_b)^3$ and $t_1 = -uv^{-1}(s_6 + 2s_3s_4 + s_2s_5)$, $t_3 = t_1t_2 - u^{-2}v(s_1^2s_3 - 2s_1s_4 - s_5)$, $t_4 = t_1t_3 + s_4 - s_1s_3$, $t_5 = u^2v^{-1}s_3 - t_1t_4$, $t_6 = t_1t_2t_4 - t_1t_3^2 - t_3t_4 - u^{-1}vs_1$, $s_4^2 + (s_3 - s_1s_2)s_5 - 2s_1s_3s_4 - s_1s_6 = u + \lambda$.

Hence $|A \cap B| = q^3(q^3+1)(q^2-1) = |SU_3(q)|$ (in fact, from [7, 5.1.14 (a)] we can see that $A \cap B \cong SU_3(q)$). Order considerations yield $G = AB$ and the factorization in (3) of the theorem is proved.

Case 4. Here $q = 3^{2n+1} > 3$. In case 3 we proved that $G = AB$, where $A \cong G_2(3^{2n+1})$, $B \cong L_4(3^{2n+1})$ and $D = A \cap B \cong SL_3(3^{2n+1})$. Take a subgroup $C \cong {}^2G_2(3^{2n+1})$ of A. Then (as shown in [9]) $A = CD$. It follows that $|C \cap B| = |C \cap D| = q - 1$. This implies $G = CB$, the factorization in (4) of the theorem.

Case 5. Suppose that $G = AB$. As A lies in a subgroup $A_1 \cong G_2(q)$ of G, we have also $G = A_1B$. Since $|A \cap (A_1 \cap B)| = |A \cap B| = q^2 - 1$, it follows (by order considerations) that $A_1 = A(A_1 \cap B)$. Now, from the list of all the factorizations of $G_2(q)$ given in [5], it follows that this is possible only if $A_1 \cap B \cong L_3^{-\varepsilon}(q)$, and $q = 3^{2n+1}$ if $\varepsilon = -$, $q = 3^{2n}$ if $\varepsilon = +$.

Conversely, with these restrictions on q, let $A_1 \cong G_2(q)$ and $B \cong L_4^{-\varepsilon}(q)$ be the subgroups of G described in case 3. As we have seen, $A_1 \cap B \cong L_3^{-\varepsilon}(q)$ and then (by [10]) there is a subgroup $A \cong L_3^{\varepsilon}(q)$ of A_1 such that $A_1 = A(A_1 \cap B)$. It follows that $|A \cap B| = |A \cap (A_1 \cap B)| = q^2 - 1$ and hence $G = AB$, the factorizations in (5) and (6) of the theorem.

This completes the proof.

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РОСТ ЦЕЛЫХ ФУНКЦИЙ, ОБРАЩАЮЩИХСЯ В НОЛЬ НА АНАЛИТИЧЕСКОМ МНОЖЕСТВЕ

МАРИЯ И. МИТРЕВА

Let f be an entire function in $\mathbb{C}^n$, V be the set of its zeroes, and $n_f(z', z_n)$ be the number of zeroes of $f(z', z_n)$ in the circle $|z_n| \le t$. We construct an entire function F such that F vanishes on V and its growth is estimated in terms of $n_f(z', t)$.

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Пусть $f(z)$ целая функция в $\mathbb{C}^n$, $X = \{z \in \mathbb{C}^n,\ f(z) = 0\}$ аналитическое множество размерности $n-1$, т. е. аналитическая гиперповерхность. Положим $u(z) = \ln|f(z)|$. Это будет плюрисубгармоническая функция в $\mathbb{C}^n$, стремящаяся к $-\infty$ на X, которая так же является субгармонической по каждой переменной. Для каждого $z = (z_1, \dots, z_{n-1}, z_n) \in \mathbb{C}^n$ положим $z' = (z_1, \dots, z_{n-1})$ и пусть $X' = \{z' = \text{const}\} \cap X$. Пусть $n_f(z', t)$ означает число корней функции f по переменной z_n при фиксированном z' в круге радиуса t, т. е. это множество тех точек из X', для которых $|z_n| \le t$ (см. [3]). Тогда известно, что n_f задается формулой

$$n_f(z', t) = \frac{4}{2\pi} \int\limits_{|z_n| \le t} \frac{\partial^2 u}{\partial z_n \partial \bar{z}_n}\, dz_n. \tag{1}$$

Наша задача будет состоять в построении целой в $\mathbb{C}^n$ функции $F(z) \not\equiv 0$, равной нулю на X и такой, что рост $\ln|F(z)|$ по переменной z_n не превышает рост функции $n_f(z', t)$ для $t = |z_n|$.

Сформулируем основную теорему:

Теорема. *Если $f(z)$ целая функция в $\mathbb{C}^n$, X ее нулевое множество, то для любого $\varepsilon > 0$ и для каждой точки $z^0 \notin X$ существует целая функция $F(z)$, для которой $\{z \in \mathbb{C}^n,\ F(z) = 0\} \supset X$, $F(z_0) \neq 0$ и*

$$\ln |F(z)| \leq C(z^0, \varepsilon, z')(1 + |z_n|^2) n_f(z', |z_n| + 2\varepsilon), \tag{2}$$

где n_f то же самое, что и выше (см. (1)), а $C(z^0, \varepsilon, z')$ функция, независящая от z_n.

Сначала докажем две леммы.

Лемма 1. *Если $f(z)$ целая функция в $\mathbb{C}^n$, n_f — дефинированная в (1) функция, считающая нулей f, то для любого $\varepsilon > 0$ существует субгармоническая по z_n функция v, такая что:*

а) *$v(z) - \ln |f(z)|$ непрерывна по z_n и равномерно по z' ограничена снизу субгармонической функцией $\alpha(z_n)$, которая равна $-\infty$ только на множестве X' при любом фиксированном z', т. е. $v(z', z_n) \geq \alpha(z_n)$, $\forall z' \in \mathbb{C}^{n-1}$.*

б) *Существует константа $C(\varepsilon)$, независящая от z' и z_n, так что*

$$v(z', z_n) \leq C(\varepsilon)|z_n|^2 n_f(z', |z_n| + \varepsilon). \tag{3}$$

Для простоты записи всюду в дальнейшем будем считать, что $n = 2$ и будем работать в пространстве $\mathbb{C}^2$ точек вида (z, w), $z \in \mathbb{C}$, $w \in \mathbb{C}$.

Доказательство. Пусть $\varepsilon > 0$ произвольно. Построим функцию

$$u_\varepsilon(z, w) = \frac{1}{\varepsilon^2} \int\limits_{\mathbb{C}_\zeta} u(z, w + \zeta) \rho\left(\frac{\zeta}{\varepsilon}\right) d\zeta, \tag{4}$$

где $\rho(x) \in C_0^\infty(\mathbb{R})$. Это положительная, с компактным носителем $\operatorname{supp}\rho \subset \{|x| < 1\}$ функция, для которой $\int\limits_0^1 \rho(x)\, dx = 1$ и $\rho(\zeta) = \rho(|\zeta|)$, $\zeta \in \mathbb{C}$. По теореме об интегрировании субгармонических функций $u_\varepsilon(z, w)$ будет субгармонической относительно w, а ввиду дальнейшего изложения представим ее в другой вид. Зафиксируем z произвольным образом и пусть $\delta(w) = u_z(w) = u(z, w)$. Тогда

$$u_\varepsilon(z, w) = \frac{1}{\varepsilon^2} \int\limits_0^\varepsilon \int\limits_{|\zeta| = t} \delta(w + \zeta) \rho\left(\frac{t}{\varepsilon}\right) d\zeta dt = \frac{2\pi}{\varepsilon^2} \int\limits_0^\varepsilon \rho\left(\frac{t}{\varepsilon}\right) dt \int\limits_{|\zeta| = t} \frac{1}{2\pi} \delta(w + \zeta)\, d\zeta$$

$$= 2\pi \int\limits_0^1 \rho(x) \mathfrak{M}_\delta(w, \varepsilon x)\, dx,$$

где $\mathfrak{M}_\delta(w, \varepsilon x)$ есть среднее значение функции δ по кругу с центром в w и радиусом εx. Ввиду свойств функции ρ и субгармоничности $\delta(w)$, получаем

$$u(\varepsilon, w) - u_\varepsilon(z, w) = 2\pi \int_0^1 [\delta(w) - \mathfrak{M}_\delta(w, \varepsilon x)]\, \rho(x)\, dx \le 0.$$

Теперь потребуем, чтобы искомая функция v имела вид

$$v(z, w) = u(z, w) - u_\varepsilon(z, w) + \gamma(z, w),$$

где неизвестная функция $\gamma(z, w)$ субгармонична по w и такая, что сама $v(z, w)$ тоже субгармонична по w. Для этого достаточно чтобы

$$\frac{\partial^2 \gamma}{\partial w \partial \bar{w}} \ge \frac{\partial^2 u_\varepsilon}{\partial w \partial \bar{w}}.$$

Поэтому дальше будем искать γ в виде

$$\gamma(z, w) = \chi(z, |w|^2), \tag{5}$$

где $\chi(z, t)$ есть выпуклая, возрастающая по t функция. От этого условия следует, что

$$\frac{\partial^2 \gamma}{\partial w \partial \bar{w}} = \frac{\partial^2 \chi}{\partial t^2}|w|^2 + \frac{\partial \chi}{\partial t} \ge \frac{\partial^2 u_\varepsilon}{\partial w \partial \bar{w}},$$

и для нас тогда достаточно чтобы

$$\left.\frac{\partial \chi}{\partial t}(z, t)\right|_{t=|w|^2} \ge \frac{\partial^2 u_\varepsilon}{\partial w \partial \bar{w}}. \tag{6}$$

Пользуясь формулами (1) и (3), оцениваем правую сторону (6) так:

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial w \partial \bar{w}} &= \frac{1}{\varepsilon^2} \int_{\mathbf{C}_\zeta} \frac{\partial^2 u}{\partial w \partial \bar{w}}(z, w + \zeta) \rho\left(\frac{\zeta}{\varepsilon}\right) d\zeta \\
&\le \frac{1}{\varepsilon^2} \max_{u \in \mathbf{C}} |\rho(u)| \int_{|\zeta| \le |w| + \varepsilon} \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}}(z, \zeta)\, d\zeta \\
&= \frac{1}{\varepsilon^2} K \frac{2\pi}{4} n_f(z, |w| + \varepsilon) = C(\varepsilon) n_f(z, |w| + \varepsilon),
\end{aligned}$$

где K и $C(\varepsilon)$ суть положительные константы, независящие от z и w. Ввиду (5) достаточно чтобы χ удовлетворяла условию

$$\frac{\partial \chi}{\partial t}(z, t) = C(\varepsilon) n_f(z, \sqrt{t} + \varepsilon),$$

откуда получаем

$$\begin{aligned}
\chi(z, t) &= \int_0^t C(\varepsilon) n_f(z, \sqrt{s} + \varepsilon)\, ds = C(\varepsilon) \int_0^{\sqrt{t}} x n_f(z, x + \varepsilon)\, dx \\
&\le C(\varepsilon) t n_f(z, \sqrt{t} + \varepsilon),
\end{aligned}$$

или

$$\chi(z,|w|^2) \le C(\varepsilon)|w|^2 n_f(z,|w|+\varepsilon),$$

и для $\gamma(z,w)$ получаем неравенство

$$\gamma(z,w) \leqq C(\varepsilon)|w|^2 n_f(z,|w|+\varepsilon),$$

которому будет удовлетворять также и функция v. Этим проверено условие б) леммы. Условие а) следует из непрерывности функций u_ε и γ по w и из свойств инфимума субгармонических функций.

Построение искомой функции $F(z)$ основано и на следующей леммы, доказательство которой использует в существенном результаты Хермандера [2] о существовании решения $\overline{\partial}$-проблемы с оценками (см. также [1]).

Лемма 2. *Пусть $\varphi(z,w)$ является субгармонической по $w \in \mathbb{C}'$ функцией в $\mathbb{C}^n$ и при фиксированном $z \in \mathbb{C}^{n-1}$ точка $w_0 = w_0(z)$ такова, что существует $r > 0$, для которого*

$$\sup_{z\in\mathbb{C}^{n-1}} \int\limits_{|w-w_0|<r} e^{-\varphi(z,w)}\,dw = M < +\infty. \tag{7}$$

Тогда существует функция $F(z,w)$ со свойствами:

$$\begin{aligned}
&1)\ F(z,w)\ \text{голоморфна по } w \text{ всюду при любом фиксированном } z \in \mathbb{C}^{n-1};\\
&2)\ F(z,w_0) \neq 0 \text{ для каждого фиксированного } z \in \mathbb{C}^{n-1};\\
&3)\ \sup_z \int\limits_{\mathbb{C}_w} |F(z,w)|^2 e^{-\varphi(z,w)}(1+|w|^2)^{-3}\,dw < +\infty;\\
&4)\ F(z,w)\ \text{голоморфна по } z \text{ в } \mathbb{C}^{n-1} \text{ при любом фиксированном } w \in \mathbb{C}^1.
\end{aligned} \tag{8}$$

Доказательство. Будем считать, что после того как зафиксировали z, точка $w_0 = w_0(z)$ перешла в ноль. Это нисколько не уменьшит общность доказательства.

В плоскости $z = \mathrm{const}$ рассмотрим множества

$$\Omega_1(z) = \{(z,w),\ |w| < r\}, \qquad \Omega_2(z) = \{(z,w),\ w \in \mathbb{C}'\}.$$

Построим функции $h_1(z,w)$, $h_2(z,w)$ со свойствами:

1) $h_i(z,w)$ голоморфны по w в $\Omega_i(z)$;

2) $h_i(z,0) \neq 0$;

3) $\sup\limits_z \int\limits_{\Omega_i} |h_i|^2 e^{-\varphi}(1+|w|^2)^{-3i+3}\,dw = N < +\infty$;

4) $h_i(z,w)$ при любом фиксированном w голоморфны по z;

5) $h_2(z,w) = h_1\psi(|w|^2) - wu(z,w)$, $\psi \in C_0^1(\mathbb{R})$, $\psi(t) = 0$, $|t| > r$, $\psi(t) = 1$, $|t| < r/2$, а $u(z,w)$ определена из условий 1)–4).

Такие функции существуют (например, можем положить $h_1(z,w)=a$, где $a\neq 0$ константа). Это требование обеспечивает сразу выполнение условий 2) и 4) для $h_1(z,w)$. Очевидно также, что h_1 голоморфна всюду по w и учитывая (7), имеем

$$\int\limits_{|w|<r}|h_1|^2e^{-\varphi}(1+|w|^2)^{-3i+3}\,dw=|a|^2\int\limits_{|w|<r}e^{-\varphi}\,dw<M<+\infty.$$

Для того, чтобы $h_2(z,w)$ выполняла условий 1)–4), необходимо подобрать функцию $u(z,w)$ так, что

$$\overline{\partial}_w h_2(z,w)=0,$$

т. е.

$$\overline{\partial}_w u(z,w)=h_1(z,w)\psi'(|w|^2)\,d\bar{w}=a\psi'(|w|^2)\,d\bar{w}.$$

Обозначим последнее выражение через $\alpha(z,w)$, оно является дифференциальной формой типа (0, 1) по w. Очевидно для нее $\overline{\partial}_w\alpha=0$ и она удовлетворяет оценке

$$\|\alpha\|=\int\limits_{\mathbb{C}_w}|\alpha|^2e^{-\varphi}\,dw\le|a|^2C\int\limits_{|w|\le r}e^{-\varphi}\,dw<M<+\infty,$$

где учли, что $\psi\in C_0^1(\mathbb{R})$, ее носитель содержится в круге радиуса r и в окрестности нуля выполнено (7). Тем самым мы находимся в условиях теоремы Хермандера (см. [2], теорема 4.4.2). Значит, существует функция $u\in L^2(\mathbb{C}_w,\mathrm{loc})$ такая, что

$$\int\limits_{\mathbb{C}_w}|u|^2e^{-\varphi}(1+|w|^2)^{-2}\,dw\le\frac{1}{2}\|\alpha\|^2. \qquad (9)$$

Тогда функция $h_2(z,w)$ будет голоморфна по w, а в силе (9), применяя для оценки неравенство Гелдера, для нее получаем

$$\int\limits_{\mathbb{C}_w}|h_2|^2e^{-\varphi}(1+|w|^2)^{-3}\,dw\le M+2\sqrt{M}\sqrt{\frac{\|\alpha\|}{2}}+\frac{1}{2}\|\alpha\|<+\infty$$

с константой M, независящей от w и z.

Кроме того,

$$\overline{\partial}_z h_2(z,w)=\overline{\partial}_z u(z,w)$$

и так как $\alpha(z,w)\not\equiv 0$ принадлежит классу $L^2(\mathbb{C}_z,\varphi)$, то уравнение $\overline{\partial}_z u(z,w)=0$ тоже имеет решение и $h_2(z,w)$ будет голоморфна и по z. Мы возьмем $F(z,w)=h_2(z,w)$, что и будет искомой функцией в силе теоремы Гартогса (см. [4]). Лемма доказана.

Доказательство теоремы. Пусть $\varepsilon>0$ произвольно, пусть $z^0=(z_1^0,\dots,z_n^0)\in X$. Зафиксируем $z'={z'}^0=\mathrm{const}$. Тогда, по лемме 1, существует субгармоническая по z_n функция v со свойствами а) и б). Применим лемму

2 к функции $\varphi(z', z_n)$, равной $v(z)$. Для нее выполнено (7), значит для $F(z)$ будет выполнено (8), а функция $F(z)/f(z)$, которая, вообще говоря, мероморфна по z_n, будет целой (имеет устранимые особенности). Она по своем выборе не тождественно равна нулю, а при интегрировании по z_n сохраняет аналитичность. Свойство (2) можно проверить при помощи стандартной оценки $|F|$ через ее среднее значение по шару с радиусом r.

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A CONSTRAINT BASED SYSTEM FOR LEXICAL KNOWLEDGE RETRIEVAL*

STOYAN MIHOV

This paper is concerned with the lexical knowledge retrieval system created at the Linguistic Modelling Laboratory. The main goal of the system is to provide a powerful and comfortable interface for lexical knowledge retrieval from large morphological dictionary. To achieve this a constraint based approach is applied that leads to a very effective algorithm. The algorithm for query building, which is also used for retrieving of general grammatical knowledge, is presented in details. In our opinion this method is very suitable for knowledge retrieval in domains with complex and irregular classifications.

