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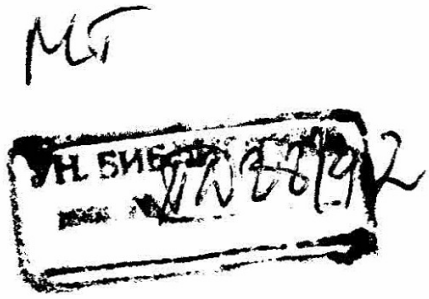

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СЪДЪРЖАНИЕ

CONTENTS

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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ВЪРХУ ПРАВОМЕРНОСТТА НА МЕТОДИТЕ С ДИФЕРЕНЦИАЛНИ И ДИФЕРЕНЧНИ МЕТОДИ ЗА ИЗСЛЕДВАНЕ НА УСТОЙЧИВОСТТА НА БЕЗНАСТАВОВ РЕЛСОВ ПЪТ

АТАНАС СМИЛЯНОВ, ЦВЕТАН МИТРОВ

Атанас Смилянов, Цветан Митров. О ПРАВОМЕРНОСТИ МЕТОДОВ С ДИФЕРЕНЦИАЛЬНЫМИ И ДИФЕРЕНЧНЫМИ УРАВНЕНИЯМИ ДЛЯ ИССЛЕДОВАНИЯ БЕЗНАСТАВНОГО РЕЛЬСОВОГО ПУТИ. **Задача исследования устойчивости рельсового пути существует очень давно. Известны более 100 теоретических методов. В работе проводится сравнение между диференциалными и диференчными методами.**

Atanas Smilijanow, Zvetan Mitrov. ON THE CORRECTNESS OF DIFFERENTIAL AND DIFFERENCE METHODS FOR ANALYSIS OF THE STABILITY OF CONTINUOUS RAIL WAYS. **The problem of the rail ways stability exists for a long time and there are about 100 theoretical methods to solve it. In this paper the connection between differential and difference methods is discussed.**

Задачата за изследване на устойчивостта на безнаставовия релсов път е известна отдавна в железопътната практика. Създадените методи за решаването ѝ се основават на различни изходни предпоставки. Това предопределя голямото разнообразие от физически постановки и механоматематически модели [1]. Големият брой съществуващи методи за теоретично решение — над 100, позволява да се прави съпоставително изследване с цел класификация и прогнозиране на направленията, в които би могло да се очаква поява на нови теоретични модели [1].

Сред намерилите приложение в практиката особен интерес представляват методите, свеждащи физическата постановка на задачата до за-

меняне на релсотраверсовата скара с безкрайно дълъг прът (заместваща греда). Характерното е, че изходните предпоставки предполагат при еднакви натоварвания с реалния случай заместващата греда да дава същите деформации (завъртания и премествания), както и изследваната релсотраверсова решетка. Обикновено взаимодействието между релсите и траверсите се отчита чрез съпротивлението срещу взаимното завъртане на траверсата спрямо релсата в даден възел. При това моментите във възлите се приемат за пропорционални на завъртането, а действието им (във възлите) се разпределя по цялата дължина на приетата заместваща греда ([2], [3], [4] и др.).

Типичен и много популярен представител е моделът, разработен от Леви [5]. Свежда се до решение на диференциално уравнение от типа

$$y^{IV} + Ay'' + Cy = 0 \tag{1}$$

с гранични условия съгласно (2):

$$\begin{aligned} y(0) &= 0; \quad y(l) = 0, \\ y''(0) &= 0; \quad y''(l) = 0. \end{aligned} \tag{2}$$

Някои по-нови изследвания оспорват коректността на физическата постановка, водеща до математически модели от типа на уравнения (1) и (2). Известни са модели, които ползват различен по сложност математически апарат за отчитане рамковидното действие на релсотраверсовата скара и реактивното въздействие на баластовото легло. Типичен в това отношение е разработеният в БДЖ метод [6]. Той отдава изключително голямо значение на изходните предпоставки с оглед максимално възможно доближаване на механоматематическия модел до реалната физическа задача. Тези основни предпоставки са:

— железният път се разглежда като безкрайно дълга рамка с деформируеми възли, положена в еластична среда. Силите от тази среда се предават върху рамката само чрез траверсите;

— всяка траверса при огъването си се третира като греда върху еластична основа;

— при напречните премествания на скарата силите от баластовото легло се приемат като съсредоточени върху челата на траверсите;

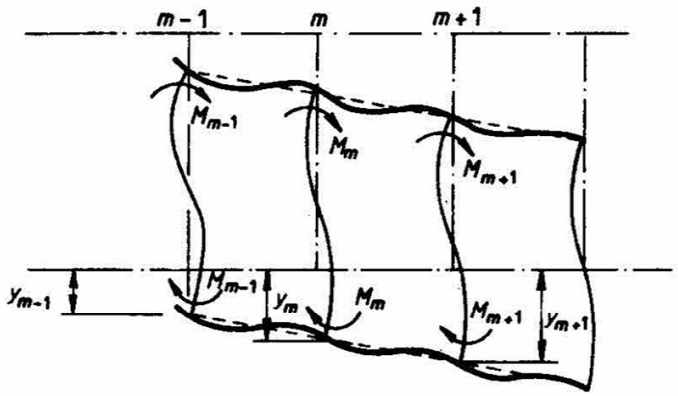
— еластичната система, с която се заменя пътят и първоначално е подложена само на действието на надлъжните натискови сили в релсите, се разглежда в равновесие при такова деформирано състояние, което се отличава от първоначалното с наличие на безкрайно малки деформации от нов вид;

— силите, въздействащи на релсотраверсовата скара, се възприемат по схемата на фиг. 1, а разрезните усилия, поддържащи системата в равновесие — по схемата на фиг. 2.

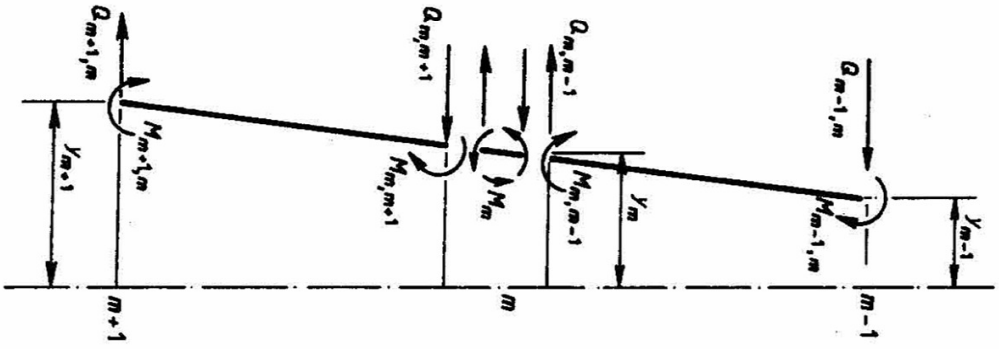
От условията за:

— взаимното завъртане между релсовия път $m-1, m$ и траверсата m, m;

— взаимното завъртане между релсовия път $m-1, m$ и траверсата $m-1$, $m-1$;


m-1
m
m+1
M_{m-1}
M_m
M_{m+1}
y_{m-1}
M_{m-1}
y_m
M_m
M_{m+1}
y_{m+1}

Фиг. 1


$Q_{m+1,m}$
$Q_{m,m+1}$
$Q_{m,m-1}$
$Q_{m-1,m}$
$M_{m+1,m}$
$M_{m,m+1}$
M_m
$M_{m,m-1}$
$M_{m-1,m}$
y_{m+1}
y_m
y_{m-1}
m+1
m
m-1

Фиг. 2

— равновесие на моментите във възел m;

— равновесие на напречните сили във възел m;

се извежда математическият модел на развития в [6] метод. Той се свежда до системата от диференчни уравнения

$$(3) \qquad \left\| \begin{array}{l} f_0 M_{m-1} + f_1 M_m + f_0 M_{m+1} + Y_{m-1} - Y_{m+1} = 0 \\ M_{m-1} - M_{m+1} + g_0 Y_{m-1} - g_1 Y_m + g_0 Y_{m+1} = 0, \end{array} \right.$$

където:

$$f_0 = k\ell \frac{\nu - \sin\nu}{\nu(1-\cos\nu)};$$

$$f_1 = \ell \frac{\nu(2ki_{\text{р}} - 1)\sin\nu - 2(ki_{\text{р}}\nu^2 + 1)\cos\nu + 2}{i_{\text{р}}\nu^2(1-\cos\nu)};$$

$$g_0 = \frac{\nu(1-\cos\nu)}{k\ell\sin\nu};$$

$$g_1 = \frac{\nu(2\nu^2 i_{\text{р}} - \frac{B\ell^2}{2})(1+\cos\nu) + B\ell^2\sin\nu}{k\ell i_{\text{р}}\nu^2\sin\nu};$$

M_m — моментът в сечение m на скарата;

Y_m — преместване в сечение m на скарата;

$$i_p = \frac{E_{\text{р}} I_{\text{р}}}{\ell};$$

$$\nu^2 = \frac{F\ell^2}{E_{\text{р}} I_{\text{р}}};$$

$E_{\text{р}}, I_{\text{р}}, E_{\text{тр}}, I_{\text{тр}}$ — модули на еластичност и инерционен момент на релсата и траверсата спрямо вертикалната ос;

$$K = \frac{K_1}{6i_{\text{тр}}} + \bar{\eta};$$

η — взаимното завъртане между релсата и траверсата, причинено от момент, числено равен на единица и действащ в скарата на мястото на свързването на релсата с траверсата;

$$i_{\text{тр}} = \frac{E_{\text{тр}} I_{\text{тр}}}{\ell_{\text{тр}}};$$

$$K_1 = \frac{3}{\omega\ell_{\text{тр}}} - (\text{sh}\omega\ell_{\text{тр}} - \sin\omega\ell_{\text{тр}})(\text{ch}^2\omega\ell + \cos^2\omega\ell)\left[(\text{sh}\omega\ell_{\text{тр}} - \sin\omega\ell_{\text{тр}})\right.$$

$$\left.\times\,(\text{sh}\omega\ell\text{ch}\omega\ell - \sin\omega\ell\cos\omega\ell) + (\text{ch}\omega\ell_{\text{тр}} - \cos\omega\ell_{\text{тр}})(\text{ch}^2\omega\ell_{\text{тр}} + \cos^2\omega\ell_{\text{тр}})\right]^{-1};$$

$$\omega = \sqrt{\frac{C}{4E_{\text{тр}} I_{\text{тр}}}};$$

C — коефициент на съпротивление на баластовото легло срещу изместване на траверсите надлъжно на железния път.

Моделът с диференчни уравнения от [6] е безусловно един връх в развитието на теорията на устойчивостта на безнастановия релсов път. Той дава интересно, а от техническа гледна точка дори елегантно решение. Получените резултати са с точност, задоволяваща изискванията на практиката. Не случайно върху него е основана цялата практическа организация на полагането и поддържането на безнастанов релсов път у нас - норми, инструкции и пр. Но разработеният метод има и някои слаби страни. Например задоволяващата изискванията на практиката точност на теоретичното решение и обосноваваните върху него норми в хронологичен порядък са създали условия на тях да се гледа като на предел на теоретичните възможности. Поради това години наред не са търсени нови, по-мощни теоретични решения. Нещо повече, на работата в тази посока се гледа като на икономически и практически никому ненужно задълбочаване в теорията, т.е. като на своеобразна еклектика.

В същото време, подложено на строг съпоставителен анализ спрямо други теоретични модели, решението от [6] не издържа на посочените претенции. Това е така, защото, от една страна, то е основано на сравнително сложен математически апарат и, от друга страна, претенциите му за по-голяма точност на резултатите могат да бъдат поставени под съмнение. Това твърдение се доказва чрез възможността моделът от [3] да се преобразува по математичен път в модел, състоящ се от едно диференциално уравнение от вид и ред, идентични с вида и реда на уравнение (1).

За целите на доказателството се разсъждава по следния начин:

Допуска се, че:

— координатата x се мени по дължината на релсовотраверсовата скара. Тогава точките m_i от модела по [6], изобразен с фиг.1 и фиг.2, могат да се означават с $x_{m_i}(x_{m-1}, x_m, x_{m+1}, \ldots)$;

— съществуват функциите $M(x)$ и $Y(x)$, с които могат да се изразят моментите и преместванията в релсотраверсовата скара и които са най-малко четири пъти диференцируеми.

Тогава може да се въведе означението

$$\begin{aligned} M(x_m) &= M_m, \\ Y(x_m) &= Y_m. \end{aligned} \tag{4}$$

Функциите (4) могат да се разложат по формулата на Тейлор до втори ред в точката x_m:

$$\begin{aligned} M(x) = M(x_m) &+ (x - x_m)M'(x_m) \\ &+ \tfrac{(x-x_m)^2}{2} M''(x_m) + O\left[(x - x_m)^2\right], \\ Y(x) = Y(x_m) &+ (x - x_m)Y'(x_m) \\ &+ \tfrac{(x-x_m)^2}{2} Y''(x_m) + O\left[(x - x_m)^2\right], \end{aligned} \tag{5}$$

където:

(6) $$O[(x-x_m)^2] = \alpha(x-x_m)(x-x_m)^2$$
$$\text{и } \alpha(x-x_m) \to 0 \text{ при } x \to x_m.$$

Ако X заеме стойностите X_{m-1} и X_{m+1}, се получава съответно:

(7)
$$\begin{aligned}
M_{(x=x_{m-1})} &= M(x_m) + (x_{m-1}-x_m)M'(x_m) \\
&\quad + \tfrac{(x_{m-1}-x_m)^2}{2}M''(x_m) + O\left[(x_{m-1}-x_m)^2\right], \\
M_{(x=x_{m+1})} &= M(x_m) + (x_{m+1}-x_m)M'(x_m) \\
&\quad + \tfrac{(x_{m+1}-x_m)^2}{2}M''(x_m) + O\left[(x_{m+1}-x_m)^2\right], \\
Y_{(x=x_{m-1})} &= Y(x_m) + (x_{m-1}-x_m)Y'(x_m) \\
&\quad + \tfrac{(x_{m-1}-x_m)^2}{2}Y''(x_m) + O\left[(x_{m-1}-x_m)^2\right], \\
Y_{(x=x_{m+1})} &= Y(x_m) + (x_{m+1}-x_m)Y'(x_m) \\
&\quad + \tfrac{(x_{m+1}-x_m)^2}{2}Y''(x_m) + O\left[(x_{m+1}-x_m)^2\right].
\end{aligned}$$

Като се отчете, че ℓ е насочена дължина, т.е.

(8) $$-\ell = x_{m-1} - x_m \qquad \text{и} \qquad +\ell = x_{m+1} + x_m,$$

и след умножаване на първите две уравнения от (7) с f_0 и f_1, както и след извършване на известни математически преобразувания от (3) и (7) се получава

(9) $$f_0\ell^2 M''(x_m) + M(x_m)(2f_0+f_1) - 2\ell Y'(x_m) + O(\ell^2) = 0.$$

Аналогично от уравнения (3) и (7) се получава второ уравнение

(10) $$g_0\ell^2 Y''(x_m) + Y(x_m)(2g_0-g_1) - 2\ell M'(x_m) + O(\ell^2) = 0.$$

Уравнения (9) и (10) образуват системата

(11)
$$\left\| \begin{aligned}
&\ell^2 g_0 Y''(x_m) - 2\ell M'(x_m) + Y(x_m)(2g_0-g_1) + O(\ell^2) = 0 \\
&\ell^2 f_0 M''(x_m) - 2\ell Y'(x_m) + M(x_m)(2f_0+f) + O(\ell^2) = 0.
\end{aligned} \right.$$

Ако се въведе нова променлива

(12) $$z = \ell x$$

и се пропусне остатъкът, отбелязан с $O(\ell^2)$ като нямащ съществено влияние върху точността на крайния резултат, може да се пристъпи към разглеждане на системата диференциални уравнения

(13)
$$\left\| \begin{aligned}
&Y''(z)g_0 - 2M'(z) + (2g_0-g_1)Y(z) = 0 \\
&M''(z)f_0 - 2Y'(z) + (2f_0+f_1)M(z) = 0.
\end{aligned} \right.$$

Ако първото уравнение от системата (13) се диференцира два пъти, а второто - един път, получават се уравненията

$$g_0 Y'^{V}(z) + (2g_0 - g_1)Y''(z) - 2M'''(z) = 0, \tag{14}$$

$$f_0 M'''(z) - 2Y''(z) + (2f_0 + f_1)M'(z) = 0. \tag{15}$$

От (15) може да се изрази $M'''(z)$ и да се замести в (14). Получава се ново уравнение. Условно може да се отбележи с E. Ако от първото уравнение на системата (13) се изрази $M'(z)$ и се замести в E, получава се съвършено ново уравнение

$$Y'^{V} + aY'' + bY = 0, \tag{16}$$

където:

$$a = \frac{4g_0 f_0 + g_0 f_1 - f_0 g_1 - 4}{f_0 g_0}, \qquad b = \frac{(2g_0 - g_1)(2f_0 + f_1)}{f_0 g_0}. \tag{17}$$

Уравнение (16) по вид и ред се идентифицира с уравнение (1). С това възможността за преход от модел с диференчни уравнения към модел с диференциални уравнения е доказана. По аналогичен път може да се извърши и обратният преход - от модел с диференциални уравнения от типа (1) към модел с диференчни уравнения от типа (3).

От приведеното доказателство могат да се направят различни изводи. Най-съществените от тях се свеждат до:

— Моделите, основани на диференциални уравнения, съдържат в себе си моделите с диференчни уравнения. Последните са само частен случай от решението с диференциални уравнения.

— По-съвършена постановка на физическия модел на решаваната задача е с ограничителни гранични условия. Това е така, защото за да се реши задачата, е необходимо безнастановият релсов път да се третира като крайна по размери скара с някаква дължина ℓ и извън този краен размер състоянието на пътя и промените в температурата не влияят на устойчивостта в разглеждания участък. В контекста на това например физическият смисъл от въвеждането на граничните условия (2) за модела (1) с най-грубо приближение е, че преместването и ускорението в началото и края на участъка е нула.

— Изложеното в горните изводи и приведеното математическо доказателство за преход на модела (3) в (1) позволяват да се очаква създаване на принципно ново теоретично решение. Може да се прогнозира, че то би обединило идеологията на модела (1) - изследване на глобалното поведение на безнастановия път при загуба на устойчивост, с възможностите на модела (3) - информация за локалното поведение на безнастановата конструкция в отделни възли. Това ново решение би съчетало предимствата и на двата метода и би позволило ползуване на съвременни числени методи и електронноизчислителна техника.

И накрая един извод с по-дълбоко методологическо значение:

Приведеното доказателство не отрича достойнствата на разработения в [6] метод. То обаче потвърждава, макар и по косвен начин, известната

максима, че авторитарните мнения в науката могат най-много да задържат, но не могат да спрат въобще развитието на нови, по-съвършени и по-съвременни решения. За конкретния пример това се оказва вярно дори и за случай, при който практиката е удовлетворена от постигнатото равнище на теорията и не стимулира търсене и създаване на нови теоретични решения.

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ПЪЛНИ СИСТЕМИ ОТ ФУНКЦИИ НА ВЕБЕР — ЕРМИТ

ПЕТЪР РУСЕВ

Петър Русев. ПОЛНЫЕ СИСТЕМЫ ФУНКЦИЙ ВЕБЕРА — ЭРМИТА. Пусть $\{D_\nu(z)\}_{\nu\in\mathbf{C}}$ обозначает систему функций Вебера — Эрмита. Доказывается, что система

$$(*) \qquad \left\{\exp\left(-t^2/4\right) D_{\omega n+\sigma}(t)\right\}_{n=0}^{\infty}$$

полна в пространстве $L_2(-\infty,+\infty)$, если $0<\omega<4/3$ и $\mathrm{Re}\,\sigma>-1/2$. Полнота функций Ермита

$$(**) \qquad \left\{\exp\left(-t^2/2\right) H_n(t)\right\}_{n=0}^{\infty}$$

в том же пространстве является частным случаем ($\omega=1$, $\sigma=0$).

Peter Rusev. COMPLETE SYSTEMS OF WEBER — HERMITE FUNCTIONS. Let $\{D_\nu(z)\}_{\nu\in\mathbf{C}}$ be the system of Weber — Hermite functions. It is proved that the system (*) is complete in the space $L_2(-\infty,+\infty)$, if $0<\omega<4/3$ and $\mathrm{Re}\,\sigma>-1/2$. The completeness of Hermite functions (**) in the same space is a particular case ($\omega=1$, $\sigma=0$).

1. ФУНКЦИИ НА ВЕБЕР — ЕРМИТ

Всяко (аналитично) решение на диференциалното уравнение

$$y''+\left(\nu+\frac{1}{2}-\frac{z^2}{4}\right)y=0, \qquad \nu\in\mathbf{C},$$

се нарича функция на параболичния цилиндър или още функция на Вебер–Ермит. Такава е например функцията D_ν, дефинирана чрез

$$D_\nu(z) = 2^{\frac{\nu}{2}} \exp\left(\frac{-z^2}{4}\right) \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\nu}{2}\right)} \Phi\left(\frac{-\nu}{2}, \frac{1}{2}; \frac{z^2}{2}\right)$$
$$+ \frac{z}{\sqrt{2}} \frac{\Gamma\left(\frac{-1}{2}\right)}{\Gamma\left(\frac{-\nu}{2}\right)} \Phi\left(\frac{1-\nu}{2}, \frac{3}{2}; \frac{z^2}{2}\right),$$

където $\Phi(a, c; z)$ е една от стандартните изродени хипергеометрични функции ([1], II, 8.2, (4)). В частност, ако $\nu = n$ е цяло неотрицателно число,

$$\exp\left(\frac{t^2}{4}\right) D_n(t) = 2^{-\frac{n}{2}} H_n\left(\frac{t}{\sqrt{2}}\right), \qquad n = 0, 1, 2, \ldots,$$

където H_n е n-тият полином на Ермит ([1], II, 8.2, (9)).

От горното съотношение получаваме, че

$$\exp\left(-\frac{t^2}{2}\right) H_n\left(\frac{t}{\sqrt{2}}\right) = 2^{\frac{n}{2}} \exp\left(-\frac{t^2}{4}\right) D_n(t), \qquad n = 0, 1, 2, \ldots,$$

и ако дефинираме $H_n^*(t) = 2^{-\frac{n}{2}} H_n\left(\frac{t}{\sqrt{2}}\right)$, идваме до равенството

$$\exp\left(-\frac{t^2}{2}\right) H_n{}^*(t) = \exp\left(-\frac{t^2}{4}\right) D_n(t), \qquad n = 0, 1, 2, \ldots \tag{1}$$

Както не е трудно да се убедим, пълнотата на системата

$$\left\{\exp\left(-\frac{t^2}{2}\right) H_n^*(t)\right\}_{n=0}^{\infty}$$

в пространството $L_2(-\infty, +\infty)$ е еквивалентна с пълнотата на системата функции на Ермит

$$\left\{\exp\left(-\frac{t^2}{2}\right) H_n(t)\right\}_{n=0}^{\infty} \tag{2}$$

в същото пространство. Това, накратко казано, е следствие от обстоятелството, че всяка от системите $\{H_n(t)\}_{n=0}^{\infty}$ и $\{H_n^*(t)\}_{n=0}^{\infty}$ е линейно независима, тъй като $\deg H_n = \deg H_n^* = n$ за всяко $n = 0, 1, 2, \ldots$ и следователно е базис на пространството на полиномите.

В сила е интегралното представяне ([1],II, 8.3, (4))

$$\sqrt{\frac{\pi}{2}} \exp\left(-\frac{t^2}{4}\right) D_\nu(t) = \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^\nu \cos\left(tu - \nu\frac{\pi}{2}\right) du,$$

което е валидно, ако $\mathrm{Re}\nu > -1$.

Да дефинираме за $t \in (-\infty, +\infty)$, $z \in \mathbb{C}$ и $\sigma \in \mathbb{C}$

$$W_\sigma(z, t) = \sqrt{\frac{\pi}{2}} \exp\left(-\frac{t^2}{4}\right) D_{z+\sigma}(t).$$

Тогава получаваме интегралното представяне

$$W_\sigma(z,t) = \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} \cos\left(tu - \pi\frac{z+\sigma}{2}\right) du,$$

което е валидно за $\mathrm{Re}z > -1 - \mathrm{Re}\sigma$. От него следва, че

$$W_\sigma(z,t) = \cos\frac{\pi(z+\sigma)}{2} U_\sigma(z,t) + \sin\frac{\pi(z+\sigma)}{2} V_\sigma(z,t), \tag{3}$$

където

$$U_\sigma(z,t) = \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} \cos tu\, du, \tag{4}$$

$$V_\sigma(z,t) = \int_0^\infty \exp\left(-\frac{u^2}{2}\right) .u^{z+\sigma} \sin tu\, du. \tag{5}$$

Каквото и да е $x_0 > -1 - \mathrm{Re}\sigma$, всеки от интегралите в (4) и (5) е равномерно сходящ в ивицата $-1 - \mathrm{Re}\sigma < \mathrm{Re}z < x_0$. Следователно всеки от тях дефинира комплексна функция, която е холоморфна в полуравнината $\mathrm{Re}z > -1 - \mathrm{Re}\sigma$. Съгласно (3) същото важи и за функцията $W_\sigma(z,t)$, разглеждана като функция на комплексната променлива z в тази полуравнина.

Функциите (4) и (5) могат да се изразят чрез функцията $\Phi(a,c;z)$, а именно в сила са представянията ([2], стр.509, 3.952, 7., 8.)

$$2^{-\frac{z+\sigma-1}{2}} U_\sigma(z,t) = \Gamma\left(\frac{z+\sigma+1}{2}\right) \Phi\left(\frac{z+\sigma+1}{2}, \frac{1}{2}; -\frac{t^2}{2}\right), \tag{6}$$

$$2^{-\frac{z+\sigma}{2}} V_\sigma(z,t) = t\Gamma\left(\frac{z+\sigma+2}{2}\right) \Phi\left(\frac{z+\sigma+2}{2}, \frac{3}{2}; -\frac{t^2}{2}\right). \tag{7}$$

От асимптотичната формула ([1], I, стр.286, 6.13)

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{(-a)}\left(1 + O(|x|^{-1})\right), \qquad \mathrm{Re}x \to -\infty$$

и от представянията (6) и (7) следва тогава, че за всяко $z = x + iy$ с $x > -1 - \mathrm{Re}\sigma$:

$$|U_\sigma(z,t)| = O(|t|^{-x-\mathrm{Re}\sigma-1}), \quad |t| \to +\infty,$$
$$|V_\sigma(z,t)| = O(|t|^{-x-\mathrm{Re}\sigma-1}), \quad |t| \to +\infty.$$

От горните съотношения следва, че каквото и да е z с $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$, всяка от функциите $U_\sigma(z,t)$ и $V_\sigma(z,t)$ е от пространството $L_2(-\infty,+\infty)$. От (3) следва, че същото важи и за $W_\sigma(z,t)$ като функция на t за всяко фиксирано z с $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$.

2. ПОМОШНИ ТВЪРДЕНИЯ

Л е м а 1. Каквито и да са $\alpha > 0$ и $\beta > 0$, съществува константа $A = A(\alpha, \beta)$, такава, че за всяко $z = x + iy$ с $x \geqq 0$ е изпълнено неравенството

$$\frac{\Gamma(\alpha x + \beta)}{\left|\Gamma\left(\alpha z + \beta + \frac{1}{2}\right)\right|} \leqq A \exp\left(\frac{\pi\alpha}{2}|y|\right).$$

Доказателство. Съгласно формулата на Стирлинг за $|\arg z| < \pi$ е в сила представянето

$$\Gamma(z) = \sqrt{2\pi} \exp\left\{\left(z - \frac{1}{2}\right)\ln z - z\right\}\{1 + \gamma(z)\}.$$

При това каквото и да е $0 < \varepsilon < \pi$, $\lim\limits_{z\to\infty} \gamma(z) = 0$, щом $|\arg z| \leqq \pi - \varepsilon$. Следователно съществува $0 < r < +\infty$, такова, че $\frac{1}{2} \leqq |1 + \gamma(z)| \leqq \frac{3}{2}$ при условие, че $\mathrm{Re} z \geqq 0$ и $|z| \geqq r$. Тогава, ако $x \geqq 0$ и $\alpha x + \beta \geqq r$,

$$\begin{aligned}\Gamma(\alpha x + \beta) &\leqq 3\sqrt{\frac{\pi}{2}} \exp\left\{\left(\alpha x + \beta - \frac{1}{2}\right)\ln(\alpha x + \beta) - \alpha x - \beta\right\} \\ &\leqq 3\sqrt{\frac{\pi}{2\beta}} \exp\{(\alpha x + \beta)\ln(\alpha x + \beta) - \alpha x - \beta\}.\end{aligned}$$

Дефинираме $l(\alpha, \beta)$ чрез

$$l(\alpha, \beta) = \max_{0\leq x\leq r} \frac{\Gamma(\alpha x + \beta)}{\exp\{(\alpha x + \beta)\ln(\alpha x + \beta) - \alpha x - \beta\}}$$

и означаваме $L(\alpha, \beta) = \max\left\{3\sqrt{\frac{\pi}{2\beta}}, l(\alpha, \beta)\right\}$. Тогава за всяко $x \geqq 0$ е изпълнено неравенството

$$\Gamma(\alpha x + \beta) \leqq L(\alpha, \beta) \exp\{(\alpha x + \beta)\ln(\alpha x + \beta) - \alpha x - \beta\}.$$

Също така за $x = \mathrm{Re} z \geqq 0$ и $|z| \geqq r$ е в сила неравенството

$$\left|\Gamma\left(\alpha z + \beta + \frac{1}{2}\right)\right|$$

$$\geqq \sqrt{\frac{\pi}{2}} \exp\left\{(\alpha x + \beta)\ln\left|\alpha z + \beta + \frac{1}{2}\right| - \alpha y \arg\left(\alpha z + \beta + \frac{1}{2}\right) - \alpha x - \beta - \frac{1}{2}\right\}$$

$$\geqq \sqrt{\frac{\pi}{2e}} \exp\left\{(\alpha x + \beta)\ln(\alpha x + \beta) - \left(\pi\frac{\alpha}{2}\right)|y| - \alpha x - \beta\right\},$$

или все едно неравенството

$$\left|\Gamma\left(\alpha z+\beta+\frac{1}{2}\right)\right|^{-1} \leqq \sqrt{\frac{2e}{\pi}}\exp\left\{-(\alpha x+\beta)\ln(\alpha x+\beta)+\frac{\pi\alpha}{2}|y|+\alpha x+\beta\right\}.$$

Дефинираме

$$m(\alpha,\beta)$$

$$=\max_{|z|\leqq r,\mathrm{Re}z\geqq 0}\left|\Gamma\left(\alpha z+\beta+\frac{1}{2}\right)\right|^{-1}\exp\left\{(\alpha x+\beta)\ln(\alpha x+\beta)-\frac{\pi\alpha}{2}|y|-\alpha x-\beta\right\}.$$

и означаваме $M(\alpha,\beta)=\max\left\{\sqrt{\frac{2e}{\pi}},m(\alpha,\beta)\right\}$. Тогава за всяко $z=x+iy$ с $x\geqq 0$ е изпълнено неравенството

$$\left|\Gamma\left(\alpha z+\beta+\frac{1}{2}\right)\right|^{-1} \leqq M(\alpha,\beta)\exp\left\{-(\alpha x+\beta)\ln(-\alpha x+\beta)+\frac{\pi\alpha}{2}|y|+\alpha x+\beta\right\}.$$

Остава да дефинираме $A(\alpha,\beta)=L(\alpha,\beta)/M(\alpha,\beta)$.

Л е м а 2. Ако $a>0$, $\lambda>0$, $\delta>-\frac{1}{2}$ и $0<\mu<2\lambda$, системата

$$\left\{\exp\left(-at^{\lambda}\right)t^{\mu n+\delta}\right\}_{n=0}^{\infty}$$

е пълна в пространството $L_2(0,+\infty)$.

Доказателство. За $\mathrm{Re}z>-\frac{\delta+\frac{1}{2}}{\mu}$ и $q\in L_2(0,+\infty)$ дефинираме

$$Q(z)=\int_0^{\infty}\exp(-at^{\lambda})t^{\mu z+\delta}q(t)\,dt. \tag{8}$$

Функцията $Q(z)$ е холоморфна в полуравнината $\mathrm{Re}z>-\frac{\delta+\frac{1}{2}}{\mu}$. Това е следствие от (абсолютно) равномерната сходимост на интеграла отдясно на (8) във всяка ивица от вида $-\frac{\delta+\frac{1}{2}}{\mu}<\mathrm{Re}z<x_0$, която от своя страна следва от неравенството на Шварц и допускането, че функцията $q\in L_2(0,+\infty)$.

Функцията $Q^*(z)$, дефинирана чрез $Q^*(z)=Q(\frac{z-\delta}{\mu})$, е холоморфна в полуравнината $\mathrm{Re}z>-\frac{1}{2}$ и в тази полуравнина е валидно интегралното представяне

$$Q^*(z)=\int_{-\infty}^{\infty}\exp(-zu)q^*(u)\,du,$$

където

$$q^*(u) = \exp\{-a\exp(-\lambda u) - u\}\, q(\exp(-u)).$$

Да означим

$$T = \max_{u\in(-\infty,+\infty)} \exp\left\{-a\exp(-\lambda u) - \frac{u}{2}\right\}.$$

Понеже за всяко $u \in (-\infty, +\infty)$ е изпълнено неравенството

$$|q^*(u)| \leqq T|q(\exp(-u))|\exp\left(-\frac{u}{2}\right),$$

функцията $q^*(u)$ е от пространството $L_2(-\infty, +\infty)$. Тогава от формулата на Парсевал за преобразуването на Лаплас на функциите от това пространство ([6], с.252, (2)) следва, че

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |q^*(u)|^2\, du = \int_{-\infty}^{\infty} |Q^*(iy)|^2\, dy. \tag{9}$$

За $z = x + iy$ с $x \geqq 0$ получаваме, че

$$|Q(z)|^2 \leqq \int_0^{\infty} \exp(-2at^\lambda)t^{2(\mu x+\delta)}\, dt \int_0^{\infty} |q(t)|^2 dt$$

$$= \left(\frac{1}{\lambda}\int_0^{\infty} |q(t)|^2 dt\right)(2a)^{-\frac{2}{\lambda}(\mu x+\delta)}\Gamma\left(\frac{2\mu}{\lambda}x + \frac{2\delta+1}{\lambda}\right).$$

Дефинираме за z с $\mathrm{Re}z > -\frac{\delta+\frac{1}{2}}{\mu}$ функцията $\tilde{Q}(z)$ чрез

$$\tilde{Q}(z) = \frac{(2a)^{\frac{1}{\lambda}(\mu z+\delta)}Q(z)}{\left\{\Gamma\left(\frac{2\mu}{\lambda}z + \frac{2\delta+1}{\lambda} + \frac{1}{2}\right)\right\}^{\frac{1}{2}}}.$$

От лема 1 следва тогава, че за $z = x + iy$ с $x \geqq 0$ е изпълнено неравенство от вида

$$|\tilde{Q}(z)| \leqq K\exp\left(\frac{\pi\mu}{2\lambda}|y|\right).$$

Да допуснем, че $Q(n) = 0$ за $n = 0, 1, 2, \ldots$ Тогава и $\tilde{Q}(n) = 0$ за $n = 0, 1, 2, \ldots$ и тъй като $\frac{\pi\mu}{2\lambda} < \pi$, от теоремата на Карлсон ([3], с. 195, 5.8.1) следва, че $\tilde{Q} \equiv 0$, т.е., $Q \equiv 0$ в полуравнината $\mathrm{Re}z > -\frac{\delta+\frac{1}{2}}{\mu}$. Но тогава $Q^* \equiv 0$ в полуравнината $\mathrm{Re}z > -\frac{1}{2}$ и от (9) получаваме, че

$$\int_{-\infty}^{\infty} |q^*(u)|^2 du = 0.$$

Това води до извода, че функцията $q^* \sim 0$ (т.е. е почти навсякъде равна на нула в интервала $(-\infty, +\infty)$, а следователно и $q \sim 0$ в интервала $(0, +\infty)$.