Keywords: lexical knowledge retrieval, morphological dictionary, query building

1991/95 Math. Subject Classification: 68T50

1. INTRODUCTION

Recently, many systems containing large amount of lexical knowledge have been built. They make use of different approaches for processing and knowledge representation. Detailed study of the problem is presented in [4] and [1].

The Linguistic Modelling Laboratory is working on a Large Morphological Dictionary that will cover most of the wordforms in modern Bulgarian (see [3]). Now the system contains grammatical information for more than 500 000 wordforms and is systematically upgraded. The grammatical information is structured in Features Structures (refer to [5] for a good introduction). It is known that Feature Structures (FSs) are *de facto* standard for representing linguistic information. That is because

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they allow comfortable description of knowledge with complex classifications and many irregularities.

The program which interfaces the lexical knowledge in our system is called kernel program. We can think the knowledge as a set containing all 500 000 feature structures corresponding to the information of wordforms. The kernel input is a Feature Structure — the query constraint that should be satisfied. The output contains all Feature Structures included in our knowledge base, unifiable with the input. See Fig. 1 for some examples of the kernel functioning. It is clear that it is very inefficient to keep all 500 000 FSs in the memory and to check every one for the query constraint satisfaction. The kernel program uses a synthesizing algorithm. The grammatical information of the output Feature Structures is built by the unification of certain *basic constraints*. Each basic constraint is a Feature Structure which corresponds to a feature-value pair of the grammatical information. In Fig. 2 there are some examples of Feature Structures[1] which present basic constraints corresponding to feature-values.

This paper reveals in details our approach for query building. It is shown how general grammatical knowledge is retrieved by our procedure. In Section 2 we describe the problem for query building. Then in Section 3 the algorithm is presented and explained. In Section 4 some details about the implementation are described. Some comments on the possibilities for generalization are given in the conclusion.

2. PROBLEM DESCRIPTION

The main goal of the system is to provide an appropriate way for knowledge retrieval. There are very different necessities concerning system interaction. For example, in syntactic analysing systems the grammar information should be extracted after the input of wordform, as it is shown in the first example in Fig. 1. In Natural Language Generation the system should find the wordform by processing input of a stem and some grammatical information. The professional linguist should be able to extract all wordforms (or stems, endings...) which satisfy a certain constraint. On the one hand, there should be no limitations on the contents of a query constraint. But, on the other hand, many of the queries are inconsistent with the grammatical knowledge. For example, there are no items in the knowledge base, which satisfy the following FS:

$$\begin{bmatrix} \text{lexeme info} & \begin{bmatrix} \text{gradability} & \text{gradable} \end{bmatrix} \\ \text{grammeme info} & \begin{bmatrix} \text{tense} & \text{imperfect} \end{bmatrix} \end{bmatrix}$$

That is because only adjectives and adverbs are gradable and they do not allow a tense characterization. It is meaningless to process a query which we can a priori

[1] For a clearer presentation, in the example we are noting the unbound anonymous variables explicitly with '_'.

Input FS (Query)	Output FSs	

[wordform 'BOEHHA']	[wordform 'BOEHHA'
	stem 'BOEHEH'
	lexeme info [part of speech adjective
	  gradability ungradable]
	grammeme info [degree positive
	  gender feminine
	  number singular
	  definiteness indefinite]
	ending 'A'
	alternation ⟨[from 'E'
	  to ''
	  pos 2]⟩]	

[stem 'CTABA'
grammeme info [tense present
  person 3]]
	[wordform 'CTABA'
	stem 'CTABA'
	lexeme info [part of speech verb
	  ...]
	grammeme info [tense present
	  person 3
	  number singular
	  ...]
	...]
		[wordform 'CTABAT'
		stem 'CTABA'
		lexeme info [part of speech verb
		  ...]
		grammeme info [tense present
		  person 3
		  number plural
		  ...]
		...]

Fig. 1. Examples of the kernel functioning

Feature-value	Corresponding basic constraint		
part of speech — noun	wordform	_	
	stem	_	
	ending	_	
	lexeme info	part of speech	noun
		noun type	_
		animateness	_
		humanness	_
		gender	_
		verb type	not defined
		gradable	not defined
		transitivity	not defined
		numeral type	not defined
		...	
	grammeme info	number	
		definiteness	_
		article form	_
		tense	not defined
		person	not defined
		case	not defined
		...	
	...		
non-finite form — participle	wordform	_	
	stem	_	
	ending	_	
	lexeme info	part of speech	verb
		verb type	_
		noun type	not defined
		gradable	not defined
		transitivity	not defined
		numeral type	not defined
		...	
	grammeme info	finiteness	non-finite
		non-finite form	participle
		number	_
		tense	_
		definiteness	not defined
		case	not defined
		...	
	...		

Fig. 2. Example of feature-value pairs and the corresponding *basic constraints*

consider as inconsistent. That is why we have to implement a more sofisticated query building algorithm.

We think that the best way is to build the query incrementally. That means that the user should be able to specify the query step by step, selecting a feature-value pair. The purpose is to receive information which feature-value pairs are acceptable (do not lead to an inconsistent query) after each step. If we switch those feature-values as unacceptable, the user would not be able to build an inconsistent query. Moreover, the user will receive general grammatical information about the consistency of some constraint combinations.

Another advantage of the system would be if the algorithm automatically selects some feature-value pairs which are derivable from the existing query specification. For example, if the user selects 'part of speech – verb' and 'gender – masculine', then the 'finiteness' has to be 'non-finite'.

To achieve the above mentioned requirements for the query building procedure, a deduction procedure has to be implemented. This procedure should check on each step the consistence of every feature-value pair with the query and should select the feature-values which are deducible from the specification of the query.

3. THE QUERY BUILDING ALGORITHM

The simplest way to fulfil the above algorithm specification is to generate a list of all possible combinations of feature-values. Unfortunately, there are thousands of possible combinations of grammatical feature-values in the system. That is why an algorithm based on this information would be very inefficient. Our algorithm is based on the *basic constraints* corresponding to feature-values which are only about 150 in the system. Those constraints are already defined in the system, because the kernel program is producing the result via their unification.

The FSs used in our application are of the classic type. That means that they are not sorted and we do not allow negation and disjunction inside the FSs. The generalization of the algorithm in order to use disjunctive FSs is not a serious problem, but if we want to use negation inside the FSs, the algorithm should be generally revised. Using negation, we loose the nice classic semantic about FS's — the interpretation that a FS represents partial knowledge. For a comprehensive study of FS semantics see [2].

Some notion preliminaries: when we write a feature-value pair, in fact we mean a pair of feature path and value, where the feature path is a list of features. In our application we are interested only in features carrying grammatical information (other features like 'stem', 'wordform', etc. could not be classificated). That is why we note only the last feature in the path and the value and call this a feature-value pair. In the application there are about 60 features (feature paths) carrying grammatical information. All feature-value pairs are about 150. This is a rather small number, hence the algorithm based on this information will be comparable effective. Bellow we present our algorithm which satisfies all requirements mentioned above.

Algorithm 1. *The Query Building Algorithm.*

```
Step 1. Set the initial query FS to the empty FS.
        Set all Feature-value pairs to 'acceptable'.
Step 2. Wait the user to select  a ('acceptable') feature-value pair.
Step 3. Unify the query FS with the basic constraint corresponding to
        the selected feature-value pair and mark it as 'sellected'.
Step 4. For every 'acceptable' basic constraint
        check the unifiability with the query
           if the basic constraint does not satisfy the query,
           then set it to 'unacceptable'.
Step 5. For every feature check
           if exactly one value for this feature is acceptable, then
              select this feature-value and go to Step 3.
Step 6. If there are no more acceptable pairs or the user has finished,
           then go to Step 7, else go to Step 2.
Step 7. Call the kernel program with the query as input.
```

This algorithm is almost self-explaining. In Step 1 the initialization is made. Step 2 and Step 3 build the query by unification of the basic constraint corresponding to the selected feature-value pair. Step 4 checks all other basic constraints for consistence with the query and switches all feature-values which are inconsistent with the query to an 'unacceptable' state. Step 5 is responsible for the automatic deduction of feature-values. Step 6 and Step 7 close the loop and invoke the kernel program respectively. We omit a detailed proof about the correctness of this algorithm, which in our point of view, is rather obvious. Maybe the only non trivial problem is the termination. The next lemma is concerned about that.

Lemma 1. Algorithm 1 *is always terminating.*

Proof. The two loops are always passing through Step 3, where a feature-value pair is set to 'selected'. There are only a finite number of pairs. Hence, after a finite number of iterations there will be no more 'acceptable' pairs, which guarantees the termination of the algorithm.

It has to be noted also that in the Algorithm is essentially used the 'Closed World Assumption'. In Step 5 we assume that there are no other values possible for the feature. In our application, where the general grammatical knowledge is fixed, this assumption is generally true. In fact, exactly this step is responsible for the automatic feature-value selection, when the value of a feature can be deduced from the information already specified by the query. Without the 'Closed World Assumption' we could not deduce anything new in our system.

4. IMPLEMENTATION

We have created a prototype version in Sicstus Prolog with Tcl/Tk on Windows environment, which is able to extract about 20 FS per second. The system is supplied with a very friendly user interface. The query is created using the Windows Graphical User Interface. By clicking on a feature in the list box, another list box is displayed, where the acceptable values for this feature are listed. By selecting a value, the unification with the corresponding constraint is invoked. If some feature or value is disabled, then the corresponding entry in the list box will be switched 'gray' (unacceptable). There are several options for the output format. The user can choose between the output of the whole FSs or only the values of some features (e.g. wordform, stem, etc.).

A more faster version will be created using C/C++ language soon. This implementation will provide a retrieval speed of about 200 FS per second. There will be no other differences between the C/C++ and the Prolog version. We hope that this system will be widely used for Bulgarian language education and research purposes.

Also, there is a World Wide Web version planned. The idea is to specify the query using the form options in HTML. Then the query will be passed to the knowledge retrieval system using a CGI-script. In this way the resources will be accessible through INTERNET.

5. CONCLUSION

The most interesting part of the algorithm, in our opinion, is the untraditional deduction procedure. It is clear that the application is very simple. That is why in fact the loop Step 3 – Step 5 will not make new changes to the acceptability of feature-value pairs. We think that this deduction procedure could be classified as a new approach to certain problems. It leads to a very elegant and effective algorithm for deduction in domains with complex classifications. In current version only classic feature structure interpretation and unification in empty theory are applied. At the moment we are working on the generalization of this approach to allow more powerful constraint based technics.

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A MOTION OF A FAST SPINNING RIGID BODY ABOUT A FIXED POINT IN A SINGULAR CASE

ABDEL ISMAIL

In this paper the problem of motion of a rigid body about a fixed point under the action of a Newtonian force field is studied for a singular value of the natural frequency ($\omega = 1/3$). This singularity deals with different bodies being classified according to the moments of inertia. Using Poincaré's small parameter method, the periodic solutions — with non-zero basic amplitudes — of the quasi-linear autonomous system are obtained in the form of power series expansions, up to the third approximation, containing assumed small parameter. Also, the quasi-linear autonomous system is integrated numerically using any of the numerical integration methods, such as the fourth order Runge - Kutta method. At the end, a comparison between the analytical and the numerical solutions is given aiming to get a small deviation between them.

Keywords: rigid body motion, small parameter method, periodic solutions

1991/95 Math. Subject Classification: 70E05

1. INTRODUCTION

In [1] the motion of a fast spinning rigid body about a fixed point in a central Newtonian force field is considered. The nonlinear differential equations of motion have been reduced to a quasi-linear autonomous system having one first integral. In the case of the rational value of the natural frequency ω (except $\omega = 1/2,\ 1,\ 2,\ 1/3,\ 3$) the periodic solutions of the initial system are obtained. Here, the analytical and the numerical solutions for the case when $\omega = 1/3$ are constructed. Let us consider a rigid body of mass (M) with one fixed point (O), whose ellipsoid of inertia is arbitrary, and acted upon by a central Newtonian force field arising from attracting centre being located on a vertical downwards axis (Z) passing through the fixed point. Let us assume ($OXYZ$) to be the fixed frame in space and ($Oxyz$) to be the moving frame (fixed of the body). It is taken into consideration that the principal axis (z) of the ellipsoid of inertia makes an angle $\theta_o \neq m\pi/2$ ($m = 0, 1, 2, \ldots$) with Z-axis and that the body spins about z-axis with a high angular velocity r_o. Without a loss of generality we select the positive branches of the z-axis and of the x-axis do not make an obtuse angle with the direction of the Z-axis. According to the restriction on θ_o and the selection of the co-ordinate system one gets

$$\gamma_o \geq 0, \quad 0 < \gamma_o'' < 1,$$

the limiting case $\gamma_o'' \approx 0$ has been studied in [2]. The following system of equations of motion and its first integral can be deduced:

$$\ddot{p}_2 + \frac{1}{9}p_2 = \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \qquad \ddot{\gamma}_2 + \gamma_2 = \mu^2 \Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \tag{1}$$

$$\gamma_o''^{-2} - 1 = \gamma_2^2 + \dot{\gamma}_2^2 + 2\mu(\nu p_2 \gamma_2 + \nu_2 \dot{p}_2 \dot{\gamma}_2 + s_{21}) + \mu^2 \Big[\nu_2^2 \dot{p}_2^2 - 2\dot{\gamma}_2 (e_2 A_1^{-1} \dot{\gamma}_2$$
$$+ A_1^{-1} \dot{p}_2 s_{21} + \frac{1}{2}\gamma_2 s_{11} - y_o' a^{-1} A_1^{-1}) + \nu^2 p_2^2 + s_{21}^2 + 2\Big(s_{22} - \frac{1}{2}s_{11}\Big)\Big] + \mu^3(\ldots), \tag{2}$$

where

$$\begin{aligned}
F &= F_2 + \mu F_3 + \cdots, & \Phi &= \Phi_2 + \mu\Phi_3 + \cdots,\\
F_2 &= f_2 - \frac{8}{9}\nu e_1 p_2, & \Phi_2 &= \phi_2 + \frac{8}{9}\nu(e + e_1\gamma_2),\\
F_3 &= f_3 - e_1\phi_2 - \frac{8}{9}\nu e_1(e + e_1\gamma_2), & \Phi_3 &= \phi_3 - \nu f_2 + \frac{8}{9}\nu^2 e_1 p_2,
\end{aligned}$$

$$\begin{aligned}
f_2 &= A_1 b^{-1} x_o' s_{21} - \frac{1}{9}p_2 s_{11} + C_1 A_1^{-1} p_2 \dot{p}_2^2 - y_o' a^{-1} p_2 \dot{\gamma}_2\\
&\quad + x_o' \dot{p}_2 \dot{\gamma}_2 - z_o' a^{-1} p_2 - y_o' A_1^{-1}(A_1 + a^{-1})\gamma_2 \dot{p}_2\\
&\quad - k[(1 - C_1)\gamma_2 \dot{p}_2 \dot{\gamma}_2 + A_1(1 + B_1)\gamma_2 s_{21} - A_1 p_2(1 - \dot{\gamma}_2^2)],\\
\phi_2 &= -\gamma_2 s_{11} + (1 + B_1)p_2 s_{21} - (1 - C_1)A_1^{-1} p_2 \dot{p}_2 \dot{\gamma}_2 + x_o' \dot{\gamma}_2{}^2 - y_o' \gamma_2 \dot{\gamma}_2\\
&\quad - z_o' b^{-1}\gamma_2 + x_o' b^{-1} - A_1^{-2}\gamma_2 \dot{p}_2^2 + k(C_1 \dot{\gamma}_2^2 - B_1)\gamma_2,\\
f_3 &= C_1 A_1^{-1} \dot{p}_2[e\dot{p}_2 + e_1\gamma_2 \dot{p}_2 - 2p_2(y_o' a^{-1} - e_2\dot{\gamma}_2)] - \frac{1}{9}(e s_{11} + e_1 \gamma_2 s_{11} + 2p_2 s_{12})\\
&\quad + A_1 b^{-1} x_o' s_{22} + x_o'[\nu_2 \dot{p}_2^2 - \dot{\gamma}_2(y_o' a^{-1} - e_2 \dot{\gamma}_2)] - y_o' a^{-1}[\dot{\gamma}_2(e + e_1\gamma_2) + \nu_2 p_2 \dot{p}_2]\\
&\quad + y_o'(1 + A_1^{-1} a^{-1})[\gamma_2(y_o' a^{-1} - e_2\dot{\gamma}_2) - \nu p_2 \dot{p}_2] + \frac{1}{2} z_o'(a^{-1} - A_1 b^{-1})\gamma_2 s_{11}\\
&\quad - z_o' a^{-1}(e + e_1\gamma_2 + p_2 s_{21}) + k\Big[(1 - C_1)(y_o' a^{-1} - e_2\dot{\gamma}_2)\gamma_2\dot{\gamma}_2\\
&\quad - \nu(1 - C_1)p_2\dot{p}_2\dot{\gamma}_2 - 2\nu_2 A_1 p_2 \dot{p}_2 \dot{\gamma}_2 - \nu_2(1 - C_1)\gamma_2 \dot{p}_2^2 - \nu A_1(1 + B_1)p_2 s_{21}\\
&\quad + 2A_1 p_2 s_{21} + \Big(\frac{1}{9} - A_1\Big)\gamma_2 s_{22} + A_1(e + e_1\gamma_2)(1 - \dot{\gamma}_2^2)\Big],
\end{aligned}$$

$$\begin{aligned}
\phi_3 \;=\;& 2x'_o\nu_2\dot{p}_2\dot{\gamma}_2 - 2\gamma_2 s_{12} - \nu p_2 s_{11} + (1+B_1)[p_2 s_{22} + (e+e_1\gamma_2)s_{21}] \\
&+(1-C_1)A_1^{-1}[p_2\dot{\gamma}_2(y'_o a^{-1} - e_2\dot{\gamma}_2) - \nu_2 p_2\dot{p}_2^2 - (e+e_1\gamma_2)\dot{p}_2\dot{\gamma}_2] \\
&-z'_o b^{-1}(\nu p_2 + \gamma_2 s_{21}) + 2x'_o b^{-1}s_{21} + A_1^{-2}[2\gamma_2\dot{p}_2(y'_o a^{-1} - e_2\dot{\gamma}_2) - \nu p_2\dot{p}_2^2] \\
&-y'_o(\nu p_2\dot{\gamma}_2 + \nu_2\gamma_2\dot{p}_2) + k[\nu p_2(C_1\dot{\gamma}_2^2 - B_1) + 2\gamma_2(\nu_2 C_1\dot{p}_2\dot{\gamma}_2 - B_1 s_{21})]; \qquad (3)
\end{aligned}$$

$$\begin{aligned}
p_2 \;=\;& p_1 - \mu e - \mu e_1\gamma_2, \qquad\qquad \gamma_2 = \gamma_1 - \mu\nu p_2, \\
q_1 \;=\;& -A_1^{-1}\dot{p}_2 + \mu A_1^{-1}(y'_o a^{-1} - e_2\dot{\gamma}_2) + \mu^2\Big[(aA_1)^{-1}y'_o s_{21} + \frac{1}{2}A_1^{-1}\dot{p}_2 s_{11} + k\dot{\gamma}_2 s_{21} \\
&-\nu_2\dot{p}_2(a^{-1}A_1^{-1}z'_o - k)\Big] + \mu^3\Big[(aA_1)^{-1}y'_o s_{22} + \frac{1}{2}A_1^{-1}e_1\dot{\gamma}_2 s_{11} + A_1^{-1}\dot{p}_2 s_{12} \\
&+a^{-1}A_1^{-2}e_1 z'_o\dot{\gamma}_2 + a^{-1}A_1^{-1}s_{11}(z'_o\dot{\gamma}_2 - y'_o) + (k - a^{-1}A_1^{-1}z'_o)(a^{-1}A_1^{-1}y'_o \\
&-a^{-1}A_1^{-1}z'_o\dot{\gamma}_2 + k\dot{\gamma}_2) + a^{-1}A_1^{-2}z'_o\dot{p}_2 s_{21} + k(\nu\dot{p}_2 s_{21} + \dot{\gamma}_2 s_{22} \\
&-A_1^{-1}e_1\dot{\gamma}_2 - \frac{3}{2}\dot{\gamma}_2 s_{11} - 2A_1^{-1}\dot{p}_2 s_{21})\Big] + \cdots, \\
r_1 \;=\;& 1 + \frac{1}{2}\mu^2 s_{11} + \mu^3 s_{12} + \cdots, \\
\gamma'_1 \;=\;& \dot{\gamma}_2 + \mu\nu_2\dot{p}_2 + \mu^2\left[(aA_1)^{-1}y'_o - A_1^{-1}(e_2\dot{\gamma}_2 + \dot{p}_2 s_{21}) - \frac{1}{2}\dot{\gamma}_2 s_{11}\right] \\
&+\mu^3\Big[-A_1^{-1}(e_1\dot{\gamma}_2 s_{21} + \dot{p}_2 s_{22}) + \frac{1}{2}(3A_1^{-1} - \nu)\dot{p}_2 s_{11} - \dot{\gamma}_2 s_{12} \\
&+\nu_2(k - a^{-1}A_1^{-1}z'_o)\dot{p}_2 + 2a^{-1}A_1^{-1}y'_o s_{21} + (2k - a^{-1}A_1^{-1}z'_o)\dot{\gamma}_2 s_{21}\Big] + \cdots, \\
\gamma''_1 \;=\;& 1 + \mu s_{21} + \mu^2\Big(s_{22} - \frac{1}{2}s_{11}\Big) - \mu^3\Big(s_{12} + \frac{1}{2}s_{11}s_{21}\Big) + \cdots; \qquad (4)
\end{aligned}$$

$$\begin{aligned}
p_1 \;=\;& p/c\sqrt{\gamma''_o}, \qquad q_1 = q/c\sqrt{\gamma''_o}, \qquad r_1 = r/r_o, \qquad \gamma_1 = \gamma/\gamma''_o, \\
\gamma'_1 \;=\;& \gamma'/\gamma''_o, \qquad \gamma''_1 = \gamma''/\gamma''_o, \qquad \tau = r_o t, \qquad (\,\dot{}\, \equiv d/d\tau); \qquad (5)
\end{aligned}$$

$$\begin{aligned}
s_{11} \;=\;& a(p_{2o}^2 - p_2^2) + b(\dot{p}_{2o}^2 - \dot{p}_2^2)/A_1^2 - 2[x'_o(\gamma_{2o} - \gamma_2) + y'_o(\dot{\gamma}_{2o} - \dot{\gamma}_2)] \\
&+k[a(\gamma_{2o}^2 - \gamma_2^2) + b(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)], \\
s_{12} \;=\;& a[e(p_{2o} - p_2) + e_1(p_{2o}\gamma_{2o} - p_2\gamma_2)] - bA_1^{-2}[y'_o a^{-1}(\dot{p}_{2o} - \dot{p}_2) \\
&-e_2(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)] - \nu x'_o(P_{2o} - p_2) - \nu_2 y'_o(\dot{p}_{2o} - \dot{p}_2) + (z'_o - k)s_{21} \\
&+k[\nu a(p_{2o}\gamma_{2o} - p_2\gamma_2) + \nu_2 b(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2)], \\
s_{21} \;=\;& a(p_{2o}\gamma_{2o} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{2o}\dot{\gamma}_{2o} - \dot{p}_2\dot{\gamma}_2), \\
s_{22} \;=\;& a[\nu(p_{2o}^2 - p_2^2) + e(\gamma_{2o} - \gamma_2) + e_1(\gamma_{2o}^2 - \gamma_2^2)] + bA_1^{-1}[-\nu_2(\dot{p}_{2o}^2 - \dot{p}_2^2) \\
&+a^{-1}y'_o(\dot{\gamma}_{2o} - \dot{\gamma}_2) - e_2(\dot{\gamma}_{2o}^2 - \dot{\gamma}_2^2)]; \qquad (6)
\end{aligned}$$

$$
\begin{aligned}
A_1 &= \frac{C-B}{A}, \quad B_1 = \frac{A-C}{B}, \quad C_1 = \frac{B-A}{C}, \quad a = \frac{A}{C}, \quad b = \frac{B}{C}, \\
c^2 &= \frac{Mg\ell}{C}, \quad \mu = \frac{c\sqrt{\gamma_o''}}{r_o}, \quad x_o = \ell x_o', \quad y_o = \ell y_o', \quad z_o = \ell z_o', \\
\ell^2 &= x_o^2 + y_o^2 + z_o^2, \quad A_1 B_1 = -\frac{1}{9}, \quad e = 9x_o' A_1 b^{-1}, \quad \nu = \frac{9}{8}(1+B_1), \\
e_1 &= \frac{9}{8}\left[k\left(A_1 - \frac{1}{9}\right) + z_o'(A_1 b^{-1} - a^{-1})\right], \quad e_2 = e_1 + a^{-1} z_o' - kA_1, \\
\nu_2 &= \nu - A_1^{-1}, \quad k = N\gamma_o''/c^2, \quad N = 3g/R, \quad g = \lambda/R^2; \qquad (7)
\end{aligned}
$$

here λ is the constant of gravity of the attracting centre, R is the distance from the fixed point to such centre, (p_o, q_o, r_o) and $(\gamma_o, \gamma_o', \gamma_o'')$ are the initial values of projections of the angular velocity vector (p, q, r) of the body on the principal axes of inertia and the direction cosines $(\gamma, \gamma', \gamma'')$ of Z-axis, respectively, A, B and C are the principal moments of inertia and x_o, y_o and z_o are the co-ordinates of the centre of mass in the moving co-ordinate system.

2. PROPOSED METHOD

In this section Poincaré's small parameter method is applied to investigate the non-zero basic amplitude periodic solutions of system (1). The generating system $(\mu = 0)$ of (1) is

$$
\ddot{p}_2^{(0)} + \frac{1}{9} p_2^{(0)} = 0, \qquad \ddot{\gamma}_2^{(0)} + \gamma_2^{(0)} = 0, \qquad (8)
$$

which admits periodic solutions in the forms

$$
p_2^{(0)} = M_1 \cos\frac{1}{3}\tau + M_2 \sin\frac{1}{3}\tau, \qquad \gamma_2^{(0)} = M_3 \cos\tau \qquad (9)
$$

with period $T_o = 6\pi$, and M_1, M_2 and M_3 are constants which have to be determined. Since the system (1) is autonomous, the condition

$$
\dot{\gamma}_2(0, \mu) = 0 \qquad (10)
$$

does not restrict the generality of the required solutions [3].

Applying Poincaré's method, the periodic solutions for system (1) are considered in the forms [4]

$$
\begin{aligned}
p_2(\tau, \mu) &= \tilde{M}_1 \cos\frac{1}{3}\tau + \tilde{M}_2 \sin\frac{1}{3}\tau + \sum_{k=2}^{\infty} \mu^k G_k(\tau), \\
\gamma_2(\tau, \mu) &= \tilde{M}_3 \cos\tau + \sum_{k=2}^{\infty} \mu^k H_k(\tau) \qquad (11)
\end{aligned}
$$

with period $T = 6\pi + \alpha(\mu)$ and initial conditions

$$
p_2(0, \mu) = \tilde{M}_1, \quad \dot{p}_2(0, \mu) = \frac{1}{3}\tilde{M}_2, \quad \gamma_2(0, \mu) = \tilde{M}_3, \quad \dot{\gamma}_2(0, \mu) = 0, \qquad (12)
$$

where $\alpha(\mu) = 0$ at $\mu = 0$ and "$\sim$" denotes the result of substitution

$$M_i \to \tilde{M}_i = M_i + \beta_i \quad (i = 1, 2, 3), \tag{13}$$

here β_1, $\frac{1}{3}\beta_2$ and β_3 denote the deviations of the initial values of p_2, $\dot{p}_2$ and γ_2 of system (1) from their initial values of the generating system (8), these deviations are functions of μ and equal zero when $\mu = 0$. Let us define the functions $G_k(\tau)$ and $H_k(\tau)$ by the operator [5]

$$U = u + \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 + \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \cdots \quad \left\{ \begin{array}{c} U = G_k, H_k \\ u = g_k, h_k \end{array} \right\}. \tag{14}$$

The functions $g_k(\tau)$ and $h_k(\tau)$ take the forms

$$\begin{aligned} g_k(\tau) &= 3\int_0^\tau F_k^{(0)}(t_1)\sin\frac{1}{3}(\tau - t_1)\,dt_1, \\ h_k(\tau) &= \int_0^\tau \Phi_k^{(0)}(t_1)\sin(\tau - t_1)\,dt_1, \quad k = 2, 3. \end{aligned} \tag{15}$$

The solutions (9) are written as follows:

$$p_2^{(0)} = E\cos\left(\frac{1}{3}\tau - \epsilon\right), \qquad \gamma_2^{(0)} = M_3\cos\tau, \tag{16}$$

where $E = \sqrt{M_1^2 + M_2^2}$, $M_1 = E\cos\epsilon$ and $M_2 = E\sin\epsilon$.

Substituting (16) into (6), one obtains

$$s_{ij}^{(0)} = s_{ij}^{(0)}\left(p_2^{(0)}, \dot{p}_2^{(0)}, \gamma_2^{(0)}, \dot{\gamma}_2^{(o)}\right), \quad i, j = 1, 2. \tag{17}$$

Making use of (16), (17) and (3), the functions $F_k^{(0)}$ and $\Phi_k^{(0)}$ are obtained, then using (15), one gets $g_k(6\pi)$, $h_k(6\pi)$, $\dot{g}_k(6\pi)$ and $\dot{h}_k(6\pi)$. Substituting the initial conditions (12) into the integral (2), evaluated at $\tau = 0$, the quantity $\tilde{M}_3$ is determined as follows:

$$\tilde{M}_3 = (\gamma_o'')^{-1}(1-\gamma_o''^2)^{1/2} - \mu a\tilde{M}_1 - 9\mu^2\nu_2^2\tilde{M}_2^2/2M_3 - 3\mu^3 y_o'\nu_2\tilde{M}_2/aA_1M_3 + \cdots, \tag{18}$$

where $\nu_2 = 9b/(9 - 8b)$ and $0 < b < 1$ or $b > 9/8$. The independent conditions [6] for periodicity of the solutions $p_2(\tau,\mu)$, $\dot{p}_2(\tau,\mu)$, $\gamma_2(\tau,\mu)$ and $\dot{\gamma}_2(\tau,\mu)$ are reduced to the forms

$$\begin{aligned} \left(\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21}\right)\tilde{M}_2 &= -\mu\tilde{M}_2\Big\{\left(\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31}\right) - \tilde{M}_1\Big[2\tilde{L}_{34} \\ &\quad + \frac{1}{9}N_{33}\tilde{M}_3^{-1}(\tilde{M}_1^2 - 3\tilde{M}_2^2)\Big]\Big\} + \cdots, \\ \left(\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21}\right)\tilde{M}_1 &= -\mu\Big\{\tilde{M}_1\left(\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31}\right) + \Big[\tilde{L}_{34}(\tilde{M}_1^2 - \tilde{M}_2^2) \\ &\quad - \frac{1}{9}N_{33}\tilde{M}_1^2(\tilde{M}_1^2 - 3\tilde{M}_2^2)\tilde{M}_3^{-1}\Big]\Big\} + \cdots; \end{aligned} \tag{19}$$

$$\alpha(\mu) = \mu^2 \tilde{M}_3^{-1}[\dot{H}_2(6\pi) + \mu \dot{H}_3(6\pi) + \cdots], \tag{20}$$

where

$$\begin{aligned}
\tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21} &= a_1(\tilde{M}_1^2 + \tilde{M}_2^2) - [a_2 + ka_3(2M_3\beta_3 + \beta_3^2)], \\
\tilde{L}_{31} - \frac{1}{9}\tilde{N}_{31} &= a_4 \tilde{M}_1 \tilde{M}_3, \\
a_1 &= (a-1)(a+b-2)/2b, \\
a_2 &= z_o'(ab)^{-1}[3(a+b) - 2(2ab+1)] + \frac{2}{9}k\left[1 - (a+b) + \frac{1}{2}bM_3^2\right], \\
a_3 &= \frac{1}{9}b, \qquad a_4 = \frac{1}{9}[20a - 9b^{-1} + ak(b-1)(32a-41)], \\
\tilde{L}_{34} &= -\frac{14}{3}\tilde{M}_3 z_o' b^{-1}(b-1)^{-1}(8b-9)^{-1}\left(b - \frac{3}{2}\right)\left(b - \frac{3}{4}\right)\left(b - \frac{33}{28}\right) \\
&\quad -\frac{1}{2}k\tilde{M}_3(b-1)(8b-9)^{-1}\Big\{(2b-3)[(9-4b) + (8b-15)(8b-9)^{-1}] \\
&\quad +\frac{1}{3}(4b-3)(4b-5)(9-7b)(b-1)^{-1}(8b-9)^{-1}\Big\}, \\
N_{33} &= \frac{1}{4}\Big\{\frac{1}{9}A_1^{-1}[\nu(A_1^{-1} + C_1) + \nu_2(1 - C_1)] - \frac{8}{9}\nu\left(\frac{1}{9}bA_1^{-2} - a\right) \\
&\quad -(1 + B_1)\left(\nu a + \frac{1}{9}bA_1^{-1}\nu_2\right)\Big\}.
\end{aligned} \tag{21}$$

Equating to zero the terms of zero power of μ for equations (19), one gets two equations for determining M_1 and M_2. Solving the resulting equations, when $M_1 M_2 = 0$, we obtain

$$\begin{aligned}
\text{(i)}\ M_1 &= M_2 = 0, \\
\text{(ii)}\ M_1 &= 0, \qquad M_2 = \pm\sqrt{\frac{a_2}{a_1}}, \\
\text{(iii)}\ M_1 &= \pm\sqrt{\frac{a_2}{a_1}}, \qquad M_2 = 0.
\end{aligned} \tag{22}$$

If $M_1 M_2 \neq 0$, subtracting from the first equation of (19), multiplied by $\tilde{M}_1$, the second equation, multiplied by $\tilde{M}_2$, and dividing by μ, we get a new form for the periodicity conditions

$$3\tilde{M}_1^2 - \tilde{M}_2^2 + \mu[\cdots] = 0, \qquad \tilde{L}_{21} - \frac{1}{9}\tilde{N}_{21} + \mu[\cdots] = 0. \tag{23}$$

The equations of the basic amplitudes of (23) are

$$3M_1^2 - M_2^2 = 0, \qquad a_1(M_1^2 + M_2^2) - a_2 = 0, \tag{24}$$

the following solutions for M_1 and M_2 are obtained:

$$M_1 = \pm\frac{1}{2}\sqrt{\frac{a_2}{a_1}}, \qquad M_2 = \pm\frac{\sqrt{3}}{2}\sqrt{\frac{a_2}{a_1}}, \tag{25}$$

where M_1 and M_2 are real under the condition

$$a_2 > 0; \tag{26}$$

this condition can be satisfied by choice of M_3, while $a_1 > 0$ is satisfied at all, since the initial fast spin r_o is assumed to be given about the major or the minor axis of the ellipsoid of inertia ($a > 1$, $b > 1$ or $a < 1$, $b < 1$). For this case β_1 and β_2 are assumed in the forms

$$\beta_1 = \sum_{k=1}^{3} \mu^k \ell_k + O(\mu^4), \qquad \beta_2 = \sum_{k=1}^{3} \mu^k m_k + O(\mu^4). \tag{27}$$

Considering (27), (23) and the substitution (13), one gets

$$\begin{aligned}
\ell_1 &= -aa_3kM_3/4a_1, \qquad m_1 = 3\ell_1 M_1/M_2, \\
\ell_2 &= \frac{1}{8}M_1^{-1}\left[-4\ell_1^2 + ka_3a_1^{-1}(a^2M_1^2 - 9M_2^2\nu_2^2)\right], \\
m_2 &= \frac{1}{2}M_2^{-1}\left(6M_1\ell_2 + 3\ell_1^2 - m_1^2\right), \\
\ell_3 &= \frac{1}{4}M_1^{-1}\Big[-4\ell_1\ell_2 + aa_1^{-1}a_3M_1k\Big(a\ell_1 + \frac{9}{2}M_3^{-1}\nu_2^2M_2^2\Big) \\
&\quad -a_1^{-1}a_3k(aM_3\ell_2 + 9M_2\nu_2^2m_1 + 3M_2y_o'\nu_2a^{-1}A_1^{-1})\Big], \\
m_3 &= M_2^{-1}\Big[aa_1^{-1}a_3M_1k\Big(a\ell_1 + \frac{9}{2}\nu_2^2M_2^2M_3^{-1}\Big) - a_1^{-1}a_3k(a\ell_2M_3 + 9\nu_2^2m_1M_2 \\
&\quad +3y_o'\nu_2M_2a^{-1}A_1^{-1}) - m_1m_2 - \ell_1\ell_2 - M_1\ell_3\Big].
\end{aligned} \tag{28}$$