З а б е л е ж к а. Твърдението на лема 2 е валидно и за по-общата система от функции $\{E(t)t^{\mu n+\delta}\}_{n=0}^{\infty}$ $(\mu > 0,\ \delta > -\frac{1}{2})$ при условие, че E е измерима и съществуват $A > 0$, $a > 0$ и $\lambda > 2\mu$, такива че

$$|E(t)| \leqq A \exp\left(-at^{\lambda}\right)$$

за $t \in (0, +\infty)$ и освен това $E(t) \neq 0$ почти навсякъде в интервала $(0, +\infty)$.

3. Пълнота в $L_2(-\infty, +\infty)$ на системата

$$\{W_\sigma(\omega n, t)\}_{n=0}^{\infty}. \tag{10}$$

Т е о р е м а. Системата (10) е пълна в пространството $L_2(-\infty, +\infty)$, ако $0 < \omega < \frac{4}{3}$ и $\mathrm{Re}\sigma > -\frac{1}{2}$.

Доказателство. Нека комплексната функция $h \in L_2(-\infty, +\infty)$. Дефинираме за z с $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$ функцията

$$H_\sigma(z) = \int_{-\infty}^{\infty} W_\sigma(z,t)h(t)\,dt.$$

От (3) следва, че

$$H_\sigma(z) = \cos\frac{\pi(z+\sigma)}{2}\int_0^{\infty} U_\sigma(z,t)\varphi(t)\,dt + \sin\frac{\pi(z+\sigma)}{2}\int_0^{\infty} V_\sigma(z,t)\psi(t)\,dt,$$

където $\varphi(t) = h(t) + h(-t)$ и $\psi(t) = h(t) - h(-t)$. Очевидно всяка от функциите φ и ψ е от пространството $L_2(0, +\infty)$.

От теорията на преобразуването на Фурие в пространството $L_2(0, +\infty$ и в частност от съответната теорема на Планшерел ([4], с. 64-65, теорема 2.3) следва съществуването на функции f и g от това пространство, такива че

$$\lim_{\lambda\to+\infty}\int_0^{\infty}\left|f(u) \doteq \sqrt{\frac{2}{\pi}}\int_0^{\lambda}\cos tu\,\varphi(t)\,dt\right|^2 du = 0,$$

$$\lim_{\lambda\to+\infty}\int_0^{\infty}\left|g(u) - \sqrt{\frac{2}{\pi}}\int_0^{\lambda}\sin tu\,\psi(t)\,dt\right|^2 du = 0,$$

Също така са изпълнени и равенствата

$$\lim_{\lambda\to+\infty}\int_0^{\infty}\left|\varphi(t) - \sqrt{\frac{2}{\pi}}\int_0^{\lambda}\cos tu\,f(u)\,du\right|^2 dt = 0, \tag{11}$$

$$\lim_{\lambda\to+\infty}\int_0^{\infty}\left|\psi(t)-\sqrt{\frac{2}{\pi}}\int_0^{\lambda}\sin tu\,g(u)\,du\right|^2 dt=0. \tag{12}$$

Дефинираме за z с $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$ функцията $\tilde{H}_\sigma$ чрез

$$\tilde{H}_\sigma(z)=\sqrt{\frac{\pi}{2}}\left\{\cos\frac{\pi(z+\sigma)}{2}\int_0^{\infty}\exp\left(-\frac{u^2}{2}\right)u^{z+\sigma}f(u)\,du\right.$$

$$\left.+\sin\frac{\pi(z+\sigma)}{2}\int_0^{\infty}\exp\left(-\frac{u^2}{2}\right)u^{z+\sigma}g(u)\,du\right\}.$$

Фактически $\tilde{H}_\sigma \equiv H_\sigma$. За да се убедим в това, дефинираме за $0 < \lambda < +\infty$ функцията $H_{\sigma,\lambda}$ чрез

$$H_{\sigma,\lambda}(z)=\int_{-\lambda}^{\lambda}W_\sigma(z,t)h(t)\,dt$$

$$=\cos\frac{\pi(z+\sigma)}{2}\int_0^{\lambda}U_\sigma(z,t)\varphi(t)\,dt+\sin\frac{\pi(z+\sigma)}{2}\int_0^{\lambda}V_\sigma(z,t)\psi(t)\,dt.$$

Ако означим

$$f_\lambda(u)=\sqrt{\frac{2}{\pi}}\int_0^{\lambda}\cos tu\varphi(t)\,dt,$$

$$g_\lambda(u)=\sqrt{\frac{2}{\pi}}\int_0^{\lambda}\sin tu\psi(t)\,dt,$$

след размяна на реда на интегрираният получаваме, че

$$H_{\sigma,\lambda}(z)=\sqrt{\frac{\pi}{2}}\left\{\cos\frac{\pi(z+\sigma)}{2}\int_0^{\infty}\exp\left(-\frac{u^2}{2}\right)u^{z+\sigma}f_\lambda(u)\,du\right.$$

$$\left.+\sin\frac{\pi(z+\sigma)}{2}\int_0^{\infty}\exp\left(-\frac{u^2}{2}\right)u^{z+\sigma}g_\lambda(u)\,du\right\}.$$

Тогава

$$\left|\tilde{H}_\sigma(z)-H_{\sigma,\lambda}(z)\right|$$

$$\leqq \sqrt{\frac{\pi}{2}} \left\{ \left| \cos \frac{\pi(z+\sigma)}{2} \right| \left| \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} (f(u) - f_\lambda(u))\, du \right| \right.$$

$$\left. + \left| \sin \frac{\pi(z+\sigma)}{2} \right| \left| \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} (g(u) - g_\lambda(u))\, du \right| \right\}$$

Но

$$\left| \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} (f(u) - f_\lambda(u))\, du \right|^2$$

$$\leqq \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2(x+\mathrm{Re}\sigma)}\, du \int_0^\infty |f(u) - f_\lambda(u)|^2 du,$$

$$\left| \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} (g(u) - g_\lambda(u))\, du \right|^2$$

$$\leqq \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2(x+\mathrm{Re}\sigma)}\, du \int_0^\infty |g(u) - g_\lambda(u)|^2 du.$$

Следователно за z с $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$ е изпълнено $\lim\limits_{\lambda \to +\infty} H_{\sigma,\lambda}(z) = \tilde{H}_\sigma(z)$. Но, от друга страна, $\lim\limits_{\lambda \to +\infty} H_{\sigma,\lambda}(z) = H_\sigma(z)$ и следователно

$$H_\sigma(z) = \cos \frac{\pi(z+\sigma)}{2} \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} f(u)\, du$$

$$+ \sin \frac{\pi(z+\sigma)}{2} \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} g(u)\, du.$$

От горното представяне може да се заключи, че функцията $H_\sigma(z)$ е холоморфна в полуравнината $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$ и освен това, че в полуравнината $\mathrm{Re}z \geqq 0$ е в сила неравенство от вида (L =const)

$$|H_\sigma(z)| \leqq L \left\{ \Gamma\left(x + \mathrm{Re}\sigma + \frac{1}{2}\right) \right\}^{\frac{1}{2}} \exp\left(\frac{\pi|y|}{2}\right) \tag{13}$$

Наистина съществува константа $B = B(\sigma) > 0$, такава че за всяко $z = x + iy$ са изпълнени неравенствата

$$\left| \cos \frac{\pi(z+\sigma)}{2} \right| \leqq B \exp\left(\frac{\pi|y|}{2}\right)$$

$$\left|\sin\frac{\pi(z+\sigma)}{2}\right| \leqq B\exp\left(\frac{\pi|y|}{2}\right)$$

Неравенството на Шварц за z с $\mathrm{Re}z \geqq 0$ дава

$$\left|\int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} f(u)\,du\right|^2$$

$$\leqq \int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2(x+\mathrm{Re}\sigma)}\,du \int_0^\infty |f(u)|^2 du = \frac{1}{2}\Gamma\left(x+\mathrm{Re}\sigma+\frac{1}{2}\right)\int_0^\infty |f(u)|^2 du$$

и аналогично

$$\left|\int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{z+\sigma} g(u)\,du\right|^2$$

$$\leqq \frac{1}{2}\Gamma\left(x+\mathrm{Re}\sigma+\frac{1}{2}\right)\int_0^\infty |g(u)|^2 du.$$

Ако дефинираме L чрез

$$L = \frac{B}{2\sqrt{2}}\max\left\{\left(\int_0^\infty |f(u)|^2 du\right)^{\frac{1}{2}}; \left(\int_0^\infty |g(u)|^2 du\right)^{\frac{1}{2}}\right\}$$

получаваме неравенството (13).

Дефинираме функцията H_σ^* в полуравнината $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$ чрез

$$H_\sigma^*(z) = \frac{H_\sigma(z)}{\{\Gamma(x+\mathrm{Re}\sigma+1)\}^{\frac{1}{2}}}.$$

От лема 1 тогава следва, че ако $\omega > 0$, в полуравнината $\mathrm{Re}z \geqq 0$ е изпълнено неравенството ($M = LA^{\frac{1}{2}}$)

$$|H_\sigma^*(\omega z)| \leqq M\exp\left(\frac{3\pi\omega}{4}|y|\right) \tag{14}$$

Да допуснем, че $\omega < \frac{4}{3}$ и че

$$\int_{-\infty}^{\infty} W_\sigma(\omega n, t)h(t)\,dt = 0$$

за всяко $n = 0, 1, 2, \ldots$ Това води до $H_\sigma^*(\omega n) = 0$ за $n = 0, 1, 2, \ldots$ От (14) и от теоремата на Карлсон следва, че $H_\sigma^* \equiv 0$, а тогава и $H_\sigma \equiv 0$ в полуравнината $\mathrm{Re}z > -\frac{1}{2} - \mathrm{Re}\sigma$.

Каквото и да е $n = 0, 1, 2, \ldots$, точката $2n - \sigma$ е от тази полуравнина, тъй като $2n > -\frac{1}{2}$ за всяко $n = 0, 1, 2, \ldots$ От $H_\sigma(2n - \sigma) = 0$ за $n = 0, 1, 2, \ldots$ следва, че

$$\int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2n} f(u)\, du = 0 \tag{15}$$

за $n = 0, 1, 2, \ldots$ и съгласно лема 2 функцията $f \sim 0$ в интервала $(0, +\infty)$.

Аналогично равенствата $H_\sigma(2(n + \frac{1}{2}) - \sigma) = 0$ $(n = 0, 1, 2, \ldots)$ или все едно

$$\int_0^\infty \exp\left(-\frac{u^2}{2}\right) u^{2n+1} g(u)\, du = 0, \quad n = 0, 1, 2, \ldots, \tag{16}$$

водят до заключението, че $g \sim 0$ в $(0, +\infty)$. Но тогава следва, че всяка от функциите φ и ψ е почти навсякъде равна на нула в интервала $(0, +\infty)$, тъй като (11) и (12) водят до равенствата

$$\int_0^\infty |\varphi(t)|^2 dt = \int_0^\infty |\psi(t)|^2 dt = 0.$$

От равенствата $2h(t) = \varphi(t) + \psi(t)$ и $2h(-t) = \varphi(t) - \psi(t)$, които са изпълнени за всяко $t \in (0, +\infty)$, следва, че $h \sim 0$ в интервала $(-\infty, +\infty)$.

Частният случай $\omega = 1$ и $\sigma = 0$ води до пълнотата в пространството $L_2(-\infty, +\infty)$ на системата (1), от която, както вече беше изтъкнато, следва пълнотата в същото пространство на системата функции на Ермит (2).

При $\omega = 1$ и $\sigma = k$ $(k = 1, 2, 3, \ldots)$ получаваме, че системата

$$\left\{\exp\left(-\frac{t^2}{2}\right) H^*_{n+k}(t)\right\}_{n=0}^\infty,$$

или все едно системата

$$\left\{\exp\left(-\frac{t^2}{2}\right) H_{n+k}\left(\frac{t}{\sqrt{2}}\right)\right\}_{n=0}^\infty,$$

е също пълна в пространството $L_2(-\infty, +\infty)$. Този последен извод е може би неочакван, тъй като каквото и да е $k = 1, 2, 3, \ldots$, системата

$$\left\{\exp\left(-\frac{t^2}{2}\right) H_{n+k}(t)\right\}_{k=0}^\infty$$

не е пълна в $L_2(-\infty, +\infty)$ - тя е истинска част от системата функции на Ермит.

З а б е л е ж к а. При доказателството на пълнотата на системата (10) в пространството $L_2(-\infty, +\infty)$ всъщност бяха използвани два частни случая на лема 2, съгласно които от равенствата (15), респ. (16) следва, че $f \sim 0$, респ. $g \sim 0$ в интервала $(0, +\infty)$. Последните заключения могат да бъдат направени и без да се привлича лема 2, а именно като се

прибегне до теоремата за единственост на класическото преобразуване на Фурие, както например при доказателството на твърдението от с. 431 на [5]. Ще изтъкнем обаче, че лема 2 изобщо не може да бъде доказана по "маниера" на доказателството на току-що споменатото твърдение и в частност това е така, когато $\frac{\lambda}{\mu} < 1$.

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О МАКСИМУМЕ КОЛИЧЕСТВА РЕБЕР ГРАФА

НИКОЛАЙ ХАДЖИИВАНОВ

Николай Хаджиивanов. О МАКСИМУМЕ КОЛИЧЕСТВА РЕБЕР ГРАФА. Находится максимум количества ребер n-вершинных графов, не содержащих полных $(s+1)$-вершинных подграфов K_{s+1}, но зато имеющих дизъюнктную фамилию, состоящую из r антиклик A_i с $|A_i| \geq \alpha_i$, где α_i — наперед заданные числа, удовлетворяющие неравенствам $\alpha_i \geq [\frac{n}{s}]$.

Nikolay Khadzhiivanov. ON THE MAXIMAL NUMBER OF EDGES IN A GRAPH. The maximal number of edges in an n-graph without K_{s+1} and with disjoint family of independent sets $A_i (i = 1, 2, \ldots, r)$, such that $|A_i| \geqq \alpha_i$, where $\alpha_i \geqq [\frac{n}{s}]$, is found.

ВВЕДЕНИЕ

Пусть $G = (V, E)$ — обыкновенный граф. Обозначим количества вершин и ребер графа G через $v(G)$ и $e(G)$, а степень вершины v — через $d_G(v)$. Множество из k попарно смежных (несмежных) вершин графа G называется k-кликой (k-антикликой). Максимальное число попарно смежных (несмежных) вершин называется кликовым (антикликовым) числом и обозначается через $\mathrm{cl}(G)(\alpha(G))$. Если вершина v смежна вершине u, иногда будем говорить, что v является соседом вершины u.

$G = (V, E)$ — полный хроматический граф, если любая его вершина смежна хотя бы одному концу произвольного ребра, т.е. когда несмежность вершин есть отношение эквивалентности в множестве всех вершин V. Классы эквивалентности (хроматические классы) являются антикликами, вершины из разных классов смежны. Таким образом полный хроматический граф вполне определен набором чисел $h_1, h_2, \ldots, h_k$, являющихся количествами вершин в хроматических классах $H_1, H_2, \ldots, H_k$,

и поэтому вполне уместно употребляемое обозначение $K(h_1, h_2, \ldots, h_k)$. Впрочем, расширим определение этого символа естественным образом, уславливаясь не обращать внимания на нулевые аргументы, иными словами, допуская наличие и пустых классов; к примеру, $K(0, h_1, h_2, h_3) = K(h_1, h_2, h_3)$.

Элементарный симметрический многочлен степени 2 обозначим через σ:

$$\sigma(x_1, x_2, \ldots, x_k) = \sum\{x_i x_j | 1 \leqq i < j \leqq k\}.$$

Очевидно,

$$e(K(h_1, h_2, \ldots, h_k)) > \sigma(h_1, h_2, \ldots, h_k)$$

и

$$\sigma(x_1 - 1, x_2 + 1, x_3, \ldots, x_k) = \sigma(x_1, x_2, x_3, \ldots, x_k) + x_1 - x_2 - 1.$$

Следовательно,

$$\sigma(x_1 - 1, x_2 + 1, \ldots, x_k) > \sigma(x_1, x_2, \ldots, x_k),$$

если $x_1 > x_2 + 1$, т.е.

$$e(K(h_1 - 1, h_2 + 1, h_3, \ldots, h_k)) > e(K(h_1, h_2, h_3, \ldots, h_k)), \tag{1}$$

если $h_1 - h_2 > 1$.

Пусть n и s — натуральные числа, а $h_1, h_2, \ldots, h_s$ — целые, и $h_1 + h_2 + \cdots + h_s = n$. Из (1) следует, что $e(K(h_1, \ldots, h_s))$ имеет максимум только тогда, когда $|h_i - h_j| \leqq 1$ для любой пары i, j. В таком случае h_i определены с точностью до порядка, к примеру $h_i = \left[\frac{n+i-1}{s}\right]$.

Граф $K(h_1, h_2, \ldots, h_s)$ с $h_i = \left[\frac{n+i-1}{s}\right]$ называется *графом Турана* и обозначается через $T_s(n)$.

Итак, $T_s(n)$ имеет максимальное количество ребер среди n-вершинных полных хроматических графов G с $\mathrm{cl}(G) \leqq s$. Туран [1941] доказал, что в этом утверждении можно отказаться от требования полной хроматичности, т.е. $T_s(n)$ имеет максимальное количество ребер и в гораздо более широком классе тех n-вершинных графов G, для которых $\mathrm{cl}(G) \leqq s$.

Обладая дополнительной информацией для G, следует ожидать более точную оценку для вопросного максимума. Эту дополнительную информацию вводим в виде требования, чтобы граф G обладал дизъюнктной системой антиклик, любая из которых имеет хотя бы $]\frac{n}{s}[$ вершин ($[x] = \max\{n | n \in \mathbf{N}, n \leqq x\}$, $]x[= \min\{n | n \in \mathbf{N}, n \geqq x\}$). Остановимся подробнее на этом.

Пусть $n \geqq s \geqq r$. n-вершинный граф $K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)$, в котором все h_i с $i > r$, отличаются между собой не более чем на 1, обозначим через $T_s(n; h_1, h_2, \ldots, h_r)$. В случае $r = 0$ определенный только что граф совпадает с $T_s(n)$.

Основным утверждением в настоящей статье является следующая

Т е о р е м а. Пусть $n, s, \alpha_1, \alpha_2, \ldots, \alpha_r$ $(r \geqq 0)$ – натуральные числа, причем $n \geqq s \geqq r, \alpha_1 + \alpha_2 + \cdots + \alpha_r \leqq n$, $\alpha_i \geqq]\frac{n}{s}[$, $i = 1, 2, \ldots, r$. Пусть G —

n-вершинный граф с $\mathrm{cl}(G) \leqq s$, в котором имеется дизъюнктная система антиклик $A_1, A_2, \ldots, A_r$ и $|A_i| \geqq \alpha_i$, $i = 1, 2, \ldots, r$. Тогда

$$e(G) \leqq e(T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)),$$

причем равенство достигается единственно когда $G = T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)$, а множества A_i являются хроматическими классами этого графа, $|A_i| = \alpha_i$, $i = 1, 2, \ldots, r$.

Замечание 1. В случае $r = 0$ утверждение совпадает с теоремой Турана.

Замечание 2. В случае $r = 1$ утверждение сформулировано более 10 лет тому назад в совместном обсуждении с моим дипломантом Вл. Никифоровым.

Отметим и следующее предложение, которое тривиальным образом вытекает из теоремы:

Следствие. Пусть $n, s, \alpha_1, \alpha_2, \ldots, \alpha_r$ ($r \geqq 1$) – натуральные числа, $n \geqq s \geqq r$; $\alpha_1 + \alpha_2 + \cdots + \alpha_r \leqq n$; $\alpha_i \geqq \left]\frac{n}{s}\right[$, $i = 1, 2, \ldots, r$. Пусть $G = (V, E)$ — n-вершинный граф с $(G) \leqq s$ и в V имеется дизъюнктная система множеств $A_1, A_2, \ldots, A_r$ со следующими свойствами: 1) $|A_i| \geqq \alpha_i$; $i = 1, 2, \ldots, r$; 2) в A_i есть вершина a_i, которая несмежна любой вершине v из A_i и $d_G(a_i) \geqq d_G(v)$ $(1 \leqq i \leqq r)$. Тогда

$$e(G) \leqq e(T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)),$$

причем равенство достигается единственно для графа $G = T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)$ и множества A_i являются хроматическими классами этого графа, $|A_i| = \alpha_i$, $i = 1, 2, \ldots, r$.

Замечание 3. Теорему (а значит и следствие) можно обобщить, заменяя количество ребер количеством p-клик. Доказательство, однако, не требует новых идей, хотя и содержит дополнительные технические трудности. Поэтому здесь его приводить не будем.

2. ГРАФ $T_s(n; h_1, h_2, \ldots, h_r)$

Граф $T_s(n; h_1, h_2, \ldots, h_r)$, где $n \geqq s \geqq r$ и $h_1 + \cdots + h_r \leqq n$, был определен в п.1 как n-вершинный граф $K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)$, такой, что $K(h_{r+1}, \ldots, h_s) = T_{s-r}(n - h_1 - \cdots - h_r)$.

В случае $h_1 + \cdots + h_r = n$ или $s = r$, очевидно, $T_s(n; h_1, \ldots, h_r) = K(h_1, \ldots, h_r)$.

Граф $T_s(n; h_1, \ldots, h_r)$ — единственный, имеющий максимальное количество ребер среди n-вершинных графов типа $K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)$. Действительно, как знаем из п.1, максимальное количество ребер среди $(n - h_1 - \cdots - h_r)$-вершинных полных хроматических графов G с $\mathrm{cl}(G) \leqq s - r$ (а $K(h_{r+1}, \ldots, h_s)$ является именно таким графом) имеет единственно граф $T_{s-r}(n - h_1 - \cdots - h_r)$, а, с другой стороны, $e(K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)) = e(K(h_1, \ldots, h_r)) + (h_1 + \cdots + h_r)(n - h_1 - \cdots - h_r) + e(K(h_{r+1}, \ldots, h_s))$.

После этого замечания никакого труда не составит доказательство следующего предложения:

Л е м м а 1. Пусть $n, s, \alpha_1, \alpha_2, \ldots, \alpha_r (r \geqq 0)$ — натуральные числа, $n \geqq s \geqq r, \alpha_1 + \alpha_2 + \cdots + \alpha_r \leqq n$; $\alpha_i \geqq]\frac{n}{s}[$, $i = 1, 2, \ldots, r$. Граф $T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)$ — единственный граф с максимальным числом ребер среди всех n-вершинных графов типа $K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s)$, для которых $h_i \geqq \alpha_i$, $i = 1, 2, \ldots, r$.

Доказательство. Согласно рассуждениям, предшествующим формулировке леммы, достаточно доказать, что

$$e(T_s(n; h_1, h_2, \ldots, h_r)) \leqq e(T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r))$$

и что равенство имеется только тогда, когда

$$T_s(n; h_1, h_2, \ldots, h_r) = T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r).$$

Это утверждение непосредственно следует из неравенства

$$e(T_s(n; h_1, h_2, \ldots, h_r)) < e(T_s(n; h_1 - 1, h_2, \ldots, h_r)) \tag{2}$$

при $h_1 > \alpha_1$.

Убедимся, что оно на самом деле выполнено.

Согласно определению,

$$T_s(n; h_1, \ldots, h_r) = K(h_1, \ldots, h_r, h_{r+1}, \ldots, h_s),$$

где $K(h_{r+1}, \ldots, h_s) = T_{s-r}(n - h_1 - \cdots - h_r)$.

Так как $h_i \geqq \alpha_i \geqq]\frac{n}{s}[i = 1, 2, \ldots, r$ и $h_1 > \alpha_1$, то $h_1 + \cdots + h_r > r]\frac{n}{s}[$. Следовательно, $h_{r+1} + \cdots + h_s = n - h_1 - \cdots - h_r < n - r]\frac{n}{s}[\leqq n - r\frac{n}{s} = (s-r)\frac{n}{s} \leqq (s-r)]\frac{n}{s}[$

Будем считать, что $h_s = \min(h_{r+1}, \ldots, h_s)$. Тогда

$$(s-r)h_s \leqq h_{r+1} + \cdots + h_s < (s-r)]\frac{n}{s}[,$$

так что $h_s <]\frac{n}{s}[$ и, значит, $h_1 - h_s > 1$. Очевидно,

$$K(h_{r+1}, \ldots, h_s + 1) = T_{s-r}(n - (h_1 - 1) - h_2 - \cdots - h_r)$$

и поэтому

$$T_s(n; h_1 - 1, h_2, \ldots, h_r) = K(h_1 - 1, \ldots, h_r, \ldots, h_s + 1).$$

Таким образом неравенство (2) следует из неравенства (1).

Доказательство леммы завершено.

3. УПОДОБЛЕНИЕ

Пусть a — вершина графа G, а B — множество вершин этого графа и все вершины из B несмежны вершине a.

Удалим из G все ребра, инцидентные множеству B, а затем введем в G новые ребра, соединяющие все вершины из B с теми вершинами графа G, которые смежны в нем вершине a. Полученный граф обозначим через

G_a^B и будем говорить, что он получен уподоблением вершин из B вершине a.

Итак, граф G_a^B определен только в том случае, когда любая вершина из B несмежна вершине a. Если B состоит из одного-единственного элемента b, вместо G_a^B будем писать G_a^b.

Очевидно $\{a\} \cup B$ является антикликой в G_a^B и $d_{G_a^B}(b) = d_{G_a^B}(a) = d_G(a)$ для любой вершины $b \in B$. Ясно, что $G_a^B = G$, если $\{a\} \cup B$ — антиклика в G и все ее вершины имеют одни и те же соседи в G.

При переходе из G к G_a^B меняются эвентуально смежность или несмежность только по отношению к вершинам $b \in B \setminus \{a\}$. Как легко сообразить, верно следующее утверждение:

Л е м м а 2. $\mathrm{cl}(G_a^B) \leqq \mathrm{cl}(G)$.

Доказательство. Пусть K — p-клика графа G_a^B. Если $K \cap B \setminus \{a\} = \phi$, тогда K есть p-клика и в G. Пусть $K \cap B \setminus \{a\} \neq \phi$. Сейчас пересечение $K \cap (B \cup \{a\})$ имеет единственную вершину $b, b \neq a$, потому что K — клика, а $B \cup \{a\}$ — антиклика в G_a^B. Множество $K \setminus \{b\}$ — клика и в G, причем вершина a смежна в G вершинам этого множества, потому что a и b имеют в G_a^B одни и те же соседи, а соседи a в G_a^B и в G одинаковы. Таким образом мы убедились, что $(K \setminus \{b\}) \cup \{a\}$ — клика в G, которая содержит ровно p вершин.

Доказательство леммы завершено.

Если B — множество вершин графа G, через $d_G(B)$ обозначим число тех ребер графа G, которые инцидентны множеству B. Таким образом $d_G(B)$ есть сумма двух слагаемых: первое — число ребер с двумя концами в B, а второе — число ребер с ровно одним концом в B. Ясно, что

$$d_G(B) \leqq \sum \{d_G(b) | b \in B\}$$

и равенство имеется тогда и только тогда, когда B является антикликой в G. Очевидно,

$$e(G_a^B) = e(G) - d_G(B) + |B| d_G(a).$$

Следовательно,

$$e(G_a^B) \geqq e(G) - \sum \{d_G(b) | b \in B\} + |B| d_G(a) \tag{3}$$

и равенство имеет место тогда и только тогда, когда B является антикликой в G. Теперь уже легко доказать следующее предложение:

Л е м м а 3. Если $d_G(a) \geqq d_G(b)$ для любой вершины b из B, тогда $e(G_a^B) \geqq e(G)$, причем равенство имеет место тогда и только тогда, когда B — антиклика в G и $d_G(b) = d_G(a)$ для любой вершины $b \in B$.

Доказательство. Из (3) следует

$$e(G_a^B) \geqq e(G) + \sum \{d_G(a) - d_G(b) | b \in B\} \geqq e(G).$$

Равенство имеется лишь тогда, когда, с одной стороны, $d_G(b) = d_G(a)$ для любой вершины $b \in B$, а с другой — когда в (3) тоже есть равенство, т.е. B — антиклика в G.

Доказательство леммы завершено.

Следующее предложение тоже легче доказать, чем сформулировать. Тем не менее, оно должно сыграть важную роль в дальнейшем.

Л е м м а 4. Если A — антиклика в G, все вершины которой имеют одни и те же соседи в G, тогда A — антиклика и в G_a^B. Притом, если $A \cap B = \phi$, тогда все вершины из A имеют одни и те же соседи и в G_a^B.

Доказательство. Множество $A \backslash B$, очевидно, есть антиклика в G_a^B. Пусть $v \in A \cap B$. Так как $v \in B$, вершины a и v — несмежны в G. Но $v \in A$ и, значит, a несмежна в G всем вершинам из A. Так как $v \in B$, то v несмежна в G_a^B всем вершинам из A. Следовательно, A — антиклика в G_a^B. Первая часть леммы доказана.

А теперь, пусть $A \cap B = \phi$. Нужно доказать, что все вершины из A имеют одни и те же соседи в G_a^B.

Пусть v — вершина, смежная в G_a^B некоторой вершине u из A. Если $v \in B$, согласно определению графа G_a^B, вершина a смежна в G вершине u из A. Тогда a смежна в G всем вершинам из A и, следовательно, v смежна в G_a^B всем вершинам из A.

Осталось рассмотреть случай, когда $v \notin B$. Имея ввиду, что $A \cap B = \phi$, очевидно, что v и u смежны и в G. Следовательно, v смежна в G всем вершинам из A. Так как $A \cap B = \phi$ и $v \notin B$, вершина v смежна и в G_a^B всем вершинам из A.

Доказательство леммы завершено.

З а м е ч а н и е 1. Множество A вовсе не обязано быть антикликой в G_a^B в случае отказа от требования, чтобы все вершины из A имели одни и те же соседи в G. Впрочем, это требование можно заменить на более слабое: чтобы в A не существовало вершины u и v, одна из которых смежна, а другая несмежна вершине a в G, причем доказательство, что A — антиклика и в G_a^B, то же самое. Следует отметить, что если поначалу $A \cap B = \phi$, тогда без всяких дополнительных требований для антиклики A в G можно утверждать, что A является антикликой и в G_a^B, однако такое утверждение не достаточно для доказательства теоремы.

З а м е ч а н и е 2. Если $A \cap B \neq \phi$, тогда отнюдь не следует, что все вершины из A имеют одни и те же соседи в G_a^B, несмотря на то, что все вершины из A имеют одни и те же соседи в G.

З а м е ч а н и е 3. Леммы 2 и 3 в том случае, когда множество B не более чем двуточечно, содержатся в статье Зыкова [1949]. Лемма 4 аналогов в этой статье не имеет.

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ ТУРАНА.

Сделаем одно отклонение — докажем теорему Турана. Приведем одну модификацию оригинального доказательства Турана, которая выгодно отличается от него тем, что она короче. Эта модификация принадлежит Зыкову [1949]. Впрочем, в работе Зыкова нет никаких ссылок на статью Турана, и поэтому можно предположить, что он дошел до формулировки и доказательства самостоятельно.

Через S обозначим множество всех n-вершинных графов G с $\mathrm{cl}(G) \leqq s$, а через $\hat{S}$ – подмножество тех из них, которые имеют максимальное ко-

личество ребер. Мы должны доказать, что граф Турана $T_s(n)$ является единственым элементом множества $\hat{S}$.

Пусть $G \in \hat{S}$.

Докажем сначала следующее утверждение:

(4) Если a и b — несмежные вершины в G, тогда $d_G(a) = d_G(b)$.

Допустим, что (4) не верно. Из-за симметрии можно считать, что $d_G(a) > d_G(b)$. Из леммы 2 следует, что $G_a^b \in S$. Из леммы 3 следует, что $e(G_a^b) > a(G)$, что невозможно.

Импликация (4) доказана. С ее помощью нетрудно установить, что G — полный хроматический граф.

Пусть a — вершина, а $[b,c]$ — ребро графа G. Нужно доказать, что a смежна хотя бы одной из вершин b, c. Допустим, что это не так. Тогда из (4) следует, что $d_G(b) = d_G(a) = d_G(c)$. Из леммы 2 следует, что $G_a^{\{b,c\}} \in S$, а из леммы 3 — что $e\left(G_a^{\{b,c\}}\right) > e(G)$, потому что $\{b,c\}$ не есть антиклика в G. Полученное противоречие завершает доказательство.

Итак, G — полный хроматический граф. Так как все полные хроматические графы с кликовым числом $\leqq s$ тоже принадлежат множеству S, то G имеет максимальное количество ребер среди них и (см. определение графа $T_s(n)$ в п.1), значит, $G = T_s(n)$.

Доказательство теоремы завершено.

5. ДОКАЗАТЕЛЬСТВО ОСНОВНОЙ ТЕОРЕМЫ

В статье Хадживанова [1976] было изложено одно доказательство теоремы Турана, которое ближе к оригинальному доказательству по сравнению с доказательством Зыкова. Однако его легче усовершенствовать таким образом, чтобы трансформировать в доказательство основной теоремы (см. ее формулировку в п. 1). Тем не менее, этому мы предпочтем другой путь. Таким образом мы предоставим возможность читателю самому судить о сходствах и отличиях доказательства основной теоремы, которое здесь изложим, по сравнению с доказательством теоремы Турана, уже изложенном в п. 4.

Через S обозначим множество всех графов, удовлетворяющих требованиям теоремы, а через $\hat{S}$ — подмножество тех графов из S, которые имеют максимальное количество ребер. Нужно доказать, что $\hat{S}$ содержит единственный элемент и это граф $T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)$.

Пусть $G \in \hat{S}$ и $A_1, A_2, \ldots, A_r$ — те антиклики в G, существование которых обеспечивается условием теоремы.

В антиклике A_1 выберем вершину a_1 с максимальной степенью в G среди вершин множества A_1. Определим граф $G_1 = G_{a_1}^{A_1}$. Согласно лемме 2, $\mathrm{cl}(G_1) \leqq s$. Очевидно, $A_1, A_2, \ldots, A_r$ — антиклики и в G_1. Согласно лемме 3, $e(G_1) \geqq (e(G)$ и, значит, $G_1 \in \hat{S}$.

Все вершины из A_1 имеют в G_1 одни и те же соседи.

Ту же самую операцию проделаем и с графом G_1, на этот раз по отношению к его антиклике A_2: выберем в ней вершину с максимальной

степенью в G_1 среди всех вершин из A_2 и введем в рассмотрение граф $G_2 = (G_1)_{a_2}^{A_2}$. Разумеется, множества A_i являются антикликами и в G_2 и снова имеем $G_2 \in \hat{S}$. Все вершины из A_2 имеют в G_2 одни и те же соседи. Следует, однако, отметить что это имеет место и по отношению к вершинам из множества A_1, которое этим свойством обладало в G_1. Чтобы убедиться в этом, достаточно применить лемму 4, не забывая, что A_1 и A_2 – дизъюнктивные антиклики в G_1.

Итак, в графе G_2 из $\hat{S}$ множества $A_1, A_2, \ldots, A_r$ являются антикликами, причем все вершины из A_i, $i = 1, 2$, имеют в G_2 одни и те же соседи.

Теперь уже совсем ясно, что, повторяя проделанную нами два раза операцию, в r шагах определим граф $\hat{G}$, принадлежащий множеству $\hat{S}$, и к тому же множества $A_1, A_2, \ldots, A_r$ являются антикликами в $\hat{G}$, причем все вершины из одного и того же A_i, $i = 1, 2, \ldots, r$, имеют в $\hat{G}$ одни и те же соседи.

Установим, что $\hat{G}$ — полный хроматический граф.

Докажем сначала следующее утверждение:

(5) Если a и b — несмежные вершины в $\hat{G}$, то $d_{\hat{G}}(a) = d_{\hat{G}}(b)$.