The equations (14) and (15) give the functions $G_k(\tau)$ and $H_k(\tau)$, then the periodic solutions (11) are obtained up to the third approximation of μ. Making use of (4) and (5), we get the required periodic solutions as follows:

$$\begin{aligned}
p &= c\sqrt{\gamma_o''}\Big\{M_1\cos\frac{1}{3}\tau + M_2\sin\frac{1}{3}\tau + \mu\Big(e + \ell_1\cos\frac{1}{3}\tau + m_1\sin\frac{1}{3}\tau + e_1M_3\cos\tau\Big) \\
&\quad +\mu^2\sum_{i=0}^{7}\Big(X_{1i}\cos\frac{i}{3}\tau + X_{1i}'\sin\frac{i}{3}\tau\Big) \\
&\quad +\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{1j}\cos\frac{j}{3}\tau + Y_{1j}'\sin\frac{j}{3}\tau\Big) + Y_{19}\cos 3\tau\Big] + \cdots\Big\}, \quad i \neq 6, \\
q &= c\sqrt{\gamma_o''}\Big\{\frac{1}{3}A_1^{-1}\Big(M_1\sin\frac{1}{3}\tau - M_2\cos\frac{1}{3}\tau\Big) + \mu A_1^{-1}\Big[y_o'a^{-1} + e_2M_3\sin\tau \\
&\quad -\frac{1}{3}\Big(\ell_1\sin\frac{1}{3}\tau + m_1\cos\frac{1}{3}\tau\Big)\Big] + \mu^2\sum_{i=0}^{7}\Big(X_{2i}\cos\frac{i}{3}\tau + X_{2i}'\sin\frac{i}{3}\tau\Big)
\end{aligned}$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{2j}\cos\frac{j}{3}\tau+Y'_{2j}\sin\frac{j}{3}\tau\Big)+Y'_{29}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 6,$$

$$r = r_o\Big\{1+\frac{1}{2}E^2\mu^2\Big[\sum_{i=0}^{3}\Big(X_{3i}\cos\frac{i}{3}\tau+X'_{3i}\sin\frac{i}{3}\tau\Big)+X_{36}\cos 2\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{4}\Big(Y_{3j}\cos\frac{j}{3}\tau+Y'_{3j}\sin\frac{j}{3}\tau\Big)+Y_{36}\cos 2\tau\Big]+\cdots\Big\},\quad i\neq 1,$$

$$\gamma = \gamma''_o\Big\{M_3\cos\tau+\mu a\Big[\Big(M_1\cos\frac{1}{3}\tau+M_2\sin\frac{1}{3}\tau\Big)-M_1\cos\tau\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{6}\Big(X_{4i}\cos\frac{i}{3}\tau+X'_{4i}\sin\frac{i}{3}\tau\Big)+X_{49}\cos 3\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{4j}\cos\frac{j}{3}\tau+Y'_{4j}\sin\frac{j}{3}\tau\Big)+Y'_{49}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 2,4,$$

$$\gamma' = \gamma''_o\Big\{-M_3\sin\tau+\mu\Big[-\frac{1}{2}\nu_2\Big(M_1\sin\frac{1}{3}\tau-M_2\cos\frac{1}{3}\tau\Big)+aM_1\sin\tau\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{6}\Big(X_{5i}\cos\frac{i}{3}\tau+X'_{5i}\sin\frac{i}{3}\tau\Big)+X'_{59}\sin 3\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{7}\Big(Y_{5j}\cos\frac{j}{3}\tau+Y'_{5j}\sin\frac{j}{3}\tau\Big)+Y'_{59}\sin 3\tau\Big]+\cdots\Big\},\quad i\neq 2,4,$$

$$\gamma'' = \gamma''_o\Big\{1+\mu M_3 E\Big[a\cos\epsilon+\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}-a\Big)\Big(\cos\epsilon\cos\frac{2}{3}\tau-\sin\epsilon\sin\frac{2}{3}\tau\Big)$$

$$-\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}+a\Big)\Big(\cos\epsilon\cos\frac{4}{3}\tau+\sin\epsilon\sin\frac{4}{3}\tau\Big)\Big]$$

$$+\mu^2\Big[\sum_{i=0}^{4}\Big(X_{6i}\cos\frac{i}{3}\tau+X'_{6i}\sin\frac{i}{3}\tau\Big)+X_{66}\cos 2\tau\Big]$$

$$+\mu^3\Big[\sum_{j=0}^{10}\Big(Y_{6j}\cos\frac{j}{3}\tau+Y'_{6j}\sin\frac{j}{3}\tau\Big)\Big]+\cdots\Big\},\quad i\neq 1,\ j\neq 9, \tag{29}$$

the correction of the period $\alpha(\mu)$ becomes

$$\alpha(\mu) = 3\pi\mu^2\Big\{-\Big(aM_1^2+\frac{bM_2^2}{9A_1^2}\Big)-\frac{1}{2}(M_1^2+M_2^2)\Big[\frac{1}{9}A_1^{-2}(1-b)+aB_1\Big]$$

$$+2M_3x'_o+\frac{8}{9}ae_1+k(M_3^2C_1-B_1)-\Big(aM_1\ell_1+\frac{1}{9}bA_1^{-2}M_2m_1\Big)$$

$$-(M_1\ell_1+m_1M_2)\Big[\frac{1}{9}A_1^{-2}(1-b)+aB_1\Big]-2z'_ob^{-1}+\frac{8}{9}ae_1$$

$$
\begin{aligned}
&-2aM_1e + \frac{2}{3}M_2y'_o(\nu_2 + ba^{-1}A_1^{-2}) + aM_1M_3[e_1(1+B_1) - z'_ob^{-1}] \\
&-2aM_1M_3(e_1 + z'_o + ak - k) + akM_1M_3[aA_1(1+B_1) - 2B_1 - 2C_1] \\
&-kB_1 + \frac{1}{36}M_1M_3^{-1}(M_1^2 - 3M_2^2)\Big[aA_1^{-1}(A_1^{-1} + C_1) - 8a\Big(\frac{1}{9}bA_1^{-2} - a\Big) \\
&+A_1^{-1}\nu_2(1 - C_1) - (1+B_1)(9a^2 + bA_1^{-1}\nu_2)\Big]\Bigg\} + \cdots, \qquad (30)
\end{aligned}
$$

where the constants X's, X''s, Y's and Y''s are determined in terms of the rigid body motion parameters and are written in about twenty pages. The symbol $(\cdots)$ means terms of order higher than $O(\mu^3)$.

3. GEOMETRIC INTERPRETATION OF MOTION

In this section the motion of the rigid body is investigated by introducing Euler's angles θ, ψ and ϕ, which can be determined through the obtained periodic solutions. Since the initial system is autonomous, then the periodic solutions still remain such if (t) is replaced by $(t + t_o)$, where (t_o) is an arbitrary interval of time. Euler's angles, in terms of time (t), take the forms [7]

$$
\begin{aligned}
&\theta = \cos^{-1}\gamma'', \quad \dot{\psi} = \frac{p\gamma + q\gamma'}{1 - \gamma''^2}, \quad \dot{\phi} = r - \dot{\psi}\cos\theta, \\
&\phi_o = \tan^{-1}\frac{\gamma_o}{\gamma'_o}, \quad (\dot{u} \equiv \frac{du}{dt}). \qquad (31)
\end{aligned}
$$

Assuming the initial instant of time corresponds to the instant $t = t_o$, substituting the solutions (29) into the equations (31), one gets

$$
\begin{aligned}
\phi_o &= \frac{\pi}{2} + r_ot_o + \cdots, \qquad \theta_o = \tan^{-1}M_3, \\
\theta &= \theta_o - \mu E[\theta_1(t+t_o) - \theta_1(t_o)] - \mu^2\cot\theta_o[\theta_2(t+t_o) - \theta_2(t_o)] \\
&\quad -\mu^3[\theta_3(t+t_o) - \theta_3(t_o)] + \cdots, \\
\psi &= \psi_o + Mg\ell C^{-1}r_o^{-1}\operatorname{cosec}\theta_o\Big(\frac{1}{2}e_1 + v_o\cot^2\theta_o\Big)t \\
&\quad +\frac{1}{2}\mu r_o\operatorname{cosec}\theta_o[\psi_1(t+t_o) - \psi_1(t_o)] \\
&\quad +\mu^2r_o\cot\theta_o\operatorname{cosec}^2\theta_o[\psi_2(t+t_o) - \psi_2(t_o)] + \cdots, \\
\phi &= \phi_o + \Big\{r_o - \frac{1}{2}e_1MglC^{-1}r_o^{-1}(\cot\theta_o + aEr_o^{-1}c\cos\epsilon\sqrt{\cos\theta_o}) \\
&\quad +\frac{1}{2}Mg\ell C^{-1}r_o^{-1}E^2X_{3o}\cos\theta_o - v_o^*\sqrt{\cos\theta_o}[r_o + acE\cos\epsilon\tan\theta_o\sqrt{\cos\theta_o} \\
&\quad +Mg\ell C^{-1}r_o^{-1}X_{6o}\cos\theta_o] - \frac{1}{6}EMg\ell C^{-1}\sqrt{\cos\theta_o}\Big[\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1} - a\Big)(X'_{72}\cos\epsilon \\
&\quad +X_{72}\sin\epsilon) - \Big(\frac{1}{3}bA_1^{-1} + a\Big)(X'_{74}\cos\epsilon - X_{74}\sin\epsilon)\Big]\Big\}t
\end{aligned}
$$

$$+\frac{1}{2}\mu\sqrt{\cos\theta_o}[\phi_1(t+t_o)-\phi_1(t_o)]+\mu^2[\phi_2(t+t_o)-\phi_2(t_o)]+\cdots, \tag{32}$$

where

$$\begin{aligned}
\theta_1(t) &= a\cos\epsilon+\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}-a\Big)\Big(\cos\epsilon\cos\frac{2}{3}r_ot-\sin\epsilon\sin\frac{2}{3}r_ot\Big)\\
&\quad -\frac{1}{2}\Big(\frac{1}{3}bA_1^{-1}+a\Big)\Big(\cos\epsilon\cos\frac{4}{3}r_ot+\sin\epsilon\sin\frac{4}{3}r_ot\Big),\\
\theta_2(t) &= \sum_{i=0}^{4}\Big(X_{6i}\cos\frac{i}{3}r_ot+X'_{6i}\sin\frac{i}{3}r_ot\Big)+X_{66}\cos 2r_ot, \quad i\neq 1,\\
\theta_3(t) &= \sum_{i=0}^{10}\Big(Y_{6i}\cos\frac{i}{3}r_ot+Y'_{6i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 9,\\
\psi_1(t) &= \sum_{i=2}^{4}\Big(X_{7i}\cos\frac{i}{3}r_ot+X'_{7i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 3,\\
\psi_2(t) &= \sum_{i=2}^{8}\Big(Y_{7i}\cos\frac{i}{3}r_ot+Y'_{7i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 5,7,\\
\phi_1(t) &= \sum_{i=2}^{4}\Big(X_{8i}\cos\frac{i}{3}r_ot+X'_{8i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 3,\\
\phi_2(t) &= \sum_{i=2}^{8}\Big(Y_{8i}\cos\frac{i}{3}r_ot+Y'_{8i}\sin\frac{i}{3}r_ot\Big), \quad i\neq 5,7,\\
v_o &= -\frac{1}{2}A_1^{-1}\Big[e\tan^2\theta_o+\frac{1}{6}\nu_2(M_1^2+M_2^2)\\
&\quad +\frac{1}{3}E(a+b)(M_1\cos\epsilon+M_2\sin\epsilon)\Big],\\
v_o^* &= Mg\ell C^{-1}r_o^{-2}v_o\operatorname{cosec}\theta_o\cot^2\theta_o,
\end{aligned} \tag{33}$$

the constants Y's, Y''s, X's and X''s are determined in terms of the motion parameters and are written in about three pages. The formula (32) shows that the expressions of Eulerian angles depend on four arbitrary constants θ_o, ψ_o, ϕ_o and r_o (sufficiently large).

4. ANALYTICAL AND NUMERICAL SOLUTIONS

This section is devoted to ascertain the accuracy of the obtained analytical solutions of the previous sections. That is the quasi-linear system (1) is integrated numerically using fourth order Runge – Kutta method [8] and the obtained results are compared with the analytical ones.

4.1. THE ANALYTICAL SOLUTIONS

In this case the analytical solutions p_2, γ_2 and their derivatives with respect to (t) are written in the following forms:

$$
\begin{aligned}
p_2 &= M_1 \cos\frac{h}{3} + M_2 \sin\frac{h}{3} + \mu\left[\ell_1 \cos\frac{h}{3} + m_1 \sin\frac{h}{3}\right] + \frac{9}{2}\mu^2\Big[2L_{2o} + i_{1o}\cos\frac{h}{3} \\
&\quad + i'_{1o}\sin\frac{h}{3} + i_2\cos\frac{2h}{3} + i'_2\sin\frac{2h}{3} + i_3\cos h + i'_3\sin h + i_4\cos\frac{4h}{3} \\
&\quad + i'_4\sin\frac{4h}{3} + i_5\cos\frac{5h}{3} + i'_5\sin\frac{5h}{3} + i_6\cos\frac{7h}{3} + i'_6\sin\frac{7h}{3}\Big], \\
x &= \frac{dp_2}{dt} = M_{1o}\sin\frac{h}{3} + M_{2o}\cos\frac{h}{3} + \frac{1}{3}\mu\left[m_1\cos\frac{h}{3} - \ell_1\sin\frac{h}{3}\right] + \frac{3}{2}\mu^2\Big[i'_{1o}\cos\frac{h}{3} \\
&\quad - i_{1o}\sin\frac{h}{3} - i_{2o}\sin\frac{2h}{3} + i'_{2o}\cos\frac{2h}{3} - i_{3o}\sin h + i'_{3o}\cos h - i_{4o}\sin\frac{4h}{3} \\
&\quad + i'_{4o}\cos\frac{4h}{3} - i_{5o}\sin\frac{5h}{3} + i'_{5o}\cos\frac{5h}{3} - i_{6o}\sin\frac{7h}{3} + i'_{6o}\cos\frac{7h}{3}\Big], \\
\gamma_2 &= M_3\cos h - \mu a M_1\cos h + \mu^2\Big[N_{2o} + v_{11}\cos\frac{h}{3} + v'_{11}\sin\frac{h}{3} + \Big(v_{12} - a\ell_1 \\
&\quad - \frac{9}{2}\nu_2^2 M_2^2 M_3^{-1}\Big)\cos h + v'_{12}\sin h + v_{13}\cos 2h + v'_{13}\sin 2h + v_{14}\cos\frac{5h}{3} \\
&\quad + v'_{14}\sin\frac{5h}{3} + v_{15}\cos 3h\Big], \\
y &= \frac{d\gamma_2}{dt} = -M_3\sin h + \mu a M_1\sin h + \mu^2\Big[\frac{1}{3}\Big(v'_{11}\cos\frac{h}{3} - v_{11}\sin\frac{h}{3}\Big) \\
&\quad - \Big(v_{12} - a\ell_1 - \frac{9}{2}\nu_2^2 M_2^2 M_3^{-1}\Big)\sin h + v'_{12}\cos h - 2v_{13}\sin 2h + 2v'_{13}\cos 2h \\
&\quad - \frac{5}{3}v_{14}\sin\frac{5h}{3} + \frac{5}{3}v'_{14}\cos\frac{5h}{3} - 3v_{15}\sin 3h\Big], \qquad (34)
\end{aligned}
$$

where

$h = iT/300$ for $i = 0$ to 300 step 5 and $T =$max value of t-variable.

Let us assume

$$
\begin{aligned}
&A = B = 1.2, \quad C = 1.6, \quad x_o = 5, \quad y_o = 6, \\
&z_o = 7, \quad R = 1500, \quad \lambda = .6, \quad M = 300, \\
&\gamma''_o = .5, \quad r_o = 1100, \quad T = 18.78775142. \qquad (35)
\end{aligned}
$$

In this case the following parameters are determined:

$$
\begin{aligned}
&A_1 = -B_1 = \omega = .3333333, \quad C_1 = 0, \quad \ell = 10.48809, \\
&x'_o = .4767313, \quad y'_o = .5720776, \quad z'_o = .6674238, \\
&g = 2.66667E-07, \quad c = 2.28999E-02, \quad \mu = 1.47206E-05, \\
&a = b = .75, \quad k = 5.08513E-07, \quad e = 1.906925, \\
&e_1 = -.667424, \quad e_2 = .222474, \quad \nu = .75, \quad \nu_2 = -2.25. \qquad (36)
\end{aligned}
$$

Consider that $p_{2a}, \gamma_{2a}, \ldots$ denote the analytical solutions $p_2, \gamma_2, \ldots$, the graphical representations and the corresponding phase plane diagrams for these solutions are given in figures 1, 4, 7, 10, 13 and 16.

For this case the system of differential equations can be rewritten as follows:

$$\frac{dp_2}{dt} = x, \qquad \frac{d\gamma_2}{dt} = y, \qquad \frac{dx}{dt} = -\frac{1}{9}p_2 + \mu^2 f_1, \qquad \frac{dy}{dt} = -\gamma_2 + \mu^2 g_1, \tag{37}$$

where

$$\begin{aligned} f_1 &= cna_1 s_{21} + p_2\Big(cna_2 - \frac{1}{9}s_{11} - y'_o y a^{-1} + C_1 x^2 A_1^{-1}\Big) \\ &\quad + \gamma_2(cna_3 x + cna_4 xy + cna_5 s_{21}) + x'_o xy, \\ g_1 &= cnb_1 + \gamma_2(cnb_2 - s_{11} + cnb_3 y^2 + cnb_4 x^2 - y'_o y) \\ &\quad + x'_o y^2 + p_2(cnb_5 s_{21} + cnb_6 xy). \end{aligned} \tag{38}$$

The constants $cna_1 \to cna_5$ and $cnb_1 \to cnb_6$ are determined by the correspondence between the above system and the system (1). Assuming the same data (35) and (36) with the initial values of the analytical solutions, the numerical solutions are obtained by using the fourth order Runge – Kutta method. Supposing $p_{2n}, \gamma_{2n}, \ldots,$ denote the numerical solutions $p_2, \gamma_2, \ldots,$ and using the computer, the numerical solutions and their phase trajectories are obtained in figures 2, 5, 8, 11, 14 and 17.

The comparison between the analytical and the numerical solutions is given in figures 3, 6, 9, 12, 15 and 18. This comparison shows that the deviation between the analytical and the numerical solutions is very small and can be neglected, that is the numerical solutions are in full agreement with the analytical ones.

5. CONCLUSIONS

Poincaré's small parameter method is applied to investigate the periodic solutions, with non-zero basic amplitudes, for the singular case of the natural frequency ($\omega = 1/3$). This problem deals with the following bodies being classified according to the moments of inertia:

1. $C > A > B,\ B < \frac{3}{4}C,\ A > \frac{3}{4}C$;
2. $C > B > A,\ A < \frac{3}{4}C,\ B > \frac{3}{4}C$;
3. $A = B = \frac{3}{4}C$, which represents rapidly spinning Lagrange's gyroscope about the axis of symmetry ($x'_o = y'_o = 0$);
4. $A = B = \frac{3}{2}C$, another Lagrange's gyroscope;
5. $\frac{9}{8}C < B < A,\ A > \frac{3}{2}C,\ B < \frac{3}{2}C$;
6. $\frac{9}{8}C < A < B,\ B > \frac{3}{2}C,\ A < \frac{3}{2}C$.

This problem is a generalization of the corresponding one in the uniform gravity field, that is the solution of the latter problem is deduced from the solution of the considered one by putting $k = 0$. The geometric interpretation of motion is considered to describe the orientation of the body at any instant (t) of time. A computer program is carried out to obtain the graphical representations for the analytical solutions. Starting the initial values of the analytical solutions, the autonomous system is solved, using the fourth order Runge – Kutta method, to obtain the numerical solutions through another program. The obtained analytical and numerical solutions are represented graphically, using the computer, to show the difference between them. The deviations between both solutions are very small, which give powerful agreement of the obtained solutions.

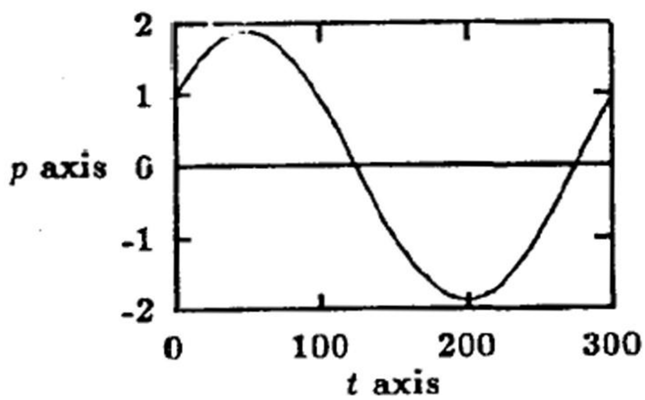

Fig. 1. $p_{2a} - t$ diagram

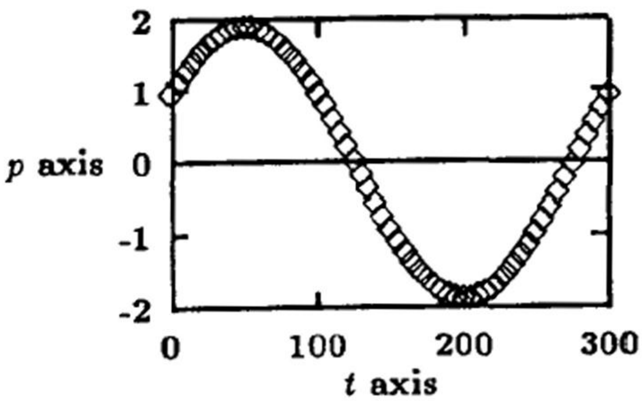

Fig. 2. $p_{2n} - t$ diagram

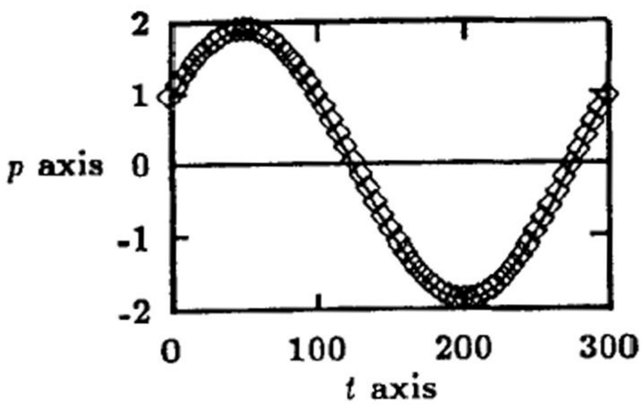

Fig. 3. $p_{2a,n} - t$ diagram

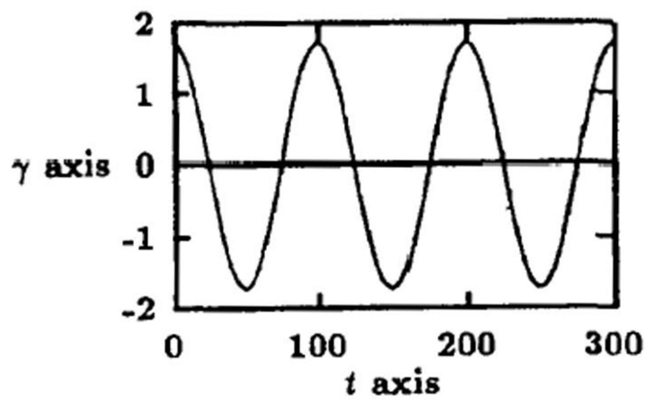

Fig. 4. $\gamma_{2a} - t$ diagram

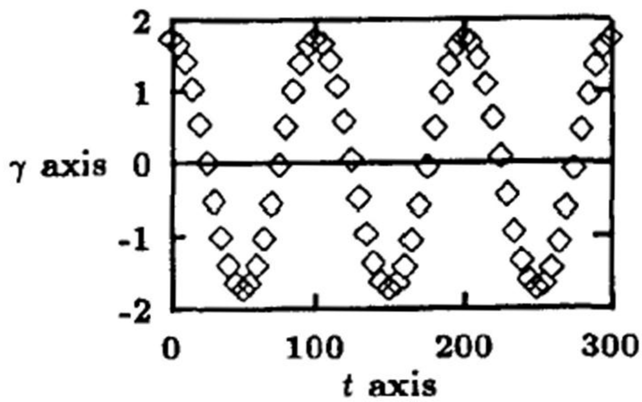

Fig. 5. $\gamma_{2n} - t$ diagram

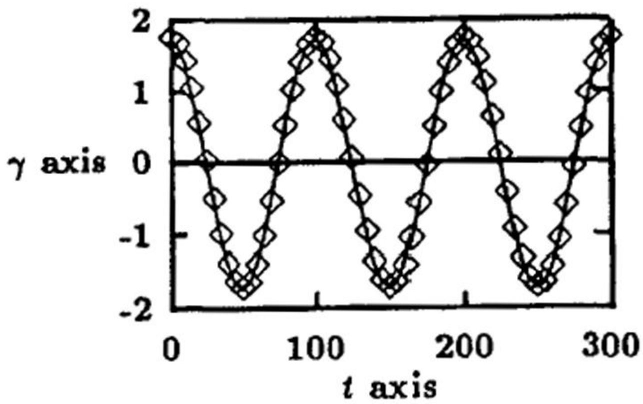

Fig. 6. $\gamma_{2a,n} - t$ diagram

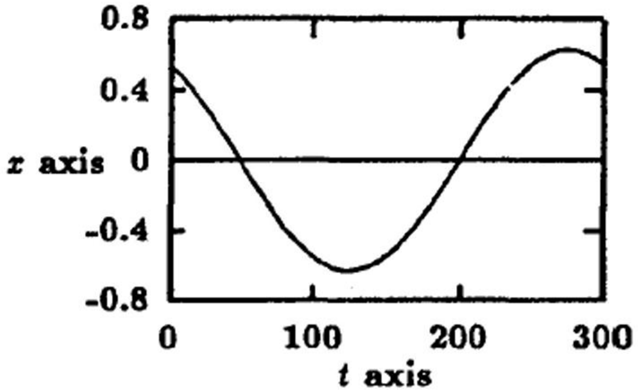

Fig. 7. $x_a - t$ diagram

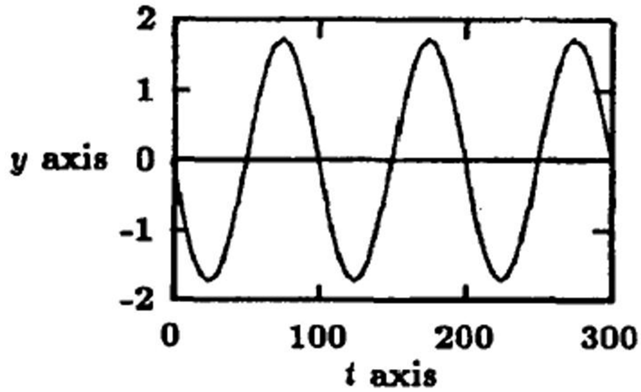

Fig. 10. $y_a - t$ diagram

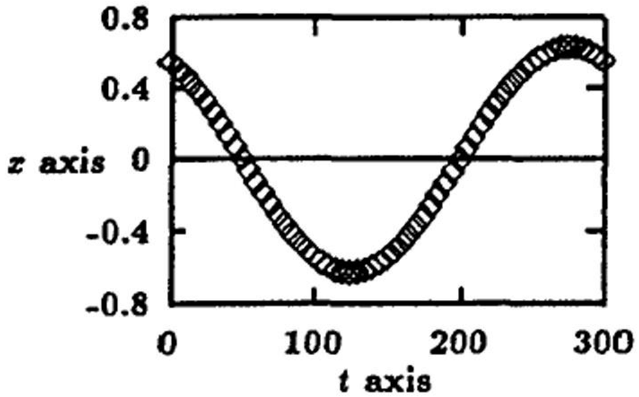

Fig. 8. $x_n - t$ diagram

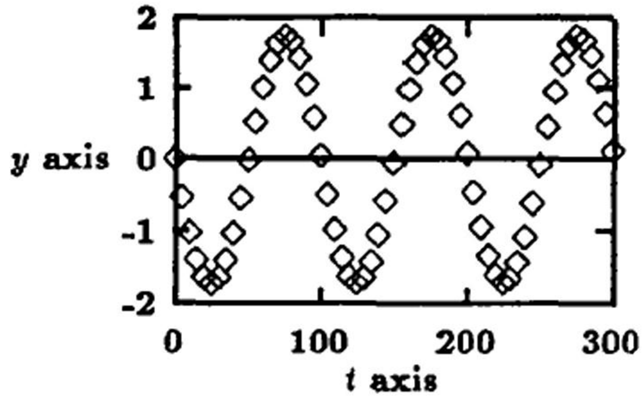

Fig. 11. $y_n - t$ diagram

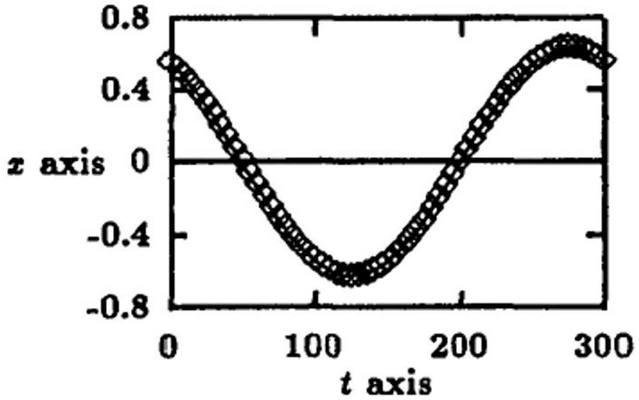

Fig. 9. $x_{a,n} - t$ diagram

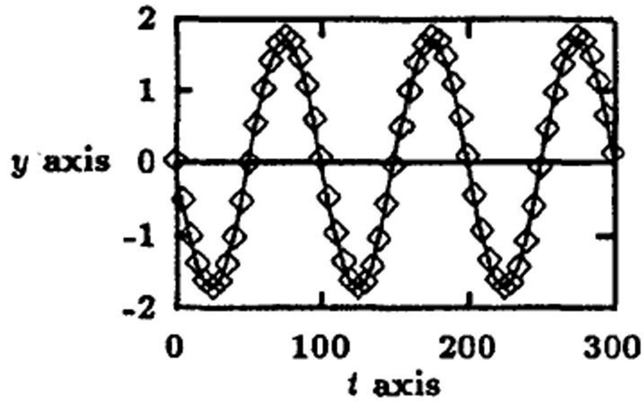

Fig. 12. $y_{a,n} - t$ diagram

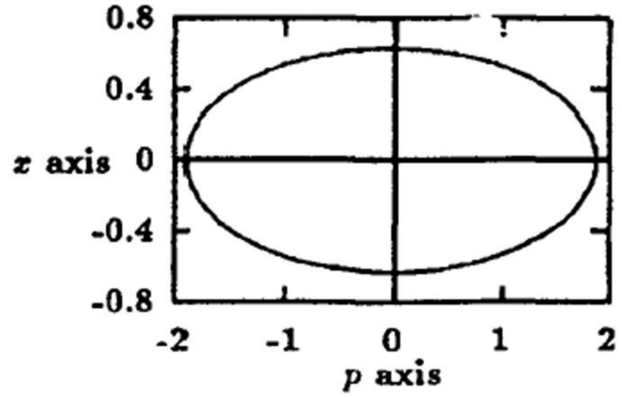

Fig. 13. $x_a - p_{2a}$ diagram

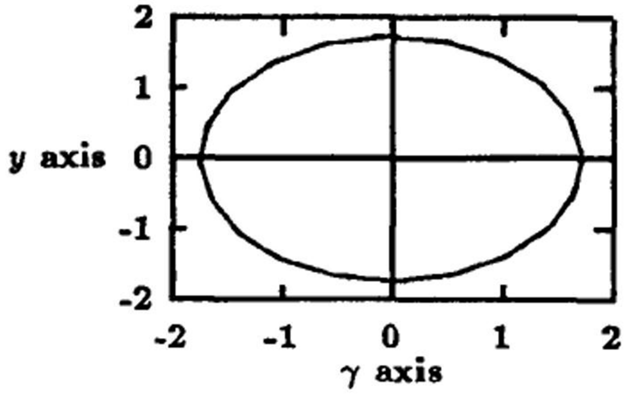

Fig. 16. $y_a - \gamma_{2a}$ diagram

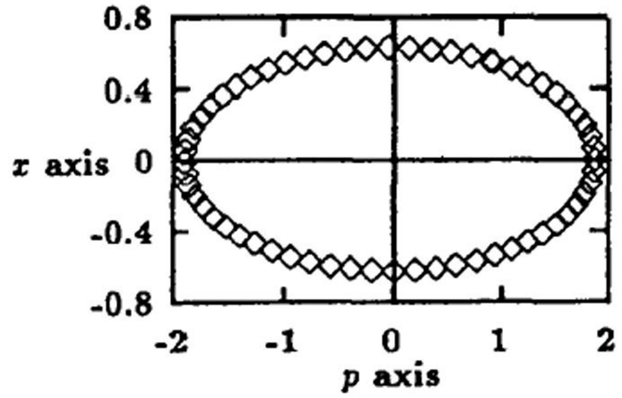

Fig. 14. $x_n - p_{2n}$ diagram

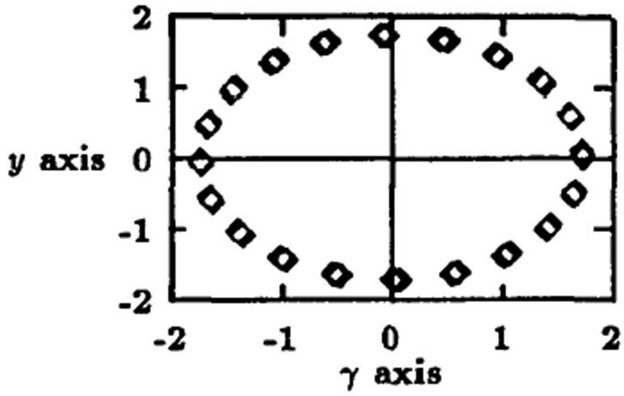

Fig. 17. $y_n - \gamma_{2n}$ diagram

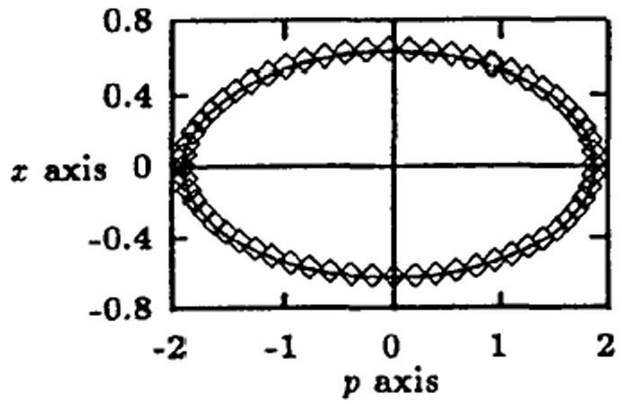

Fig. 15. $x_{a,n} - p_{2a,n}$ diagram

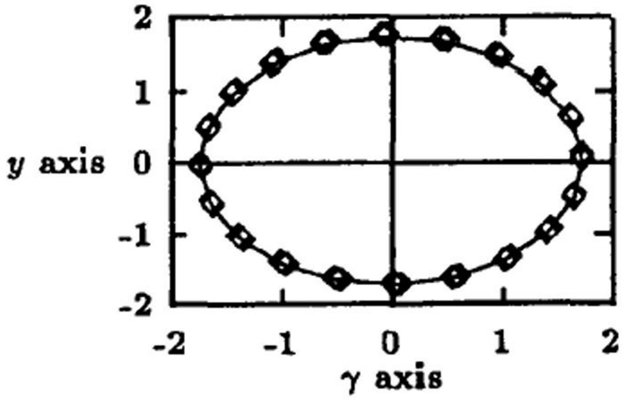

Fig. 18. $y_{a,n} - \gamma_{2a,n}$ diagram

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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KAM — A KNOWLEDGE-BASED TOOL FOR DEVELOPING COMPUTER ALGEBRA SYSTEMS

MARIA M. NISHEVA-PAVLOVA

The paper presents a description of KAM — a knowledge-based tool for building computer algebra systems developed at the Faculty of Mathematics and Informatics, Sofia University. The main features of KAM are analyzed. The architecture of KAM and the knowledge representation formalisms supported by the tool transformation rules, frames, rewrite rules, generalized rules, are briefly described. A presentation of the experimental computer algebra system STRAMS being under development as an application of KAM is given.

Keywords: intelligent computer algebra system, knowledge representation tools

1991/95 Math. Subject Classification: main 68T35; secondary 68T30, 68T05

1. INTRODUCTION

In the last 2–3 decades Computer Algebra Systems (CAS) have been successfully used in many fields of science and engineering. These systems can help in the solution of different types of problems connected with the execution of complicated and labour-consuming transformations of mathematical expressions. "Classical" CAS like Reduce, Macsyma, Maple, Mathematica etc. provide thousands of sophisticated algebraic algorithms, but sometimes they are difficult for use. On the one hand, it is often hard to select the appropriate algorithm from the amount of available algorithms. On the other hand, the interpretation of the solution sometimes needs significant efforts, because the system does not give any information about the solution steps. In other words, the user does not receive any explanation or information about the problem solving process (for instance, how the solution of the problem is found, or why the output is the solution of the given problem). Moreover, the extension of the functional facilities of most of the "classical" CAS is a hard job that usually needs a great amount of programming efforts.