Допустим, что (5) не верно. Без ограничения общности можно считать, что $d_{\hat{G}}(a) > d_{\hat{G}}(b)$. Согласно лемме 2, $\mathrm{cl}(\hat{G}_a^b) \leqq s$. Согласно лемме 4, $A_1, A_2, \ldots, A_r$ являются антикликами в $\hat{G}_a^b$. Следовательно, $\hat{G}_a^b \in S$. Из леммы 3 следует, что $e(\hat{G}_a^b) > e(\hat{G})$, а это невозможно.

Импликация (5) доказана.

Из (5) полная хроматичность графа $\hat{G}$ следует непосредственно. Пусть a – вершина, а $[b, c]$ — ребро графа $\hat{G}$. Нужно доказать, что a смежна в $\hat{G}$ хотя бы одной из вершин b, c. Допустим, что это не так. Из леммы 2 вытекает $\mathrm{cl}(\hat{G}_a^{\{b,c\}}) \leqq s$. Из леммы 4 следует, что $A_1, A_2, \ldots, A_r$ являются антикликами в $\hat{G}_a^{\{b,c\}}$. Следовательно, $\hat{G}_a^{\{b,c\}} \in S$. Из (5) следует, что $d_{\hat{G}}(b) = d_{\hat{G}}(a) = d_{\hat{G}}(c)$. Так как $\{b, c\}$ не есть антиклика в $\hat{G}$, то из леммы 3 следует, что $e(\hat{G}_a^{\{b,c\}}) > e(G)$.

Полученное противоречие показывает, что $\hat{G}$ — полный хроматический граф, принадлежащий множеству $\hat{S}$. Обозначим через $H_1, H_2, \ldots, H_s$ его хроматические классы и положим $h_i = |H_i|$, так что $\hat{G} = K(h_1, h_2, \ldots, h_s)$.

Множества A_i являются антикликами в $\hat{G}$, и поэтому любое из них содержится в некотором H_j. Докажем, что разные A_i содержатся в разных H_j.

Допустим, что это не так и пусть, например, $A_1 \subset H_1 \supset A_2$. Тогда можно считать, что

$$A_1 \cup A_2 \cup \ldots \cup A_r \subset H_1 \cup H_2 \cup \ldots \cup H_{r-1}.$$

В таком случае выполнены неравенства

$$h_1 \geqq \alpha_1 + \alpha_2, h_1 + \cdots + h_{r-1} \geqq \alpha_1 + \alpha_2 + \ldots + \alpha_r \geqq r\left[\frac{n}{s}\right[,$$

Положим $h_s = \min(h_r, h_{r+1}, \ldots, h_s)$. Тогда

$$(s-r+1)h_s \leqq h_r + \cdots + h_s = n - h_1 - \cdots - h_{r-1} \leqq n - r]\frac{n}{s}[\leqq (s-r)\frac{n}{s} < (s-r+1)]\frac{n}{s}[$$

и, следовательно, $h_s <]\frac{n}{s}[$. Так как $\alpha_1 \geqq]\frac{n}{s}[> h_s$, то

$$h_1 - h_s \geqq \alpha_1 + \alpha_2 - h_s > \alpha_2.$$

Введем граф

$$G' = K(h_1 - \alpha_2, h_2, \ldots, h_s + \alpha_2).$$

Очевидно, $G' \in S$, так как $\mathrm{cl}(G') \leqq s$ и в G' имеется дизъюнктная система антиклик, количества вершин которых соответственно равны $|A_1|, |A_2|, \ldots, |A_r|$. Докажем, что $e(G') > e(\hat{G})$. Действительно,

$$e(G') - e(\hat{G}) = \sigma(h_1 - \alpha_2, h_2, \ldots, h_s + \alpha_2) - \sigma(h_1, \ldots, h_s) = \alpha_2(h_1 - h_s - \alpha_2),$$

а, как было установлено только что, $h_1 - h_s - \alpha_2 > 0$.

Итак, $G' \in S$ и $e(G') > e(\hat{G})$, а это противоречит включению $\hat{G} \in \hat{S}$.

Таким образом доказано, что различные A_i содержатся в различных H_j. Поэтому можно считать, что $h_i \geqq |A_i| \geqq \alpha_i$, $i = 1, \ldots, r$. Теперь уже с помощью леммы 1 заключаем, что $\hat{G} = T_s(n; \alpha_1, \ldots, \alpha_r)$.

Докажем, что антиклики $A_1, \ldots, A_r$ совпадают с r из самых больших хроматических классов графа $T_s(n; \alpha_1, \ldots, \alpha_r)$. Нетрудно сообразить, что за исключением первых r хроматических классов этого графа, любой из остальных имеет не больше $]\frac{n}{s}[$ элементов. Следовательно, сумма количеств вершин произвольных r хроматических классов не превосходит числа $\alpha_1 + \cdots + \alpha_r \leqq |A_1| + \cdots + |A_r|$. Поэтому $|A_i| = \alpha_i$; $i = 1, 2, \ldots, r$.

Итак, $\hat{G} = T_s(n; \alpha_1, \ldots, \alpha_r)$, причем A_i – хроматические классы этого графа и $|A_i| = \alpha_i$; $i = 1, 2, \ldots, r$.

В процессе построения графа $\hat{G}$ смежность или несмежность тех пар вершин, которые лежат вне $A_1 \cup \ldots \cup A_r$, сохраняется. Что касается тех пар вершин, которые имеют хотя бы одну вершину в объединении $A_1 \cup \ldots \cup A_r$, то там могут произойти изменения. Однако такие пары, которые, разумеется, не содержатся в одном и том же A_i, смежны в $\hat{G}$. Таким образом оказывается, что G — подграф графа $\hat{G}$. Если допустить, что $G \neq \hat{G}$, тогда $e(G) < e(\hat{G})$, что противоречит принадлежности G к $\hat{S}$. Следовательно, $G = \hat{G}$.

Окончательно, доказано, что $G = T_s(n; \alpha_1, \ldots, \alpha_r)$ и множества A_1, A_2, ..., A_r являются хроматическими классами этого графа, причем $|A_i| = \alpha_i$; $i = 1, 2, \ldots, r$.

Доказательство теоремы завершено.

6. ДОКАЗАТЕЛЬСТВО СЛЕДСТВИЯ

Положим $G_1 = G_{a_1}^{A_1}, G_2 = (G_1)_{a_2}^{A_2}, \ldots, G_r = (G_{r-1})_{a_r}^{A_r}$. Множества $A_1, A_2, \ldots, A_r$ являются антикликами в G_r и, согласно лемме 3, $e(G_r) \geqq$

$e(G)$, причем равенство влечет, что A_i — антиклики в G. Так как $\mathrm{cl}(G_r) \leqq s$ (см. лемму 2), то G_r удовлетворяет всем требованиям теоремы и, следовательно, $e(G_r) \leqq e(T_s(n;\alpha_1,\alpha_2,\ldots,\alpha_r))$. Таким образом

$$e(G) \leqq e(T_s(n;\alpha_1,\ldots,\alpha_r)).$$

Если в этом неравенстве имеет место знак равенства, тогда $e(G) = e(G_r)$ и, значит, $A_1, A_2, \ldots, A_r$ — антиклики в G. Тогда можно применить теорему и заключить, что $G = T_s(n;\alpha_1,\ldots,\alpha_r)$.

Следствие доказано.

7. ОБСУЖДЕНИЕ ТЕОРЕМЫ

В лемме 1 можно ослабить предположение $\alpha_i \geqq]\frac{n}{s}[$, заменив его неравенством $\alpha_i \geqq [\frac{n}{s}]$. Это вносит, однако, некоторые минимальные осложнения в ее доказательстве. Так, например, неравенство (2) может и не выполняться, если $\alpha_1 = [\frac{n}{s}]$ и $h_1 = [\frac{n}{s}] + 1$. Дело в том, что сейчас нельзя утверждать, что $h_1 - h_s > 1$, а только, что $h_1 - h_s \geqq 1$, как нетрудно убедиться. Если $h_1 - h_s = 1$, т.е. $h_s = [\frac{n}{s}]$, тогда вместо (2) имеет место равенство. Тем не менее, все в порядке, потому что в этом случае $T_s(n; h_1 - 1, h_2, \ldots, h_r) = T_s(n; h_1, \ldots, h_r)$, как это нетрудно установить.

Итак, лемма 1 верна при более слабом ограничении $\alpha_i \geqq [\frac{n}{s}]$, $i = 1,\ldots,r$. С незначительными осложнениями в доказательстве теоремы можно установить, что она тоже остается верной при менее ограничительных предположениях $\alpha_i \geqq [\frac{n}{s}]$, $i = 1,2,\ldots,r$. То же самое относится, конечно, и к следствию.

Невозможно дальнейшее ослабление в этом духе предположений теоремы. Так, еще при $r = 1$ нельзя утверждать, что если $\alpha_1 < [\frac{n}{s}]$, граф $T_s(n;\alpha_1)$ имеет максимальное количество ребер среди графов G, для которых $v(G) = n, \mathrm{cl}(G) \leqq s, \alpha(G) \geqq \alpha_1$. Это просто неверно. Действительно, граф $T_s(n)$ в этом случае удовлетворяет всем перечисленным требованиям, однако $e(T_s(n)) > e(T_s(n;\alpha_1))$. Чтобы охватить и случай $\alpha_1 < [\frac{n}{s}]$, формулировку теоремы (при $r = 1$) нужно видоизменить следующим образом:

Среди графов G, для которых $v(G) = n, \mathrm{cl}(G) \leqq s, \alpha(G) \geqq \alpha_1$, максимальное количество ребер при $\alpha_1 \geqq [\frac{n}{s}]$ имеет только граф $T_s(n;\alpha_1)$, а при $\alpha_1 < [\frac{n}{s}]$ — только граф $T_s(n)$.

Аналогичным образом можно обобщить основную теорему.

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ON THE MAXIMAL NUMBER OF EDGES IN A GRAPH

Nikolay G. Khadzhiivanov

(Summary)

Let $K(h_1, h_2, \ldots, h_s)$ be the complete s-partite graph with h_i vertices in i-th class. For $n \geqq s \geqq r$ we denote by $T_s(n; h_1, h_2, \ldots, h_r)$ the n-graph $K(h_1, h_2, \ldots, h_s)$ with $|h_i - h_j| \leqq 1$ for each pair $i, j : r < i < j \leqq s$. When $r = 0$ this is the Turan's graph $T_s(n)$.

The number of edges of a graph G is denoted by $e(G)$.

Main theorem. Let $n, s, \alpha_1, \alpha_2, \ldots, \alpha_r$ $(r \geqq 0)$ be natural numbers and $n \geqq s \geqq r$, $\alpha_1 + \alpha_2 + \cdots + \alpha_r \leqq n$, $\alpha_i \geqq [\frac{n}{s}]$, $i = 1, 2, \ldots, r$. Let G be an n-graph without K_{s+1}, having disjoint family of independent vertex sets $A_1, A_2, \ldots, A_r$, $|A_i| \geqq \alpha_i$, $i = 1, 2, \ldots, r$. Then

$$e(G) \leqq e(T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r))$$

and the equality holds iff

$$G = T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r) \text{ and } |A_i| = \alpha_i;\ i = 1, 2, \ldots, r.$$

For $r = 0$ this is equivalent to the Turan s theorem [3]. For $r = 1$ it was stated together with V. Nikiforov more than 10 years ago.

Corollary. Let $n, s, \alpha_1, \alpha_2, \ldots, \alpha_r$ $(r \geqq 1)$ be natural numbers, $n \geqq s \geqq r$, $\alpha_1 + \alpha_2 + \cdots + \alpha_r \leqq n$, $\alpha_i \geqq [\frac{n}{s}]$, $i = 1, 2, \ldots, r$. Let G be an n-vertex graph without K_{s+1} and having disjoint family of vertex sets $A_1, A_2, \ldots, A_r$ with the following properties: 1) $|A_i| \geqq \alpha_i$ and 2) in A_i there is a vertex a_i such that each of the vertices in A_i has at most the degree of a_i and is non-adjacent to a_i. Then

$$e(G) \leqq e(T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r))$$

and the equality holds iff $G = T_s(n; \alpha_1, \alpha_2, \ldots, \alpha_r)$ and the sets A_i are its chromatic classes.

The asssumption $\alpha_i \geqq [\frac{n}{s}]$ is essential in the above two propositions. So, for $r = 1$ it is not true that if $\alpha_1 < [\frac{n}{s}]$ then the graph $T_s(n, \alpha_1)$ has maximal number of edges among the n-graphs without K_{s+1} and having an independent set A_1 with at least α_1 vertices. Only the Turan's graph $T_s(n)$ has maximal number of edges in this class of graphs.

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О МАКСИМУМЕ КОЛИЧЕСТВА ТРЕУГОЛЬНИКОВ С ОБЩИМ РЕБРОМ

НИКОЛАЙ ХАДЖИИВАНОВ

Николай Хаджииванов. О МАКСИМУМЕ КОЛИЧЕСТВА ТРЕУГОЛЬНИКОВ С ОБЩИМ РЕБРОМ. Исследуется максимум $\hat{t}$ числа треугольников с общим ребром в графах, принадлежащих некоторым естественным и важным классам графов, в частности, классу n-графов с хотя бы $\left[\frac{n^2}{4}\right]$ ребрами и имеющих треугольник. Обсуждается статья [2], в которой решена проблема Эрдеша о $\hat{t}$, и выводятся новые следствия из основной теоремы этой статьи. Доказана одна гипотеза Эдвардса, сформулированная в дополнительном замечании к его работе [3], и подробно обсуждается эта его статья.

Nikolay Khadzhiivanov. ON THE MAXIMAL NUMBER OF TRIANGLES WITH A COMMON EDGE. The maximal number $\hat{t}$ of triangles with a common edge in graphs of some natural and important classes (in particular the class of n-graphs with at least one triangle and $\left[\frac{n^2}{4}\right]$ edges) is investigated. Paper [2], where a problem of P. Erdös about $\hat{t}$ is solved, is discussed and some new consequences of the main theorem, proved in [2], are deduced. The Edwards' conjecture is proved and his paper [3] is analysed in details.

1. ОПРЕДЕЛЕНИЯ, ВСПОМОГАТЕЛЬНЫЕ СООТНОШЕНИЯ И ПЕРВАЯ ТЕОРЕМА

Рассматриваем только обыкновенные графы. Если G – граф, множества его вершин, ребер и треугольников обозначим через V, E и T, а числа их элементов – через n, e и t. Число тетраэдров графа обозначим через q. Если v – вершина графа, а $A(v)$ – множество всех смежных ей вершин, тогда $d(v) = |A(v)|$ – степень этой вершины. Числа

$$t[u,v] = |A(u) \cap A(v)| \text{ и } \bar{t}[u,v] = |V \setminus (A(u) \cup A(v))|$$

в том случае, когда $[u,v] \in E$, являются соответственно количеством треугольников со стороной $[u,v]$ и количеством вершин, одновременно несмежных u и v. Ясно, что

$$3t = \sum \{t[u,v] | [u,v] \in E\}, \tag{1}$$

а

$$\bar{t} = \sum \{\bar{t}[u,v] | [u,v] \in E\}, \tag{2}$$

есть число тех 3-вершинных множеств, любое из которых имеет ровно одну пару смежных вершин.

Положим

$$\begin{aligned} t[u,v,w] &= t[u,v] + t[u,w] + t[w,u], \\ \bar{t}[u,v,w] &= \bar{t}[u,v] + \bar{t}[v,w] + \bar{t}[w,u], \\ q[u,v,w] &= |A(u) \cap A(v) \cap A(w)|, \\ \bar{q}[u,v,w] &= |V \setminus (A(u) \cup A(v) \cup A(w))|. \end{aligned}$$

Если $[u,v,w] \in T$, то $q[u,v,w]$ – число всех тетраэдров, имеющих в качестве грани треугольник $[u,v,w]$, а $\bar{q}[u,v,w]$ – число тех вершин, которые одновременно несмежны вершинам u, v, w. Ясно, что

$$4q = \sum \{q(u,v,w) | [u,v,w] \in T\} \tag{3}$$

а

$$\bar{q} = \sum \{\bar{q}(u,v,w) | [u,v,w] \in T\} \tag{4}$$

есть количество тех 4-вершинных множеств, любое из которых состоит из треугольника и одной вершины, несмежной всем вершинам этого треугольника.

Легко сообразить верность следующих двух равенств:

$$\sum \{d(u) + d(v) | [u,v] \in E\} = \sum \{d^2(v) | v \in V\}, \tag{5}$$

$$\sum \{\bar{t}[u,v,w] | [u,v,w] \in T\} = \sum \{\bar{t}[u,v]t[u,v] | [u,v] \in E\}. \tag{6}$$

Действительно, в левой сумме (5) $d(v_0)$ встречается во всех слагаемых, для которых $[u,v_0] \in E$, т.е. $d(v_0)$ раз, а в левой сумме (6) $\bar{t}[u_0,v_0]$ встречается во всех слагаемых, для которых $[u_0,v_0,w] \in T$, т.е. $t[u_0,v_0]$ раз.

Если $[u,v,w] \in T$, $t[u,v] - q[u,v,w]$ есть число всех вершин, смежных вершинам u,v, однако несмежных вершине w. Поэтому $t[u,v,w] - 3q[u,v,w]$ – число тех вершин, любая из которых смежна ровно двум из вершин

треугольника $[u,v,w]$. Присоединяя к ним вершины, которые одновременно смежны всем вершинам треугольника $[u,v,w]$, а их число – $q[u,v,w]$, в результате получим, что $\alpha = t[u,v,w] - 2\bar{q}[u,v,w]$ есть число тех вершин графа, любая из которых смежна хотя бы двум вершинам треугольника $[u,v,w]$.

Аналогично, $\beta = \bar{t}[u,v,w] - 2\bar{q}[u,v,w]$ – число тех вершин, любая из которых несмежна хотя бы двум вершинам треугольника $[u,v,w]$.

Разумеется, любая вершина графа смежна или несмежна хотя бы двум вершинам треугольника $[u,v,w]$ и, значит, $\alpha+\beta = n$. Таким образом доказано, что если $[u,v,w] \in T$, тогда

$$t[u,v,w] + \bar{t}[u,v,w] = n + 2q[u,v,w] + 2\bar{q}[u,v,w]. \tag{7}$$

Просуммировав почленно (7) по всевозможным треугольникам $[u,v,w] \in T$, получим (см.(3), (4), (6)):

$$\sum\{t[u,v,w] | [u,v,w] \in T\} + \sum\{\bar{t}[u,v]t[u,v] | [u,v] \in E\} = nt + 8q + 2\bar{q}. \tag{8}$$

Через $\hat{t}$ обозначим максимум числа треугольников с общей стороной, т.е.

$$\hat{t} = \max\{t[u,v] | [u,v] \in E\}. \tag{9}$$

Очевидно,

$$t[u,v,w] \leqq 3\hat{t}, \text{ если } [u,v,w] \in T. \tag{10}$$

Следовательно,

$$\sum\{t[u,v,w] | [u,v,w] \in T\} \leqq 3t\hat{t}. \tag{11}$$

Равенство в (11) имеет место тогда и только тогда, когда есть равенство в (10) для любого $[u,v,w] \in T$, т.е. когда верна импликация:

(*) Если $[u,v] \in E$ и $t[u,v] > 0$, тогда $t[u,v] = \hat{t}$.

Из (6) следует

$$\sum\{\hat{t}[u,v,w] | [u,v,w] \in T\} \leqq \bar{t}\hat{t}. \tag{12}$$

Чтобы убедиться в этом, достаточно принять ввиду (2) и (9).

Равенство в (12) имеет место тогда и только тогда, когда верна импликация:

(**) Если $[u,v] \in E$ и $\bar{t}[u,v] > 0$, тогда $t[u,v] = \hat{t}$.

Из (8), (11) и (12) следует

$$(3t + \bar{t})\hat{t} \geqq nt + 8q + 2\bar{q}. \tag{13}$$

К тому же ясно, что равенство в (13) имеет место тогда и только тогда, когда верны импликации (*) и (**)

Т е о р е м а 1. Для любого графа G выполнено неравенство

$$(3t + \bar{t})\hat{t} \geqq nt, \tag{14}$$

причем равенство достигается тогда и только тогда, когда $q = \bar{q} = 0$ и, кроме того, для всех $[u, v] \in E$, для которых $t[u, v] + \bar{t}[u, v] > 0$, число $t[u, v]$ одно и то же.

Неравенство (14) очевидно вытекает из (13). Для равенства в (14) необходимо и достаточно $q = \bar{q} = 0$ и равенство в (13), т.е. чтобы (*) и (**) одновременно выполнялись, а это означает: если $[u, v] \in E$ и хотя бы одно из чисел $t[u, v]$ и $\bar{t}[u, v]$ положительно, тогда $t[u, v] = \hat{t}$.

Неравенство (14) доказано немного иначе в [2].

Построим два экстремальные графа для теоремы 1, т.е. такие графы, для которых в (14) имеет место равенство.

П р и м е р 1. Пусть a_i – вершины треугольной призмы, $n \equiv 0 (\mathrm{mod}\ 6)$ и A_i - дизъюнктные множества с $|A_i| = \frac{n}{6}$, $1 \leqq i \leqq 6$. Положим $V' = \bigcup_{i=1}^{6} A_i$. Через E' обозначим совокупность всех двухэлементных подмножеств $[u, v]$ множества V', для которых $u \in A_i$ и $v \in A_j$, где $[a_i, a_j]$ – ребро призмы. Для графа $G' = (V', E')$, как нетрудно проверить с помощью теоремы, в (14) имеет место равенство и $e = \frac{n^2}{4}$, а $\hat{t} = \frac{n}{6}$.

П р и м е р 2. Возьмем три треугольника $[a_1, a_2, a_3]$, $[a_4, a_5, a_6]$, $[a_7, a_8, a_9]$ и любые две из 9 вершин a_i, индексы которых сравнимы по модулю 3, соединим ребром. Полученный граф Г будет иметь 9 вершин и 18 ребер – все ребра данных трех треугольников и, к тому же, все ребра треугольников $[a_1, a_4, a_7]$, $[a_2, a_5, a_6]$, $[a_3, a_6, a_9]$. Пусть $n \equiv 0 (\mathrm{mod}\ 9)$ и A_i – девять дизъюнктных множеств с $|A_i| = \frac{n}{9}$. Положим $V'' = \bigcup_{i=1}^{9} A_i$, а E'' определим как совокупность всех двухэлементных подмножеств $[u, v]$ множества V'', для которых $u \in A_i$ и $v \in A_j$, где $[a_i, a_j]$ – ребро графа Г. Для графа $G'' = (V'', E'')$, как нетрудно установить с помощью теоремы, в (14) есть равенство и $e = \frac{2n^2}{9}$, а $\hat{t} = \frac{n}{9}$.

Граф G' имеет ребра, которые не являются сторонами треугольников, а G'' таких ребер не имеет.

Нордхауз и Стьюарт [8] установили, что

$$3t = \sum \{d^2(v) | v \in V\} - ne + \bar{t}. \tag{15}$$

Впрочем, это интересное равенство можно доказать нетрудно. Действительно,

$$|A(u) \cup A(v)| = |A(u)| + |A(v)| - |A(u) \cap A(v)|. \tag{16}$$

В случае, когда $[u, v] \in E$, равенство (16) можно переписать в следующей форме:

$$t[u, v] = d(u) + d(v) - n + \bar{t}[u, v], [u, v] \in E, \tag{17}$$

потому что $t[u,v] = |A(u) \cap A(v)|$, $d(u) = |A(u)|$, $d(v) = |A(v)|$, $\bar{t}[u,v] = n - |A(u) \cup A(v)|$.

Складывая почленно всевозможные равенства (17), получим

$$\sum\{t[u,v] | [u,v] \in E\} = \sum\{d(u) + d(v) | [u,v] \in E\} - ne + \sum\{\bar{t}[u,v] | [u,v] \in E\}. \tag{18}$$

Чтобы убедиться в том, что равенство (18) совпадает с (15), достаточно принять ввиду равенства (1), (5) и (2).

С помощью равенства (15) неравенству (14) можно придать следующую эквивалентную форму:

$$\left(6t + ne - \sum\{d^2(v) | v \in V\}\right) \hat{t} \geqq nt. \tag{19}$$

Отсюда выведем следующие два следствия.

Следствие 1. Если

$$\sum\{d^2(v) | v \in V\} > ne, \tag{20}$$

тогда

$$\hat{t} > \frac{n}{6}. \tag{21}$$

Доказательство этого утверждения моментально следует из (19), так как, согласно (15), $3t > \bar{t}$ и, значит, $t > 0$ (разумеется, и $\hat{t} > 0$).

Следствие 2. Если $t > 0$ и

$$\sum\{d^2(v) | v \in V\} \geqq ne, \tag{22}$$

тогда

$$\hat{t} \geqq \frac{n}{6}. \tag{23}$$

Неравенство (23) следует из (19) и (22), так как $t > 0$. Последнее требование обязательно, потому что, если $t = \bar{t} = 0$, тогда, согласно (15), (22) выполнено, однако, (23) нет.

2. ПОЛНЫЕ ХРОМАТИЧЕСКИЕ ГРАФЫ И ВТОРАЯ ТЕОРЕМА. ПРОБЛЕМА ЭРДЕША И РЕШЕНИЕ

$G = (V, E)$ - полный хроматический граф точно тогда, когда любая его вершина смежна хотя бы одному из концов любого ребра, т.е. если $\bar{t} = 0$.

Очевидно, $G = (V, E)$ – полный хроматический граф тогда, когда отношение несмежности вершин является эквивалентностью в V; классы эквивалентности называются хроматическими классами. В случае, когда их количество равняется r, граф называется полным r-хроматическим графом. Полный r-хроматический граф имеет полные r-вершинные подграфы K_r, однако не имеет полных подграфов K_{r+1}.

G называем почти регулярным графом, если $|d(u)-d(v)| \leqq 1$ для любой пары вершин u, v и регулярным – если степени всех его вершин равны.

Система целых чисел $x_1, x_2, \ldots, x_r$ определена однозначно, если $x_1 \geqq x_2 \geqq \ldots \geqq x_r$, $x_1 + x_2 + \cdots + x_r = s$ и $|x_i - x_j| \leqq 1$ для любой пары индексов i, j.

Действительно, положим $x_r = x$. Тогда $x_i \leqq x+1$ и если ν – количество тех i, для которых $x_i = x+1$, то $x_1 = \cdots = x_\nu = x+1$ и $x_{\nu+1} = \cdots = x_r = x$, а $0 \leqq \nu < r$. Тогда $\nu(x+1) + (r-\nu)x = s$, т.е. $s = rx + \nu$, где $0 \leqq \nu < r$. Таким образом $x = \left[\frac{s}{r}\right]$ – частное, а ν – остаток от деления s на r.

Эти элементарные рассуждения показывают, что для любых n и r, $r \leqq n$, имеется единственный почти регулярный полный хроматический граф с n вершинами и r классами; обозначаем его через $T_r(n)$ и называем графом Турана.

Очевидно, $e\,(T_r(n)) = \frac{r-1}{2r}n^2$, если $r|n$ (т.е. когда граф $T_r(n)$ регулярен), а $e\,(T_r(n)) \leqq \frac{r-1}{2r}n^2$ для любого r ($r \leqq n$, но не обязательно $r|n$), см. [1].

Туран [9] доказал, что единственный граф с максимальным количеством ребер среди n-вершинных графов без K_{r+1} является граф $T_r(n)$.

Прочее, задолго до него Мантель [7] установил то же самое, однако только в частном случае $r = 2$.

Из (1) и (9) следует

$$\hat{te} \geqq 3t. \tag{24}$$

Равенство в (24) достигается только тогда, когда $t[u, v]$ одно и то же для любого ребра $[u, v]$.

Л е м м а 1. Выполнено неравенство

$$e\hat{t} \geqq \sum \left\{d^2(v) | v \in V\right\} - ne. \tag{25}$$

Равенство имеет место только тогда, когда G – полный r-хроматический граф, который к тому же и регулярен, если $r \geqq 3$.

Неравенство (25) следует из (15) и (24). Равенство достигается только тогда, когда $\bar{t} = 0$, т.е. G – полный r-хроматический граф и, кроме того, $t[u, v]$ – постоянно. Последнее не является ограничением при $r \leqq 2$, так как тогда $t[u, v] = 0$, однако при $r \geqq 3$ означает, что граф регулярен. Действительно, если $[u, v] \in E$, тогда u и v принадлежат разным хроматическим классам и $t[u, v]$ – количество вершин в объединении остальных хроматических классов. Итак, объединение любых $r-2$-хроматических классов в G имеет постоянное количество элементов, а это означает, что все хроматические классы имеют одинаковое число вершин и, следовательно, граф регулярен.

Л е м м а 2. Имеет место неравенство

$$\hat{t} \geqq \frac{4e}{n} - n \tag{26}$$

и оно переходит в равенство только тогда, когда G - регулярный полный r-хроматический граф и $r \geqq 2$.

Действительно, если $e = 0$, (26) выполнено по тривиальным соображениям и к тому же оно строго. Если $e > 0$, оно следует из (25) и очевидного неравенства

$$\sum \{d^2(v) | v \in V\} \geqq \frac{1}{n} \left(\sum \{d(v) | v \in V\} \right)^2 = \frac{4e^2}{n}. \tag{27}$$

Равенство в (26) достигается только одновременно с равенствами в (25) и (27). Равенство в (27) имеет место только для регулярных графов. Применяя лемму 1, заключаем, что равенство в (26) достигается только тогда, когда G – регулярный полный r-хроматический граф и $r \geqq 2$ (потому что $e > 0$).

З а м е ч а н и е. Доказательства этих двух лемм излагаются здесь исключительно для полноты. Дело в том, что они по существу известны еще из статьи Нордхауза и Стьюарта [8]. В этой статье число $\hat{t}$ не участвует, однако вместо неравенств (25) и (26) там встречаются более точные неравенства

$$3t \geqq \sum \{d^2(v) | v \in V\} - ne, \tag{28}$$

$$3t \geqq e \left(\frac{4e}{n} - n \right) \tag{29}$$

Неравенство (28) следует из (15), а (29) – из (28) и (27). Случай равенства в (28) или (29) тоже рассмотрен в [8]. Неравенства (25) и (26) следуют из (28) и (29) с помощью тривиального неравенства (24). Они приводятся в [3], как это ни странно, без всякой ссылки на [8]. Притом в [3] есть дополнительное и совершенно лишнее предположение $\sum \{d^2(v) | v \in V\} \geqq ne$.

Если $e > \frac{r-1}{2r} n^2$, тогда $e > e(T_r(n))$ и, согласно теореме Турана, в G имеется K_{r+1}, так что $\hat{t} \geqq r - 1$. Значительно более сильное утверждение при том же самом предположении содержится в следующем предложении:

Т е о р е м а 2. Если

$$e \geqq \frac{r-1}{2r} n^2, \tag{30}$$

тогда

$$\hat{t} \geqq \frac{r-2}{r} n, \tag{31}$$

причем равенство в (31) достигается тогда и только тогда, когда $2 \leqq r | n$ и $G = T_r(n)$.

Неравенство (31) следует из (26) и (30). Равенство в (31) имеется только одновременно с равенствами в (26) и (30), т.е. (см. лемму 2) когда $G = T_p(n)$, где $2 \leqq p | n$, и $e = \frac{r-1}{2r} n^2$. Но в этом случае $e = \frac{p-1}{2p} n^2$ и, следовательно, $p = r$.

Доказательство завершено.

Эта элементарная теорема при $r = 2$ принимает следующий вид: если $e \geqq \frac{n^2}{4}$, тогда $\hat{t} \geqq 0$ и $\hat{t} = 0$ точно тогда, когда n – четно и $G = T_2(n)$.

Разумеется, если $e > \frac{n^2}{4}$, тогда $\hat{t} > 0$. Эрдеш [4] доказал значительно больше: существует такая константа $c > 0$, что если $e > \frac{n^2}{4}$, тогда $\hat{t} > cn$. Несколько лет спустя Эрдеш установил, что здесь можно взять $c = 30^{-18}$. В [4] и [5] Эрдеш высказал гипотезу: если $e > \frac{n^2}{4}$, тогда $\hat{t} \geqq \frac{n}{6} + O(1)$. В [6] гипотеза принимает более точную форму:

Проблема Эрдеша. Если $e > \frac{n^2}{4}$, тогда $\hat{t} \geqq \frac{n}{6}$.

В [2] эту проблему мы решили полностью совместно с моим дипломантом В. Никифоровым. Чтобы и здесь рассказать об этом решении, нам понадобится следующая

Л е м м а 3. Пусть n – натуральное, а s – целое число. Функция $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$ целочисленных аргументов $x_1, \ldots, x_n$, удовлетворяющих условию $\sum_{i=1}^{n} x_i = s$, принимает свое минимальное значение только тогда, когда $|x_i - x_j| \leqq 1$ для любых двух индексов i и j.

Высказанное утверждение, очевидно, следует причислять к математическому фольклору, но мы докажем его здесь для полноты изложения, тем более что на это понадобятся всего несколько строк.

Функция f, очевидно, достигает свой минимум, и пусть это происходит например в $(\bar{x}_1, \ldots, \bar{x}_n)$. Надо доказать, что $|\bar{x}_i - \bar{x}_j| \leqq 1$, $i, j = 1, 2, \ldots, n$. Допустим противное. Тогда без ограничения общности можно считать, что $\bar{x}_1 > \bar{x}_2 + 1$. Имеем $(\bar{x}_1 - 1)^2 + (\bar{x}_2 + 1)^2 = \bar{x}_1^2 + \bar{x}_2^2 - 2(\bar{x}_1 - \bar{x}_2 - 1) < \bar{x}_1^2 + \bar{x}_2^2$ и, следовательно,

$$f(\bar{x}_1 - 1, \bar{x}_2 + 1, \bar{x}_3, \ldots, \bar{x}_n) < f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n),$$

что является противоречием.

Лемма доказана. Выведем из нее одно важное для нас следствие:

Л е м м а 4. Если

$$e \geqq \left[\frac{n^2}{4}\right], \tag{32}$$

тогда

$$\sum \{d^2(v) | v \in V\} \geqq ne, \tag{33}$$

причем, если неравенство (32) строгое, то и неравенство (33) является таковым.

Если неравенство (32) строгое, тогда $e > \frac{n^2}{4}$ и из (27) следует, что выполнено строгое неравенство (33). Если $e = \left[\frac{n^2}{4}\right]$ и n – четно, тогда неравенство (33) следует снова из (27), потому что сейчас $e = \frac{n^2}{4}$. Наконец, пусть $e = \left[\frac{n^2}{4}\right]$ и n - нечетно, $n = 2k + 1$.

Итак, надо доказать, что если $n = 2k + 1$ и $e = k(k + 1)$, то

$$\sum \{d^2(v) | v \in V\} \geqq (2k + 1)k(k + 1). \tag{34}$$

Заметим, что

$$\sum \{d(v) | v \in V\} = 2e = 2k(k + 1)$$

и положим $s = 2k(k+1)$. Согласно лемме 3, функция f принимает свое минимальное значение только тогда, когда $|x_i - x_j| \leqq 1$. Однако, как было установлено в начале пункта, в этом случае числа x_i определены однозначно с точностью до порядка. Теперь $\left[\frac{s}{n}\right] = k$ и остаток ν от деления s на n тоже равняется k, $\nu = k$, так что k из чисел x_i равны $k+1$, а остальные $k+1$ равны k. Это показывает, что вопросный минимум равен числу $k(k+1)^2 + (k+1)k^2 = (2k+1)k(k+1)$.

Таким образом неравенство (34), а заодно с ним и лемма 4 доказаны.

Из следствия 1 и леммы 4 получаем

С л е д с т в и е 3. Если $e > \left[\frac{n^2}{4}\right]$, тогда $\hat{t} > \frac{n}{6}$.

Из следствия 2 и леммы 4 получаем

С л е д с т в и е 4. Если $e \geqq \left[\frac{n^2}{4}\right]$ и $t > 0$, то $\hat{t} \geqq \frac{n}{6}$.

Следствия 3 и 4 подтверждают с избытком гипотезу Эрдеша. В то же время следствия 1 и 2 представляют более сильные утверждения. Дело в том, что из (33) не следует (32), включительно и тогда, когда $t > 0$. В этом убедимся на следующем примере.