The problem here is that "classical" CAS have no mathematical knowledge about the properties of the functions and the problem solving methods represented in an explicit, declarative way. Their mathematical knowledge is embedded implicitly in the algorithms and is inaccessible to the user.

For that reason a series of successful attempts has been made to integrate the classical methods for developing CAS with Artificial Intelligence methods and tools. These attempts have been made in different directions.

On the one hand, many famous large CAS have been supplied with intelligent user interfaces. For example, the Praxis [5] system is implemented as a rule-based expert system for the computer algebra system Macsyma.

On the other hand, several systems like AXIOM [6] and MAGMA [1] are aimed at the integration of means for description of algebraic structures (and, more general, of database and deductive facilities) with computer algebra algorithms.

In the systems of the type of LP [11], APS [7, 8] etc. the emphasis basically falls on the representation of knowledge about the methods for mathematical problem solving.

The hybrid knowledge representation system MANTRA [2, 3] is a next step of such integration. It combines different formalisms for specification of mathematical domains and provides a computational environment for solving problems combining the strong mathematical algorithms with heuristic search for solutions.

Thus the term "intelligent CAS" becomes quite popular in the last years. In general, intelligent CAS are systems that are capable to manipulate different types of mathematical knowledge and use a large set of Artificial Intelligence methods and techniques.

A set of projects aimed at the investigation of different aspects of building intelligent CAS has been under development at the Faculty of Mathematics and Informatics, Sofia University. An approach to building CAS has been developed with the purpose of creating flexible, "open" CAS that:

— can easily be integrated with other software packages and can be used in the development of CAD systems, intelligent tutoring systems etc.;

— solve in a satisfactory way the problems discussed above, in particular, are able to do some kind of learning and explanation generation.

This approach is based on the representation of knowledge about the properties of the functions and the methods for mathematical problem solving defined by these properties.

The knowledge-based tool KAM described in this paper has been developed with a mainly experimental purpose. It is a software system supporting our approach to building CAS. KAM has been used as a tool for performing experiments with the suggestions presented in this paper and in [9], and as an instrument for building CAS that could have concrete applications. The Common Lisp interpreter from the integrated environment POPLOG, version 14.5, has been used for its implementation [12].

2. MAIN CHARACTERISTICS OF THE APPROACH TO BUILDING CAS SUPPORTED BY KAM

The suggested approach is based on the conception for the knowledge representation in CAS discussed in Section 2.1. This conception can be considered as a modification and further development of some ideas and mechanisms suggested by B. Silver in [11].

2.1. KNOWLEDGE REPRESENTATION

The formalism we suggest for the representation of mathematical knowledge in CAS is a hybrid one. It includes several levels of representation.

The knowledge about the properties of the functions manipulated in the corresponding CAS can be described using a special type of rules called rewrite rules. The structure of each rewrite rule contains a description of a correct transformation of a definite class of mathematical expressions and a formulation of some general preconditions for its performance (if there are any).

The description of the methods for transformation of expressions and equations in the corresponding definition domain can be realized by the so-called generalized rules (methods). Each generalized rule describes a sequence of transformations of the given expression (equation) aimed at its conversion into a particular form. Usually, generalized rules contain sequences of properly grouped rewrite rules. Depending on their contents and application mode, they are classified as declarative, procedural and hybrid. Another classification criterion of the generalized rules is the role they play in the problem solving process of a given, relatively complex task (equation solving, symbolic integration etc.). In this sense they are classified as key and non-key ones. The key generalized rules play a significant role in the control of the search in the state graph of the corresponding problem.

The knowledge about the problem solving methods for the included types of tasks can be described either directly by proper generalized rules or using specific constructions called schemata. A schema is a sequence of non-key generalized rules. It describes a definite step in the problem solving process of a relatively complex task. Schemata are a natural generalization of methods (generalized rules). The precondition of a schema is the applicability of its first generalized rule. The goal is to solve the problem or to be able to apply a key generalized rule after the application of the schema.

During its working cycle each particular CAS realizes a search in the state space of the user's problem. In the role of operators it uses the schemata and generalized rules available at the current moment. Some additional search control knowledge is used with the purpose of avoiding the possible cycles and focusing the attention of the system on certain situations.

2.2. LEARNING FACILITIES

The formalism for the knowledge representation in CAS described in Section 2.1 is a good basis for the realization of some mechanisms for unsupervised learning that could considerably increase the effectiveness of these systems. In general, the unsupervised learning in the discussed type of CAS is based on the capability for discovering and memorizing the schemata used in the problem solving process of some complex tasks (such as factorization, equation solving or symbolic integration). These new schemata could be directly used in solving further problems. More precisely, the suggested unsupervised learning mechanism can be summarized as follows. Before the first run of the corresponding CAS the set of schemata included in its knowledge base is empty. During its working cycle the system uses the schemata and generalized rules available in the knowledge base at the moment and the built-in search control knowledge. The discovering and the application of a proper schema can considerably speed up the problem solving process. Whenever a given problem is successfully solved, the system can analyze the used sequence of generalized rules, construct the new schemata candidates and merge them with the set of existing schemata. In this way the CAS can perform some kind of self-perfection, i.e. some type of unsupervised learning.

2.3. EXPLANATION GENERATION

It is advisable for an intelligent CAS to be able to generate various kinds of explanations. The minimal requirement in this respect is the capability to explain the mode in which a given problem is solved. To ensure this capability, a CAS of the discussed type can keep in a special record the history of the current session. The history of a given session may contain the sequence of problems solved at the time of this session and the main steps of their problem solving process. These steps correspond to the methods (generalized rules) used by the system. The processing of a given explanation request can be performed in two steps: extracting the corresponding information from the record and generating the text of the explanation. This text ought to be in a natural language and to contain a description of the consequent steps of the problem solving process.

2.4. COMPARISON WITH SILVER'S APPROACH

As it was mentioned above, the approach to building knowledge-based CAS supported by KAM is a modification and further development of the ideas of the so-called Precondition Analysis suggested by B. Silver and realized in his system LP [11]. The main differences between our approach and Silver's one can be briefly brought to the following:

— our approach is intended for the development of general-purpose CAS with varied functional facilities (while LP is a special-purpose CAS for symbolic equation solving). Therefore our generalized rule mechanism is considerably more complicated;

— schemata in our formalism realize the separate steps in the problem solving process of the given mathematical task (but, in contrast with LP schemata, they do not realize the entire problem solving process of the given task from the beginning to the end). Thus our knowledge representation formalism is more flexible and relevant to the human problem solving process;

— our approach is aimed at the development of means for unsupervised learning while the learning process in LP is a supervised one.

3. ARCHITECTURE OF KAM

The final version of KAM will include the following functional components:

— a mathematical problem solving engine;

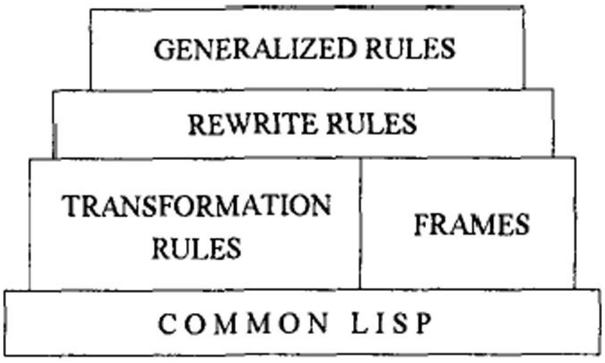
— an explanation module;

— a rule editor;

— an interface module.

The mathematical problem solving engine consists of two modules: an inference engine and a learning module realizing respectively the problem solving and unsupervised learning mechanisms discussed in Section 2.

From the implementation point of view the structure of the inference engine of KAM seems as follows:



Transformation rules are a declarative formalism intended for the description of list transformations. The transformation rule and the frame manipulation mechanisms have been used in the implementation of the rewrite rule interpreter. The generalized rule interpreter uses some basic means provided by the frame interpreter and the rewrite rule interpreter. The control block realizes the search process in the state space of the given problem. In particular, the search for proper schemata and generalized rules and their application are realized by the control block using the generalized rule interpreter. When a problem is successfully solved, the learning module can be activated. It discovers the new schemata used by the inference engine (if there are any) and adds them to the set of known schemata.

The explanation module realizes the explanation generation mechanisms described in Section 2.3.

The develpoment of the rule editor and the interface module of KAM is still at the designing phase.

The rule editor will be a tool assisting the users of KAM in building the knowledge bases of the concrete CAS they intend to create. It will suggest means for convenient input and editing of rewrite rules and generalized rules and procedures for automatical translation of these rules into the internal form that is "understandable" for the corresponding interpreter.

The interface module will be the component of KAM the typical users will be in touch with. It will enable the user to choose from a special menu the variuos functions of KAM relevant to the development of concrete CAS: using the rule editor (i.e. building or modification of the knowledge base of a given CAS), building the functional modules (the inference engine, the learning module or the explanation module) of a new CAS by adjustment of copies of the corresponding modules of KAM or their proper subsets, designing the interface module of a CAS etc.

As it was mentioned above, the current version of KAM includes working prototypes of the mathematical problem solving engine and the explanation module only. Nowadays the rule editor and the interface module are still at the designing phase of development. Therefore the current version of KAM is intended for users with good skills in Common Lisp programming. The completion of proper working versions of the interface module and the rule editor will help mathematicians, engineers, teachers and other subject specialists with no Lisp programming experience to work conveniently and successfully with KAM.

The knowledge representation formalisms supported by the inference engine of KAM are briefly described in Sections 4–7.

4. TRANSFORMATION RULES

Transformation rules are a formalism for description of list transformations. Each transformation rule consists of two parts — a left-hand part and a right-hand one. The left-hand part describes the class of lists the rule can be applied to. It is called the pattern. The right-hand part describes the method for the new list construction. It is called the constructor. The rule is applied to a list called the object list (the object). The application of a given transformation rule to a given object list is accomplished in the following way:

— an attempt for matching the object list and the pattern is made;

— if the matching succeeds, some elements of the object list are extracted in order to be used in the new list construction. Then the result list is constructed in a way described by the constructor;

— if the matching fails, the rule is not applicable to the given object. In this case *nil* is returned as a result.

Hence the transformation rule interpreter performs two main operations: pattern matching and construction of the result list.

Pattern matching is an operation in which each element of the pattern matches one or several elements of the object list in accordance with a definite set of rules. The pattern consists of two types of elements — ordinary elements and special ones. An ordinary element of the pattern matches only an equal to it element of the object list. Special elements of the pattern match the corresponding elements of the object list using specific rules. Examples of admissible types of special elements:

(*?* ⟨*var*⟩).

If the variable ⟨*var*⟩ is not bound, this special element successfully matches the corresponding element of the object list and this element of the object list is assigned to ⟨*var*⟩. In the other case, the matching succeeds if the value of ⟨*var*⟩ is equal to the corresponding element of the object.

(*?* ⟨*var*⟩ ⟨*predicate-name*⟩).

Here ⟨*predicate-name*⟩ is a name of a Common Lisp function. This special element is interpreted as the previous one with one additional condition for the success of the matching: the evaluation of the function ⟨*predicate-name*⟩ with an argument equal to the corresponding element of the object must be different from *nil*.

The construction of the result list is an evaluation of some of the constructor elements. In other words, the result list can be obtained from the constructor by an evaluation of some of its elements and a substitution of these elements with the corresponding values.

5. FRAMES

As a module of KAM a frame system called FS is developed. FS gives a set of standard frame manipulation means that can be divided into the following groups: frame definition, creation of a frame instance, providing an access to the slots of a frame instance.

To define a frame, one has to determine the name of the frame and to create a description of the structure of this frame. The names of the frames and the slots are Lisp symbols. The possible facet names in FS are the following: *value, default, if-needed, if-added, if-removed*. The semantics of these facets is identical to the conventional one. The values associated to them are Lisp expressions (in particular, Common Lisp functions can be used).

A Common Lisp function called a service procedure or a method can be used as a slot value as well. Slots supplied with method values have no facets. Methods are performed by sending messages to them. A special type of methods are the so-called auto-methods. Auto-methods contain calls to the slots of the frame they belong to.

FS provides some means for inheritance of properties that is reduced to the possibility of appending a given subset of the slots and facets of the parent frame to the structure of the inheritor frame.

As it was mentioned above, the frame manipulation mechanisms in KAM have been used in the implementation of the rewrite rule interpreter and the generalized rule interpreter. Each particular rule of these two types has been described as an instance of a frame with proper structure and methods.

6. REWRITE RULES

Rewrite rules have been used to describe the properties of the functions that can be manipulated in the corresponding CAS. Each rewrite rule includes a description of a correct transformation of a given class of mathematical expressions and, if necessary, a general precondition for its performance. Examples:

$c_1 a + c_2 a = (c_1 + c_2)a,$

$e^a e^b = e^{a+b},$

$\mathrm{tg}(a+b) = \dfrac{\mathrm{tg}\, a + \mathrm{tg}\, b}{1 - \mathrm{tg}\, a\, \mathrm{tg}\, b}$ with precondition a, b, $a+b$ different from $\dfrac{(2k+1)\pi}{2}$.

The implementation of the means for rewrite rule description and application uses mainly the mechanisms provided by the frame and the transformation rule formalisms. A possibility for procedural implementation of some rewrite rules aimed at reaching better effectiveness is provided as well.

7. GENERALIZED RULES

The generalized rule formalism has been used for the description of the transformation methods applicable to the types of expressions and equations the corresponding CAS can manipulate. A generalized rule can be considered as a description of a sequence of transformations of the given expression aimed at its conversion into a definite form. In this sense, most often generalized rules are sets of properly grouped rewrite rules. According to the contents of their description and the method of their application generalized rules can be classified as declarative, procedural and hybrid (combined).

Each generalized rule consists of two parts — a precondition and a body. The precondition is a predicate whose satisfaction is a necessary condition for the application of the generalized rule and for achieving its purpose. The evaluation of the precondition of a given generalized rule is the first step of its application. If the precondition is true, then the body of the generalized rule is performed. Depending on its type, the body of a generalized rule may contain:

— *in the case of a declarative rule*: a sequence of rewrite rules. Each of them can include some additional control information about the correct direction(s) of its application;

— *in the case of a procedural rule*: the code of a procedure realizing the application of this rule;

— *in the case of a hybrid rule*: a set of pairs (*pattern, procedure*). When the examined expression matches one of the patterns, the corresponding procedure is executed.

Most often declarative generalized rules have been used in building CAS. The body of such a rule consists of a sequence of rewrite rules that can be divided in three groups: pre-rules, basic rules, post-rules.

The pre-rules are intended to prepare the given expression for the performance of the basic rules. The post-rules are used to remove some "defects" remaining after the performance of the basic rules.

There are three basic types of declarative generalized rules according to the mode of application of their bodies: normal, cyclic and recursive. The body of a normal generalized rule is performed in the following way: First the pre-rules are consecutively applied to the given expression. Each of them is executed on the result returned by the previous one. Then the basic rules are applied in the same way on the result of the execution of the pre-rules. At last the post-rules are applied in the described way.

The body of a cyclic generalized rule contains only one basic rule. It is performed in the following way: First the pre-rules are executed as in the case of a normal rule. Then the basic rule is executed. If it has not changed its argument, the execution of the body of the generalized rule stops and the current result is returned. In the other case, the corresponding post-rules are performed and then a cyclic execution of the described sequence of steps is carried out until the basic rule returns its argument unchanged.

The body of a recursive generalized rule is first executed on the subexpressions of the given expression and then it is applied to the obtained new argument.

It is possible to construct some combinations between the basic types of declarative generalized rules. For example, very attractive are the so-called cyclic recursive generalized rules that can be used as a proper mean for the description of some methods for expression simplification.

8. ONE APPLICATION OF KAM: THE STRAMS COMPUTER ALGEBRA SYSTEM

An experimental CAS named STRAMS [9, 10] has been under development at the Faculty of Mathematics and Informatics, Sofia University, using the current version of KAM. STRAMS is a knowledge-based system for symbolic manipulations of expressions that may contain numbers, symbols and the functions: $+$, $-$, $*$, $/$, power function, exponential, logarithmic and trigonometric functions. It is intended for solving the following main problem types:

— expression simplification;

— symbolic equation solving;

— symbolic differentiation;

— symbolic integration.

The formalism discussed in Section 2 is used for the knowledge representation in STRAMS. The knowledge of STRAMS about the properties of the manipulated functions is described by rewrite rules. Examples of such rules can be found in Section 6.

The description of the methods for transformation of expressions and equations in STRAMS is realized by a set of generalized rules. Here we give several examples of generalized rules (generalized rules are called methods in STRAMS) used in the symbolic equation solving subsystem of STRAMS.

Example 1. Isolation.

Let an equation $eq : expr_1 = expr_2$ be given and let f be the outermost function in $expr_1$. The method consists in the application of the inverse of f to $expr_1$ and $expr_2$. The precondition of the method is: the unknown occurs in only one of the arguments of f and $expr_2$ does not contain the unknown. The goal is in the left-hand side of eq to remain only the argument containing the unknown.

The method is a key one and is implemented procedurally due to effectiveness considerations.

Example 2. Collection.

The goal of this method is to reduce the number of occurrences of the unknown. Collection is a non-key method with no explicit precondition. STRAMS applies it only if none of the key methods can be applied. So the precondition of Collection (and of all non-key methods) is: there is no key method with satisfied preconditions.

The method is declarative, normal. One of its rewrite rules is

$AB + AC = A(B + C)$ with precondition A must contain the unknown.

Example 3. Attraction.

Attraction is a non-key method with no explicit precondition. The goal here is to move the occurrences of the unknown "closer" together in hope that another method (for example Collection) will then be applicable. One of the rewrite rules of Attraction is

$AC + BC = (A + B)C$.

In this rule the expressions A and B are attracted, so they must contain the unknown.

The problem solving process of some complex tasks (such as equation solving or symbolic integration) consists of a series of steps realized by the STRAMS schemata. According to the ideology of KAM a schema is a sequence of non-key generalized rules (methods) whose application directs to a definite aim in the problem solving process.

STRAMS has some means for unsupervised learning and explanation generation that entirely correspond to the ideas and mechanisms described in Section 2.