П р и м е р 3. Пусть граф G''' имеет вершины $v_1, v_2, \ldots, v_n$ и ребра $[v_1, v_2]$, $[v_1, v_i]$ и $[v_2, v_i]$, где $i = 3, \ldots, n$.

Очевидно, $t > 0$, $e = 2n - 3$ и $e < \frac{n^2}{4}$, если $n > 6$ С другой стороны, $\sum\limits_{i=1}^{n} d(v_i)^2 = 2n^2 - 6$ и $\sum\limits_{i=1}^{n} d(v_i)^2 > ne$.

Надо заметить тоже, что в следствии 4 требование $t > 0$ нельзя отставить, так как для любого полного 2-хроматического графа имеют место равенства $\sum \{d^2(v) | v \in V\} = ne$ и $\hat{t} = 0$, а для графа $T_2(n) - e = \left[\frac{n^2}{4}\right]$ и $\hat{t} = 0$.

Положим

$$t(n) = \min\left\{\hat{t} | e \geqq [\frac{n^2}{4}], t > 0\right\}.$$

Итак, $\hat{t}(n)$ – минимальное $\hat{t}(G)$, где G – n-вершинный граф, для которого $e(G) \geqq \left[\frac{n^2}{4}\right]$ и $t(G) > 0$.

Для любого вещественного числа x положим

$$]x[= \min\{k | k \in \mathbf{N}, k \geqq x\}.$$

Естественным дополнением к следствию 4 является следующее

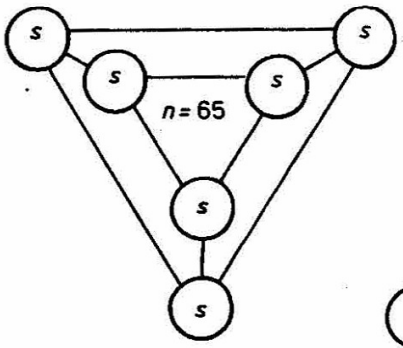
С л е д с т в и е 5. Для любого натурального числа n $(n \geqq 4)$

$$\hat{t}(n) = \left]\frac{n}{6}\right[. \tag{35}$$

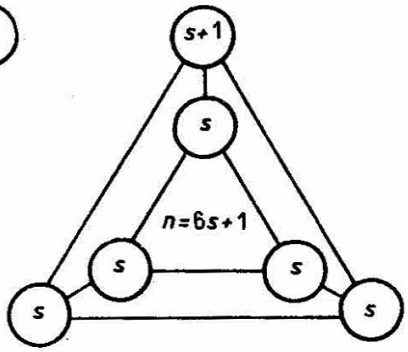
Полный 2-хроматический граф с числами вершин в классах p и q изображен символично на фиг. 1, которая дает пояснение к пониманию следующих фигур. Фиг. 2–7 – примеры графов с $e = \left[\frac{n^2}{4}\right]$, $t > 0$ и $\hat{t} = \left]\frac{n}{6}\right[$. При помощи этих примеров и следствия 4 справедливость равенства (35) при $n \geqq 6$ устанавливается тривиально.

Следствие 5 показывает, что следствие 4 не подлежит усилению. То же самое верно и по отношению к следствию 3. Чтобы убедится в этом, достаточно посмотреть на фиг. 8, где изображен граф с $e = \left[\frac{n^2}{4}\right] + 1$ и $\hat{t} = \left]\frac{n}{6}\right[$.

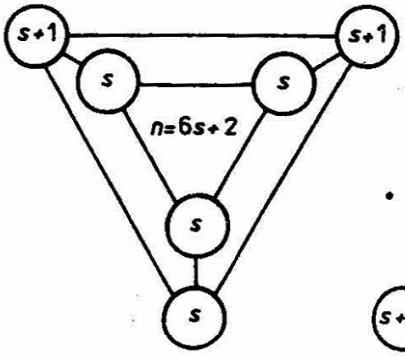
Фиг. 1



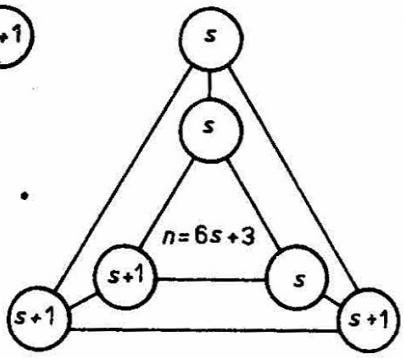
Фиг. 2



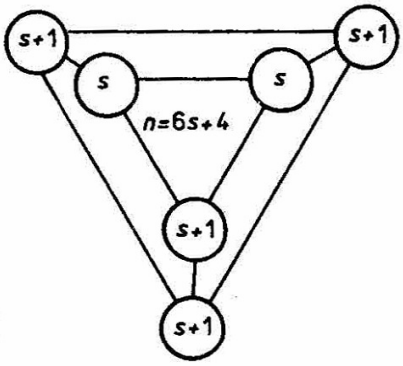
Фиг. 3



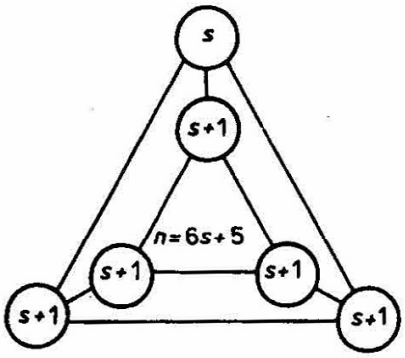
Фиг. 4



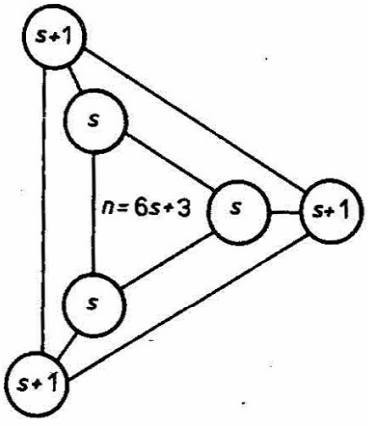
Фиг. 5



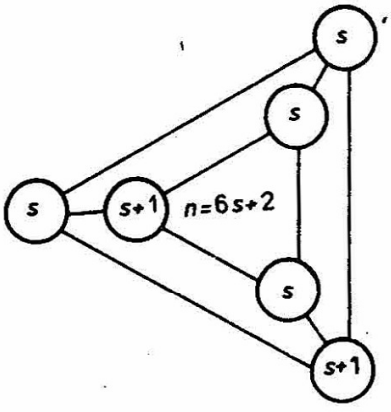
Фиг. 6



Фиг. 7



Фиг. 8



Фиг. 9

Граф, изображенный на фиг. 9, имеет $e = \frac{n^2}{4} - 1$ и $\hat{t} = \left]\frac{n}{6}\right[$, что показывает, что требование $e \geqq \left[\frac{n^2}{4}\right]$ отнюдь не необходимо для выполнения

неравенства $\hat{t} \geqq \frac{n}{6}$.

3. О СТАТЬЕ ЭДВАРДСА И ТЕОРЕМА 3

Квадратный трехчлен

$$\varphi(x) = 2ex^2 - \left(6t + \frac{ne}{3}\right)x + nt$$

имеет корни $x_1 = \frac{n}{6}$ и $x_2 = \frac{3t}{e}$.

Докажем, что если $\hat{t} \geqq \frac{n}{6}$, тогда $\varphi(\hat{t}) \geqq 0$.

Действительно, согласно (24), $\hat{t} \geqq x_2$, так что если $x_2 \geqq x_1$, тогда, очевидно, $\varphi(\hat{t}) \geqq 0$, а если $x_1 \geqq x_2$, тогда неравенство $\hat{t} \geqq \frac{n}{6} = x_1$ снова показывает, что $\varphi(\hat{t}) \geqq 0$.

Таким образом доказана следующая

Л е м м а 5. Если $\hat{t} \geqq \frac{n}{6}$, тогда

$$6t + ne - \frac{nt}{\hat{t}} \leqq 2e\hat{t} + \frac{2}{3}ne. \tag{36}$$

Для любого графа G, согласно теореме 1 (см. и (19)), имеет место неравенство

$$\sum\{d^2(v) | v \in V\} \leqq 6t + ne - \frac{nt}{\hat{t}}. \tag{37}$$

С помощью (36) и (37) тривиально получаем:

Т е о р е м а 3. Если $\hat{t} \geqq \frac{n}{6}$, тогда

$$\hat{t} \geqq \frac{1}{2e}\sum\{d^2(v) | v \in V\} - \frac{n}{3}. \tag{38}$$

Естественно, неравенство (38) представляет интерес только тогда, когда его правая часть $\geqq \frac{n}{6}$, т.е. $\sum\{d^2(v) | v \in V\} \geqq en$. С другой стороны, если последнее неравенство выполнено и $t > 0$, согласно следствию 2 будем иметь $\hat{t} \geqq \frac{n}{6}$ и, значит, (38) имеет место.

Итак, мы доказали следующее предложение:

С л е д с т в и е 6. Если $\sum\{d^2(v) | v \in V\} \geqq ne$ и $t > 0$, тогда справедливо неравенство (38).

Если $e \geqq \left[\frac{n^2}{4}\right]$ и $t > 0$, согласно следствию 4 имеем $\hat{t} \geqq \frac{n}{6}$ и, значит, (38) выполнено.

Итак, доказано следующее предложение:

С л е д с т в и е 7. Если $e \geqq \left[\frac{n^2}{4}\right]$ и $t > 0$, тогда имеет место неравенство (38).

Формулировка следствия 7 опубликована в качестве дополнительного замечания к статье Эдвардса [3], однако его доказательство не было опубликовано до сих пор. Как мы убедились, это было бы лишним, потому

что с помощью нашей теоремы 1 оно выводится в качестве простого следствия из справедливости гипотезы Эрдеша, доказанная, как и теорема 1, еще в [2]:

В самой статье [3] ее автор анонсировал 5 теорем и несколько других результатов, намереваясь опубликовать их доказательства впоследствии, что не произошло в прошедшие 10 лет. Все результаты относятся к гипотезе Эрдеша, однако, несмотря на это, нет никакого сдвига к ее решению.

Автор считает основной свою пятую теорему. Формулировку ее здесь приводить не будем, потому что ее предположения абсурдны: между ними значится и отрицание гипотезы Эрдеша.

Его первая теорема гласит: Если $[u,v] \in E$ и $d(u)+d(v) \geqq \frac{7}{6}n$, тогда $\hat{t} \geqq \frac{n}{6}$.

Эта теорема тривиально следует из очевидного неравенства (см. (17))

$$t[u,v] \geqq d(u)+d(v)-n.$$

Теорему 2 здесь приводить не будем, потому что она тривиально извлекается из только что упомянутой теоремы 1.

Из очевидного равенства

$$|A(u)\cup A(v)\cup A(w)| = |A(u)|+|A(v)|+|A(w)|$$
$$-|A(u)\cap A(v)|-|A(v)\cap A(w)|-|A(w)\cap A(u)|+|A(u)\cap A(v)\cap A(w)|$$

следует тривиально, что

$$3\hat{t} \geqq t[u,v,w] \geqq d(u)+d(v)+d(w)-n, \text{ если } [u,v,w]\in T. \tag{39}$$

Последнее неравенство с избытком содержит теоремы 3 и 4 Эдвардса. К примеру рассмотрим теорему 4, о которой мы упомянули и в [2] и которая тривиально следует из неравенства (39): если регулярный граф степени d содержит хотя бы один треугольник, тогда $\hat{t} \geqq d-\frac{n}{3}$. В частности, если $d \geqq \frac{n}{2}$, то $\hat{t} \geqq \frac{n}{6}$.

Таким образом закончен разбор теорем из [3]. В [3] выводятся и две следствия из теоремы 5. Мы уже отметили, что эта теорема лишена смысла, так что их на самом деле нельзя вывести из нее.

Следствие 1 гласит: Если

$$\frac{4e}{n}-n \geqq \frac{n}{9+6\sqrt{2}}, \tag{40}$$

тогда

$$\hat{t} \geqq \max\left(\frac{n}{6}, \frac{4e}{n}-n\right). \tag{41}$$

На самом деле достаточно доказать, что $\hat{t} \geqq \frac{n}{6}$, так как неравенство $\hat{t} \geqq \frac{4e}{n}-n$ выполнено для любого графа (см. лемму 2). А для доказательства неравенства $\hat{t} \geqq \frac{n}{6}$ совершенно излишне предполагать, что выполнено неравенство (40), а только что $\frac{4e}{n}-n>0$ (см. следствие 3 из п.2).

Следствие 2 гласит: Если

$$0 < \frac{4e}{n} - n < \frac{n}{9+6\sqrt{2}}, \tag{42}$$

тогда

$$\hat{t} > \frac{n}{8} + \frac{3+\sqrt{74}}{16}\left(\frac{4e}{n} - n\right). \tag{43}$$

Из левой части (42) следует неравенство $\hat{t} > \frac{n}{6}$ (см. следствие 3 из п.2), а

$$\frac{n}{6} > \frac{n}{8} + \frac{3+\sqrt{74}}{16}\frac{n}{9+6\sqrt{2}},$$

так что неравенство (43) действительно имеет место, но явно не интересно.

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ON THE MAXIMAL NUMBER OF TRIANGLES WITH A COMMON EDGE

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(Summary)

§1. Let $G = (V, E)$ be a graph, $|V| = n$, $|E| = e$, T is the set of triangles in G, $|T| = t$, and q is the number of tetrahedrons in G. For a vertex v we denote by $A(v)$ the set of its neighbours, so that $d(v) = |A(v)|$ is the degree of v. Put

$$t[u,v] = |A(u) \cap A(v)|,\ \bar{t}[u,v] = |V \setminus (A(u) \cup A(v))|,\ \bar{t} = \sum_{[u,v] \in E} \bar{t}[u,v].$$

Put also

$$\begin{aligned} q[u,v,w] &= |A(u) \cap A(v) \cap A(w)|, \\ \bar{q}[u,v,w] &= |V \setminus (A(u) \cup A(v) \cup A(w))|, \\ \bar{q} &= \sum \bar{q}[u,v,w],\ \hat{t} = \max\{t[u,v] | [u,v] \in E\}. \end{aligned}$$

T h e o r e m 1. For any graph G we have the inequality

$$(3t + \bar{t})\hat{t} \geqq nt \tag{15}$$

and the equality holds iff $q = \bar{q} = 0$ and $t[u,v]$ is the same for all $[u,v] \in E$, such that $t[u,v] + \bar{t}[u,v] > 0$.

There is another proof of (14) in [2]. For $n \equiv 0 (\mathrm{mod}\ 6)$ and $n \equiv 0 (\mathrm{mod}\ 9)$ two extremal graphs (i.e. graphs for which the equality in (14) holds) are constructed.

The following two consequences are deduced from (14):

C o r o l l a r y 1. If

$$\sum_{v \in V} d^2(v) > ne, \tag{20}$$

then

$$\hat{t} > \frac{n}{6}. \tag{21}$$

C o r o l l a r y 2. If $t > 0$ and

$$\sum_{v \in V} d^2(v) \geqq ne, \tag{22}$$

then

$$\hat{t} \geqq \frac{n}{6}. \tag{23}$$

§2. L e m m a 1. The inequality

$$e\hat{t} \geqq \sum_{v \in V} d^2(v) - ne, \tag{25}$$

is true and the equality holds iff G is a complete r-partite graph and G is regular in addition for $r \geqq 3$.

L e m m a 2. The inequality

$$\hat{t} \geqq \frac{4e}{n} - n \tag{26}$$

is true and the equality holds iff G is a regular complete r-partite graph with $r \geqq 2$.

By applying Lemmas 1 and 2 the following Theorem is proved:

Theorem 2. If

$$e \geqq \frac{r-1}{2r} n^2, \tag{30}$$

then

$$\hat{t} \geqq \frac{r-2}{r} n \tag{31}$$

and the equality holds in (31) iff $2 \leqq r|n$ and G is a regular complete r-partite graph.

Lemma 4. If

$$e \geqq \left[\frac{n^2}{4}\right] \tag{32}$$

then

$$\sum_{v \in V} d^2(v) \geqq ne \tag{33}$$

and if (32) is strict, then (33) is strict also.

From Corollary 1 and Lemma 4 we obtain:

Corollary 3. If $e > \left[\frac{n^2}{4}\right]$, then $\hat{t} > \frac{n}{6}$.

This corollary is obtained in [2]. It gives a positive solution on the following

Erdös' conjecture. If $e > \frac{n^2}{4}$, then $\hat{t} \geqq \frac{n}{6}$.

Closely connected with Corollary 3 is the following assertion, proved in [2], which is immediately deduced from Corollary 2 and Lemma 4.

Corollary 4. If $e \geqq \left[\frac{n^2}{4}\right]$ and $t > 0$, then $\hat{t} \geqq \frac{n}{6}$.

Put

$$\hat{t}(n) = \min\left\{\hat{t}(G) | e(G) \geqq [\frac{n^2}{4}],\ t(G) > 0\right\}.$$

The conclusion in Corollary 4 cannot be improved under the assumption of this corollary, since from it and by using appropriate examples we obtain:

Corollary 5. For each n $(n \geqq 4)$ we have

$$\hat{t}(n) = \left]\frac{n}{6}\right[. \tag{35}$$

Note. $]x[= \min\{k | k \in \mathbf{N}, k \geqq x\}$.

§3. From Theorem 1 we obtain:

Theorem 3. If $\hat{t} \geqq \frac{n}{6}$, then

$$\hat{t} \geqq \frac{1}{2e} \sum_{v \in V} d^2(v) - \frac{n}{3}. \tag{38}$$

From Corollary 2 and Theorem 3 we obtain:

Corollary 6. If $\sum_{v \in V} d^2(v) \geqq ne$ and $t > 0$, then (38) holds.

From Corollary 4 and Corollary 6 we obtain:

Corollary 7. If $e \geqq \left[\frac{n^2}{4}\right]$ and $t > 0$, then (38) holds.

Corollary 7 confirms an Edwards' conjecture, given in added note of [3]. Since Corollary 7 is easily deduced from Theorem 3, we can say that the Edwards' conjecture is a consequence of Erdös' one. The contrary is also true since if the assumptions of Corollary 7 are satisfied, then by Lemma 4 we have $\sum_{v \in V} d^2(v) \geqq ne$ and (38) implies $\hat{t} \geqq \frac{n}{6}$. Finally, the Erdös' and Edwards' conjectures are equivalent.

In §3 a detailed analysis of Edwards' paper [3] is done also. It is shown that the essential assertions in this paper are trivial consequences of the inclusion-exclusion principle for two or three sets and have nothing to do with the solution of Erdös' conjecture.

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THE CREATION OF WEAK SINGULARITIES IN REFLECTING SEMILINEAR WAVES *

MARK WILLIAMS **

Марк Уилямс. ЗАРОЖДЕНИЕ СЛАБЫХ ОСОБЕННОСТЕЙ ДЛЯ ОТРАЖЕННЫХ ПОЛУЛИНЕЙНИХ ВОЛН. **Рассматривается смешанная задача для полулинейного волнового уравнения. Дается описание процесса зарождения новых особенностей. Ставятся новые проблемы.**

Mark Williams. THE CREATION OF WEAK SINGULARITIES IN REFLECTING SEMILINEAR WAVES. **The process of creation of new singularities in mixed problems for semilinear wave equation is described. Some open problems are stated.**

INTRODUCTION

For solutions to nonlinear hyperbolic equations, it is well-known that interactions between singularity-bearing rays can lead to the appearance of "anomalous" singularities, that is, singularities not present in the solutions to corresponding linear problems. The mechanisms by which this happens, crossing and self-spreading, have been understood for some time in the case when interactions occur in free space (e.g., [1], [2], [7]). Here we shall describe how anomalous singularities are produced in mixed problems for semilinear wave equations $\Box u = f(u)$ on the half-space $\mathbf{R}^{n+1}_+$, due to crossing and self-spreading at boundary points. Several new phenomena connected to the boundary appear. For example we will show that the analogue of Beals' "3s-theorem" fails for reflection in second-order mixed problems. Complete proofs of the results presented here may be found in [4], [8], [9].

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For $n \geq 2$ let $\mathbf{R}^n_+ = \{(x,y) : x > 0\}$, $\Omega = (-\infty, +\infty) \times \overline{\mathbf{R}^n_+}$, $\Omega_T = (-T,T) \times \overline{\mathbf{R}^n_+}$, and $\mathbf{b}\Omega_T = \Omega_T \cap \{x = 0\}$. We state the following problems for a function $u(t,x,y) \in H^s_{\mathrm{loc}}(\Omega_T)$, $s > \frac{n+2}{2}$, which satisfies

$$\Box u = (D_t^2 - \Delta)u = \beta(t) f(u),\ u|_{\mathbf{b}\Omega_T} \in C^\infty, \tag{1}$$

where $\beta(t) \in C_0^\infty$ and $\mathrm{supp}\beta \subset \{|t| < \delta\}$ for some $\delta < T$. Here $z = (t,x,y)$ and $\zeta = (\tau, \xi, \eta)$ denotes the dual variables.

P r o b l e m I (*Crossing*). Let $\Gamma_1 = \{(z_1(t), \zeta_1)\}$, $\Gamma_2 = \{(z_2(t), \zeta_2)\}$ be incoming null bicharacteristics such that $\{z_1(t)\}$, $\{z_2(t)\}$ strike $\mathbf{b}\Omega_T$ transversally when $t = 0$ and cross there (i.e., $z_1(0) = z_2(0) \in \mathbf{b}\Omega_T$, $\zeta_1 \neq \pm\zeta_2$). Suppose that $\mathbf{WF}\, u|_{t<-\delta} = \Gamma_1|_{t<-\delta} \cup \Gamma_2|_{t<-\delta}$ (where $\Gamma_1|_{t<-\delta}$ means $\{(z_i(t), r\zeta_i) : r > 0,\ t < -\delta)\}$. Determine the location and strength of the anomalous singularities that may arise.

P r o b l e m II (*Self-spreading*). Let $\Gamma_\pm = \{(z(t), \pm\zeta)\}$ be incoming null bicharacteristics such that $\{z(t)\}$ strikes $\mathbf{b}\Omega_T$ transversally when $t = 0$. Suppose that $\mathbf{WF}\, u|_{t<-\delta} = \Gamma_+|_{t<-\delta} \cup \Gamma_-|_{t<-\delta}$. Determine the location and strength of the anomalous singularities that may arise.

From [4; Theorem 1.3] it is known that for u as in (1), microlocal H^r regularity for $r <\sim 2s - \frac{n}{2}$ propagates along generalized bicharacteristics. Thus, anomalous singularities in Problems I and II can have strength at most $\sim 2s - \frac{n}{2}$. But can singularities of this strength actually appear? When there is no boundary, Beals' 3s-theorem ([2], [3]) implies that a solution $u \in H^s_{\mathrm{loc}}$ to $\Box u = \beta f(u)$ can have anomalous singularities of strength at most $\sim 3s - n$. When the domain is a half-space, Theorem 1 below shows that for certain choices of $\beta(t)$, $f(u)$, and incoming singularities, new singularities of strength $\sim 2s - \frac{n}{2}$ do actually appear at the moment of reflection. So for u as in (1), although microlocal H^r-regularity for $r <\sim 3s - n$ propagates along null bicharacteristics in free space, for $r >\sim 2s - \frac{n}{2}$ it does not in general reflect (see Fig. 2).

MAIN RESULTS

Choose $\beta(t)$ as above, but now also such that $\widehat{\beta} \geq 0$ and $\widehat{\beta}(0) > 0$, and consider the mixed problem on Ω_T:

$$\Box u = \beta u^2,\ u|_{\mathbf{b}\Omega_T} \in C^\infty,\ u|_{t=-T} = w_0,\ u_t|_{t=-T} = w_1, \tag{2}$$

where w_0, w_1 are the Cauchy data of a function $w(t,x,y) \in H^s_{\mathrm{loc}}(\mathbf{R}^{n+1})$ defined as follows.

Fixing $\rho > 1$ and setting $\langle \xi, \eta \rangle = (1 + |\xi, \eta|^2)^{\frac{1}{2}}$, for any $\omega \in S^{n-1}$ we denote by $f_\omega(x,y)$ a function in $H^s(\mathbf{R}^n)$ such that

(3)
a) $\mathbf{WF}\, f_\omega = \{(0, r_\omega) : r > 0\}$;

b) $0 \leq \widehat{f}_\omega(\xi,\eta) \leq C\langle \xi, \eta \rangle^{-(s + \frac{n}{2} + a(\rho))}$, where $a(\rho) > 0$ and $a(\rho) = O(\rho - 1)$;

c) For (ξ, η) such that $|(\xi,\eta) - |\xi,\eta|\omega| \leq |\xi,\eta|^{\frac{1}{\rho}}$, $\widehat{f}_\omega(\xi,\eta) \geq C'\langle \xi, \eta \rangle^{-(s + \frac{n}{2} + a(\rho))}$.

(Such functions are constructed in [1].) Now define $f_1, f_2, g \in H^s(\mathbf{R}^n)$ by taking ω in (3) equal to $\omega_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\omega_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$, and $\tilde{\omega} = (1,0)$ respectively. We shall consider three possibilities for w:

$$
\begin{aligned}
&\text{a)} \quad w^{\wedge}(t,\xi,\eta) &&= e^{it|\xi,\eta|}[\hat{f}_1(\xi,\eta) + \hat{f}_2(\xi,\eta)];\\
(4) \qquad &\text{b)} \quad w^{\wedge}(t,\xi,\eta) &&= e^{it|\xi,\eta|}\hat{f}_1(\xi,\eta) + e^{-it|\xi,\eta|}\hat{f}_2(-\xi,-\eta);\\
&\text{c)} \quad w^{\wedge}(t,\xi,\eta) &&= e^{it|\xi,\eta|}\hat{g}(\xi,\eta) + e^{-it|\xi,\eta|}\hat{g}(-\xi,-\eta).
\end{aligned}
$$

Hence $\Box w = 0$ and $\mathbf{WF}\, w$ in the three cases is, respectively,

$$
\begin{aligned}
&\text{a)} \quad \bigcup_{i=1,2} \{(t, -t\omega_i, r, r\omega_i) : t \in \mathbf{R},\ r > 0\};\\
(5) \qquad &\text{b)} \quad \{(t, -t\omega_1, r, r\omega_1) : t \in \mathbf{R},\ r > 0\}\\
&\qquad\quad \bigcup \{(t, -t\omega_2, -r, -r\omega_2) : t \in \mathbf{R},\ r > 0\};\\
&\text{c)} \quad \{(t, -t\tilde{\omega}, r, r\tilde{\omega}) : t \in \mathbf{R},\ r \in \mathbf{R}\backslash 0\}.
\end{aligned}
$$

We proceed to define the sets that will carry anomalous singularities. For $w \in S^{n-1}$ let $K^{\omega}_{\pm}$ be the rays through $\pm(1,\omega)$ in $\mathbf{R}^{n+1}\backslash 0$($(\tau,\xi,\eta)$-space) and set

$$
\begin{aligned}
&\text{a)} \quad B^{\omega_1,\omega_2} &&= \overline{K^{\omega_1}_{+} + K^{\omega_2}_{+}} \text{ (closure in } \mathbf{R}^{n+1}\backslash 0);\\
(6) \qquad &\text{b)} \quad B^{\omega_1,-\omega_2} &&= \overline{K^{\omega_1}_{+} + K^{\omega_2}_{-}};\\
&\text{c)} \quad B^{\tilde{\omega}} &&= \{ \text{ tangent plane to } \tau^2 = |\xi,\eta|^2 \text{ at } \pm(1,\tilde{\omega})\}.
\end{aligned}
$$

With $\pi(\tau,\xi,\eta) = (\tau,\eta)$ set

$$
\begin{aligned}
&\text{a)} \quad C_1 &&= \pi B^{\omega_1,\omega_2} \bigcap \{\tau^2 \geq |\eta|^2\} = \pi B^{\omega_1,\omega_2};\\
(7) \qquad &\text{b)} \quad C_2 &&= \pi B^{\omega_1,-\omega_2} \bigcap \{\tau^2 \geq |\eta|^2\};\\
&\text{c)} \quad C_3 &&= \pi B^{\tilde{\omega}} \bigcap \{\tau^2 \geq |\eta|^2\} = \{\tau^2 \geq |\eta|^2\}.
\end{aligned}
$$

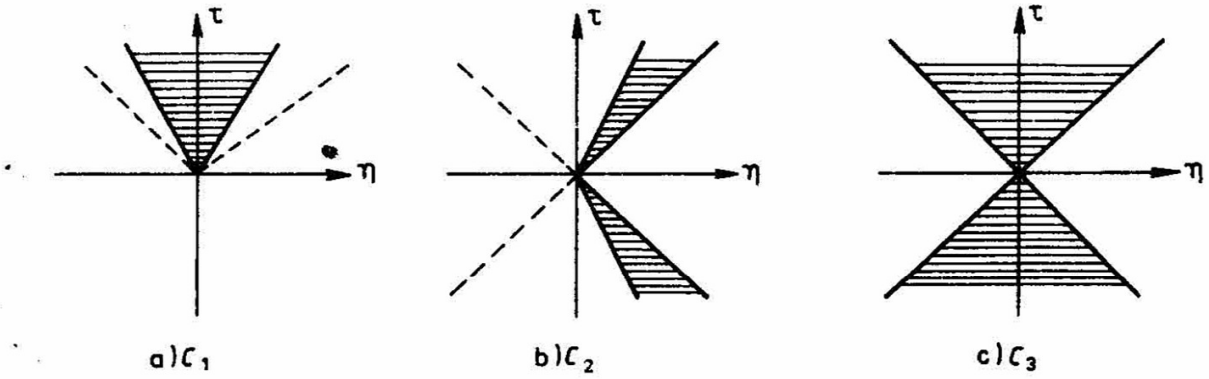

Fig. 1

Next, for $i = 1,2,3$ we define

$$
\Lambda_i = \{\lambda \doteq (t,x,y,\tau,\xi,\eta) : t > 0,\ (t,x,y) \in \mathring{\Omega}_T, \tag{8}
$$

and λ lies on the outgoing $\left(\frac{dx}{dt} > 0\right)$ null bicharacteristic that passes over $(0, (\tau, \eta))$, for some $(\tau, \eta) \in \overset{\circ}{C}_i\}$.

Assuming T is small enough so that a solution $u \in H^s(\Omega_T)$ of (2) exists, we can now state

Theorem 1. Fix $\varepsilon > 0$.

i) *(Crossing)* Define w_0, w_1 in (2) as the Cauchy data at $t = -T$ of w as in (4a) (resp. (4b)). Then if ρ in (3) is chosen close enough to 1, $u \notin H^{2s-\frac{n}{2}+2+\varepsilon}(\lambda)$ for all $\lambda \in \Lambda_1$ (resp. Λ_2) (Fig.2a and 2b);

ii) *(Self-spreading)* The statement is the same, eexept w as in 4c is used and Λ_3 is substituted for Λ_1, Λ_2 (Fig. 2c).

The fact that u is C^∞ in the regions indicated in Fig. 2a and 2b is a consequence of Theorem 1.7 of [8]. This theorem shows that for solutions $u \in H^s(\Omega_T)$ whose incoming singularities are confined to proper cones in $T \star \overset{\circ}{\Omega}_T \cup T \star \mathrm{b}\Omega_T$, it is possible to propagate microlocal $H^{s',t'}$ regularity, for $s' < 2s + \frac{1}{2}$ and arbitrarily large t', outside an appropriate outgoing family of proper cones. This enables one to identify regions into which singularities arising from interactions at the boundary cannot spread.

For semilinear wave equations in free space, anomalous singular support is never produced by the interaction of fewer than three bicharacteristics, unless self-spreading occurs [1]. Part i) of Theorem 1 shows that when a boundary is involved, two crossing rays suffice.

Fig. 2

In the shaded areas $u \notin H^{2s-n/2+2+\varepsilon}$.

SKETCH OF THE PROOF OF THEOREM 1.

The proof is based on an idea like that used by Beals [1, 2] in his studies of spreading in free space. We write $u = v + R\beta u^2$ where $v \in H^s_{\mathrm{loc}}(\Omega)$ satisfies

$$\Box v = 0,\ v|_{\mathrm{b}\Omega} \in C^\infty,\ v|_{t=-T} = w_0,\ v_t|_{t=-T} = w_1, \tag{9}$$

and R is the forward solution operator such that for any U with support in $t > -\delta$, RU satisfies (in Ω)

$$\Box RU = U,\ RU|_{\mathbf{b}\Omega} = 0,\ RU = 0 \text{ in } t < -\delta. \tag{10}$$

Rewriting u as $u = v + R\beta v^2 + R\beta(u^2 - v^2)$, we see that it will suffice to find singularities of strength $2s - \frac{n}{2} + 2 + \varepsilon$ in $R\beta v^2$ on Λ_i, as long as any singularities of $R\beta(u^2 - v^2)$ on Λ_i can be shown to be strictly weaker. So we proceed to study $R\beta v^2$ in the case where w as in (4a) is used to define w_0, w_1.

We can write

$$R\beta v^2 = (E\beta v^2)|_{\overline{\Omega}} - C((E\beta v^2)|_{\mathbf{b}\Omega}), \tag{11}$$

where E is the solution operator in $\mathbf{R}^{n+1}$ such that for any U with support in $t > -\delta$

$$\Box EU = U \text{ and } EU = 0 \text{ in } t < -\delta, \tag{12}$$

and C is such that, for any V on $\mathbf{b}\Omega$ with support in $t > -\delta$, CV satisfies (in Ω)

$$\Box CV = 0,\ (CV)|_{\mathbf{b}\Omega} = V,\ CV = 0 \text{ in } t < -\delta. \tag{13}$$

The term $(E\beta v^2)|_{\overline{\Omega}}$ contributes nothing anomalous to $R\beta v^2$. To see this note first that v is simply the restriction to $x > 0$ of the function in $H^s_{\mathrm{loc}}(\mathbf{R}^{n+1})$, which we will also call v, given by

$$v = v_a - v_b, \text{ where } \hat{v_a}(t,\xi,\eta) = \hat{w}(t,\xi,\eta) \text{ and } \hat{v_b}(t,\xi,\eta) = \hat{w}(t,-\xi,\eta). \tag{14}$$

Observe that the singularities of v_a and v_b lie on incoming $\left(\frac{dx}{dt} < 0\right)$ and outgoing $\left(\frac{dx}{dt} > 0\right)$ rays, respectively. Recalling (4a), we may write $v_a = v_{a1} + v_{a2}$, $v_b = v_{b1} + v_{b2}$, where

$$\hat{v_{ai}} = e^{it|\xi,\eta|}\,\widehat{f_i}(\xi,\eta) \text{ and } \hat{v_{bi}} = e^{it|\xi,\eta|}\,\widehat{f_i}(-\xi,\eta). \tag{14'}$$

The assertion about $E\beta v^2$ follows immediately from the fact that since (with $\pi_2(t,x,y,\tau,\xi,\eta) = (\tau,\xi,\eta)$)

$$\pi_2\mathbf{WF}v^2 \cap \mathrm{char}\Box = \pi_2\mathbf{WF}v \cap \mathrm{char}\Box, \tag{15}$$

v^2 provides nothing new for E to propagate. To verify (15) just note that $\pi_2\mathbf{WF}\,v_{a1}^2 = K_+^{\omega_1}$; $\pi_2\mathbf{WF}\,v_{a1}v_{a2} \subset B^{\omega_1,\omega_2}$ (which satisfies $B^{\omega_1,\omega_2} \cap \mathrm{char}\Box = K_+^{\omega_1} \cup K_+^{\omega_2}$), and similarly for the remaining terms constituting v^2. So we have reduced to considering $C((E\beta v^2)|_{\mathbf{b}\Omega})$. It is worth noting that the above argument indicates the main reason why singularities of strength $\sim 2s - \frac{n}{2}$ never appear in solutions to $\Box u = f(u)$ due to interactions in free space.

Writing $v^2 = (v_a^2 - v_a v_b) + (v_b^2 - v_a v_b)$, we examine first

$$C((E\beta v_a^2)|_{\mathbf{b}\Omega}) - C((E\beta v_a v_b)|_{\mathbf{b}\Omega}). \tag{16}$$

After expanding (16) in terms of the v_{ai}, v_{bi}, one must consider differences like

$$C((E\beta v_{a1}v_{a2})|_{\mathbf{b}\Omega}) - C((E\beta v_{a1}v_{b2})|_{\mathbf{b}\Omega}) \tag{17}$$

We claim that each term in (17) has singularities of strength $\sim 2s - \frac{n}{2}$ on Λ_1. This illustrates an interesting feature of spreading at the boundary. Anomalous singularities of strength $\sim 2s - \frac{n}{2}$ arise at the boundary from three sources: interactions of incoming rays with incoming rays, incoming rays with outgoing (or reflected) rays, and outgoing rays with outgoing rays. We will see that the singularities produced by the incoming-outgoing interactions (represented in (17) by the second term) are strictly weaker than the other two types, so although some cancellations occur, singularities of strength $\sim 2s - \frac{n}{2}$ remain.

Return now to the first term in (17). It is not hard to show that $\beta v_{a1} v_{a2}$ and therefore also $E\beta v_{a1} v_{a2}$ have singularities of strength $\sim 2s - \frac{n}{2}$ throughout $\overset{\circ}{B}^{\omega_1,\omega_2}$ over $(t,x,y) = 0$. Consequently, $(E\beta v_{a1} v_{a2})|_{\mathrm{b}\Omega}$ has singularities of strength $\sim 2s - \frac{n}{2}$ on $\overset{\circ}{C}_1 = \pi \overset{\circ}{B}^{\omega_1,\omega_2}$ over $0 \in \mathrm{b}\Omega$. Since $C_1 \subset \{\tau^2 > |\eta|^2\}$, all of these singularities are propagated by the operator C. Hence $C((E\beta v_{a1} v_{a2})|_{\mathrm{b}\Omega})$ has singularities of strength $\sim 2s - \frac{n}{2}$ throughout Λ_1. Since $\pi_2 \mathbf{WF} \beta v_{a1} v_{b2} \doteq B^{\omega_1,\omega_2^*}$, where $\omega_2^* = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$, and $\pi B^{\omega_1,\omega_2^*}$ also equals C_1, the same argument shows that the second term in (17) also has singularities of strength $\sim 2s - \frac{n}{2}$ throughout Λ_1. We will now show that these latter singularities are weaker by a factor strictly less than one.

Let us focus attention on $\gamma_+(0,(\overline{\tau},\overline{\eta}))$, the outgoing null bicharacteristic passing over a fixed point $(0,(\overline{\tau},\overline{\eta})) \in T^*\mathrm{b}\Omega \backslash 0$ with $(\overline{\tau},\overline{\eta}) \in \overset{\circ}{C}_1$ (so $\gamma_+(0,(\overline{\tau},\overline{\eta})) \subset \Lambda_1$). Setting $\overline{\mu} = (\overline{\tau},\overline{\eta})$, without loss of generality we may take $\overline{\mu} = (1,0)$. Our starting point is the observation that for the purpose of studying the singularities on $\gamma_+(0,\overline{\mu})$ of either term in (17), the rather awkward operator E can be replaced by a simple multiplier.

L e m m a 1. Let $H(\tau,\xi,\eta)$ be the characteristic function of a small conic neighborhood of $A \equiv (1, \frac{1}{\sqrt{2}}, 0) = \frac{1}{2}(1,\omega_1) + \frac{1}{2}(1,\omega_2) \in B^{\omega_1,\omega_2}$, and set $\widehat{U}(\tau,\xi,\eta) = \frac{H(\tau,\xi,\eta)}{\tau^2 - |\xi,\eta|^2} \widehat{\beta v_{a1} v_{a2}}$. Then $\mathbf{WF}[C((E\beta v_{a1} v_{a2})|_{\mathrm{b}\Omega}) - C(U|_{\mathrm{b}\Omega})] \cap \gamma_+(0,\overline{\mu}) = \phi$.

Proof. Set $V = E\beta v_{a1} v_{a2} - C((E\beta v_{a1} v_{a2})|_{\mathrm{b}\Omega})$ and $\tilde{V} = U - C(U|_{\mathrm{b}\Omega})$. Thus $V - \tilde{V}$ satisfies (in Ω)

$$\Box(V - \tilde{V}) = (1 - H(D))\beta v_{a1} v_{a2}, \quad (V - \tilde{V})|_{\mathrm{b}\Omega} = 0, \tag{18}$$

$$V - \tilde{V} \in C^\infty \text{ in } t < -\delta.$$

Now $\widehat{\beta v_{a1} v_{a2}}$ is rapidly decreasing outside B^{ω_1,ω_2}. Thus the fact that $H \equiv 1$ near A implies $\mathbf{WF}(1 - H(D))\beta v_{a1} v_{a2} \cap [\gamma_-(0,\overline{\mu}) \cup \gamma_+(0,\overline{\mu})] = \phi$ (where γ_- denotes the incoming bicharacteristic) and $(0,\overline{\mu}) \notin \mathbf{WF}_{\mathrm{b}}(1 - H(D))\beta v_{a1} v_{a2}$. Classical results on reflection of $\mathbf{WF}_{\mathrm{b}}$ (e.g. [5]) then imply $\mathbf{WF}(V - \tilde{V}) \cap \gamma_+(0,\overline{\mu}) = \phi$, since this clearly holds for γ_-. Because $E\beta v_{a1} v_{a2}$ and U have no singularities on γ_+, the lemma follows.

Of course an analogous result holds for $C((E\beta v_{a1} v_{a2})|_{\mathrm{b}\Omega})$, where now one takes $\tilde{H}(\tau,\xi,\eta)$ to be the characteristic function of a small conic neighborhood of $B \equiv (1,0,0) \in B^{\omega_1,\omega_2^*}$ (recall, $\omega_2^* = (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$) and sets $\widehat{\tilde{U}} = \frac{H(\tau,\xi,\eta)}{\tau^2 - |\xi,\eta|^2} \widehat{\beta v_{a1} v_{b2}}$. In Fig. 3 we have drawn the crosssection of $\tau^2 = |\xi,\eta|^2$ at $\tau = 1$, indicating A, B, B^{ω_1,ω_2}, B^{ω_1,ω_2^*} and the singular directions of the v_{ai}, v_{bi}.

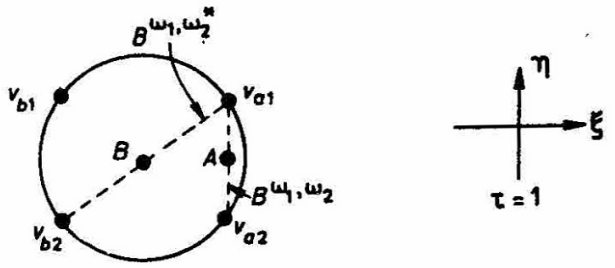

Fig. 3

In view of Lemma 1 it suffices to compare $C(U|_{b\Omega})$ with $C(\tilde{U}|_{b\Omega})$. Now $\widehat{U|_{b\Omega}} = \int \frac{H}{\tau^2-|\xi,\eta|^2}\beta\widehat{v_{a1}v_{a2}}\,d\xi$ while $\widehat{\tilde{U}|_{b\Omega}} = \int \frac{\tilde{H}}{\tau^2-|\xi,\eta|^2}\beta\widehat{v_{a1}v_{b2}}\,d\xi$. Since $\mathrm{supp}H$ lies closer to $\tau^2 = |\xi,\eta|^2$ than $\mathrm{supp}\tilde{H}$ does, it follows that for $(\tau,\xi,\eta) \in \mathrm{supp}H(\tau,\xi',\eta) \in \mathrm{supp}\tilde{H}$, and $\langle\tau,\eta\rangle$ sufficiently large, we have

$$(\tau^2 - |\xi',\eta|^2)^{-1} \leq \varepsilon(\tau^2 - |\xi,\eta|^2)^{-1} \text{ for an } \varepsilon < 1. \tag{19}$$

A comparison of the above two integrals using (19) yields easily that singularities of $C(\tilde{U}|_{b\Omega})$ on $\gamma_+(0,\overline{\mu})$ are strictly weaker than those of $C(U|_{b\Omega})$ ([9], Lemma 3.18), so the difference still has singularities of strength $\sim 2s - \frac{n}{2}$ on γ_+. Since the same argument applies to the terms like $C((E\beta v_{b1}v_{b2})|_{b\Omega}) - C((E\beta v_{a1}v_{b2})|_{b\Omega})$ in $R\beta v^2$, we can summarize the above as follows:

(20) The anomalous singularities on a fixed ray in Λ_1 produced by interactions of incoming rays with incoming rays (v_a, v_a) , or by interactions of outgoing rays with outgoing rays (v_b, v_b) , are strictly stronger than those produced by incoming-outgoing interactions (v_a, v_b). Moreover, as the incoming rays approach being gliding rays, the difference in strength decreases, and so cancellations become increasingly significant.

Essentially the same arguments applied to the remaining terms constituting $R\beta v^2$ show that they contribute singularities of the same strength and sign as those of (17). So this ends our discussion of $R\beta v^2$ in the case when v is defined using w as in (4a). When w as in (4b) (resp.(4c)) is used, the analysis follows the same outline with the plane $B^{\omega_1,-\omega_2}$ (resp. $B^{\tilde{\omega}}$) in the role of B^{ω_1,ω_2}. Note that C_2 (resp. C_3) is properly contained in $\pi B^{\omega_1,-\omega_2}$ (resp. $\pi B^{\tilde{\omega}}$). We restrict to $\tau^2 \geq |\eta|^2$, of course, because the only singularities in $(E\beta v^2)|_{b\Omega}$ that C can propagate are those at points $(0,(\tau,\eta))$ with $\tau^2 \geq |\eta|^2$. In each of the three cases one obtains $R\beta v^2 \notin H^{2s-\frac{n}{2}+2+\epsilon}$ on the appropriate Λ_i.

THE REMAINDER $R\beta(u^2 - v^2)$.

Recalling that $u = v + R\beta u^2$, we have $R\beta(u^2 - v^2) = R\beta(2vR\beta u^2) + R\beta(R\beta u^2)^2$. Since each application of R smooths by one derivative, it is reasonable to expect that any singularities of $R\beta(u^2 - v^2)$ are strictly weaker than those of $R\beta v^2$ on Λ_i. The proof that this is so (see[9]) uses the following results:

(21) a) a microlocal H^s algebra lemma for interior points (namely, Rauch's Lemma [6]);

b) a microlocal $H^{s,s'}$ algebra lemma for boundary points analogous to Rauch's Lemma ([8, Lemma 1.10]);

c) the fact that for u as in (1), microlocal H^r-regularity for $r < 2s - \frac{n}{2} + \frac{1}{2}$ propagates along generalized bicharacteristics ([4, Theorem 1.3]);

d) a theorem describing propagation of microlocal $H^{s,s'}$ regularity along generalized bicharacteristics for second-order, linear, mixed problems with Dirichlet boundary conditions ([8, Theorem 1.3]).

The first step is to obtain a careful estimate of the regularity of v^2 on Λ_i. With this (21 a-d) yield fairly easily that the term $R\beta(R\beta u^2)^2$ is strictly weaker than $R\beta v^2$ on Λ_i, but such an argument does not quite yield sufficient regularity of $R\beta(vR\beta u^2)$. (The difference arises because $R\beta u^2 \in H^{s+1}_{\text{loc}}$, while $v \in H^s_{\text{loc}}$.) To find the extra smoothness needed, we write $R\beta u^2 = R\beta Au^2 + R\beta(I - A)u^2$, where A is a tangential pseudo-differential operator equal to 1 and supported near the strong (s) singularities of u^2. Since the incoming singularity-bearing rays meet $\mathbf{b}\Omega$ transversally, A can be chosen with support in the hyperbolic region of $T \star \mathbf{b}\Omega$. This permits one to extend $R\beta Au^2$ (originally defined only in $x \geq 0$) across the boundary as the solution U of a wave equation in free space. Using this property of U, estimates like $\langle \tau, \xi, \eta \rangle^{s+1} \phi \widehat{R\beta u^2} \in L^2$ can be improved to estimates like $\langle \tau - |\xi, \eta| \rangle \langle \tau, \xi, \eta \rangle^{s+1} \widehat{U} \in L^2$. These in turn are used to obtain a more refined estimate of $vR\beta u^2$ on Λ_i. A final application of (21d) then shows that $R\beta(vR\beta u^2)$ is strictly weaker than $R\beta v^2$ on Λ_i.

OPEN PROBLEMS

It is interesting to consider the analogues of Problems I and II when the incoming rays are gliding rays. The remark (20) suggests, if gliding rays are thought of as limits of transversal rays, that anomalous singularities arising from the crossing of two gliding rays may be significantly weaker than $\sim 2s - \frac{n}{2}$. In Problem II note that if $\Gamma_\pm = \{(z(t), \pm\zeta)\}$ are gliding rays, the tangent plane to char$\Box$ at $\pm\zeta$ projects under $(\tau, \xi, \eta) \to (\tau, \eta)$ to a set that misses the hyperbolic region. In view of the role played by $B^{\tilde{\omega}}$ in the preceding analysis, this suggests that new singularities due to the self-spreading of a gliding ray may be much weaker than in the transversal case, perhaps only of strength $\sim 3s - n$.

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INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, II

IVAN CHOBANOV

Das Wesen der Mathematik liegt eben in ihrer Freiheit.

Georg Cantor

Иван Чобанов. ВВЕДЕНИЕ В АЛГЕБРАИЧЕСКУЮ ТЕОРИЮ СКОЛЬЗЯЩИХ ВЕКТОРОВ, II. Эта работа является второй частью статьи [1] под тем же наименованием, опубликованной несколько лет тому назад в том же *Ежегоднике*, в которой предложена алгебраическая теория реальных скользящих векторов на основе аксиоматически введенных реальных стандартных векторов. Между тем автором опубликована комплексная версия [2] реалных стандартных векторных пространств и развита соответствующая комплексная версия [3] реальной трехмерной линейной аналитической геометрии [4]. Настоящая работа является комплексной версией конструкций, изложенных в работе [1]. Как известно, традиционной механической интерпретáцией реальных скользящих векторов являются концентрированные силы аналитической статики и аналитической динамики. Возможность построения комплексных скользящих векторов имеет глубокие последствия. Ее главным результатом является возможность построения комплексной аналитической механики со всеми проистекающими от этого консеквенциями для логического фундамента этой науки и для решения шестой проблемы Гильберта об ее аксиоматической консолидации.

Ivan Chobanov. INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, II. This paper represents the second part of the article [1] under the same title published some years ago in this *Annual*, in which an algebraic theory of arrows or sliding vectors has been proposed, based on the axiomatically defined real standard vectors. Meanwhile the author has proposed a complex version [2] of the real standard vector spaces and has developed the corresponding complex version [3] of the real 3-dimensional linear analytic geometry [4]. This paper represents a complex version of the constructions exposed in [1]. As it is wellknown, the traditional mechanical interpretation of the real sliding vectors are the concentrated forces in analytical statics and analytical dynamics. The possibility of defining complex arrows has far reaching consequences. Its main result consists

in the potenciality to develop a complex analytical mechanics with all the after-effects this fact implies for the logical foundations of this science and for the solution of Hilbert's sixth problem concerning its axiomatical consolidation.

The present paper represents the second part of the article [1] under the same title published in this *Annual* about ten years ago. In the latter an algebraic theory of the real arrows (or sliding vectors, *vecteurs glissants, gleitende Vectoren, скользящие векторы*) has been proposed, based on the axiomatically defined real standard vectors.

Meanwhile some important development has taken place. It has been discovered [2] that, by the aid of *mot-á-mot* the same system of 15 axioms, by means of which the real standard vector space may be described axiomatically, it is possible to define a complex standard vector space (infinitely many such spaces, as a matter of fact), by introducing a fourth operation vector multiplication in an Hermitean space (which turns out to be probably 3-dimensional), this operation being characterized by two only specific axioms. At that, as it turned out to be, these complex standard vector spaces possess verbatim the same algebraic properties as the real one, *mutatis mutandis*, as it is clear by itself.

This mathematical phenomenon has far-reaching consequences.

First of all, analytic geometries in complex standard vector spaces may be developed, as it has been manifested in the article [3]. This mathematical process provides the geometry, necessary as well as sufficient, for all the following constructions.

Second, an algebraic theory of arrows in complex standard vector spaces may be developed, as this paper and its continuations display. This fact is important in the following two respects.

On the one hand, the real arrows interpret mathematically the (sometimes) so-called *concentrated forces*, i.e. those active and passive forces that are specific for analytical statics and analytical dynamics. Now the possibility to define complex forces is a *conditio sine qua non* for the potentiality to develop a *complex analytical mechanics* (i.e. an analytical mechanics in complex standard vector spaces), and this condition is satisfied by the complex arrows proposed in this paper.

On the other hand, by means of the so-called *statical-kinematical analogy*, the arrows have a direct relationship with the rigid body kinematics. Strictly speaking, a dictionary may be composed (that may be called the *statical-kinematical dictionary*), by the aid of which a bijection may be established between the mathematical facts in the algebra of arrows, on the one hand, and of the kinematics of rigid bodies, on the other hand. In the presence of this dictionary it is out and out superfluous to seek and prove theorems of rigid body kinematics which concern the velocity distribution in a moving rigid body: it is perfectly sufficient to point out the terms of the arrow-algebra that correspond to the respective kinematical terms involved in the kinematical theorems in question, and to prove corresponding theorems for these arrow-algebraic terms. Afterwards, the conclusions of these theorems have to be translated into the kinematical language by means of the statical-kinematical dictionary. In such a way, to any proposition of the algebra of arrows there corresponds automatically a true proposition of the analysis of rigid body kinematics. At that, theorems are discovered and proved out and out easier in the former than in the latter. In such a manner, the possibility to define complex arrows has a direct

bearing to the potentiality to develop a rigid body kinematics in complex standard vector spaces (established, by the way, earlier, in the more general mathematical situation proposed by Hermitean spaces, at that, see [5]), and a complex analytical dynamics in the long run, as it will be immediately seen.

Namely, complex forces and rigid body kinematics in complex standard vector spaces being once developed, all that remains to be done for the building-up of a complex rigid body dynamics is to form those fundamental for this science quantities, the *momentum* and the *moment of momentum* (alias *kinetical moment*) of a rigid body in complex standard vector spaces, and to formulate the corresponding *Eulerian dynamical axioms* (or *laws*, or *principles*), viz. those *of momentum* and *of moment of momentum* of rigid bodies, without which in analytical dynamics *terra, aqua, aere et igni interdicti sumus.*

The importance of all these constructions is predominantly an ideological one, since all they result in a mathematical *Weltanschauung* which affects profoundly the logical foundations of the great science of analytical mechanics.

A wide-spread prejudice even today and even among professional mathematicians is that rational mechanics, in general, and analytical mechanics, in particular, are not mathematics — al least not in the sense this term is accepted nowadays. This bias is supported bilaterally.

On the one hand, there is the multitudinous army of mechanicians and of the ingratiated themselves to mechanics physicists and engineers with such a level of mathematical schooling that, at the best, cannot but call forth condescending smiles on the part of the modern professional mathematicians. For these mechanicians the standpoint that rational mechanics is mathematics, and not applied mathematics at that, is a rather disadvantageous — we should say, unprofitable, unproductive, even contraproductive — one. This attitude once adopted, all mechanical writings should unconditionally satisfy the severe modern mathematical criteria of logical rigour — a demand that goes many times beyond the possibilities of their authors. Herefrom the myth of those would-be specific peculiarities of rational mechanics that presumably does not permit its insertion in the confined frames of pure mathematics, in the downright sense of the word. If we try to persuade these people in the contrary, then we purely and simply *canimus surdis*, putting it mildly.

On the other hand, there is the not lesser army of those mathematicians who have wound up with rational mechanics on the very student's desks and, in their *horror vacui*, regard it as a little short of a *monstrum horrendum, informe, ingens.* The fear of the unknown is instinctive; it is proverbial, too: *ignoti nulla supido ... damnant quod non intellegunt.* As to mechanical ignorance of some pure mathematicians, it is comparable only with the mathematical ignorance of some applied mechanicians: Banach, for instance, went as far as to write in his *Mechanics* [6] neither more nor less than "if a rigid body is at rest, we shall say that it is in equilibrium" (p. 234) ?!?

Things being as they are, is it strange that rational mechanics is nowadays a *persona non grata* in the United Kingdom of Mathematical Sciences?

There are lucky exceptions, though. One of them was Hilbert. Another one is Truesdell.

As a pure mathematician, Hilbert needs no recommendations. *Corvo quoque rarior albo*, however, he was one of those few pure mathematicians who are complete strangers to the very idea of mathematical chauvinism. Hilbert was a mathematical cosmopolitan. The final chord of his famous *Mathematische Probleme* [7] is a vivid

incarnation of his mathematical credo:

"... und es drängt sich uns die Frage an, ab der Mathematik einst bevorsteht, was anderen Wissenschaften längst widerfahren ist, nämlich daß sie in einzelne Teilwissenschaften zerfällt, deren Vertreter sich kaum noch einmal verstehen und deren Zusammenhang daher immer loser wird. Ich glaube und wünsche dies nicht. Die mathematische Wissenschaft ist meiner Ansicht nach ein unteilbares Ganzes, ein Organismus, dessen Lebensfähigkeit durch den Zusammenhang seiner Teile bedingt wird. Denn bei aller Verschiedenheiten des mathematischen Wissenstoffes im einzelnen, gewahren wir doch sehr deutlich die Gleichheit der logischen Hilfsmitteln, die Verwandschaft der Ideenbildungen in der ganzen Mathematik, und die zahlreichen Analogien in ihren verschiedenen Wissensgebieten. Auch bemerken wir: je weiter eine mathematische Theorie ausgebildet wird, desto harmonischer und einheitlicher gestaltet sich ihr Aufbau, und ungeahnte Beziehungen zwischen bisher getrennten Wissenszweigen werden entdeckt. So kommt es, daß mit der Ausdehnung der Mathematik ihr einheitlicher Charakter nicht verlorengeht, sondern desto deutlicher offenbar wird.

Aber — so fragen wir — wird es bei der Ausdehnung des mathematischen Wissens für den einzelnen Forscher nicht schließlich unmöglich, alle Teile dieses Wissens zu umfassen? Ich möchte als Antwort darauf hinweisen, wie sehr es im Wesen der mathematischen Wissenschaft liegt, daß jeder wirkliche Fortschritt stets Hand in Hand geht mit der Auffindung schärferen Hilfsmittel und einfacheren Methoden, die zugleich das Verständnis früheren Theorien erleichtern und umständliche ältere Entwicklungen beseitigen, and daß es daher dem einzelnen Forscher, indem er sich diese schärferen Hilfsmitter und einfacheren Methoden zu eigen macht, leichter gelingt, sich in den verschiedenen Wissenszweigen der Mathematik zu orientieren, als dies für irgend eine andere Wissenschaft der Fall ist.

Der einheitliche Charakter der Mathematik liegt im inneren Wesen dieser Wissenschaft begründet ..."

Like Archimedes, Hilbert stood firm on his physical legs. Although he has never taught mechanics and has written not a single specific line on rational mechanics, he nevertheless did not deny its purely mathematical core and believed steadily in its potentialities of being developable as an axiomatically deducible structure. In point of fact, Hilbert included in his list of 23 mathematical problems [7] the nineteenth century bequeaths to the twentieth to solve, as problem number six, that of the axiomatical foundation of rational mechanics.

As a pure mechanician Truesdell needs no recommendations either. Thirty years ago he performed such a bright mathematical apology of rational mechanics that I shall never get tired of quoting it over and over again:

"... *rational mechanics is a part of mathematics.* It is a mathematical science, and in its relations to experience, intuition, abstraction, and everyday life it does not differ in essence from other branches of mathematics ...

Is rational mechanics a part of pure mathematics? To most mathematicians today pure mathematics means topology, abstract algebra, or analysis in abstract spaces. These, most certainly, rational mechanics makes no attempt to imitate. While in spirit it is nearest to geometry, its problems, its aims, and its methods bear little evident similarity to those of other parts of mathematics. A theorem in topology is not evaluated in terms of its bearing on the theory of numbers. It is equally ridiculous, though unfortunately not infrequent, to deprecate theorems of rational mechanics when they do not also contribute to the more popular branches

of pure mathematics.

Is rational mechanics a part of applied mathematics? Most certainly not" [8, p. 335, 337].

Tormented words, it is true. *Nuda veritas*, though. Alas, one swallow does not make a summer. *Sunt verba et voces, praetereque nihil*: the physical, the engineering, the antimathematical mental constitution — mathematics in no wise means formulas only — of the prevalent majority of contemporary mechanicians that has driven *Ilias malorum* in rational mechanics, persevers in being the predominating ideology, the retrograde philosophy in this great science, in which the *certamen pro aris et focis* has not yet begun.

However modest, the present paper contributes my mite in the noble struggle against present-day obscurantism in mechanics conceived by Lagrange's *ipsissima verba* "la maniere dont j'ai tâché de remplir cet objet ne laissera rien à desirer". At the same time, it incarnates Hilbert's dictim in [7], *nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien berücksichtigen zu haben.*

§ 1. PRAELIMINARIA

The complex standard vector space being *principium ab Jove* for the whole following exposition, the most important moments of their introduction will be now reminded.

The following notations are permanently used throughout this paper.

The symbols Ax, Df, Pr, Dm, Sch, Sgn, and sgn: replace the words *axiom, definition, proposition, proof, scholium, notation* and *denote* respectively.

The letters R and C are reserved for the *fields of all real* and *all complex numbers* respectively.

The letters F amd P are reserved for any *ordered field* and for any *Pythagorean field* respectively. An ordered field P is called *Pythagorean* iff $0 \leqq \alpha \in P$ implies the existance of a $\beta \in P$ with $0 \leqq \beta$ and $\beta^2 = \alpha$. Then β is called the *square root* of α and is denoted by $\sqrt{\alpha}$.

The symbol $C(F)$ is reserved for the *complex extension* of F. In other words, $C(F) = F^2$ supplied with the two operations

$$(1) \qquad (x_1, x_2) + (y_1, y_2) \qquad \text{sgn}: \quad (x_1 + y_1, x_2 + y_2)$$

(*addition* in $C(F)$) and

$$(2) \qquad (x_1, x_2)(y_1, y_2) \qquad \text{sgn}: \quad (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1)$$

(*multiplication* in $C(F)$). Besides, by definition

$$(3) \qquad \overline{(x_1, x_2)} \qquad \text{sgn}: \quad (x_1 - x_2) \qquad ((x_1, x_2) \in F^2),$$

$$(4) \qquad |(x_1, x_2)| \qquad \text{sgn}: \quad \sqrt{x_1^2 + x_2^2} \qquad ((x_1, x_2) \in P^2).$$

The symbols in the left-hand sides of (3) and (4) are called the *conjugate number* of (x_1, x_2) and the *module* of (x_1, x_2) respectively. Obviously, (2)–(4) imply

$$(5) \qquad |(x_1, x_2)|^2 = (x_1, x_2)\overline{(x_1, x_2)} \qquad ((x_1, x_2) \in P^2),$$

the "real" element $(x, 0)$ of $C(F)$ being identified with the element x of F by means of the traditional convention

(6) $\qquad x \quad \text{sgn}: \quad (x, 0).$

Quotations are made in the following manner (the example is a fictitious one): Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and relation (6) of §7, for instance, are cited by Sgn 1, Ax 2, Df 3, Pr 4, Sch 5, and (6) respectively in §7 itself, but by 7Sgn 1, 7Ax 2, 7Df 3, 7Pr 4, 7Sch 5, and 7(6) respectively anywhere else.

The whole of the following exposition is based on the following definition. •

Df 1. S denoting F or $C(F)$, a *standard vector space over* S (an *S-standard vector space*) is called any set V_S for which mappings

(7) $\qquad m_1 : \quad V_S^2 \longrightarrow V_S$

(*addition* in V_S),

(8) $\qquad m_2 : \quad S \times V_S \longrightarrow V_S$

(*multiplication* of the elements of S and V_S),

(9) $\qquad m_3 : \quad V_S^2 \longrightarrow S$

(*scalar multiplication* of the elements of V_S), and

(10) $\qquad m_4 : \quad V_S^2 \longrightarrow V_S$

(*vector multiplication* in V_S) are defined, such that, if

(11) $\qquad \boldsymbol{a}+\boldsymbol{b} \quad \text{sgn}: \quad m_1((\boldsymbol{a}, \boldsymbol{b}))$

(*sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$),

(12) $\qquad \alpha\boldsymbol{a} \quad \text{sgn}: \quad m_2((\alpha, \boldsymbol{a}))$

(*product* of α and $\boldsymbol{a}$),

(13) $\qquad \boldsymbol{ab} \quad \text{sgn}: \quad m_3((\boldsymbol{a}, \boldsymbol{b}))$

(*scalar product* of $\boldsymbol{a}$ and $\boldsymbol{b}$),

(14) $\qquad \boldsymbol{a} \times \boldsymbol{b} \quad \text{sgn}: \quad m_4((\boldsymbol{a}, \boldsymbol{b}))$

(*vector product* of $\boldsymbol{a}$ and $\boldsymbol{b}$), and

(15) $\qquad \boldsymbol{a}-\boldsymbol{b} \quad \text{sgn}: \quad \boldsymbol{a}+(-\boldsymbol{b})$

(*difference* of $\boldsymbol{a}$ and $\boldsymbol{b}$), then the following conditions are satisfied:

Ax 1S. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$.

Ax 2S. There exists $\boldsymbol{o} \in V_S$ with $\boldsymbol{a} \in V_S$ implies $\boldsymbol{a}+\boldsymbol{o} = \boldsymbol{a}$.

Ax 3S. $\boldsymbol{a} \in V_S$ implies: there exists $-\boldsymbol{a} \in V_S$ with $\boldsymbol{a}+(-\boldsymbol{a}) = \boldsymbol{o}$.

Ax 4S. $\boldsymbol{a} \in V_S$ implies $1\boldsymbol{a} = \boldsymbol{a}$.

Ax 5S. $\lambda, \mu \in S$, $\boldsymbol{a} \in V_S$ imply $(\lambda\mu)\boldsymbol{a} = \lambda(\mu\boldsymbol{a})$.

Ax 6S. $\lambda, \mu \in S$, $\boldsymbol{a} \in V_S$ imply $(\lambda+\mu)\boldsymbol{a} = \lambda\boldsymbol{a}+\mu\boldsymbol{a}$.

Ax 7S. $\lambda \in S$, $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\lambda(\boldsymbol{a}+\boldsymbol{b}) = \lambda\boldsymbol{a}+\lambda\boldsymbol{b}$.

Ax 8S. $a, b \in V_S$ imply $ab = \overline{ba}$.
Ax 9S. $\lambda \in S$, $a, b \in V_S$ imply $(\lambda a)b = \lambda(ab)$.
Ax 10S. $a, b, c \in V_S$ imply $(a+b)c = ac + bc$.
Ax 11S. $a \in V_S$ implies $aa \geqq 0$.
Ax 12S. $a \in V_S$, $aa = 0$ imply $a = o$.
Ax 13S. $a, b, c \in V_S$ imply $a \times b \cdot c = b \times c \cdot a$.
Ax 14S. $a, b, c \in V_S$ imply $(a \times b) \times c = (ac)b - (bc)a$.
Ax 15S. There exist $a, b \in V_S$ with $a \times b \neq o$.
Df 2. The elements of V_S are called *standard vectors over* S (*S-standard vectors*).
Sch 1. The conditions Ax 1S–15S are called *axioms of a standard vector space over* S (*of a S-standard vector space*).
Sch 2. The symbols 0 in Ax 11S, Ax 12S and 1 in Ax 4S denote the *zero-element* and the *unit-element* of S respectively.
Df 3. o is called the *zero-vector*.
Df 4. $-a$ is called the *opposite vector* of a.
Df 5. $a \times b \cdot c$ is called the *right-hand compound product* of a, b, c.
Df 6. $a \cdot b \times c$ is called the *left-hand compound product* of a, b, c.
Df 7. $(a \times b) \times c$ is called the *right-hand double vector product* of a, b, c.
Df 8. $a \times (b \times c)$ is called the *left-hand double vector product* of a, b, c.
Sgn 1. a^2 sgn: aa if $a \in V_S$.
Df 9. a^2 is called the *scalar square* of a.
Sgn 2. a, $|a|$, mod$a$ sgn: $\sqrt{a^2}$ if $a \in V_P$ or $a \in V_{C(P)}$.
Df 10. a, $|a|$, mod$a$ is called the *module* of a.
Sgn 3. V sgn: V_R.
Df 11. V is called the *real standard vector space*.
Df 12. The elements of V are called *real standard vectors*.
Df 13. V_C is called the *complex standard vector space*.
Df 14. The elements of V_C are called *complex standard vectors*.
Sgn 4. a^0 sgn: $\frac{1}{a}a$ if $a \in V_P$ or $a \in V_{C(P)}$ and $a \neq o$.
Df 15. a^0 is called the *unit-vector* (the *ort*) of a.
Sch 2. The basic algebraic properties of V_S and especially of $V_{C(F)}$ and $V_{C(P)}$ are discussed at length in the article [2], see also [9, 10]. Therefrom we shall not dwell on this question in details here and, if necessary, we shall refer the reader to these sources. Yet, a compendium of the basic situations of V_S-algebra will be found to be useful. Therefore, such one is exposed immediately below.
Pr 1. The system of axioms Ax 1S–15S is consistent.
Sch 3. Pr 1 is proved by constructing a model of V_S. It is proposed by S^3, supplied with the following operations corresponding to the mappings (7)–(10) respectively:

$$(16) \qquad (x_1, x_2, x_3) + (y_1, y_2, y_3) \quad \text{sgn}: \quad (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

$$(17) \qquad \lambda(x_1, x_2, x_3) \quad \text{sgn}: \quad (\lambda x_1, \lambda x_2, \lambda x_3),$$

$$(18) \qquad (x_1, x_2, x_3)(y_1, y_2, y_3) \quad \text{sgn}: \quad \sum_{\nu=1}^{3} x_\nu \overline{y}_\nu,$$

(19) $(x_1, x_2, x_3) \times (y_1, y_2, y_3)$ sgn :

$$(\overline{x}_2\overline{y}_3 - \overline{x}_3\overline{y}_2, \overline{x}_3\overline{y}_1 - \overline{x}_1\overline{y}_3, \overline{x}_1\overline{y}_2 - \overline{x}_2\overline{y}_1)$$

Now it is verified that (16)–(19) satisfy Ax 1S–15S.

Pr 2. The system of axioms Ax 1S–15S is categorical.

Sch 4. Pr 2 is an immediate corollary from Pr 5 below and from the fact, well-known from the algebra of Hermitean spaces, that the theory of any finite-dimensional Hermitean space is *categorical*, i.e. any two of its models are *isomorphic*

Pr 3. V_S is a *group* with respect to the operation (7).

Pr 4. V_S is a *linear space* over S with respect to the mappings (7), (8).

Pr 5. V_S is a 3-*dimensional Hermitean space* over S with respect to the mappings (7)–(9).

Pr 4 implies

Pr 6. V_S is a *commutative group.*

Sch 5. Pr 3–6 represent, as the saying is, a global characteristic of the S-standard vector spaces. A local picture is proposed by the following propositions.

Pr 7. $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a}(\lambda\boldsymbol{b}) = \overline{\lambda}(\boldsymbol{ab})$.

Pr 8. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \overline{\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}}$.

Pr 9. $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $(\boldsymbol{a} \times \boldsymbol{b})^2 = \boldsymbol{a}^2\boldsymbol{b}^2 - (\boldsymbol{ab})$.

Pr 10. $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply: $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent iff $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{o}$.

Pr 11. $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply

$$(20) \qquad (\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})\overline{(\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c})} = \begin{vmatrix} \boldsymbol{a}^2 & \boldsymbol{ab} & \boldsymbol{ac} \\ \boldsymbol{ba} & \boldsymbol{b}^2 & \boldsymbol{bc} \\ \boldsymbol{ca} & \boldsymbol{cb} & \boldsymbol{c}^2 \end{vmatrix}.$$

Pr 12. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply: $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent iff $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} \neq 0$.