The mathematical problem solving engine and the explanation module of KAM were used without any effort for the implementation of STRAMS. The knowledge base of STRAMS was built by direct recording of the corresponding rewrite rules and generalized rules in internal form. The interface module of STRAMS that analyzes the user requests and realizes the general control of the system's work was developed especially for the purpose.

9. CONCLUSION

Our experience in developing and using STRAMS demonstrates that the current version of KAM and the approach to building CAS supported by it do really work. Our current activities are oriented to the development of working versions of the rule editor and the interface module of KAM.

At the same time some research activities have been carried out with the purpose of extending our approach to building CAS supported by KAM in the following directions:

— including some mechanisms for supervised learning. It is useful at that to provide means for learning both new schemata and new generalized rules;

— improvement of the explanation generation mechanisms. It is necessary to extend the range of explanations that can be generated by the discussed type of CAS and to develop some means for generating explanations with different degrees of circumstantiality;

— including some proper formalisms for problem solving and learning by analogy. Our hypothesis is that the methodology suggested by J. Carbonell in [4] can be used for this purpose.

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NUMERICAL SOLUTIONS FOR STEADY FLOW PAST A CIRCULAR CYLINDER VIA THE METHOD OF VARIATIONAL IMBEDDING

CHRISTO I. CHRISTOV, ROSSITZA S. MARINOVA

A numerical investigation of the two-dimensional flow around a circular cylinder is performed using a primitive-variable approach. Steady (but unstable) solutions have been calculated up to Re = 200. The imbedding system is solved numerically by a difference scheme of splitting type. A staggered non-uniform grid is used. The obtained results are in good agreement with the available data.

Keywords: Navier-Stokes equations, viscous flow past a circular cylinder, high Reynolds numbers, method of variational imbedding, difference schemes

1991/95 Math. Subject Classification: 76D05, 76M30, 65M06

1. INTRODUCTION

The numerical treatment of high-Reynolds number viscous flows is of considerable interest for the applications because of the fact that the predominant part of the practically important flows take place either in large scales and high speeds or with small viscosity. Classical examples of such a kind are, above all, geophysical flows and flows around vehicles and vessels. The steady-state solution to Navier-Stokes (N-S) equations for high Reynolds numbers is unstable and cannot be treated as an initial value problem for the unsteady N-S equations. At the same time, the above mentioned problem is of crucial fundamental importance in the sense of answering the question of which is the limiting pattern for the solution of N-S equations when the coefficient Re^{-1} of the highest-order derivatives approaches zero.

The problem of steady-state viscous incompressible flow past bluff bodies has over a long time received much attention, both theoretically and numerically. The circular cylinder is the simplest two-dimensional bluff body shape and the flow past it has been the subject of considerable experimental and numerical study. The flow round this shape has the attraction of being the source of intriguing transitions. Many of the numerical treatments are concerned with low Reynolds number flows. In spite of the many numerical calculations on flow past a circular cylinder, accurate results have been obtained only for Reynolds number ($\mathrm{Re} = U_\infty d/\nu$) up to about 700, see Fornberg [14, 15]. The Reynolds number Re is the governing dimensionless parameter. The cylinder diameter $d = 2a$ is the characteristic length; velocity at infinity U_∞ is the characteristic velocity; and ν is the kinematic coefficient of viscosity. Fornberg has found that the wake bubble (region of recirculating flow) has eddy length $L \propto \mathrm{Re}$, width $W \propto \sqrt{\mathrm{Re}}$ up to $\mathrm{Re} = 300$, and $W \propto \mathrm{Re}$ beyond that. Smith [18] has developed an asymptotic theory which agrees with Fornberg's results up to $\mathrm{Re} \approx 300$ only. Smith [20] and Peregrine [17] have performed theoretical work which gives a fresh interpretation of Fornberg's results. There are several differences between the theories of Smith and Peregrine, some of which are a matter of interpretation. These are unlikely to be resolved without further analysis and computational work.

The problem of viscous steady-state flow past a circular cylinder at high Reynolds numbers represents one of the classical problems in fluid mechanics. Although some agreement between theoretical, numerical and experimental results exists, there is a need for further work in all these aspects of this fundamental and classical problem. In the present paper the steady-state Navier-Stokes equations are solved using the so-called Method of Variational Imbedding.

2. BASIC EQUATIONS AND METHOD OF SOLUTION

The N-S equations are given in dimensionless form, corresponding to a cylinder of radius $r = 1$ in an uniform stream of unit magnitude with direction along the positive axis of x. Polar co-ordinates (r, φ) are used. The N-S equations governing the steady-state motion then read

$$u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi u_r}{r} = -\frac{1}{r}\frac{\partial p}{\partial \varphi} + \frac{1}{\mathrm{Re}}\left[D u_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right], \tag{2.1}$$

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{\mathrm{Re}}\left[D u_r - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right], \tag{2.2}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} = 0, \tag{2.3}$$

where $u_r = u(r,\varphi)$ and $u_\varphi = v(r,\varphi)$ are the velocity components parallel respectively to the polar axes r and φ; $p = p(r,\varphi)$ is the pressure. Respectively,

$$D \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

is the so-called *Stokesian*.

In terms of dimensionless variables, the cylinder surface is represented by $r = 1$, while the velocity at infinity — by unity.

The boundary conditions reflect the non-slipping at the cylinder surface

$$u_r(1,\varphi) = u_\varphi(1,\varphi) = 0, \tag{2.4}$$

on the one hand, and the asymptotic matching with the uniform outer flow at infinity, on the other. Numerically one has to pose the asymptotic condition at a certain large enough value of the radial co-ordinate, called "actual infinity," say, r_∞. Then the dimensionless boundary conditions read

$$u_r(r_\infty,\varphi) = \cos\varphi, \quad u_\varphi(r_\infty,\varphi) = -\sin\varphi. \tag{2.5}$$

Due to the obvious flow symmetry with respect to the line $\varphi = 0, \pi$, the computational domain may be reduced to $0 \le \varphi \le \pi$, $r \ge 1$ and additional boundary conditions on the lines $\varphi = 0$ and $\varphi = \pi$ are added to acknowledge the mentioned symmetry, namely:

$$u_\varphi = \frac{\partial u_r}{\partial \varphi} = 0 \quad \text{for} \quad \varphi = 0, \pi. \tag{2.6}$$

2.1. APPLYING THE METHOD OF VARIATIONAL IMBEDDING

For tackling inverse and incorrect problems, Christov [4 - 6] has developed the already mentioned Method of Variational Imbedding (MVI) which is a special implementation of the Least Square Method to ODE and PDE.

Consider the imbedding functional

$$\mathcal{J} = \int_0^\pi \int_1^\infty \left(\Phi^2 + \Omega^2 + X^2\right) r\,dr d\varphi, \tag{2.7}$$

where

$$\Phi = u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi u_r}{r} + \frac{1}{r}\frac{\partial p}{\partial \varphi} - \frac{1}{\mathrm{Re}}\left(Du_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial \varphi}\right),$$

$$\Omega = u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r} + \frac{\partial p}{\partial r} - \frac{1}{\mathrm{Re}}\left(Du_r - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial \varphi}\right),$$

$$X = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi}.$$

As far as the boundary value problem for the N–S equations possesses a solution, then the global minimum of the functional (2.7) is equal to zero, which is the value the functional assumes on the solutions of N–S. This allows us to seek a local minimum of the functional $\mathcal{J}$ and to check afterwards whether this is the global minimum.

The necessary conditions for minimizing of a functional are the Euler-Lagrange equations (see [8]). After some simplification these equations of Euler-Lagrange for the velocity components and pressure take the form of a conjugated system for Φ, u_φ, Ω, u_r and p:

$$\frac{1}{\mathrm{Re}}\left(D\Phi + \frac{2}{r^2}\frac{\partial\Omega}{\partial\varphi}\right) + \left(u_r\frac{\partial\Phi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial\Phi}{\partial\varphi} + \frac{2u_\varphi\Omega}{r} + \frac{1}{r}\frac{\partial \mathrm{X}}{\partial\varphi}\right) + \Phi\frac{\partial u_r}{\partial r} - \frac{\Omega}{r}\frac{\partial u_r}{\partial\varphi} = 0,$$

$$\frac{1}{\mathrm{Re}}\left(Du_\varphi + \frac{2}{r^2}\frac{\partial u_r}{\partial\varphi}\right) - \left(u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_\varphi u_r}{r} + \frac{1}{r}\frac{\partial p}{\partial\varphi}\right) + \Phi = 0,$$

$$\frac{1}{\mathrm{Re}}\left(D\Omega - \frac{2}{r^2}\frac{\partial\Phi}{\partial\varphi}\right) + \left(u_r\frac{\partial\Omega}{\partial r} + \frac{u_\varphi}{r}\frac{\partial\Omega}{\partial\varphi} - \frac{u_\varphi\Phi}{r} + \frac{\partial \mathrm{X}}{\partial r}\right) - \Phi\frac{\partial u_\varphi}{\partial r} - \Omega\frac{\partial u_r}{\partial r} = 0,$$

$$\frac{1}{\mathrm{Re}}\left(Du_r - \frac{2}{r^2}\frac{\partial u_\varphi}{\partial\varphi}\right) - \left(u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi^2}{r} + \frac{\partial p}{\partial r}\right) + \Omega = 0,$$

$$\Delta p - \frac{2}{r}\left(\frac{\partial u_\varphi}{\partial\varphi}\frac{\partial u_r}{\partial r} - \frac{\partial u_\varphi}{\partial r}\frac{\partial u_r}{\partial\varphi} + u_\varphi\frac{\partial u_\varphi}{\partial r} + u_r\frac{\partial u_r}{\partial r}\right) = 0.$$

All five equations above are of elliptic type and of second order on each boundary point. Therefore, five boundary conditions are needed. We already posed two of them when formulating the problem, see Eqs. (2.4) — (2.6). The remaining three are the natural conditions for minimization of the functional (2.7), which are nothing else but $\Phi = \Omega = \mathrm{X} = 0$. From the continuity equation $\mathrm{X} = 0$ we have $\partial u_r/\partial r = 0$ at the boundaries $r = 1$ and $r = r_\infty$. Respectively, the symmetry conditions at the lines of symmetry $\varphi = 0, \pi$ are $\partial p/\partial\varphi = 0$, which is equivalent to the condition on the function $u_\varphi(r,\varphi)$ at the same lines, namely $-\partial^2 u_\varphi/\partial\varphi^2 = 0$. Thus we have a correctly posed boundary problem for the set of functions we are looking for.

It is clear that if we find a solution of the imbedding system for which Φ and Ω are equal to zero, then u_φ, u_r and p form the solution of the original problem.

Here we consider the same problem that was outlined in [8]. The difference is that we treat the Imbedding system differently. In [8] the autors have solved numerically the Imbedding system of Euler–Lagrange equations for functions u_φ, u_r which are of forth order, and pressure equation for p of second order. This system looks apparently much more complicated (together with boundary conditions) than the system of five equations for Φ, u_φ, Ω, u_r and p.

We introduce the notations (E is the unitary operator)

$$\vec{\theta} = \begin{pmatrix} \Phi \\ u_\varphi \\ \Omega \\ u_r \\ p \end{pmatrix}, \quad F^{\vec{\theta}} = \begin{pmatrix} F^\Phi \\ F^{u_\varphi} \\ F^\Omega \\ F^{u_r} \\ F^p \end{pmatrix}, \quad I = \begin{pmatrix} E & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & E \end{pmatrix},$$

where

$$F^\Phi = \frac{2}{\mathrm{Re}\cdot r^2}\frac{\partial \Omega}{\partial \varphi} + u_r\frac{\partial \Phi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial \Phi}{\partial \varphi} + \frac{2u_\varphi \Omega}{r} + \frac{1}{r}\frac{\partial \mathrm{X}}{\partial \varphi} + \Phi\frac{\partial u_r}{\partial r} - \frac{\Omega}{r}\frac{\partial u_r}{\partial \varphi},$$

$$F^{u_\varphi} = \frac{2}{\mathrm{Re}\cdot r^2}\frac{\partial u_r}{\partial \varphi} - \left(u_r\frac{\partial u_\varphi}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\varphi u_r}{r}\right),$$

$$F^\Omega = -\frac{2}{\mathrm{Re}\cdot r^2}\frac{\partial \Phi}{\partial \varphi} + u_r\frac{\partial \Omega}{\partial r} + \frac{u_\varphi}{r}\frac{\partial \Omega}{\partial \varphi} - \frac{u_\varphi \Phi}{r} + \frac{\partial \mathrm{X}}{\partial r} - \Phi\frac{\partial u_\varphi}{\partial r} - \Omega\frac{\partial u_r}{\partial r},$$

$$F^{u_r} = -\frac{2}{\mathrm{Re}\cdot r^2}\frac{\partial u_\varphi}{\partial \varphi} - \left(u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r}\right),$$

$$F^p = -\frac{2}{r}\left(\frac{\partial u_\varphi}{\partial \varphi}\frac{\partial u_r}{\partial r} - \frac{\partial u_\varphi}{\partial r}\frac{\partial u_r}{\partial \varphi} + u_\varphi\frac{\partial u_\varphi}{\partial r} + u_r\frac{\partial u_r}{\partial r}\right),$$

$$\Lambda_{rr} = \frac{1}{\mathrm{Re}}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\right), \quad \Lambda_{\varphi\varphi} = \frac{1}{\mathrm{Re}\cdot r^2}\frac{\partial^2}{\partial \varphi^2}, \quad \Lambda_r = \frac{\partial}{\partial r}, \quad \Lambda_\varphi = \frac{1}{r}\frac{\partial}{\partial \varphi},$$

$$\Lambda_{rr}^{[p]} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}, \quad \Lambda_{\varphi\varphi}^{[p]} = \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

Upon denoting by

$$\Lambda_1 = \begin{pmatrix} \Lambda_{rr} & 0 & 0 & 0 & 0 \\ \frac{1}{2}E & \Lambda_{rr} & 0 & 0 & 0 \\ 0 & 0 & \Lambda_{rr} & 0 & 0 \\ 0 & 0 & \frac{1}{2}E & \Lambda_{rr} & -\Lambda_r \\ 0 & 0 & 0 & 0 & \Lambda_{rr}^{[p]} \end{pmatrix}$$

and

$$\Lambda_2 = \begin{pmatrix} \Lambda_{\varphi\varphi} & 0 & 0 & 0 & 0 \\ \frac{1}{2}E & \Lambda_{\varphi\varphi} & 0 & 0 & -\Lambda_\varphi \\ 0 & 0 & \Lambda_{\varphi\varphi} & 0 & 0 \\ 0 & 0 & \frac{1}{2}E & \Lambda_{\varphi\varphi} & 0 \\ 0 & 0 & 0 & 0 & \Lambda_{\varphi\varphi}^{[p]} \end{pmatrix},$$

we render the equations for Φ, u_φ, Ω, u_r and p to the following vectorial form for $\vec{\theta}$:

$$(\Lambda_1 + \Lambda_2)\vec{\theta} + F^{\vec{\theta}} = 0. \tag{2.8}$$

Upon adding derivatives with respect to a fictitios time t, we get

$$\frac{\partial \vec{\theta}}{\partial t} = (\Lambda_1 + \Lambda_2)\vec{\theta} + F^{\vec{\theta}}. \tag{2.9}$$

Note that in the physically unsteady case the time derivatives are present only in the equations for the velocity components (the original system is not of *Cauchy-Covalewska* type).

2.3. THE SPLITTING SCHEME

The system under consideration is non-linear. It can be solved by means of an iterational process in which at each stage the equations are linearized. In the present work we make use of the iterative procedure based on the co-ordinate-splitting method because of its computational efficiency. We employ the method of fractional steps, namely the second scheme of Douglas and Rachford [12], sometimes called the scheme of "stabilizing correction" [22]. The stabilizing correction scheme reads (τ is the increment of the fictitious time)

$$\frac{\vec{\theta}^{n+\frac{1}{2}} - \vec{\theta}^{n}}{\tau} = \Lambda_1 \vec{\theta}^{n+\frac{1}{2}} + \Lambda_2 \vec{\theta}^{n} + F^{\vec{\theta}^n}, \qquad \frac{\vec{\theta}^{n+1} - \vec{\theta}^{n+\frac{1}{2}}}{\tau} = \Lambda_2 \vec{\theta}^{n+1} - \Lambda_2 \vec{\theta}^{n},$$

or, which is the same,

$$(I - \tau\Lambda_1)\vec{\theta}^{n+\frac{1}{2}} = (I + \tau\Lambda_2)\vec{\theta}^{n} + \tau F^{\vec{\theta}^n}, \quad (I - \tau\Lambda_2)\vec{\theta}^{n+1} = \vec{\theta}^{n+\frac{1}{2}} - \tau\Lambda_2\vec{\theta}^{n}.$$

The approximation with respect to fictitious time can be assessed after excluding the half time–step variable $\vec{\theta}^{n+\frac{1}{2}}$. After some obvious manipulations, we obtain the equation

$$(I + \tau^2\Lambda_1\Lambda_2)\,\frac{\vec{\theta}^{n+1} - \vec{\theta}^{n}}{\tau} = (\Lambda_1 + \Lambda_2)\vec{\theta}^{n+1} + F^{\vec{\theta}^n}. \tag{2.10}$$

The splitting scheme is implicit for the linear terms and explicit for the non-linear convective terms.