Pr 13. $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$.

Pr 14. $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $(\lambda\boldsymbol{a}) \times \boldsymbol{b} = \overline{\lambda}(\boldsymbol{a} \times \boldsymbol{b})$.

Pr 15. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$.

Pr 16. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{ca})\boldsymbol{b} - (\boldsymbol{ba})\boldsymbol{c}$.

Pr 17. $\lambda \in S$; $\boldsymbol{a}, \boldsymbol{b} \in V_S$ imply $\boldsymbol{a} \times (\lambda\boldsymbol{b}) = \overline{\lambda}(\boldsymbol{a} \times \boldsymbol{b})$.

Pr 18. $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V_S$ imply $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$.

Sch 6. A most important role in V_S-algebra play the so-called *Gibbs' vectors.* They are defined in the following manner.

Let

$$(21) \qquad \boldsymbol{a}_\nu \in V_S \qquad (\nu = 1, 2, 3),$$

$$(22) \qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \neq 0.$$

Then

$$(23) \qquad \boldsymbol{a}_\nu^{-1} \quad \text{sgn} : \quad \frac{\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}}{\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3} \qquad (\nu = 1, 2, 3)$$

provided

$$(24) \qquad \boldsymbol{a}_{\nu+3} \quad \text{sgn} : \quad \boldsymbol{a}_\nu \qquad (\nu = 1, 2)$$

are called *Gibbs'* or *reciprocal vectors* of the vectors (21).

The basic properties of (23) are described by the following propositions.

Pr 19. (21), (22) imply

$$\boldsymbol{a}_\mu^{-1}\boldsymbol{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1, 2, 3). \tag{25}$$

Pr 20. (21), (22) imply

$$\boldsymbol{a}_1^{-1} \times \boldsymbol{a}_2^{-1} \cdot \boldsymbol{a}_3^{-1} \neq 0. \tag{26}$$

Pr 21. (21), (22) imply

$$(\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a})(\boldsymbol{a}_1^{-1} \times \boldsymbol{a}_2^{-1} \cdot \boldsymbol{a}_3^{-1}) = 1. \tag{27}$$

Pr 22. (21), (22) imply

$$(\boldsymbol{a}_\nu^{-1})^{-1} = \boldsymbol{a}_\nu \qquad (\nu = 1, 2, 3). \tag{28}$$

Pr 23. (21), (22) imply

$$\boldsymbol{a}_\nu^{-1} = \boldsymbol{a}_\nu \qquad (\nu = 1, 2, 3). \tag{29}$$

iff

$$\boldsymbol{a}_\mu\boldsymbol{a}_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1, 2, 3). \tag{30}$$

Pr 24. (21), (22),

$$\boldsymbol{r} \in V_S \tag{31}$$

imply

$$\boldsymbol{r} = \sum_{\nu=1}^{3} (\boldsymbol{r}\boldsymbol{a}_\nu^{-1})\boldsymbol{a}_\nu. \tag{32}$$

Pr 25. (21), (22), (31) imply

$$\boldsymbol{r} = \sum_{\nu=1}^{3} (\boldsymbol{r}\boldsymbol{a}_\nu)\boldsymbol{a}_\nu^{-1}. \tag{33}$$

Pr 26. (21), (22),

$$\alpha_\nu \in S \qquad (\nu = 1, 2, 3) \tag{34}$$

imply: there exists exactly one (31) with

$$\boldsymbol{r}_\nu\boldsymbol{a}_\nu = \alpha_\nu \qquad (\nu = 1, 2, 3), \tag{35}$$

namely

$$\mathfrak{r} = \sum_{\nu=1}^{3} \alpha_\nu \mathfrak{a}_\nu^{-1}. \tag{36}$$

Pr 27. (21), (22) imply: there exists exactly one (31) with

$$\mathfrak{r}\mathfrak{a}_\nu = 0 \qquad (\nu = 1, 2, 3), \tag{37}$$

namely

$$\mathfrak{r} = \mathfrak{o}. \tag{38}$$

Pr 28. (31),

$$\mathfrak{a}_\nu \in V_S \qquad (\nu = 1, 2), \tag{39}$$

$$\mathfrak{b}_\nu \in V_S \qquad (\nu = 1, 2), \tag{40}$$

$$\mathfrak{r} \times \mathfrak{a}_\nu = \mathfrak{b}_\nu \qquad (\nu = 1, 2) \tag{41}$$

imply

$$\mathfrak{a}_\mu \mathfrak{b}_\nu + \mathfrak{a}_\nu \mathfrak{b}_\mu = 0 \qquad (\mu, \nu = 1, 2). \tag{42}$$

Pr 29. (31), (39),

$$\mathfrak{a}_1 \times \mathfrak{a}_2 \neq \mathfrak{o}, \tag{43}$$

$$\mathfrak{r} \times \mathfrak{a}_\nu = \mathfrak{o} \qquad (\nu = 1, 2) \tag{44}$$

imply (38).

Pr 30. (21), (22),

$$\mathfrak{b}_\nu \in V_S \qquad (\nu = 1, 2, 3), \tag{45}$$

$$\mathfrak{a}_\mu \mathfrak{b}_\nu + \mathfrak{a}_\nu \mathfrak{b}_\mu = 0 \qquad (\mu, \nu = 1, 2). \tag{46}$$

imply: there exists exactly one (31) with

$$\mathfrak{r} \times \mathfrak{a}_\nu = \mathfrak{b}_\nu \qquad (\nu = 1, 2, 3), \tag{47}$$

namely

$$\mathfrak{r} = \frac{1}{2} \sum_{\nu=1}^{3} \mathfrak{a}_\nu^{-1} \times \mathfrak{b}_\nu. \tag{48}$$

Pr 31. (39), (40), (42), (43) imply: there exists exactly one (31) with (41), namely (48) provided

$$\mathfrak{a}_3 \quad \text{sgn} : \quad \mathfrak{a}_1 \times \mathfrak{a}_2, \tag{49}$$

$$\mathfrak{b}_3 \quad \text{sgn} : \quad (\mathfrak{b}_1 \cdot \mathfrak{a}_2 \times \mathfrak{a}_1)\mathfrak{a}_1^{-1} + (\mathfrak{b}_2 \cdot \mathfrak{a}_2 \times \mathfrak{a}_1)\mathfrak{a}_2^{-1}. \tag{50}$$

Sch 7. There are four basic systems of vector-algebraic equations which are routinely applied to V_S-algebra and in its applications to various problems, mainly in geometry and mechanics. Three of them are the systems (35), (41), and (47). With regard to the fourth one, it is regarded in the following propositions Pr 33–Pr 35.

Pr 32, (31),

(51) $\quad \boldsymbol{a}, \boldsymbol{b} \in V_S,$

(52) $\quad \boldsymbol{a} \neq \boldsymbol{o},$

(53) $\quad \boldsymbol{ab} = 0$

imply:

(54) $\quad \boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}$

iff there exists

(55) $\quad \alpha \in S$

with

(56) $\quad \boldsymbol{r} = \alpha \boldsymbol{a} + \dfrac{\boldsymbol{a} \times \boldsymbol{b}}{\boldsymbol{a}^2}.$

Pr 33. (51), (53), (55),

(57) $\quad \boldsymbol{c} \in V_S,$

(58) $\quad \boldsymbol{ac} \neq 0$

imply: there exists exactly one (31) with (54) and

(59) $\quad \boldsymbol{rc} = \alpha,$

namely

(60) $\quad \overset{*}{\boldsymbol{r}} = \dfrac{\alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b}}{\boldsymbol{ac}}.$

Pr 34. (51), (55), (57),

(61) $\quad \boldsymbol{ac} = 0,$

(62) $\quad \alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} \neq \boldsymbol{o}$

imply: there exists no (31) with (54), (59).

35. (51)–(53), (55), (57), (61),

(63) $\quad \boldsymbol{c} \neq \boldsymbol{o},$

(64) $\quad \alpha \boldsymbol{a} + \boldsymbol{c} \times \boldsymbol{b} = \boldsymbol{o}$

imply: any (31) satisfying (54) is satisfying (59) too, but there exists one at least (31) satisfying (59) which does not satisfy (54).

§ 2. BASIC DEFINITIONS

This paragraph contains the basic definitions relating mainly to a single arrow and its fundamental attributes.

Sgn 1. W_S sgn: $\{(s, m) \in V_S^2 : s \neq o, sm = 0 \vee s = m = o\}$.

Df 1. The elements of W_S are called *arrows in* V_S or *S-arrows.*

Df 2. s is called the *basis* of $\overrightarrow{s}$ if

(1) $\qquad \overrightarrow{s} \in W_S,$

(2) $\qquad \overrightarrow{s} = (s, m).$

Df 3. m is called the *moment* of $\overrightarrow{s}$ if (1), (2).

Sgn 2. $\overrightarrow{o}$ sgn: (o, o).

Df 4. $\overrightarrow{o}$ is called the *zero-arrow.*

Df 5. $\overrightarrow{s}$ is called a *non-zero arrow* if (1),

(3) $\qquad \overrightarrow{s} \neq \overrightarrow{o}.$

Pr 1. (2) implies (3) iff

(4) $\qquad s \neq o, \qquad sm = 0.$

Dm. Sgn 1, Sgn 2.

Sgn 3. Λ_S sgn: $\{(s, m) \in V_S^2 : s \neq o, sm = 0\}$.

Pr 2. (1) implies (3) iff $\overrightarrow{s} \in \Lambda_S$.

Dm. Sgn 1, Pr 1, Sgn 3.

Pr 3. If

(5) $\qquad \overrightarrow{o} \neq \overrightarrow{s} \in W_S,$

then there exists exactly one $l \in L_S$ with

(6) $\qquad \overrightarrow{s}$ & $l.$

Dm. Pr 2, Sgn 3, [3] 1, Pr 19.

Sgn 4. dir $\overrightarrow{s}$ sgn: $l \in L_S$ with (6) if (5).

Df 6. dir $\overrightarrow{s}$ is called the *directrix* of $\overrightarrow{s}$.

Pr 4. If

(7) $\qquad \overrightarrow{s} \in \Lambda_S,$

then

(8) $\qquad \overrightarrow{s}$ & dir $\overrightarrow{s}$.

Dm. Sgn 3, Sgn 1, Pr 2, Pr 3, Sgn 4.

Sgn 5. $\mathrm{mom}_r \overrightarrow{s}$ sgn: $m + s \times r$ if (1), (2),

(9) $\qquad r \in V_S.$

Df 7. $\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s}$ is called the *r-moment* of $\overrightarrow{s}$.

Df 8. r is called the *pole* of $\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s}$.

Pr 5. (1), (2), (9) imply $s \cdot \mathrm{mom}_{\mathfrak{r}} \overrightarrow{s} = 0$.

Dm. Sgn 5, Sgn 1.

Pr 6. (1), (2) imply $m = \mathrm{mom}_{\mathfrak{o}} \overrightarrow{s}$.

Dm. Sgn 5.

Pr 7. (9) implies $\mathrm{mom}_{\mathfrak{r}} \overrightarrow{o} = o$.

Dm. Sgn 5, Sgn 2.

Pr 8. (5), (9) imply

(10) $$\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s} = \mathfrak{o}$$

iff

(11) $$r \; Z \; \mathrm{dir} \; \overrightarrow{s}.$$

Dm. Sgn 5 implies (10) iff

(12) $$r \times s = m.$$

Pr 3, Sgn 4 imply: dir $\overrightarrow{s}$ exists. Now Sgn 4, [3] 4 Sgn 1 imply (11) iff (12).

Pr 9. (5), (9),

(13) $$\overline{\rho} \in V_S,$$

(14) $$\overline{\rho} \; Z \; \mathrm{dir} \; \overrightarrow{s}$$

imply

(15) $$\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s} = (\overline{\rho} - r) \times s.$$

Dm. (13), (14), Sgn 4, [3] 4Sgn 1 imply

(16) $$\overline{\rho} \times s = m.$$

Now (16), Sgn 5 imply (15).

Sch 1. Traditionally text-books on analytical mechanics define $\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s}$ (in the case of V-arrows, naturally) by (15) rather than by

(17) $$\mathrm{mom}_{\mathfrak{r}} \overrightarrow{s} = m + s \times r.$$

The definition (17) obviously surpasses the definition (15) in being more economical.

Pr 10. (1), (2),

(18) $$r_\nu \in V_S \qquad (\nu = 1, 2)$$

imply

(19) $$\mathrm{mom}_{\mathfrak{r}_1} \overrightarrow{s} - \mathrm{mom}_{\mathfrak{r}_2} \overrightarrow{s} = s \times (r_1 - r_2).$$

Dm. Sgn 5.

Sch 2. The relation (19) is usually called the *connection between the moments of an arrow with respect to two poles.*

Sch 3. (19) implies

$$(20) \qquad \boldsymbol{s} \cdot \mathrm{mom}_{\mathfrak{r}_1} \overrightarrow{s} = \boldsymbol{s} \cdot \mathrm{mom}_{\mathfrak{r}_2} \overrightarrow{s}.$$

The inference (20) from (19) is, however, a trivial one, in the light of Pr 5.

Pr 11. (1), (18) imply

$$(21) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathfrak{r}_1} \overrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathfrak{r}_2} \overrightarrow{s}.$$

Dm. Pr 10.

Pr 12. (1), (18),

$$(22) \qquad \boldsymbol{r}_1 \neq \boldsymbol{r}_2,$$

$$(23) \qquad S = P \quad \text{or} \quad S = C(P)$$

imply

$$(24) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \cdot \mathrm{mom}_{\mathfrak{r}_1} \overrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \ \mathrm{mom}_{\mathfrak{r}_2} \overrightarrow{s}$$

Dm. Pr 11, 1 Sgn 4.

Sch 4. The relation (24) gives an utterance of the fact that if (23) holds and if (18) are different poles, then the projections of $\mathrm{mom}_{\mathfrak{r}_\nu} \overrightarrow{s}$ $(\nu = 1, 2)$ on the line l connecting them, i.e. defined by

$$(25) \qquad (\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{r}_2 \times \boldsymbol{r}_1) \ \& \ l,$$

are equal.

Pr 13. (1),

$$(26) \qquad \boldsymbol{r}_\nu \in V_S \qquad (\nu = 1, 2, 3),$$

$$(27) \qquad \boldsymbol{r}_1 \times \boldsymbol{r}_2 + \boldsymbol{r}_2 \times \boldsymbol{r}_3 + \boldsymbol{r}_3 \times \boldsymbol{r}_1 \neq \mathfrak{o},$$

$$(28) \qquad \mathrm{mom}_{\mathfrak{r}_\nu} \overrightarrow{s} = \mathfrak{o} \qquad (\nu = 1, 2, 3)$$

imply

$$(29) \qquad \overrightarrow{s} = \overrightarrow{o}.$$

Dm. (2), (28), Sgn 5 imply

$$(30) \qquad \boldsymbol{r}_\nu \times \boldsymbol{s} = \boldsymbol{m} \qquad (\nu = 1, 2, 3),$$

whence

$$(31) \qquad (\boldsymbol{r}_\nu - \boldsymbol{r}_3) \times \boldsymbol{s} = \mathfrak{o} \qquad (\nu = 1, 2, 3).$$

On the other hand, (27) is equivalent to

(32) $(r_1 - r_3) \times (r_2 - r_3) \neq o$

and (31), (32), 1 Pr 29 imply

(33) $s = o.$

Now (33), (30) imply

(34) $m = o$

and (2), (33), (34), Sgn 2 imply (29).

Sch 5. The condition (27) implies that (26) are not colinear, i.e. that there exists no line $l \in L_S$ with

(35) $r_\nu \; \& \; l \qquad (\nu = 1, 2, 3).$

In such a manner, $\overrightarrow{s}$ is cetainly the zero-arrow if there exist three non-colinear poles (26) with (28). The inverse statement (i.e: that if (29), then there exist (26) and (27), (28)) is trivial in the light of Pr 7.

Sch 6. Pr 13 admits the following inversion.

Pr 14. (1), (26), (27),

(36) $n \in V_S,$

(37) $\mathrm{mom}_{r_\nu} \overrightarrow{s} = n \qquad (\nu = 1, 2, 3)$

imply

(38) $n = o$

and (29).

Dm. (2), (37), Sgn 5 imply

(39) $r_\nu \times s = m - n \qquad (\nu = 1, 2, 3),$

whence (31). Since (27) is equivalent to (32), the relations (31), (32), 1 Pr 29 imply again (33). Now (33), (39) imply

(40) $m = n.$

On the orher hand, (2), (33), Sgn 1 imply (34), and (34), (40) imply (38).

Sch 7. Pr 13 and Pr 14 imply that the zero-arrow is the only arrow, the moments of which with respect to three non-colinear poles are invariable with respect to the latter.

The following two propositions give an idea of the distribution of the moments of an arrow in space.

Pr 15. (1), (2), (18),

(41) $s \times (r_1 - r_2) = o$

imply

(42) $\mathrm{mom}_{r_1} \overrightarrow{s} = \mathrm{mom}_{r_2} \overrightarrow{s}.$

Dm. (41) implies

(43) $$m + s \times r_1 = m + s \times r_2,$$

whence (42) (Sgn 5).

Pr 16. (1), (2), (18),

(44) $$s \times (r_1 - r_2) \neq \mathbf{o}$$

imply

(45) $$\mathrm{mom}_{r_1} \overrightarrow{s} \neq \mathrm{mom}_{r_2} \overrightarrow{s}.$$

Dm. (44) implies

(46) $$m + s \times r_1 \neq m + s \times r_2,$$

whence (45) (Sgn 6).

Sch 8. Let (25) hold good. Then (41) implies that l is coherent to dir $\overrightarrow{s}$ (Sgn 4, [3] 1 Sgn 6), i.e. l is parallel or coincides with dir $\overrightarrow{s}$, while (44) implies that l is non-coherent to dir $\overrightarrow{s}$ (Sgn 4, [3] 1 Sgn 7). Now Pr 15 and Pr 16 display that a necessary and sufficient condition for the equality of the moments of a non-zero arrow with respect to two different poles is the coincidence or the parallelism of dir $\overrightarrow{s}$ with the line incident with these poles.

Pr 17. (1), (2) imply $(-s, -m) \in W_S$.

Dm. Sgn 1.

Sgn 6. $-\overrightarrow{s}$ sgn: $(-s, -m)$ if (1), (2).

Df 9. $-\overrightarrow{s}$ is called the *opposite arrow* of $\overrightarrow{s}$.

Pr 18. (1) implies $-(-\overrightarrow{s}) = \overrightarrow{s}$.

Dm. Sgn 6.

Pr 19. $-\overrightarrow{o} = \overrightarrow{o}$.

Dm. Sgn 6, Sgn 2.

Pr 20. (1) implies (3) iff $-\overrightarrow{s} \neq \overrightarrow{o}$.

Dm. Pr 18, Pr 19.

Pr 21. (1) implies (3) iff $-\overrightarrow{s} \in \Lambda_S$.

Dm. Pr 20, Pr 2.

Pr 22. (1) implies (29) iff

(47) $$-\overrightarrow{s} = \overrightarrow{s}.$$

Dm. Pr 19 and: (2), Sgn 6 imply that (47) is equivalent to

(48) $$-s = s, \qquad -m = m.$$

Now (48) imply (33), (34), whence (29) (Sgn 2).

Pr 23. (1), (2),

(49) $$\lambda \in S$$

imply $(\lambda s, \overline{\lambda} m) \in W_S$.

Dm. Sgn 1.

Sgn 7. $\lambda \overrightarrow{s}$ sgn: $(\lambda s, \overline{\lambda} m)$ if (1), (2), (49).

Df 10. $\lambda \overrightarrow{s}$ is called the ***product*** of λ and $\overrightarrow{s}$.

Pr 24. (1) implies $0 \overrightarrow{s} = \overrightarrow{o}$.

Dm. Sgn 7, Sgn 2.

Pr 25. (49) implies $\lambda \overrightarrow{o} = \overrightarrow{o}$.

Dm. Sgn 7, Sgn 2.

Pr 26. (1) implies $1 \overrightarrow{s} = \overrightarrow{s}$.

Dm. Sgn 7.

Pr 27. (1) implies $(-1) \overrightarrow{s} = -\overrightarrow{s}$.

Dm. Sgn 7, Sgn 6.

Pr 28. (1), (49) imply $(-\lambda) \overrightarrow{s} = -(\lambda \overrightarrow{s})$.

Dm. Sgn 7, Sgn 6.

Pr 29. (1), (49) imply $\lambda(-\overrightarrow{s}) = -(\lambda \overrightarrow{s})$.

Dm. Sgn 6, Sgn 7.

Pr 30. (1), (49) imply $(-\lambda) \overrightarrow{s} = \lambda(-\overrightarrow{s})$.

Dm. Pr 28, Pr 29.

Pr 31. (1), (49) imply $(-\lambda)(-\overrightarrow{s}) = \lambda \overrightarrow{s}$.

Dm. Sgn 6, Sgn 7.

Pr 32. (1), (49),

(50) $\mu \in S$

imply $(\lambda\mu) \overrightarrow{s} = \lambda(\mu \overrightarrow{s})$.

Dm. Sgn 7.

Pr 33. (1),

(51) $0 \neq \lambda \in S$,

(52) $\lambda \overrightarrow{s} = \overrightarrow{o}$

imply (29).

Dm. Pr 32, Pr 26, Pr 25.

Pr 34. (5), (49), (52) imply $\lambda = 0$.

Dm. Pr 33.

Pr 35. (5), (51) imply $\lambda \overrightarrow{s} \neq \overrightarrow{o}$.

Dm. Pr 33, Pr 34.

Pr 36. (5), (51) imply $\lambda \overrightarrow{s} \in \Lambda_S$.

Dm. Pr 35, Pr 2.

Pr 37. (51),

(53) $\overrightarrow{s}_\nu \in W_S$ $(\nu = 1, 2)$,

(54) $\overrightarrow{s}_1 = \lambda \overrightarrow{s}_2$

imply $\vec{s}_2 = \frac{1}{\lambda}\vec{s}_1$.

Dm. Pr 32, Pr 26.

Pr 38. (5), (51) imply $\vec{s} \sim \lambda\vec{s}$.

Dm. Pr 2, Pr 36, Sgn 7, [3] 1 Sgn 2.

Pr 39. (5), (51) imply: dir $\lambda\vec{s}$ exists.

Dm. Pr 35, Sgn 4.

Pr 40. (5), (51) imply dir $\lambda\vec{s}$ = dir $\vec{s}$.

Dm. Sgn 4, Pr 39, Pr 38, [3] 1 Ax 3.

Pr 41. (5) implies dir $(-\vec{s})$ = dir $\vec{s}$.

Dm. Pr 27, Pr 40.

Pr 42. If

$$\vec{0} \neq \vec{s}_\nu \in W_S \qquad (\nu = 1, 2), \tag{55}$$

$$\text{dir } \vec{s}_1 = \text{dir } \vec{s}_2, \tag{56}$$

then there exists (51) with (54).

Dm. (56), Sgn 4, Pr 2, [3] 1 Ax 5 imply

$$\vec{s}_1 \sim \vec{s}_2. \tag{57}$$

Now (57), [3] 1 Sgn 2 imply that there exists (51) with (54).

Pr 43. (55) imply (56) iff (54).

Dm. Pr 40, Pr 42.

Pr 44. (49), (53), (54), (9) imply

$$\text{mom}_r\, \vec{s}_1 = \bar{\lambda}\text{mom}_r\, \vec{s}_2 \tag{58}$$

Dm. If

$$\vec{s}_\nu = (s_\nu, m_\nu) \qquad (\nu = 1, 2), \tag{59}$$

then

$$s_1 = \lambda s_2, \qquad m_1 = \bar{\lambda} m_2 \tag{60}$$

(Sgn 7). Now (60), Sgn 5, 1 Pr 14 imply

$$\begin{aligned} \text{mom}_r\, \vec{s}_1 &= m_1 + s_1 \times r = \bar{\lambda} m_2 + (\lambda s_2) \times r \\ &= \bar{\lambda} m_2 + \bar{\lambda}(s_2 \times r) = \bar{\lambda}\ \text{mom}_r\, \vec{s}_2. \end{aligned} \tag{61}$$

Sgn 8. mom $(\vec{s}_1, \vec{s}_2)$ sgn: $s_1 m_2 + s_2 m_1$ if (53), (59).

Df 11. mom $(\vec{s}_1, \vec{s}_2)$ is called the *mutual moment* of $\vec{s}_\nu$ $(\nu = 1, 2)$.

Pr 45. (53) imply mom $(\vec{s}_1, \vec{s}_2)$ = mom $(\vec{s}_2, \vec{s}_1)$.

Dm. Sgn 8.

Pr 46. (1) implies mom $(\vec{s}, \vec{o}) = 0$.

Dm. Sgn 2, Sgn 8.

Pr 47. (1) implies mom $(\overrightarrow{s}, \overrightarrow{s}) = 0$.
Dm. Sgn 8, Sgn 1.
Pr 48. (1), (49) imply mom $(\overrightarrow{s}, \lambda\overrightarrow{s}) = 0$.
Dm. Sgn 8, Sgn 7, Ax 9S, 1 Pr 7, Sgn 1.
Pr 49. (1) implies mom $(\overrightarrow{s}, -\overrightarrow{s}) = 0$.
Dm. Pr 48, Pr 27.
Pr 50. (55), (59), (18),

(62) $\qquad r_\nu \; Z \;\text{ dir }\; \overrightarrow{s}_\nu \qquad (\nu = 1, 2)$

imply

(63) $\qquad s_1 \cdot \mathrm{mom}_{r_1} \overrightarrow{s}_2 = s_2 \cdot \mathrm{mom}_{r_2} \overrightarrow{s}_1.$

Dm. Pr 9, 1 Pr 9, 1 Ax 8S, 1 Pr 13 imply

(64) $\qquad s_1 \cdot \mathrm{mom}_{r_1} \overrightarrow{s}_2 = s_1 \cdot (r_2 - r_1) \times s_2 = \overline{s_1 \times (r_2 - r_1) \cdot s_2}$

$$= s_2 \cdot s_1 \times (r_2 - r_1) = s_2 \cdot (r_1 - r_2) \times s_1 = s_2 \cdot \mathrm{mom}_{r_2} \overrightarrow{s}_1.$$

Pr 51. (53), (59), (9) imply

(65) $\qquad \mathrm{mom}(\overrightarrow{s}_1, \overrightarrow{s}_2) = s_1 \cdot \mathrm{mom}_r \overrightarrow{s}_2 + s_2 \cdot \mathrm{mom}_r \overrightarrow{s}_1.$

Dm. Sgn 5 implies

(66) $\qquad s_1 \cdot \mathrm{mom}_r \overrightarrow{s}_2 + s_2 \cdot \mathrm{mom}_r \overrightarrow{s}_1 = s_1(m_2 + s_2 \times r) + s_2(m_1 + s_1 \times r).$

On the other hand, 1Pr 8, 1 Ax 8S, 1 Pr 13 imply

(67) $\qquad s_1 \cdot s_2 \times r + s_2 \cdot s_1 \times r = \overline{s_1 \times s_2 \cdot r} + \overline{s_2 \times s_1 \cdot r}.$

$$= r \cdot s_1 \times s_2 + r \cdot s_2 \times s_1 = r(s_1 \times s_2 + s_2 \times s_1) = r0 = 0.$$

Now (66), (67), Sgn 8 imply (65).
Pr 52. (18), (53), (59), (62) imply

(68) $\qquad \mathrm{mom}(\overrightarrow{s}_1, \overrightarrow{s}_2) = (r_1 - r_2) \cdot s_1 \times s_2.$

Dm. (62), Sgn 4, [3] 4 Sgn 1 imply

(69) $\qquad r_\nu \times s_\nu = m_\nu \qquad (\nu = 1, 2).$

Now Sgn 8, (69), 1 Ax 8S, 1Pr 8, 1 Pr 13 imply (68).

Sgn 1. $\overrightarrow{s}_1 \mid \overrightarrow{s}_2$ sgn: $s_1 \times s_2 = \mathfrak{o}$ if 2(53), 2(59).
Df 1. $\overrightarrow{s}_1$ is called *adherent* to $\overrightarrow{s}_2$ if $\overrightarrow{s}_1 \mid \overrightarrow{s}_2$.
Sgn 2. $\overrightarrow{s}_1 \overline{\mid} \overrightarrow{s}_2$ sgn: $s_1 \times s_2 \neq \mathfrak{o}$ if 2(53), 2(59).
Df 2. $\overrightarrow{s}_1$ is called *non-adherent* to $\overrightarrow{s}_2$ if $\overrightarrow{s}_1 \overline{\mid} \overrightarrow{s}_2$.
Pr 1. 2(53) imply: exactly one of the relations

(1) $\overrightarrow{s}_1 \mid \overrightarrow{s}_2$

or

(2) $\overrightarrow{s}_1 \overline{\mid} \overrightarrow{s}_2$

holds.
Dm. Sgn 1, Sgn 2.
Pr 2. 2(53), (1) imply $\overrightarrow{s}_2 \mid \overrightarrow{s}_1$.
Dm. Sgn 1, 1 Pr 13.
Pr 3. 2(53), (2) imply $\overrightarrow{s}_2 \overline{\mid} \overrightarrow{s}_1$.
Dm. Pr 1, Pr 2.
Pr 4. 2(1) implies $\overrightarrow{s} \mid \overrightarrow{s}$.
Dm. Sgn 1.
Pr 5. 2(1) implies $\overrightarrow{s} \mid \overrightarrow{o}$.
Dm. Sgn 1, 2 Sgn 2.
Pr 6. 2(1) implies $\overrightarrow{s} \mid -\overrightarrow{s}$.
Dm. Sgn 1, 2 Sgn 6.
Pr 7. 2(1), 2(49) imply $\overrightarrow{s} \mid \lambda \overrightarrow{s}$.
Dm. Sgn 1, 2 Sgn 7.
Sgn 3. $\overrightarrow{s_1} \mid\mid\mid \overrightarrow{s}_2$ sgn: $\overrightarrow{s_1} \mid \overrightarrow{s}_2$ iff 2(55).
Df 3. $\overrightarrow{s}_1$ is called *coherent* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \mid\mid\mid \overrightarrow{s}_2$.
Sgn 4. $\overrightarrow{s_1} \overline{\mid\mid\mid} \overrightarrow{s}_2$ sgn: $\overrightarrow{s_1} \overline{\mid} \overrightarrow{s}_2$ or $\overrightarrow{s}_\nu = \overrightarrow{o}$ $(1 \leqq \nu \leqq 2)$ iff 2(53).
Df 4. $\overrightarrow{s}_1$ is called *non-coherent* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\mid\mid\mid} \overrightarrow{s}_2$.
Pr 8. 2(53) imply: exactly one of the relations

(3) $\overrightarrow{s}_1 \mid\mid\mid \overrightarrow{s}_2$

or

(4) $\overrightarrow{s}_1 \overline{\mid\mid\mid} \overrightarrow{s}_2$

holds.
Dm. Sgn 3, Sgn 4.
Pr 9. 2(55), (3) imply $\overrightarrow{s}_2 \mid\mid\mid \overrightarrow{s}_1$.
Dm. Sgn 3, Pr 2.
Pr 10. 2(53), (4) imply $\overrightarrow{s}_2 \overline{\mid\mid\mid} \overrightarrow{s}_1$.
Dm. Pr 8, Pr 9.

Pr 11. 2(5) implies $\overrightarrow{s} \mid\mid\mid \overrightarrow{s}$.
Dm. Sgn 3, Pr 4.
Pr 12. 2(5) implies $\overrightarrow{s} \mid\mid\mid -\overrightarrow{s}$.
Dm. Sgn 3, 2 Pr 21, Pr 4.
Pr 13. 2(5), 2(51) imply $\overrightarrow{s} \mid\mid\mid \lambda\overrightarrow{s}$.
Dm. Sgn 3, 2 Pr 35, Pr 7.
Pr 14. 2(1) implies $\overrightarrow{s} \overline{\mid\mid\mid} \overrightarrow{o}$.
Dm. Sgn 4.
Sgn 5. $\overrightarrow{s_1} \mid\mid \overrightarrow{s}_2$ sgn: $\overrightarrow{s_1} \mid\mid\mid \overrightarrow{s}_2$, $\overrightarrow{s}_1 \neq \lambda\overrightarrow{s}_2$ iff 2(53), 2(51).
Df 5. $\overrightarrow{s}_1$ is called *parallel* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \mid\mid \overrightarrow{s}_2$.
Sgn 6. $\overrightarrow{s_1} \overline{\mid\mid} \overrightarrow{s}_2$ sgn: $\overrightarrow{s_1} \overline{\mid\mid\mid} \overrightarrow{s}_2$ or $\overrightarrow{s}_1 = \lambda\overrightarrow{s}_2$ iff 2(53), 2(51).
Df 6. $\overrightarrow{s}_1$ is called *non-parallel* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\mid\mid} \overrightarrow{s}_2$.
Pr 15. 2(53) imply: exactly one of the relations

(5) $$\overrightarrow{s}_1 \mid\mid \overrightarrow{s}_2$$

or

(6) $$\overrightarrow{s}_1 \overline{\mid\mid} \overrightarrow{s}_2$$

holds.
Dm. Sgn 5, Sgn 6.
Pr 16. 2(53), (5) imply $\overrightarrow{s}_2 \mid\mid \overrightarrow{s}_1$.
Dm. Sgn 5, Pr 9, 2 Pr 37.
Pr 17. 2(53), (6) imply $\overrightarrow{s}_2 \overline{\mid\mid} \overrightarrow{s}_1$.
Dm. Pr 15, Pr 16.
Pr 18. 2(1) implies $\overrightarrow{s} \overline{\mid\mid} \overrightarrow{s}$.
Dm. Sgn 6, 2 Pr 26.
Pr 19. 2(1) implies $\overrightarrow{s} \overline{\mid\mid} -\overrightarrow{s}$.
Dm. Sgn 6, 2 Pr 27.
Pr 20. 2(1) implies $\overrightarrow{s} \overline{\mid\mid} \overrightarrow{o}$.
Dm. Sgn 6, Pr 14.
Sgn 7. $\overrightarrow{s_1} \uparrow\downarrow \overrightarrow{s}_2$ sgn: $s_1 + s_2 = \mathfrak{o}$, $m_1 + m_2 \neq \mathfrak{o}$ iff 2(53), 2(59).
Df 7. $\overrightarrow{s}_1$ is called *dipolar* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \uparrow\downarrow \overrightarrow{s}_2$.
Sgn 8. $\overrightarrow{s_1} \overline{\uparrow\downarrow} \overrightarrow{s}_2$ sgn: $s_1 + s_2 \neq \mathfrak{o}$ or $m_1 + m_2 = \mathfrak{o}$ iff 2(53), 2(59).
Df 8. $\overrightarrow{s}_1$ is called *non-dipolar* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\uparrow\downarrow} \overrightarrow{s}_2$.
Pr 21. 2(53) imply: exactly one of the relations

(7) $$\overrightarrow{s}_1 \uparrow\downarrow \overrightarrow{s}_2$$

or

(8) $$\overrightarrow{s}_1 \overline{\uparrow\downarrow} \overrightarrow{s}_2$$

holds.

Dm. Sgn 7, Sgn 8.

Pr 22. 2(53), (7) imply $\overrightarrow{s}_2 \uparrow\downarrow \overrightarrow{s}_1$.

Dm. Sgn 7.

Pr 23. 2(53), (8) imply $\overrightarrow{s}_2 \overline{\uparrow\downarrow} \overrightarrow{s}_1$.

Dm. Pr 21, Pr 22.

Pr 24. 2(53), (7) imply (1).

Dm. Sgn 7, Sgn 1.

Pr 25. 2(53), (7) imply 2(55).

Dm. Let 2(59) hold and let, for instance, $\overrightarrow{s}_1 = \overrightarrow{o}$. Then 2 Sgn 2 implies $s_1 = m_1 = \mathfrak{o}$. Now $s_1 = \mathfrak{o}$ and Sgn 7 imply $s_2 = \mathfrak{o}$, whence $m_2 = \mathfrak{o}$ according to 2 Sgn 1. Then $m_1 + m_2 = \mathfrak{o}$ contrary to Sgn 7.

Pr 26. 2(1) implies $\overrightarrow{s} \overline{\uparrow\downarrow} \overrightarrow{o}$.

Dm. Pr 25, Pr 21.

Pr 27. 2(1) implies $\overrightarrow{s} \overline{\uparrow\downarrow} \overrightarrow{s}$.

Dm. Pr 26, 2 Pr 1, Sgn 8.

Pr 28. 2(1) implies $\overrightarrow{s} \overline{\uparrow\downarrow} -\overrightarrow{s}$.

Dm. Sgn 6, Sgn 8.

Pr 29. 2(1), $-1 \neq \lambda \in S$ imply $\overrightarrow{s} \overline{\uparrow\downarrow} \lambda\overrightarrow{s}$.

Dm. If $\overrightarrow{s} = \overrightarrow{o}$, then 2 Pr 25, Pr 26. If $\overrightarrow{s} \neq \overrightarrow{o}$, then 2(2) implies $s \neq \mathfrak{o}$ (2 Pr 1) whence $s + \lambda s \neq \mathfrak{o}$ by virtue of $\lambda \neq -1$. Now Sgn 8.

Pr 30. 2(53), (7) imply (3).

Dm. Sgn 3, Pr 24, Pr 25.

Pr 31. 2(53), (7) imply (5).

Dm. Sgn 5, Pr 28–Pr 30, 2 Pr 24.

Pr 32. 2(18), 2(53), (7), 2(62) imply

$$\text{mom}_{\mathfrak{r}_1} \overrightarrow{s}_2 = \text{mom}_{\mathfrak{r}_2} \overrightarrow{s}_1. \tag{9}$$

Dm. 2(62), 2 Sgn 4, [3] 4 Sgn 1 imply 2(69). 2 Sgn 5 implies

$$\text{mom}_{\mathfrak{r}_1} \overrightarrow{s}_2 = m_2 + s_2 \times r_1, \tag{10}$$

$$\text{mom}_{\mathfrak{r}_2} \overrightarrow{s}_1 = m_1 + s_1 \times r_2. \tag{11}$$

Sgn 7, (7) imply

$$s_1 + s_2 = \mathfrak{o}. \tag{12}$$

Now 2(69) and (10)–(12) imply (9).

Pr 33. 2(53), (1) imply

$$\text{mom}(\overrightarrow{s}_1, \overrightarrow{s}_2) = 0. \tag{13}$$

Dm. 2 Pr 46 or 2 Pr 52, Sgn 1.

Pr 34. 2(53), (3) imply (13).

Dm. Sgn 3, Pr 33.

Pr 35. 2(53), (5) imply (13).
Dm. Sgn 5, Pr 33.
Pr 36. 2(53), (7) imply (13).
Dm. Sgn 7, Pr 33.

§ 4. PERPENDICULARITY

Sgn 1. $\vec{s_1} \top \vec{s_2}$ sgn: $s_1 s_2 = 0$ iff 2(53), 2(59).
Df 1. $\vec{s}_1$ is called *normal* to $\vec{s}_2$ if $\vec{s_1} \top \vec{s}_2$.
Sgn 2. $\vec{s_1} \overline{\top} \vec{s}_2$ sgn: $s_1 s_2 \neq 0$ iff 2(53), 2(59).
Df 2. $\vec{s}_1$ is called *non-normal* to $\vec{s}_2$ if $\vec{s_1} \overline{\top} \vec{s}_2$.
Pr 1. 2(53) imply: exactly one of the relations

(1) $$\vec{s}_1 \top \vec{s}_2$$

or

(2) $$\vec{s}_1 \overline{\top} \vec{s}_2$$

holds.
Dm. Sgn 1, Sgn 2.
Pr 2. 2(53), (1) imply $\vec{s}_2 \top \vec{s}_1$.
Dm. Sgn 1.
Pr 3. 2(53), (2) imply $\vec{s}_2 \overline{\top} \vec{s}_1$.
Dm. Pr 1, Pr 2.
Pr 4. 2(53), (1) imply 3(4).
Dm. If $\vec{s}_\nu = \vec{o}$ $(1 \leqq \nu \leqq 2)$, then 3 Sgn 4. If $\vec{s}_\nu \neq \vec{o}$ $(\nu = 1, 2)$, then $s_\nu \neq o$ $(\nu = 1, 2)$ (2 Pr 1). Now $s_1 s_2 = 0$ implies $(s_1 \times s_2)^2 = s_1^2 s_2^2$ (1 Pr 9) whence $s_1 \times s_2 \neq o$ (1 Ax 12S). Then 3 Sgn 2, 3 Sgn 4.
Pr 5. 2(53), (1) imply 3(6).
Dm. Pr 4, Sgn 6.
Pr 6. 2(53), (1) imply 3(8).
Dm. 3 Pr 21, 3 Pr 31, Pr 5, 3 Pr 15.
Pr 7. 2(1) implies $\vec{s} \top \vec{o}$.
Dm. 2 Sgn 2, Sgn 1.
Pr 8. 2(5) implies $\vec{s} \overline{\top} \vec{s}$.
Dm. 2 Pr 1, 1 Ax 12S, Sgn 2.
Pr 9. 2(5) implies $\vec{s} \overline{\top} -\vec{s}$.
Dm. 2 Pr 1, 2 Sgn 6, 1 Ax 12S, Sgn 2.
Pr 10. 2(5), 2(51) imply $\vec{s} \overline{\top} \lambda \vec{s}$.
Dm. 2 Pr 1, 2 Sgn 7, 1 Ax 12S, Sgn 2.
Pr 11. 2(5), 2(53), $\vec{s} \mid \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).
Dm. 2(2), 2(59) imply $s \neq o$ (2 Pr 1), $s \times s_1 = o$ (2 Sgn 1), $s s_2 = 0$ (Sgn 1). Then there exists 1(55) with $s_1 = \alpha s$ (1 Pr 13, 1 Pr 32), hence $s_1 s_2 = 0$. Now Sgn 1.
Pr 12. 2(1), 2(53), $\vec{s} \mid\mid\mid \vec{s}_1$, $\vec{s} \top \vec{s}_2$ imply (1).

Dm. 2 Sgn 3, Pr 11.

Pr 13. 2(1), 2(53), $\overrightarrow{s} \parallel \overrightarrow{s}_1$, $\overrightarrow{s} \top \overrightarrow{s}_2$ imply (1).

Dm. 2 Sgn 5, Pr 12.

Pr 14. 2(1), 2(53), $\overrightarrow{s} \uparrow\downarrow \overrightarrow{s}_1$, $\overrightarrow{s} \top \overrightarrow{s}_2$ imply (1).

Dm. 2 Pr 31, Pr 13.

Sgn 3. $\overrightarrow{s_1} \perp \overrightarrow{s}_2$ sgn: $s_1 s_2 = 0$ iff 2(55), 2(59).

Df 3. $\overrightarrow{s}_1$ is called *perpendicular* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \perp \overrightarrow{s}_2$.

Sgn 4. $\overrightarrow{s_1} \overline{\perp} \overrightarrow{s}_2$ sgn: $s_1 s_2 \neq 0$ or $s_\nu = \mathfrak{o}$ $(1 \leqq \nu \leqq 2)$ iff 2(53), 2(59).

Df 4. $\overrightarrow{s}_1$ is called *non-perpendicular* to $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\perp} \overrightarrow{s}_2$.

Pr 15. 2(53) imply: exactly one of the relations

(3) $$\overrightarrow{s}_1 \perp \overrightarrow{s}_2$$

or

(4) $$\overrightarrow{s}_1 \overline{\perp} \overrightarrow{s}_2$$

holds.

Dm. Sgn 3, Sgn 4.

Pr 16. 2(53), (3) imply $\overrightarrow{s}_2 \perp \overrightarrow{s}_1$.

Dm. Sgn 3.

Pr 17. 2(53), (4) imply $\overrightarrow{s}_2 \overline{\perp} \overrightarrow{s}_1$.

Dm. Pr 15, Pr 16.

Pr 18. 2(53), (3) imply (1).

Dm. Sgn 3, Sgn 1.

Pr 19. 2(53), (3) imply 3(2).

Dm. 2(59) imply $s_\nu \neq \mathfrak{o}$ $(\nu = 1, 2)$ (Sgn 3, 2 Pr 1). Now $s_1 s_2 = 0$ implies $(s_1 \times s_2)^2 = s_1^2 s_2^2$ (1 Pr 9) whence $s_1 \times s_2 \neq \mathfrak{o}$ (1 Ax 12S). Then 3 Sgn 2.

Pr 20. 2(53), (3) imply 3(4).

Dm. Pr 18, Pr 4.

Pr 21. 2(53), (3) imply 3(6).

Dm. Pr 18, Pr 5.

Pr 22. 2(53), (3) imply 3(8).

Dm. Pr 18, Pr 6.

Pr 23. 2(1) implies $\overrightarrow{s} \overline{\perp} \overrightarrow{o}$.

Dm. Sgn 4, 2 Sgn 2.

Pr 24. 2(1) implies $\overrightarrow{s} \overline{\perp} \overrightarrow{s}$.

Dm. Sgn 4, 2 Sgn 2, 2 Pr 1, 1 Ax 12S.

Pr 25. 2(1) implies $\overrightarrow{s} \overline{\perp} -\overrightarrow{s}$.

Dm. Sgn 4, 2 Sgn 6, 2 Sgn 2, 2 Pr 1, 1 Ax 12S.

Pr 26. 2(1), 2(49) imply $\overrightarrow{s} \overline{\perp} \lambda \overrightarrow{s}$.

Dm. Sgn 4, 2 Pr 24, 2 Pr 25, 2 Pr 1, 1 Ax 12S.

Pr 27. 2(1), 2(53), $\overrightarrow{s} \mid \overrightarrow{s}_1$, $\overrightarrow{s} \perp \overrightarrow{s}_2$ imply (1).

Dm. Pr 18, Pr 11.

Pr 28. 2(1), 2(53), $\overrightarrow{s} \parallel\!\mid \overrightarrow{s}_1$, $\overrightarrow{s} \perp \overrightarrow{s}_2$ imply (3).

Dm. 3 Sgn 3, Pr 12, Sgn 1, Sgn 3.

Pr 29. 2(1), 2(53), $\overrightarrow{s} \parallel \overrightarrow{s}_1$, $\overrightarrow{s} \perp \overrightarrow{s}_2$ imply (3).
Dm. 2 Sgn 5, Pr 28.
Pr 30. 2(1), 2(53), $\overrightarrow{s} \uparrow\downarrow \overrightarrow{s}_1$, $\overrightarrow{s} \perp \overrightarrow{s}_2$ imply (3).
Dm. 2 Pr 31, Pr 29.

§ 5. OTHER RELATIONS

Sgn 1. $\overrightarrow{s_1} \wedge \overrightarrow{s}_2$ sgn: $s_1 \times s_2 \neq o$, $s_1 m_2 + s_2 m_1 = 0$ iff 2(53), 2(59).
Df 1. $\overrightarrow{s}_1$ is called *intersecting* $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \wedge \overrightarrow{s}_2$.
Sgn 2. $\overrightarrow{s_1} \overline{\wedge} \overrightarrow{s}_2$ sgn: $s_1 \times s_2 = 0$, $s_1 m_2 + s_2 m_1 \neq 0$ iff 2(53), 2(59).
Df 2. $\overrightarrow{s}_1$ is called *non-intersecting* $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\wedge} \overrightarrow{s}_2$.
Pr 1. 2(53) imply: exactly one of the relations

(1) $$\overrightarrow{s}_1 \wedge \overrightarrow{s}_2$$

or

(2) $$\overrightarrow{s}_1 \overline{\wedge} \overrightarrow{s}_2$$

holds.
Dm. Sgn 1, Sgn 2.
Pr 2. 2(53), (1) imply $\overrightarrow{s}_2 \wedge \overrightarrow{s}_1$.
Dm. Sgn 1.
Pr 3. 2(53), (2) imply $\overrightarrow{s}_2 \overline{\wedge} \overrightarrow{s}_1$.
Dm. Pr 1, Pr 2.
Pr 4. 2(53), (1) imply

(3) $$\overrightarrow{s}_\nu \neq \overrightarrow{o} \qquad (\nu = 1, 2).$$

Dm. Sgn 1, 2 Pr 1.
Pr 5. 2(53), (1) imply 3(2).
Dm. Sgn 1, 3 Sgn 2.
Pr 6. 2(53), (1) imply 3(4).
Dm. 3 Sgn 4, Pr 5.
Pr 7. 2(53), (1) imply 3(6).
Dm. 3 Sgn 6, Pr 6.
Pr 8. 2(53), (1) imply 3(8).
Dm. Sgn 1, 3 Sgn 8.
Pr 9. 2(1) implies $\overrightarrow{s} \overline{\wedge} \overrightarrow{s}$.
Dm. Sgn 2.
Pr 10. 2(1) implies $\overrightarrow{s} \wedge \overrightarrow{o}$.
Dm. Pr 4.
Pr 11. 2(1) implies $\overrightarrow{s} \overline{\wedge} -\overrightarrow{s}$.
Dm. 2 Sgn 6, Sgn 2.
Pr 12. 2(1), 2(49) imply $\overrightarrow{s} \overline{\wedge} \lambda \overrightarrow{s}$.
Dm. 2 Sgn 7, Sgn 2.
Pr 13. 2(53), (1) imply $\overrightarrow{s}_1 \wedge -\overrightarrow{s}_2$.

Dm. 2 Sgn 6, Sgn 1.
Pr 14. 2(53), (1), 2(51) imply $\overrightarrow{s}_1 \wedge \lambda\overrightarrow{s}_2$.
Dm. 2 Sgn 7, Sgn 1.
Sgn 3. $\overrightarrow{s_1} \otimes \overrightarrow{s}_2$ sgn: $s_1 \times s_2 \neq \mathfrak{o}, s_1 m_2 + s_2 m_1 \neq 0$ iff 2(53), 2(59).
Df 3. $\overrightarrow{s}_1$ is called *crossed* with $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \otimes \overrightarrow{s}_2$.
Sgn 4. $\overrightarrow{s_1} \overline{\otimes} \overrightarrow{s}_2$ sgn: $s_1 \times s_2 = \mathfrak{o}$ or $s_1 m_2 + s_2 m_1 = 0$ iff 2(53), 2(59).
Df 4. $\overrightarrow{s}_1$ is called *non-crossed* with $\overrightarrow{s}_2$ if $\overrightarrow{s_1} \overline{\otimes} \overrightarrow{s}_2$.
Pr 15. 2(53) imply: exactly one of the relations

(4) $$\overrightarrow{s}_1 \otimes \overrightarrow{s}_2$$

or

(5) $$\overrightarrow{s}_1 \overline{\otimes} \overrightarrow{s}_2$$

holds.
Dm. Sgn 3, Sgn 4.
Pr 16. 2(53), (4) imply $\overrightarrow{s}_2 \otimes \overrightarrow{s}_1$.
Dm. Sgn 3.
Pr 17. 2(53), (5) imply $\overrightarrow{s}_2 \overline{\otimes} \overrightarrow{s}_1$.
Dm. Pr 15, Pr 16.
Pr 18. 2(53), (4) imply (3).
Dm. Sgn 3, 2 Pr 1.
Pr 19. 2(53), (4) imply 3(2).
Dm. Sgn 3, 3 Sgn 2.
Pr 20. 2(53), (4) imply 3(4).
Dm. 3 Sgn 4, Pr 19.
Pr 21. 2(53), (4) imply 3(6).
Dm. 3 Sgn 6, Pr 20.
Pr 22. 2(53), (4) imply 3(8).
Dm. 3 Pr 21, 3 Sgn 7, Sgn 3.
Pr 23. 2(53), (4) imply (2).
Dm. Sgn 1, Sgn 4.
Pr 24. 2(53), (1) imply (5).
Dm. Sgn 1, Sgn 4.
Pr 25. 2(1) implies $\overrightarrow{s} \overline{\otimes} \overrightarrow{s}$.
Dm. 2 Sgn 1, Sgn 4.
Pr 26. 2(1) implies $\overrightarrow{s} \overline{\otimes} \overrightarrow{o}$.
Dm. 2 Sgn 2, Sgn 4.
Pr 27. 2(1) implies $\overrightarrow{s} \overline{\otimes} -\overrightarrow{s}$.
Dm. 2 Sgn 6, Sgn 4.
Pr 28. 2(1), 2(49) imply $\overrightarrow{s} \overline{\otimes} \lambda\overrightarrow{s}$.
Dm. 2 Pr 24, 2 Pr 25, Pr 26, 2 Sgn 7, Sgn 4.
Pr 29. 2(53), (4) imply $\overrightarrow{s}_1 \otimes -\overrightarrow{s}_2$.
Dm. 2 Sgn 6, Sgn 3.
Pr 30. 2(53), (4), 2(51) imply $\overrightarrow{s}_1 \otimes \lambda\overrightarrow{s}_2$.
Dm. 2 Sgn 7, Sgn 3.
Pr 31. 2(53), (1) imply: there exists exactly one 2(9) with

(6) $\quad r \; Z \text{ dir } \overrightarrow{s}_\nu \qquad (\nu = 1, 2),$

namely

(7) $$r = \frac{1}{2} \sum_{\nu=1}^{3} s_\nu^{-1} \times m_\nu$$

provided 2(59),

(8) $\quad s_3 \text{ sgn} : s_1 \times s_2,$

(9) $$m_3 \text{ sgn} : (m_1 \cdot s_2 \times s_1) s_1^{-1} + (m_1 \cdot s_2 \times s_1) s_2^{-1}.$$

Dm. dir $\overrightarrow{s}_\nu$ $(\nu = 1, 2)$ exist by virtue of Pr 4, 2 Sgn 4. The relations (6) are equivalent to

(10) $\quad r \times s_\nu = m_\nu \qquad (\nu = 1, 2)$

respectively, provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). In view of Sgn 1, 1 Pr 31 the system (10) has exactly one solution 2(9), namely (7) provided (8), (9).

Pr 32. 2(53), (4) imply: there exists no 2(9) with (6).

Dm. dir $\overrightarrow{s}_\nu$ $(\nu = 1, 2)$ exist by virtue of Pr 18, 2 Sgn 4. The relations (6) are equivalent to (10) respectively provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). In view of Sgn 3, 1 Pr 28, there exists no 2(9) with (10), since the necessary for the consistency of the system of vector equations (10) condition $s_1 m_2 + s_2 m_1 = 0$ is violated.

§ 6. ARMS & FEET

Sch 1. Let $r \in V_S$ and $\overrightarrow{s} = (s, m) \in W_S$ be given, $\overrightarrow{s} \neq \overrightarrow{o}$. Let $\bar{\rho} \in V_S$ be wanted satisfying

(1) $\quad \bar{\rho} \; Z \text{ dir } \overrightarrow{s},$

(2) $\quad (\bar{\rho} - r)s = 0.$

The condition (1) is equivalent to

(3) $\quad \bar{\rho} \times s = m$

(2 Sgn 4, [3] 4 Sgn 1). In other words, $\bar{\rho}$ is sought as a solution of the system of vector equations (3) and

(4) $\quad \bar{\rho} s = rs$

provided $s \neq o$, $sm = 0$. According to 1 Pr 33, this system has exactly one solution

(5) $$\bar{\rho} = \frac{(rs)s + s \times m}{s^2}.$$

It is called the *foot* or r on $\overrightarrow{s}$ and is denoted by $\text{foot}_r \overrightarrow{s}$. On the other hand, (5) implies

$$\overline{\rho} - r = \frac{s \times (m + s \times r)}{s^2} \tag{6}$$

The right-hand side of (6) is called the *arm* of $\overrightarrow{s}$ with respect to r and is denoted by $\text{arm}_r \overrightarrow{s}$. These and other circumstances are formalized in the present paragraph.

Sgn 1. $\text{arm}_r \overrightarrow{s}$ sgn: $\dfrac{s \times (m + s \times r)}{s^2}$ iff 2(5), 2(2), 2(9).

Df 1. $\text{arm}_r \overrightarrow{s}$ is called the *arm* of $\overrightarrow{s}$ with respect to r (the *r-arm* of $\overrightarrow{s}$).

Pr 1. 2(5), 2(2), 2(9) imply

$$\text{arm}_r \overrightarrow{s} = \frac{s \times \text{mom}_r \overrightarrow{s}}{s^2}. \tag{7}$$

Dm. Sgn 1, 2 Sgn 5.

Pr 2. 2(5), 2(2), 2(9) imply $s \cdot \text{arm}_r \overrightarrow{s} = 0$.

Dm. Sgn 1.

Pr 3. 2(5), 2(2), 2(9) imply $\text{mom}_r \overrightarrow{s} \cdot \text{arm}_r \overrightarrow{s} = 0$.

Dm. Pr 1.

Pr 4. 2(5), 2(2), 2(9) imply $s \times \text{mom}_r \overrightarrow{s} = s^2 \text{arm}_r \overrightarrow{s}$.

Dm. Pr 1.

Pr 5. 2(5), 2(2), 2(9) imply $\text{mom}_r \overrightarrow{s} = \text{arm}_r \overrightarrow{s} \times s$.

Dm. Pr 1, Pr 5.

Pr 6. 2(5), 2(2), 2(9) imply

$$\text{arm}_r \overrightarrow{s} = \frac{s \times m + (rs)s}{s^2} - r. \tag{8}$$

Dm. Sgn 1, Pr 16.

Pr 7. 2(5), 2(9) imply $\text{arm}_r(-\overrightarrow{s}) = \text{arm}_r \overrightarrow{s}$.

Dm. 2 Sgn 6, Sgn 1.

Pr 8. 2(5), 2(9), 2(51) imply $\text{arm}_r(\lambda \overrightarrow{s}) = \text{arm}_r \overrightarrow{s}$.

Dm. 2 Sgn 7, Sgn 1.

Pr 9. 2(5), 2(2), 2(9) imply $\left(\text{mom}_r \overrightarrow{s}\right)^2 = s^2 \left(\text{arm}_r \overrightarrow{s}\right)^2$.

Dm. Pr 5, Pr 2, 1 Pr 9.

Pr 10. 2(5), 2(9) imply 2(10) iff

$$\text{arm}_r \overrightarrow{s} = \mathbf{o}. \tag{9}$$

Dm. Pr 9, 1 Ax 12S.

Pr 11. 2(5), 2(9) imply (1) iff (9).

Dm. Pr 10, 2 Pr 8.

Pr 12. 2(53), 3(5), 2(18), 2(62) imply

$$\text{arm}_{r_2} \overrightarrow{s}_1 + \text{arm}_{r_1} \overrightarrow{s}_2 = \mathbf{o}. \tag{10}$$

Dm. 3(5) imply 5(3) (3 Pr 20, 2 Pr 15), whence dir $\overrightarrow{s}_\nu$ and $\mathrm{arm}_{\mathfrak{r}} \overrightarrow{s}_\nu$ ($\nu = 1, 2$) exist for any 2(9) (2 Sgn 4, Sgn 1). The relations 2(62) are equivalent with 2(69) provided 2(59) (2 Sgn 4, [3] 4 Sgn 1). On the other hand, Sgn 1 implies

$$\mathrm{arm}_{\mathfrak{r}_2} \overrightarrow{s}_1 = \frac{s_1 \times (m_1 + s_1 \times r_2)}{s_1^2}, \tag{11}$$

$$\mathrm{arm}_{\mathfrak{r}_1} \overrightarrow{s}_2 = \frac{s_2 \times (m_2 + s_2 \times r_1)}{s_2^2} \tag{12}$$

and (11), (12), 2(69) imply

$$\mathrm{arm}_{\mathfrak{r}_2} \overrightarrow{s}_1 = \frac{s_1 \times ((r_1 - r_2) \times s_1)}{s_1^2}, \tag{13}$$

$$\mathrm{arm}_{\mathfrak{r}_1} \overrightarrow{s}_2 = \frac{s_2 \times ((r_2 - r_1) \times s_2)}{s_2^2} \tag{14}$$

respectively. At last, 3(5) and 2(59) imply

$$s_1 \times s_2 = \mathfrak{o} \tag{15}$$

(3 Sgn 5, 3 Sgn 3, 3 Sgn 1) and (15), 5(3), 2 Pr 1 imply that there exists (51) with

$$s_2 = \lambda s_1. \tag{16}$$

Now (16), (14), 1Ax 8S, 1 Pr 7, 1 Sgn 1, 1 Pr 14, 1 Pr 17 imply

$$\mathrm{arm}_{\mathfrak{r}_1} \overrightarrow{s}_2 = \frac{s_1 \times ((r_2 - r_1) \times s_1)}{s_1^2} \tag{17}$$

and (13), (17) imply (10).

Pr 13. 2(53), 3(7), 2(18), 2(62) imply (10).

Dm. 3 Pr 31, Pr 12.

Sgn 2. $\mathrm{foot}_{\mathfrak{r}} \overrightarrow{s}$ sgn: $\dfrac{s \times m + (rs)s}{s^2}$ iff 2(5), 2(2), 2(9).

Df 2. $\mathrm{foot}_{\mathfrak{r}} \overrightarrow{s}$ is called the *foot* of r on $\overrightarrow{s}$.

Pr 14. 2(5), 2(2), 2(9) imply $\mathrm{foot}_{\mathfrak{r}} \overrightarrow{s} = r + \mathrm{arm}_{\mathfrak{r}} \overrightarrow{s}$.

Dm. Sgn 2, Pr 6.

Pr 15. 2(5), 2(9) imply $\mathrm{foot}_{\mathfrak{r}}(-\overrightarrow{s}) = \mathrm{foot}_{\mathfrak{r}} \overrightarrow{s}$.

Dm. Pr 14, Pr 7.

Pr 16. 2(5), 2(9), 2(51) imply $\mathrm{foot}_{\mathfrak{r}}(\lambda \overrightarrow{s}) = \mathrm{foot}_{\mathfrak{r}} \overrightarrow{s}$.

Dm. Pr 14, Pr 8.

Pr 17. 2(5), 2(9),

$$r \; Z \; \mathrm{dir} \; \overrightarrow{s} \tag{18}$$

imply

(19) $\left(\mathrm{arm}_{\mathfrak{r}}\overrightarrow{s}, r \times \mathrm{arm}_{\mathfrak{r}}\overrightarrow{s}\right) \in \Lambda_S.$

Dm. 2(5) implies: dir $\overrightarrow{s}$ exists (2 Sgn 4). Now (18) is equivalent to

(20) $\mathrm{arm}_{\mathfrak{r}}\overrightarrow{s} \neq \mathfrak{o}$

(Pr 11) whence (19) (2 Sgn 3).

Pr 18. 2(5), 2(9), (18) imply: there exists exactly one $l \in L_S$ with

(21) $\left(\mathrm{arm}_{\mathfrak{r}}\overrightarrow{s}, r \times \mathrm{arm}_{\mathfrak{r}}\overrightarrow{s}\right)$ & l.

Dm. Pr 17, [3] 1 Pr 19.

Sgn 3. $\mathrm{axis}_{\mathfrak{r}}\overrightarrow{s}$ sgn: $l \in L_S$ with (21) iff 2(5), 2(9), (18).

Df 3. $\mathrm{axis}_{\mathfrak{r}}\overrightarrow{s}$ is called the *r-axis* of $\overrightarrow{s}$.

Pr 19. 2(5), 2(9), (18) imply r Z $\mathrm{axis}_{\mathfrak{r}}\overrightarrow{s}$.

Dm. [3] 4 Sgn 1.

Pr 20. 2(5), 2(9), (18) imply $\mathrm{axis}_{\mathfrak{r}}(-\overrightarrow{s}) = \mathrm{axis}_{\mathfrak{r}}\overrightarrow{s}$.

Dm. Sgn 3, Pr 7.

Pr 21. 2(5), 2(9), (18), 2(51) imply $\mathrm{axis}_{\mathfrak{r}}(\lambda\overrightarrow{s}) = \mathrm{axis}_{\mathfrak{r}}\overrightarrow{s}$.

Dm. Sgn 3, Pr 8.

Sch 2. Let

(22) $(a_\nu, b_\nu) \in \Lambda_S, \quad l_\nu \in L_S, \quad (a_\nu, b_\nu)$ & $l_\nu, \quad (\nu = 1, 2),$

(23) $a_1 \times a_2 \neq \mathfrak{o}.$

Then

(24) $$\left(a_1 \times a_2, \frac{(a_1 \times a_2 \cdot b_2)a_1 + (a_2 \times a_1 \cdot b_1)a_2}{(a_1 \times a_2)^2} \times (a_1 \times a_2)\right) \in \Lambda_S.$$

Let $l \in L_S$ be defined by

(25) $$\left(a_1 \times a_2, \frac{(a_1 \times a_2 \cdot b_2)a_1 + (a_2 \times a_1 \cdot b_1)a_2}{(a_1 \times a_2)^2} \times (a_1 \times a_2)\right)$$ & l.

Then it is proved [3, p. 122] that there exists exactly one couple $r_\nu \in V_S$ $(\nu = 1, 2)$ with r_ν Z l, r_ν Z l_ν $(\nu = 1, 2)$ respectively, namely

(26) $$r_1 = \frac{(a_1 \times a_2 \cdot b_2)a_1 + (a_2 \times a_1 \cdot b_1)a_2 + (a_2 b_1)a_1 \times a_2}{(a_1 \times a_2)^2},$$

(27) $$r_2 = \frac{(a_1 \times a_2 \cdot b_2)a_1 + (a_2 \times a_1 \cdot b_1)a_2 - (a_1 b_2)a_1 \times a_2}{(a_1 \times a_2)^2}.$$

Now (26), (27) imply

(28) $$r_1 - r_2 = \frac{a_1 b_2 + a_2 b_1}{(a_1 \times a_2)^2} a_1 \times a_2.$$

These geometrical facts give rise to the following considerations concerning V_S-arrows.

Pr 22. $s_\nu, m_\nu \in V_S$ $(\nu = 1, 2)$,

$$s_1 \times s_2 \neq \mathfrak{o}, \tag{29}$$

$$s \text{ sgn} : \frac{s_1 m_2 + s_2 m_1}{(s_1 \times s_2)^2} s_1 \times s_2, \tag{30}$$

$$m \text{ sgn} : \frac{s}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2)s_1 + (s_1 \times s_2 \cdot m_1)s_2), \tag{31}$$

2(2) imply 2(1).

Dm. 2 Sgn 1.

Sgn 4. $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2)$ sgn: (s, m) iff 1(53), 2(59), (29) — (31).

Df 4. $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2)$ is called the *axis* of $\overrightarrow{s}_1, \overrightarrow{s}_2$.

Pr 23. 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \neq \overrightarrow{o}$ iff 5(4).

Dm. 2Pr 1, Sgn 4, 5 Sgn 3.

Pr 24. 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) = -\text{ax}(\overrightarrow{s}_2, \overrightarrow{s}_1)$.

Dm. Sgn 4, 2 Sgn 6.

Pr 25. 2(53), 2(59), 3(2) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \top \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

Dm. Sgn 4, 4 Sgn 1.

Pr 26. 2(53), 2(59), 5(4) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \perp \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

Dm. Sgn 4,.4 Sgn 3, Pr 23, 5 Pr 18.

Pr 27. 2(53), 2(59), 5(4) imply $\text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \wedge \overrightarrow{s}_\nu$ $(\nu = 1, 2)$.

Dm. Sgn 4, 5 Sgn 1, [3] 1 Pr 106.

Pr 28. 2(53), 2(59), 5(4) imply: there exists exactly one couple $r_\nu \in V_S$ $(\nu = 1, 2)$ with 2(62) and

$$r_\nu \text{ Z dir } \text{ax}(\overrightarrow{s}_1, \overrightarrow{s}_2) \qquad (\nu = 1, 2), \tag{32}$$

namely

$$r_1 = \frac{1}{(s_1 \times s_2)^2} ((s_1 \times s_2 m_2)s_1 + (s_2 \times s_1 m_1)s_2 + (s_2 m_1)s_1 \times s_2), \tag{33}$$

$$r_2 = \frac{1}{(s_1 \times s_2)^2} ((s_1 \times s_2 m_2)s_1 + (s_2 \times s_1 m_1)s_2 - (s_1 m_2)s_1 \times s_2). \tag{34}$$

Dm. [3] 4 Pr 47.

Pr 29. 2(53), 2(59), 5(4), (30), (38), (39) imply $s = r_1 - r_2$.

Dm. Clear.

§ 7. ADDITION

Pr 1. If

$$(s_\nu, m_\mu) \in W_S \qquad (\nu = 1, 2), \tag{1}$$

then

(2) $(s_1 + s_2, m_1 + m_2) \in W_S$

iff

(3) $s_1 + s_2 = o, \quad m_1 + m_2 = o$

or

(4) $s_1 + s_2 \neq o, \quad s_1 m_2 + s_2 m_1 = 0$:

Dm. 2 Sgn 1.

Pr 2. (1) imply (2) iff one of the following conditions is satisfied:

(5) $s_1 = -s_2, \quad m_1 = -m_2$

or

(6) $s_1 \neq -s_2, \quad s_1 \times s_2 = o$

or

(7) $s_1 \times s_2 \neq o, \quad s_1 m_2 + s_2 m_1 = 0.$

Dm. By virtue of Pr 1 it must be proved that the systems of conditions (3) and (4), on the one hand, and (5) — (7), on the other hand, are equivalent. Since (3) and (5) are equivalent, it remains to be proved that the conditions (4), on the one hand, and (6) and (7), on the other hand, are equivalent.

The first condition (4), i.e.

(8) $s_1 + s_2 \neq o,$

is consistent with the alternative

(9) $s_1 \times s_2 = o$

or

(10) $s_1 \times s_2 \neq o.$

The case (8), (10) with

(11) $s_1 m_2 + s_2 m_1 = 0,$

i.e. the case (7), is equivalent to (4).

As regards the case (8), (9), the following two subcases are possible: at least one of the vectors s_1 and s_2 is zero; or none is zero.

In the first subcase, let for instance

(12) $s_1 = o.$

Now (12), (1), 2 Sgn 1 imply $m_1 = o$, and (11) is satisfied trivially.

In the second subcase, (9) implies that there exist $\lambda_\nu \in S$ $(\nu = 1, 2)$ with $\lambda_\nu \neq 0$ $(\nu = 1, 2)$ and

(13) $\qquad s_1 = \lambda_1 s_2, \qquad s_2 = \lambda_2 s_1.$

On the other hand, (1) and 2 Sgn 1 imply

(14) $\qquad s_\nu m_\nu = 0 \qquad (\nu = 1, 2).$

Now (13), (14) imply (11) again.

In such a manner, it is proved that the conditions (4) are equivalent to the conditions (6) in the case (9), and with (7), in the case (10), *q.e.d.*

Pr 3. (1) imply

(15) $\qquad (s_1 + s_2, m_1 + m_2) \notin W_S$

iff

(16) $\qquad s_1 = -s_2, \qquad m_1 \neq -m_2$

or

(17) $\qquad s_1 m_2 + s_2 m_1 \neq 0.$

Dm. Pr 1, Pr 2.

Pr 4. (1) imply

(18) $\qquad (s_1 + s_2, m_1 + m_2) \in \Lambda_S$

iff (4).

Dm. 2Sgn 3, 2 Sgn 1.

Pr 5. (1) imply (18) iff (6) or (7).

Dm. Pr 2, 2 Sgn 3.

Sgn 1. $\vec{s}_1 + \vec{s}_2$ sgn: $\vec{s}$ with

(19) $\qquad \vec{s} = (s_1 + s_2, m_1 + m_2)$

provided

(20) $\qquad \vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2)$

iff (2).

Df 1. $\vec{s}_1 + \vec{s}_2$ is called the *sum* of $\vec{s}_1$ and $\vec{s}_2$.

Df 2. The operation in W_S defined by means of Sgn 1, is called *addition* in W_S.

Pr 6. (20), (2) imply

(21) $\qquad (s_1, m_1) + (s_2, m_2) = (s_1 + s_2, m_1 + m_2).$

Dm. Sgn 1.