2.4. GRID PATTERN AND APPROXIMATIONS

The flow field shows a mixture of different scales for high Re. There is a thin boundary layer close to the body, which separates and extends downstream. Neither Cartesian nor polar co-ordinate systems are adequate enough for describing the topology of the flow when the separation takes place. These problems are aggravated with the increase of the Reynolds number. The usual polar co-ordinate

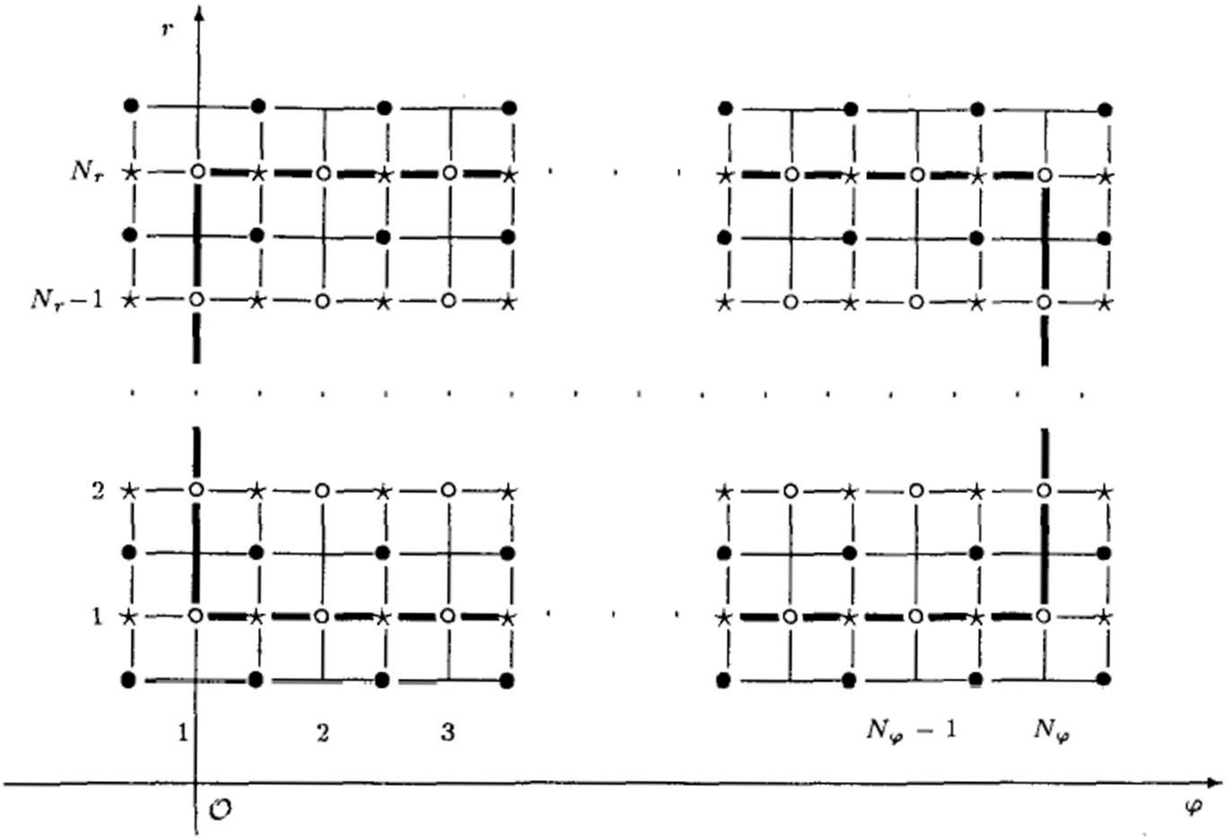

Fig. 1. Grid pattern

system, dense enough to resolve the wake far out, will be very wasteful in other directions. For this reason our mesh labelled by 'o' is chosen to be non-uniform. The spacings are given by

$$h_r = \frac{R-1}{N_r-1}, \quad h_\varphi = \frac{\pi}{N_\varphi - 1},$$

where N_r stands for the number of points in the direction r and N_φ — in the direction φ, respectively. The mesh is staggered for p in direction φ. For u_r and Ω it is staggered in both directions. In Fig. 1 the mesh is depicted, where the thick lines represent the borders of the region of computations. The co-ordinates of a point of the mesh are defined as follows:

$$r_i = \exp\left[(i-1)h_r\right], \; \varphi_j = \frac{1}{\pi}\left[(j-1)h_\varphi\right]^2, \text{ where } i = 1, \ldots, N_r; \; j = 1, \ldots, N_\varphi.$$

The points in Fig. 1, which are labelled by 'o', are those where the functions u_φ and Φ are calculated. The pressure is calculated in the points labelled by '$\star$', and functions u_r and Ω are calculated in the grid point labelled by '$\bullet$'.

We employ the following two-point and three-point approximations for the first and second derivatives (equivalent to the central differences scheme on uniform mesh). The derivatives are approximated as

$$\left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2}} \approx \frac{u_i - u_{i-1}}{h_{i-1}},$$

$$\left.\frac{\partial u}{\partial x}\right|_i \approx \frac{h_{i-1}}{h_i(h_i+h_{i-1})}u_{i+1} + \frac{h_i - h_{i-1}}{h_i h_{i-1}}u_i - \frac{h_i}{h_{i-1}(h_i+h_{i-1})}u_{i-1},$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i \approx \frac{2}{h_i+h_{i-1}}\left[\frac{u_{i+1}-u_i}{h_i} - \frac{u_i - u_{i-1}}{h_{i-1}}\right],$$

where u stands for Φ, u_φ, Ω, u_r or p and x stands for r or φ. Respectively, h_{i-1} and h_i are the values of the spacing on the left or on the right from the reference point (Fig. 2).

Fig. 2. A point of the non-uniform mesh

The staggered mesh allows to use second order approximations for the boundary conditions. All boundary conditions are imposed implicitly.

2.5. IMPLEMENTATION OF THE SCHEME

The algebraic problem is coupled with the difference approximations of the boundary conditions. The boundary conditions for the pressure equation stem from the additional condition on function u_r from the continuity equation. The idea consists in treating the system for different half–time steps as conjugated (see [19, 2]). On the first half-time step (the operators with derivatives with respect to r) we solve the equations for the "vector" $\{\Phi, u_\varphi\}$. Respectively, the equations for the "vector" $\{\Omega, u_r, p\}$ are solved simultaneously. On the second half-time step (derivatives with respect to φ) the respective equations for the vectors $\{p, u_\varphi, \Phi\}$ and $\{u_r, \Omega\}$ are solved. The arguments for selecting the "pairs" and "triplets" of equations are obvious: Φ enters the equation for u_φ, while Ω enters the equation for u_r. The resulting systems are either five- or seven-diagonal and can be treated by the solver described in [7]. The method of the so-called non–monotonous *progonka* is a kind of Gaussian elimination with pivoting and it is highly efficient for multidiagonal cases. The solution algorithm allows for complete coupling of the boundary conditions.

We solve the system governing the functions $\Phi(r,\varphi)$, $u_\varphi(r,\varphi)$, $\Omega(r,\varphi)$, $u_r(r,\varphi)$ and $p(r,\varphi)$ in the following iterational manner:

(i) The initial conditions Φ^0, u_φ^0, Ω^0, u_r^0, p^0 for small Reynolds numbers (Re $\approx 2 \div 4$) are defined as

$$\Phi|_{i,j} = 0,\ u_\varphi|_{i,j} = \frac{r_i - 1}{r_\infty - 1}\cos\varphi_j,\ \Omega|_{i,j} = 0,\ u_r|_{i,j} = \frac{r_i - 1}{r_\infty - 1}\sin\varphi_j,\ p|_{i,j} = 0.$$

For larger values of Reynolds number the solution for the closest smaller Re is used as the initial condition for the iterations for the current Re.

The counter of time steps is set $n = 0$;

(ii) On the first half–step for the line $\varphi = \varphi_j$ we solve two systems for the unknowns $\Phi^{n+\frac{1}{2}}$, $u_\varphi^{n+\frac{1}{2}}$, $\Omega^{n+\frac{1}{2}}$, $u_r^{n+\frac{1}{2}}$ and $p^{n+\frac{1}{2}}$ — with a seven-diagonal matrix for $\{\Omega, u_r, p\}$ and a five–diagonal matrix for $\{\Phi, u_\varphi\}$;

(iii) Similarly, for the vectors Φ^{n+1}, u_φ^{n+1}, Ω^{n+1}, u_r^{n+1} and p^{n+1}, we solve for the lines $r = r_i$ on the second half-step two systems — with a five-diagonal matrix for $\{\Omega, u_r\}$ and a seven–diagonal matrix for $\{\Phi, u_\varphi, p\}$;

(iv) The norm of the difference between two consecutive iterations $(n+1)$ and (n) (time steps with respect to fictitious time)

$$\max_{i,j} |\vec{\theta}^{n+1} - \vec{\theta}^{n}|$$

is calculated. If this norm is lesser than a prior prescribed value, then the calculations are terminated. Otherwise the index of iterations is stepped up $n := n+1$ and the algorithm is returned to step (ii).

3. RESULTS AND DISCUSSION

In order to assess the approximation of the proposed scheme and the performance of the algorithm, a number of numerical experiments have been conducted.

The accuracy of the developed here difference scheme and algorithm is checked with the mandatory tests involving different increments of the fictitious time τ and mesh parameters: N_φ, N_r and r_∞.

First of all, we check that the approximation of the steady-state solution does not depend of the fictitious time increment of the splitting scheme. Theoretically, it follows from equation (2.10) and provides a good check for the correctness of the algorithm if it is respected in practice. We have calculated the flow with Re $= 40$ with three different fictitious-time increments: $\tau = 0.1, 0.01, 0.001$. We have found that the iterative solution of the steady-state problem does not depend on the magnitude of the time increment τ.

The second important verification is the spatial approximation of the scheme. We have conducted a number of calculations with different values of mesh parameters and verified the practical convergence and the approximation of the difference scheme. In Table 1 and Table 2 we present the obtained numerical results for some parameters like positions of separation point φ_{sep} from the rear stagnation point, difference between the pressure at front and at rear of stagnation point. The values of the drag coefficient C_D are computed from

$$C_D = -2 \int_0^\pi p(1,\varphi) \cos\varphi \, d\varphi - \frac{4}{\text{Re}} \int_0^\pi \left. \frac{\partial u_\varphi}{\partial r} \right|_{r=1} \sin\varphi \, d\varphi.$$

The first term on the right gives the pressure drag coefficient and the second — the friction drag coefficient. In these tests Re = 100, $r_\infty \approx 53$. For $N_\varphi = 125$ and $N_\varphi = 195$ the difference between the present results for $\varphi_{\rm sep}$ and $p(1,\pi) - p(1,0)$ is indistinguishable within the accuracy of calculations with ordinary precision. But for $N_r = 126$ and $N_r = 161$ the flow picture changes significantly. These calculations illustrate the convergence of the difference approximation to the solution of the differential problem under study.

TABLE 1. Results of calculation for different N_φ, Re = 100, $N_r = 126$ and $r_\infty \approx 53$

N_φ	$\varphi_{\rm sep}$	C_D	$p(1,\pi) - p(1,0)$
100	1.013	0.89820	0.56900
125	1.031	0.89289	0.59805
195	1.031	0.89081	0.59805

TABLE 2. Results of calculation for different N_r, Re = 100, $N_\varphi = 126$ and $r_\infty \approx 53$

N_r	$\varphi_{\rm sep}$	C_D	$p(1,\pi) - p(1,0)$
126	1.031	0.89289	0.59805
161	1.049	0.94539	0.63609
199	1.065	0.96055	0.64400

The values of r_∞ are obtained as the results of experience [13, 11].

We have successful calculations for $2 \leq \text{Re} \leq 200$. The numbers of grid points N_φ and N_r cannot be very large (although it is desired) due to computer limitations. The tests have shown that for Re ≤ 100 the mesh size with $N_\varphi = 150$, $N_r = 251$, $r_\infty \approx 88$ is probably safe. The calculations at Re = 200 presented bellow are carried out by using the mesh size $N_\varphi = 125$, $N_r = 199$, $r_\infty \approx 80$.

Some of our results along with the results of other authors are given in Table 3 for comparison. The values of C_D, L and $p(1,\pi) - p(1,0)$ for Re = 20, 40, 50, 100 are in good agreement with those of Fornberg [13] and reasonably agree with those of Takami and Keller [21] and Dennis and Chang [11].

TABLE 3. Calculated values for C_D, L and $p(1,\pi) - p(1,0)$

Ref.	Re	C_D	L	$p(1,\pi) - p(1,0)$
[21]	20	2.003	2.87	
	40	1.536	5.65	
	50	1.418	7.10	
[11]	20	2.045	2.88	0.9290
	40	1.522	5.69	0.8265
	100	1.056	14.11	0.7265
[13]	20	2.000	2.82	0.910
	40	1.498	5.48	0.801
	100	1.060	13.20	0.693
	200	0.831	26.20	0.589
Present work	20	2.0095	2.84	0.9215
	40	1.4877	5.47	0.8163
	50	1.3470	6.61	0.7843
	100	1.0052	13.33	0.6956
	200	0.6769	17.81	0.4740

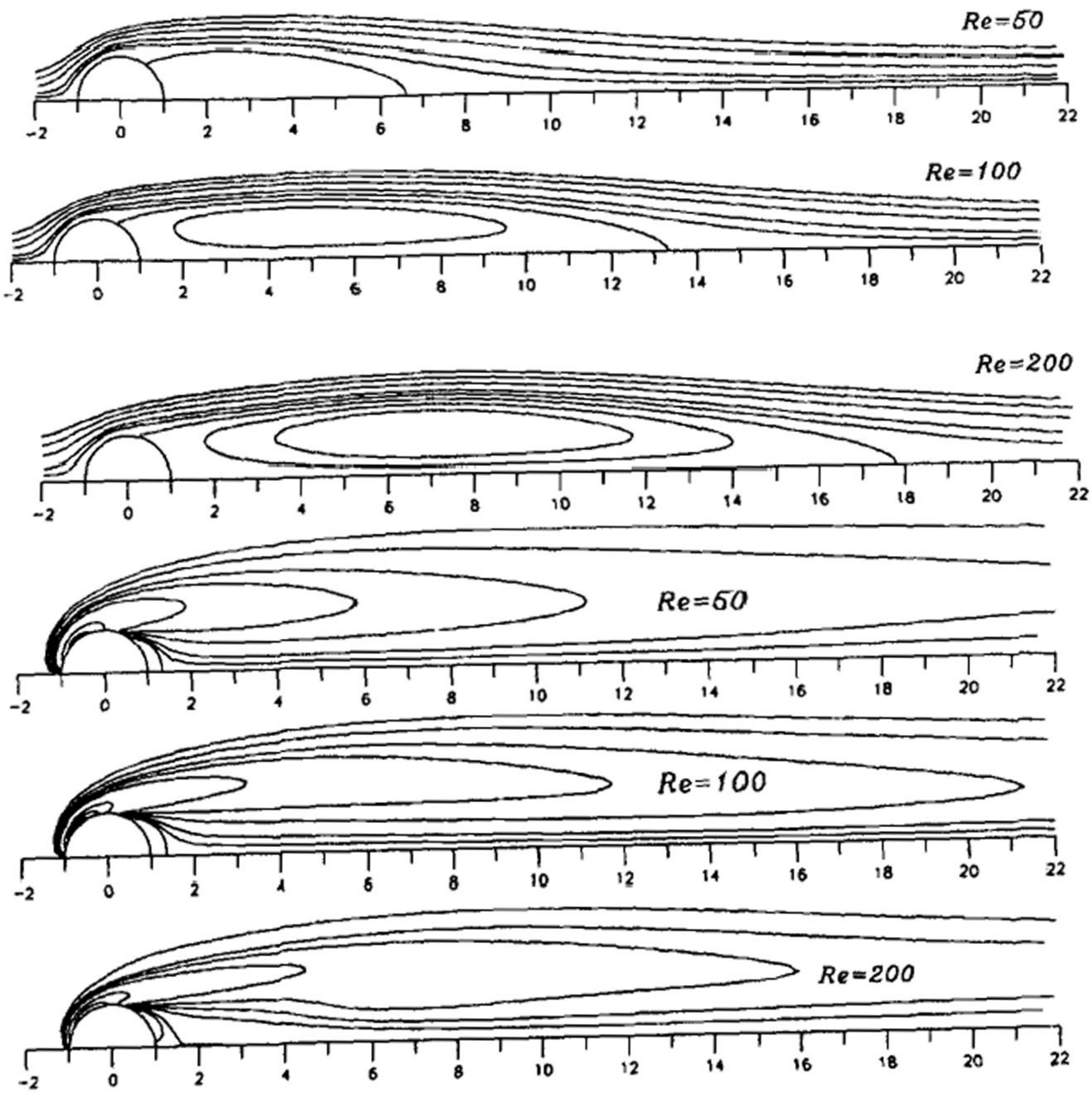

Fig. 3. Streamlines and vorticity fields

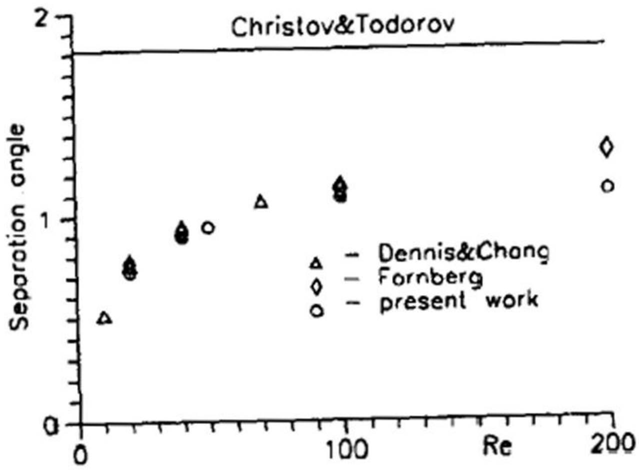

Fig. 4. The separation angle

Streamlines and vorticity isolines of the flow for Reynolds numbers 50, 100 and 200 are shown in Fig. 3. For the stream function the contour values, starting from the top, are {0.4, 0.3, 0.2, 0.1, 0.05}; enclosed streamlines, starting from the centre, are {−0.1, −0.05, 0}, and for the vorticity the contour values are {0.1, 0, −0.2, −0.4, −0.6, −1, −3, ...}. Fig. 4 gives the calculated values of φ_{sep} measured from the rear stagnation point. They are in good agreement with the calculations of Fornberg [13] and of Dennis and Chang [11]. Theory based on the Helmholtz-Kirchhoff model predicts that as Re goes to infinity [1], the separation point may move forward to an angle of 125°. In [10] this angle is $\varphi = 1.815(104°)$. Our results and this possible limit are shown in Fig. 4.

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The manuscripts must be *typed* on one side of the paper in double spacing with wide margins. On the *first* page the author should provide: a title, name(s) of the author(s), a short abstract, a list of keywords and the appropriate 1995 Mathematical Subject Classification codes (primary and secondary, if necessary). The affiliation(s), including the electronic address, is given at the end of the manuscripts.

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Tables should be inserted in the text as close to the point of reference as possible. Some space should be left above and below the table.

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References must be cited in the text in square brackets, like [3], or [5, 7], or [11, p. 123], or [16, Ch. 2.12]. They have to be numbered either in the order they appear in the text or alphabetically. Examples (please note the order, style and punctuation):

For books: Obreshkoff, N. Higher algebra. Nauka i Izkustvo, 2nd edition, Sofia, 1963 (in Bulgarian).

For journal articles: Frisch, H. L. Statistics of random media. *Trans. Soc. Rheology*, 9, 1965, 293–312.

For articles in edited volumes or proceedings: Friedman, H. Axiomatic recursive function theory. In: *Logic Colloquium 95*, eds. R. Gandy and F. Yates, North-Holland, 1971, 188–195.