Pr 7. 2(53) imply:

(22) $\qquad \vec{s}_1 + \vec{s}_2$ exists

iff

(23) $\overrightarrow{s}_1 | \overrightarrow{s}_2,\ \overrightarrow{s}_1 \overrightarrow{\uparrow\downarrow} \overrightarrow{s}_2$

or

(24) $\overrightarrow{s}_1 \wedge \overrightarrow{s}_2.$

Dm. Sgn 1, Pr 2, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1.

Pr 8. 2(53) imply:

(25) $\overrightarrow{s}_1 + \overrightarrow{s}_2$ does not exist

iff

(26) $\overrightarrow{s}_1 \uparrow\downarrow \overrightarrow{s}_2$

or

(27) $\overrightarrow{s}_1 \otimes \overrightarrow{s}_2.$

Dm. Sgn 1, Pr 3, 3 Sgn 7, Pr 2, 5 Sgn 3.

Pr 9. 2(53), (22) imply

(28) $\overrightarrow{s}_1 + \overrightarrow{s}_2 = \overrightarrow{s}_2 + \overrightarrow{s}_1.$

Dm. The right-hand side of (28) exists (Pr 7, 3 Pr 2, 3 Pr 23, 5 Pr 16). Then Sgn 1.

Pr 10. 2(1) implies

(29) $\overrightarrow{s} + \overrightarrow{o} = \overrightarrow{s}.$

Dm. The left-hand side of (29) exists (2 Sgn 2, Sgn 1, 2 Sgn 1). Then Sgn 1.

Pr 11. 2(1) implies

(30) $\overrightarrow{s} + (-\overrightarrow{s}) = \overrightarrow{o}.$

Dm. The left-hand side of (30) exists (2 Sgn 6, Sgn 1, 2 Sgn 1). Then Sgn 1, 2 Sgn 2.

Pr 12. If

(31) $\overrightarrow{s}_\nu \in W_S \qquad (\nu = 1, 2, 3),$

then

(32) $(\overrightarrow{s}_1 + \overrightarrow{s}_2) + \overrightarrow{s}_3 = \overrightarrow{s}_1 + (\overrightarrow{s}_2 + \overrightarrow{s}_3),$

provided all sums exist.

Dm. Sgn 1, 1 Ax 1S.

Sgn 2. $L(\overrightarrow{s})$ sgn: $\{\lambda \overrightarrow{s} : \lambda \in S\}$ iff $\overrightarrow{o} \neq \overrightarrow{s} \in W_S$.

Df 3. $L(\overrightarrow{s})$ is called the *linear span* of $\overrightarrow{s}$.

Pr 13. 2(55), 2(49), $\overrightarrow{s}_1 = \lambda \overrightarrow{s}_2$ imply $L(\overrightarrow{s}_1) = L(\overrightarrow{s}_2)$.

Dm. Sgn 2, 2 Sgn 7.

Pr 14. If

(33) $\overrightarrow{s}_\nu \in L(\overrightarrow{s})$ $(\nu = 1, 2),$

then (22).

Dm. (33), Sgn 2 imply: there exist

(34) $\lambda_\nu \in S$ $(\nu = 1, 2)$

with

(35) $\overrightarrow{s}_\nu = \lambda_\nu \overrightarrow{s}$ $(\nu = 1, 2).$

Then (35) and 2(2) imply

(36) $\overrightarrow{s}_\nu = (\lambda_\nu s, \overline{\lambda}_\nu m)$ $(\nu = 1, 2),$

in view of 2 Sgn 7, and (36), 2(59) imply

(37) $s_1 + s_2 = (\lambda_1 + \lambda_2)s, \qquad m_1 + m_2 = (\overline{\lambda}_1 + \overline{\lambda}_2)m.$

If $\lambda_1 + \lambda_2 = 0$, then (37) imply (3) whence (22) (Sgn 1, Pr 1). If $\lambda_1 + \lambda_2 \neq 0$, then (8) (Sgn 2). On the other hand, (36), 2(59) imply

(38) $s_1 m_2 + s_2 m_1 = 2\lambda_1 \overline{\lambda}_2 (sm) = 0.$

(1 Ax 9S, 1 Pr 7, 2 Sgn 1), i.e. (11), and (8), (11) imply (22) (Pr 1, Sgn 1).

Pr 15. $L(\overrightarrow{s})$ is a group with respect to the addition in W_S.

Dm. Pr 14, Pr 12, Sgn 2 with $\lambda = 0$, 2 Pr 24, Pr 10, Sgn 2 with $\lambda = -1$, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\overrightarrow{s})$ instead of V_S).

Pr 16. 2(1),

(39) $\lambda, \mu \in S$

imply

(40) $(\lambda + \mu)\overrightarrow{s} = \lambda \overrightarrow{s} + \mu \overrightarrow{s}.$

Dm. Both sides of (40) exist: in case of $\overrightarrow{s} = \overrightarrow{o}$ all members in (40) are the zero-arrows (2 Pr 25); in case of $\overrightarrow{s} \neq \overrightarrow{o}$ see Pr 14. Now (40) follows from 2 Sgn 7 and Sgn 1.

Pr 17. $\lambda \in S$, 2(53), (22) imply

(41) $\lambda(\overrightarrow{s}_1 + \overrightarrow{s}_2) = \lambda \overrightarrow{s}_1 + \lambda \overrightarrow{s}_2.$

Dm. Both sides of (41) exist: in case of $\lambda = 0$, all members in (41) are the zero-arrows (2 Pr 24); in case of $\lambda \neq 0$ see Pr 1 and 2 Sgn 7. Now (41) follows from 2 Sgn 7 and Sgn 1.

Pr 18. $L(\overrightarrow{s})$ is a 1-dimensional linear space over S with respect to the addition in W_S and to the multiplication 2 Sgn 7 of the elements of S and W_S.

Dm. As regards the addition see Pr 15. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied (with $L(\overrightarrow{s})$ instead of V_S).

As regards the dimension of the linear space $L(\overrightarrow{s})$ over S, let uz note, first, that there exists a linearly independent element of $L(\overrightarrow{s})$, namely $\overrightarrow{s}$; and, second, that any two elements of $L(\overrightarrow{s})$, are linearly dependent. Indeed, let (33) hold. Then Sgn 2 implies that there exist (34) with (35). If both of (34) are zeroes, then both of (33) are zeroes too (2 Pr 24), and they are, therefrom, trivially linearly dependent; if at least one of (34) iz non-zero, then the linear combination

$$\lambda_2 \overrightarrow{s}_1 + (-\lambda_1)\overrightarrow{s}_2 = \lambda_1\lambda_2(\overrightarrow{s} - \overrightarrow{s}) = \lambda_1\lambda_2\overrightarrow{o} \tag{42}$$

vanishes with non-zero coefficients (34): consequently, (33) are linearly dependent.

Sch 1. The record (42) is not a quite orthodoxal one: it exploits the undefined still notion of a *difference* of two arrows (the difference $\overrightarrow{s} - \overrightarrow{s}$, as a matter of fact). In actually, (42) may be rewritten in the almost equivalent form

$$\lambda_3 \overrightarrow{s}_1 + (-\lambda_1)\overrightarrow{s}_2 = \lambda_2(\lambda_1 \overrightarrow{s}) + (-\lambda_1)(\lambda_2 \overrightarrow{s}) \tag{43}$$

$$= (\lambda_2\lambda_1)\overrightarrow{s} + (-\lambda_1\lambda_2)\overrightarrow{s}_2 = (\lambda_2\lambda_1 + (-\lambda_1\lambda_2))\overrightarrow{s} = 0\overrightarrow{s} = \overrightarrow{o}$$

(2 Pr 32, Pr 16, 2 Pr 24). As it is, the *difference* of two arrows (if it exists) is defined immediately below.

Sgn 3. $\overrightarrow{s}_1 - \overrightarrow{s}_2$ sgn: $\overrightarrow{s}_1 + (-\overrightarrow{s}_2)$ iff 2(53) and $\overrightarrow{s}_1 + (\div \overrightarrow{s}_2)$ exist.

Df 4. $\overrightarrow{s}_1 - \overrightarrow{s}_2$ is called the *difference* of $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$.

Df 5. The operation in W_S defined by means of Sgn 3 is called *subtraction* in W_S.

Pr 19. 2(53) imply

$$\overrightarrow{s}_1 - \overrightarrow{s}_2 \quad \text{exists} \tag{44}$$

iff

$$\overrightarrow{s}_1 | \overrightarrow{s}_2, \qquad \overrightarrow{s}_1 \overrightarrow{\uparrow\downarrow} - \overrightarrow{s}_2 \tag{45}$$

or

$$\overrightarrow{s}_1 \wedge \overrightarrow{s}_2. \tag{46}$$

Dm. Sgn 3, Pr 7, 2 Sgn 6, 3 Sgn 1, 5 Sgn 1.

Pr 20. 2(53) imply

$$\overrightarrow{s}_1 - \overrightarrow{s}_2 \quad \text{does not exist} \tag{47}$$

iff

$$\overrightarrow{s}_1 | \overrightarrow{s}_2, \qquad \overrightarrow{s}_1 \uparrow\downarrow - \overrightarrow{s}_2 \tag{48}$$

or

$$\overrightarrow{s}_1 \otimes \overrightarrow{s}_2. \tag{49}$$

Dm. Sgn 3, Pr 8, 2 Sgn 6, 5 Sgn 3.
Pr 21. (20)

(50) $$(s_1 - s_2, m_1 - m_2) \in W_S$$

imply

(51) $$(s_1, m_1) - (s_2, m_2) = (s_1 - s_2, m_1 - m_2).$$

Dm. Sgn 3, 2 Sgn 6, Pr 6.
Pr 22. 2(53), (44) imply

(52) $$\vec{s}_1 - \vec{s}_2 = -\vec{s}_2 + \vec{s}_1.$$

Dm. Sgn 3, Pr 9.
Pr 23. 2(1)) implies

(53) $$\vec{s} - \vec{o} = \vec{s}.$$

Dm. Sgn 3, 2 Pr 19, Pr 10
Pr 24. 2(1) implies

(54) $$\vec{s} - \vec{s} = \vec{o}.$$

Dm. Sgn 3, Pr 11.
Sch 2. Pr 24 has been used on the sly in (42).
Pr 25. 2(53), (22) imply

(55) $$-(\vec{s}_1 + \vec{s}_2) = -\vec{s}_1 - \vec{s}_2.$$

Dm. The right-hand side of (55) exists (Pr 7, Sgn 3, 2 Sgn 6, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1). Then Pr 6, 2 Sgn 6, Pr 21.
Pr 26. 2(53), (22) imply

(56) $$-(\vec{s}_1 - \vec{s}_2) = -\vec{s}_1 + \vec{s}_2.$$

Dm. Pr 25, Sgn 3, 2 Sgn 6.
Pr 27. (31) imply

(57) $$(\vec{s}_1 - \vec{s}_2) + \vec{s}_3 = \vec{s}_1 + (-\vec{s}_2 + \vec{s}_3)$$

provided all sums and differences exist.
Dm. Sgn 3, Pr 12.
Pr 28. (31) imply

(58) $$(\vec{s}_1 + \vec{s}_2) - \vec{s}_3 = \vec{s}_1 + (\vec{s}_2 - \vec{s}_3)$$

provided all sums and differences exist.
Dm. Sgn 3, Pr 12.
Pr 29. (31) imply

(59) $$(\vec{s}_1 - \vec{s}_2) - \vec{s}_3 = \vec{s}_1 - (\vec{s}_2 + \vec{s}_3)$$

provided all sums and differences exist.

Dm. Sgn 3, Pr 12.

Pr 30. (39), 2(1) imply

(60) $$(\lambda - \mu)\vec{s} = \lambda\vec{s} - \mu\vec{s}.$$

Dm. If $\vec{s} = \vec{o}$, then all members in (60) are zeroes (2 Pr 25). If $\vec{s} \neq \vec{o}$, then (60) is simplified by Pr 18.

Pr 31. 2(49), 2(53), (44) imply

(61) $$\lambda(\vec{s}_1 - \vec{s}_2) = \lambda\vec{s}_1 - \lambda\vec{s}_2.$$

Dm. The right-hand side of (61) exists: if $\lambda = 0$, then all members in (61) are zeroes (2 Pr 24); if $\lambda \neq 0$, then Pr 19, 2 Sgn 7, 3 Sgn 1, 3 Sgn 8, 5 Sgn 1. Now 2 Sgn 7 and Pr 21.

Pr 32. 2(53), (24), (34),

(62) $$\lambda_1\lambda_2 \neq 0$$

imply

(63) $$\lambda_1\vec{s}_1 \wedge \lambda_2\vec{s}_2.$$

Dm. (24) implies (10), (11) (5 Sgn 1). If 2(59), then

(64) $$\lambda_\nu\vec{s}_\nu = (\lambda_\nu s_\nu, \overline{\lambda}_\nu m_\nu) \qquad (\nu = 1, 2)$$

(2 Sgn 7). Now (64), 1 Pr 14, 1 Pr 17, I Ax 8S, 1 Pr 7, (62), (10), (11) imply

(65) $$(\lambda_1 s_1) \times (\lambda_2 s_2) = (\overline{\lambda_1\lambda_2}) s_1 \times s_2 \neq o,$$

(66) $$(\lambda_1 s_1)(\overline{\lambda_2} m_2) + (\lambda_2 s_2)(\overline{\lambda_1} m_1) = (\lambda_1\lambda_2)(s_1 m_2 + s_2 m_1) = 0,$$

and (65), (66) imply (63) (5 Sgn 1).

Pr 33. 2(53), (24), (34) imply

(67) $$\lambda_1\vec{s}_1 + \lambda_2\vec{s}_2 \quad \text{exists.}$$

Dm. If $\lambda_1 = 0$ or $\lambda_2 = 0$, then (67) is implied by 2 Pr 24, Pr 10, Pr 9. If (62), then Pr 32, Pr 7.

Sgn 4. $L(\vec{s}_\nu)_{\nu=1}^2$ sgn: $\left\{\lambda_1\vec{s}_1 + \lambda_2\vec{s}_2 : \lambda_\nu \in S(\nu = 1, 2)\right\}$ Iff $\vec{s}_\nu \in W_S$ $(\nu = 1, 2)$, $\vec{s}_1 \wedge \vec{s}_2$.

Df 6. $L(\vec{s}_\nu)_{\nu=1}^2$ is called the *linear span* of $\vec{s}_\nu$ $(\nu = 1, 2)$.

Pr 34. 2(53), (24),

(68) $$\vec{o}_\mu \in L(\vec{s}_\nu)_{\nu=1}^2 \qquad (\mu = 1, 2)$$

imply

(69) $\overrightarrow{\sigma}_1 + \overrightarrow{\sigma}_2$ exists.

Dm. (68), Sgn 4 imply: there exist

(70) $\lambda_{\mu\nu} \in S$ $(\mu, \nu = 1; 2)$

with

(71) $\overrightarrow{\sigma}_\mu = \lambda_{\mu 1} \overrightarrow{s}_1 + \lambda_{\mu 2} \overrightarrow{s}_2$ $(\mu = 1, 2)$

Now (71) and 2(59) imply

(72) $\overrightarrow{\sigma}_\mu = (\lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2, \overline{\lambda}_{\mu 1} m_1 + \overline{\lambda}_{\mu 2} m_2)$ $(\mu = 1, 2).$

(2 Sgn 7, Pr 6), and (72), 1 Ax 8S, 1 Pr 7, 1 Pr 14, 1 Pr 17 imply

(73) $(\lambda_{11} s_1 + \lambda_{12} s_2) \times (\lambda_{21} s_1 + \lambda_{22} s_2) = (\overline{\lambda}_{11}\overline{\lambda}_{22} - \overline{\lambda}_{12}\overline{\lambda}_{21}) s_1 \times s_2,$

(74) $(\lambda_{11} s_1 + \lambda_{12} s_2)(\overline{\lambda_{21}} m_1 + \overline{\lambda_{22}} m_2) + (\lambda_{21} s_1 + \lambda_{22} s_2)(\overline{\lambda_{11}} m_1 + \overline{\lambda_{12}} m_2)$

$= (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})(s_1 m_2 + s_2 m_1) = 0.$

in view of (14) (2 Sgn 1) and (11) (5 Sgn 1).

The alternative

(75) $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \neq 0$

or

(76) $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 0$

now arises

If (75), then (73) and (10) (5 Sgn 1) imply

(77) $(\lambda_{11} s_1 + \lambda_{12} s_2) \times (\lambda_{21} s_1 + \lambda_{22} s_2) \neq o$

and (72), (77), (74), 5 Sgn 1 imply

(78) $\overrightarrow{\sigma}_1 \wedge \overrightarrow{\sigma}_2.$

consequently (69) (Pr 7).

If (76), then (73) implies

(79) $(\lambda_{11} s_1 + \lambda_{12} s_2) \times (\lambda_{21} s_1 + \lambda_{22} s_2) = o,$

and(79), (72) imply

(80) $\overrightarrow{\sigma}_1 | \overrightarrow{\sigma}_2$

in view of 3 Sgn 1.

The supposition

(81) $\overrightarrow{\sigma}_1 \uparrow\downarrow \overrightarrow{\sigma}_2$

is wrong. Indeed, (72) and 3 Sgn 7 imply that (81) is equivalent to

(82) $\lambda_{11}s_1 + \lambda_{12}s_2 = -(\lambda_{21}s_1 + \lambda_{22}s_2),$

(83) $\overline{\lambda_{11}}m_1 + \overline{\lambda_{12}}m_2 \neq -(\overline{\lambda_{21}}m_1 + \overline{\lambda_{22}}m_2)$

i.e. with

(84) $(\lambda_{11} + \lambda_{21})s_1 + (\lambda_{12} + \lambda_{22})s_2 = o,$

(85) $(\overline{\lambda_{11}} + \overline{\lambda_{21}})m_1 + (\overline{\lambda_{12}} + \overline{\lambda_{22}})m_2 \neq o.$

Now (84) and (10) (5 Sgn 1) imply

(86) $\lambda_{11} + \lambda_{21} = 0, \qquad \lambda_{12} + \lambda_{22} = 0,$

and (86) imply

(87) $(\overline{\lambda_{11}} + \overline{\lambda_{21}})m_1 + (\overline{\lambda_{12}} + \overline{\lambda_{22}})m_2 = 0,$

contrary to (85). In such a manner,

(88) $\overrightarrow{\sigma}_1 \uparrow\downarrow \overrightarrow{\sigma}_2$

(3 Pr 21) and (80), (88) imply (69) (Pr 7).

Pr 35. $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ is a group with respect to the addition in W_S.

Dm. Pr 34, Pr 12, Sgn 4 with $\lambda_1 = \lambda_2 = 0$, 2 Pr 24, Pr 10, Sgn 4 with λ_1, λ_2, on the one hand, and $-\lambda_1$, $-\lambda_2$, on the other hand, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ instead of V_S).

Pr 36. $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ is a 2-dimensional linear space over S with respect to the addition in W_S and to the multiplication 2 Sgn 7 of the elements of S and W_S.

Dm. As regards the addition, see Pr 35. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied (with $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ instead of V_S).

As regards the dimension of the linear space $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ over S, let us note, first, that there exist two linearly independent elements of $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ namely $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$; and, second, that any three elements of $L(\overrightarrow{s}_\nu)_{\nu=1}^2$ are linearly dependent.

Indeed, $\overrightarrow{s}_\mu \in L(\overrightarrow{s}_\nu)_{\nu=1}^2$ $(\mu = 1, 2)$ since $\overrightarrow{s}_1 = 1\overrightarrow{s}_1 + 0\overrightarrow{s}_2$, $\overrightarrow{s}_2 = 0\overrightarrow{s}_1 + 1\overrightarrow{s}_2$ (2 Pr 24, 2 Pr 26, Pr 10, Pr 9). Let now (34) and

(89) $\lambda_1\overrightarrow{s}_1 + \lambda_2\overrightarrow{s}_2 = \overrightarrow{o}$

hold. If 2(59), then (89) is equivalent to

(90) $(\lambda_1 s_1 + \lambda_2 s_2, \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2) = (o, o)$

(2 Sgn 7, Pr 6, 2 Sgn 2), whence

(91) $\lambda_1 s_1 + \lambda_2 s_2 = o.$

Now (91) and (100 (5 Sgn 1) imply $\lambda_1 = \lambda_2 = 0$, hence the linear independency of $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$.

On the other hand, let

(92) $\qquad \overrightarrow{\sigma}_\mu \in L(\overrightarrow{s}_\nu)_{\nu=1}^2 \qquad (\mu = 1, 2, 3).$

Now (92) and Sgn 4 imply that there exist

(93) $\qquad \lambda_{\mu\nu} \in S \qquad (\mu = 1, 2, 3; \nu = 1, 2)$

with

(94) $\qquad \overrightarrow{\sigma}_\mu = \lambda_{\mu 1}\overrightarrow{s}_1 + \lambda_{\mu 2}\overrightarrow{s}_2 \qquad (\mu = 1, 2, 3).$

If 2(59), then (94) is equivalent to

(95) $\qquad \overrightarrow{\sigma}_\mu = (\lambda_{\mu 1}s_1 + \lambda_{\mu 2}s_2, \overline{\lambda}_{\mu 1}m_1 + \overline{\lambda}_{\mu 2}m_2) \qquad (\mu = 1, 2, 3).$

Let 1(34) be a non-zero solution of the system of equations

(96) $\qquad \sum_{\mu=1}^{3} \alpha_\mu \lambda_{\mu 1} = 0, \qquad \sum_{\mu=1}^{3} \alpha_\mu \lambda_{\mu 2} = 0,$

i.e. with

(97) $\qquad \sum_{\mu=1}^{3} \alpha_\mu \overline{\alpha}_\mu \neq 0.$

Now (96) imply

(98) $\qquad \sum_{\mu=1}^{3} \overline{\alpha}_\mu \overline{\lambda}_{\mu 1} = 0, \qquad \sum_{\mu=1}^{3} \overline{\alpha}_\mu \overline{\lambda}_{\mu 2} = 0$

and (95), (96), (98),

(99) $\qquad \sum_{\mu=1}^{3} \alpha_\mu \overrightarrow{\sigma}_\mu \quad \text{sgn} : \quad (\alpha_1 \overrightarrow{\sigma}_1 + \alpha_2 \overrightarrow{\sigma}_2) + \alpha_3 \overrightarrow{\sigma}_3,$

Pr 6, 2 Sgn 2 imply

(100) $\qquad \sum_{\mu=1}^{3} \alpha_\mu \overrightarrow{\sigma}_\mu = \overrightarrow{o}$

with (97), i.e. the linear dependency of (92).

Pr 37. If

(101) $\qquad \overrightarrow{s} \in L(\overrightarrow{s}_\nu)_{\nu=1}^2.$

(102) $\qquad \overrightarrow{s} \mid \overrightarrow{s}_\nu \qquad (\nu = 1, 2),$

then

(103) $\qquad \overrightarrow{s} \wedge \overrightarrow{s}_\nu \qquad (\nu = 1, 2).$

Dm. (101), Sgn 4 imply: there exist (34) with

(104) $\quad \vec{s} = \lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2.$

If 2(2), 2959), then (102), 3 Sgn 2 imply

(105) $\quad s \times s_\nu \neq \mathfrak{o} \qquad (\nu = 1, 2).$

Besides, (104), 2(2), 2(59), 2 Sgn 7, Pr 6 imply

(106) $\quad s = \lambda_1 s_1 + \lambda_2 s_2, \qquad m = \bar{\lambda}_1 m_1 + \bar{\lambda}_2 m_2.$

now (106), 2 Sgn 1, 1 Ax 8S, 1 Pr 7, Sgn 4, 5 Sgn 1 imply

(107) $\quad s m_1 + s_1 m = \lambda_2 (s_1 m_2 + s_2 m_1) = 0,$

(108) $\quad s m_2 + s_2 m = \lambda_1 (s_1 m_2 + s_2 m_1) = 0,$

and (105), (107), (108), 5 Sgn 1 imply (103).

Pr 38. (68),

(109) $\quad \vec{\sigma}_1 \overline{|} \vec{\sigma}_2$

imply

(110) $\quad L(\vec{\sigma}_\nu)_{\nu=1}^2 = L(\vec{s}_\nu)_{\nu=1}^2.$

Dm. As in proof of Pr 34 it is proved that (73), (74) hold good. The case (76) is impossible since it implies (80) contrary to (109) (3 Pr 1). Now as in the proof of Pr 34 it is proved that (78) holds. Hence the left-hand side of (110) exists (Sgn 4). Then Pr 36 and Sgn 4.

Pr 39. If

(111) $\quad \vec{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2, 3),$

(112) $\quad s_1 \times s_2 \cdot s_3 \neq 0,$

(113) $\quad s_\mu m_\nu + s_\nu m_\mu = 0 \qquad (\mu, \nu = 1, 2, 3),$

then there exists exactly one 1(31) with

(114) $\quad r \; Z \text{ dir } \vec{s}_\nu \qquad (\nu = 1, 2, 3),$

namely 5(7).

Dm. (112) imply $s_\nu \neq \mathfrak{o}$ $(\nu = 1, 2, 3)$, whence

(115) $\quad \vec{s}_\nu \neq \vec{o} \qquad (\nu = 1, 2, 3)$

(2 Pr 1). Now (115) and 2 Sgn 4 imply: dir $\vec{s}_\nu$ $(\nu = 1, 2, 3)$ exist and (114) is equivalent to

(116) $\quad r \times s_\nu = m_\nu \qquad (\nu = 1, 2, 3).$

Then (112), (113), 1 Pr 30 imply that there exists exactly one 1(31) with (116), namely 5(7).

Pr 40. (111) — (113) imply

$$\overrightarrow{s}_\mu \wedge \overrightarrow{s}_\nu \qquad (\mu, \nu = 1, 2, 3;\ \mu \neq \nu). \tag{117}$$

Dm. 5 Sgn 1.

Pr 41. (111) — (113),

$$\lambda_\nu \in S \qquad (\nu = 1, 2, 3), \tag{118}$$

$$\lambda_1\lambda_2\lambda_3 \neq 0 \tag{119}$$

imply

$$\lambda_\mu \overrightarrow{s}_\mu \wedge \lambda_\nu \overrightarrow{s}_\nu \qquad (\mu, \nu = 1, 2, 3;\ \mu \neq \nu). \tag{120}$$

Dm. Pr 40, Pr 32.

Pr 42. (111) — (113), (118), (119) imply

$$\lambda_1 \overrightarrow{s}_1 + \lambda_2 \overrightarrow{s}_2 \wedge \lambda_3 \overrightarrow{s}_3. \tag{121}$$

Dm. Pr 40, Pr 33 imply (67). Besides, (111), 2 Sgn 7, Pr 6 imply

$$\lambda_1 \overrightarrow{s}_1 + \lambda_2 \overrightarrow{s}_2 = (\lambda_1 s_1 + \lambda_2 s_2, \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2). \tag{122}$$

On the other hand, (111) and 2 Sgn 7 imply

$$\lambda_3 \overrightarrow{s}_3 = (\lambda_3 s_3, \overline{\lambda}_3 m_3). \tag{123}$$

Now (122), (123) imply

$$(\lambda_1 s_1 + \lambda_2 s_2) \times (\lambda_3 s_3) = \overline{\lambda}_3(\overline{\lambda}_1(s_1 \times s_3) + \overline{\lambda}_2(s_2 \times s_3)) \tag{124}$$

(1 Pr 14, 1 Pr 17), whence

$$(\lambda_1 s_1 + \lambda_2 s_2) \times (\lambda_3 s_3) \neq \mathfrak{o}. \tag{125}$$

Indeed, otherwise (124), (119) imply

$$\overline{\lambda}_1(s_1 \times s_3) + \overline{\lambda}_2(s_2 \times s_3) = \mathfrak{o} \tag{126}$$

and scalar multiplication of (126) with s_1 implies

$$\lambda_2(s_1 \cdot s_2 \times s_3) = 0 \tag{127}$$

by virtue of 1 Pr 7, contrary to (119), (112) in view of 1 Pr 8. On the other hand, 1 Pr 7, (113) imply

(128) $$(\lambda_1 s_1 + h_2 s_2)m_3 + s_3(\bar{\lambda}_1 m_1 + \bar{\lambda}_2 m_2)$$

$$= \lambda_1(s_1 m_3 + s_3 m_1) + \lambda_2(s_2 m_3 + s_3 m_2) = 0,$$

and (122), (123), (125), 5 Sgn 1 imply (121).

Pr 43. (111) — (113), (118) imply

(129) $$(\lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2) + \lambda_2 \vec{s}_3 \quad \text{exists.}$$

Dm. If $\lambda_1\lambda_2\lambda_3 = 0$, then (129) is implied by 2 Pr 24, Pr 10, Pr 9. if (119) then Pr 33, Pr 42, Pr 7.

Sgn 5. $\vec{s}_1 + \vec{s}_2 + \vec{s}_3$ sgn: $(\vec{s}_1 + \vec{s}_2) + \vec{s}_3$ if (31).

Sgn 6. $L(\vec{s}_\nu)_{\nu=1}^3$ sgn: $\{\lambda_1 \vec{s}_1 + \lambda_2 \vec{s}_2 + \lambda_3 \vec{s}_3 \colon \lambda_\nu \in S \ (\nu = 1, 2, 3)\}$ iff (111) — (113).

Df 7. $L(\vec{s}_\nu)_{\nu=1}^3$ is called the *linear span* of $\vec{s}_\nu$ $(\nu = 1, 2, 3)$.

Pr 44. (111) — (113),

(130) $$\vec{\sigma}_\mu \in L(\vec{s}_\nu)_{\nu=1}^3 \qquad (\mu = 1, 2)$$

imply (69).

Dm. (130), Sgn 6 imply: there exist

(131) $$\lambda_{\mu\nu} \in S \qquad (\mu = 1, 2; \nu = 1, 2, 3)$$

with

(132) $$\vec{\sigma}_\mu = \lambda_{\mu 1} \vec{s}_1 + \lambda_{\mu 2} \vec{s}_2 + \lambda_{\mu 3} \vec{s}_3 \qquad (\mu = 1, 2).$$

Now (132) and (111) imply

(133) $$\vec{\sigma}_\mu = (\lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3, \bar{\lambda}_{\mu 1} m_1 + \bar{\lambda}_{\mu 2} m_2 + \bar{\lambda}_{\mu 3} m_3)(\mu = 1, 2)$$

(2 Sgn 7, Pr 6), and (133), 1 Ax 8S, 1 Pr 7, 1 Pr 14, 1 Pr 17 imply

(134) $$(\lambda_{11} s_1 + \lambda_{12} s_2 + \lambda_{13} s_3) \times (\lambda_{21} s_1 + \lambda_{22} s_2 + \lambda_{23} s_3)$$

$$= (\bar{\lambda}_{11}\bar{\lambda}_{22} - \bar{\lambda}_{12}\bar{\lambda}_{21}) s_1 \times s_2 + (\bar{\lambda}_{12}\bar{\lambda}_{23} - \bar{\lambda}_{13}\bar{\lambda}_{22}) s_2 \times s_3 + (\bar{\lambda}_{13}\bar{\lambda}_{21} - \bar{\lambda}_{11}\bar{\lambda}_{23}) s_3 \times s_1$$

and

(135) $$(\lambda_{11} s_1 + \lambda_{12} s_2 + \lambda_{13} s_3)(\bar{\lambda}_{21} m_1 + \bar{\lambda}_{22} m_2 + \bar{\lambda}_{23} m_3)$$

$$+ (\lambda_{21} s_1 + \lambda_{22} s_2 + \lambda_{23} s_3)(\bar{\lambda}_{11} m_1 + \bar{\lambda}_{12} m_2 + \bar{\lambda}_{13} m_3)$$

$$= (\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21})(s_1 m_2 + s_2 m_1) + (\lambda_{12}\lambda_{23} + \lambda_{13}\lambda_{22})(s_2 m_3 + s_3 m_2)$$

$$+ (\lambda_{13}\lambda_{21} + \lambda_{11}\lambda_{23})(s_3 m_1 + s_1 m_3) = 0$$

in view of

(136) $$s_\nu m_\nu = 0 \qquad (\nu = 1, 2, 3)$$

(2 Sgn 1) and (113) (Sgn 6).

The alternative

(137) $$|\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}|^2 + |\lambda_{12}\lambda_{21} - \lambda_{13}\lambda_{22}|^2 + |\lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{23}|^2 \neq 0$$

or

(138) $$\lambda_{11}\lambda_{22}-\lambda_{12}\lambda_{21} = 0, \qquad \lambda_{12}\lambda_{21}-\lambda_{13}\lambda_{22} = 0 \qquad \lambda_{13}\lambda_{21}-\lambda_{11}\lambda_{23} = 0$$

now arises.

If (137), then (1340 implies

(139) $$(\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3) \times (\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3) \neq o.$$

Indeed, the right-hand side of (134) represents a linear combination of Gibbs' vectors $\overrightarrow{s}_\nu^{-1}$ ($\nu = 1, 2, 3$) defined by 1(23) provided 1(24); they exist by virtues of (112) (Sgn 6): in view of 1 Pr 20 and 1 Pr 12 such a combination may be zerro if, and only if, all coefficients are zeroes, and this is not the case if (137) holds. Now (132), (139), (135), 5 Sgn 1 imply (78), consequently (69) (Pr 7).

If (138), then (134) implies

(140) $$(\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3) \times (\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3) = o,$$

and (140), (132) imply (80) in view of 3 Sgn 1.

The supposition (81) is wrong. Indeed, (132) and 3 Sgn 7 imply that (81) is equivalent to

(141) $$\lambda_{11}s_1 + \lambda_{12}s_2 + \lambda_{13}s_3 = -(\lambda_{21}s_1 + \lambda_{22}s_2 + \lambda_{23}s_3),$$

(142) $$\overline{\lambda}_{11}m_1 + \overline{\lambda}_{12}m_2 + \overline{\lambda}_{13}m_3 \neq -(\overline{\lambda}_{21}m_1 + \overline{\lambda}_{22}m_2 + \overline{\lambda}_{23}m_3),$$

i.e. to

(143) $$(\lambda_{11} + \lambda_{21})s_1 + (\lambda_{12} + \lambda_{22})s_2 + (\lambda_{13} + \lambda_{23})s_3 = o,$$

(144) $$(\overline{\lambda}_{11} + \overline{\lambda}_{21})m_1 + (\overline{\lambda}_{12} + \overline{\lambda}_{22})m_2 + (\overline{\lambda}_{13} + \overline{\lambda}_{23})m_3 \neq o.$$

Now (143) and (112), 1 Pr 12 imply

(145) $$\lambda_{11} + \lambda_{21} = 0, \qquad \lambda_{12} + \lambda_{22} = 0, \qquad \lambda_{13} + \lambda_{23} = 0,$$

and (145) imply

(146) $$(\overline{\lambda}_{11} + \overline{\lambda}_{21})m_1 + (\overline{\lambda}_{12} + \overline{\lambda}_{22})m_2 + (\overline{\lambda}_{13} + \overline{\lambda}_{23})m_3 = o.$$

contrary to (144). In such a manner, (88) holds (3 Pr 21) and (80), (88) imply (69) (Pr 7).

Pr 45. $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ is a group with respect to the addition in W_S.

Dm. Pr 44, Sgn 6 with $\lambda_\nu = 0$ ($\nu = 1, 2, 3$), 2 Pr 24, Pr 10, Sgn 6 with λ_1, λ_2, λ_3, on the one hand, and $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, respectively, on the other hand, 2 Pr 27, Pr 11 display that 1 Ax 1S — 1 Ax 3S are satisfied (with $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ instead of V_S).

Pr 46. $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ is a 3-dimensional linear space over S with respect to the addition in W_S and to the multiplication 2Sgn 7 of the elements of S and W_S.

Dm. As regards the addition, see Pr 45. Now 2 Sgn 7, 2 Pr 26, 2 Pr 32, Pr 16, Pr 17 display that 1 Ax 4S — 1 Ax 7S are satisfied with $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ instead of V_S).

As regards the dimension of the linear space $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ over S, let us note, first, that there exist three linearly independent elements of $L(\overrightarrow{s}_\nu)_{\nu=1}^3$, namely $\overrightarrow{s}_\nu$ $(\nu = 1, 2, 3)$; and second, that any four elements of $L(\overrightarrow{s}_\nu)_{\nu=1}^3$ are linearly dependent.

Indeed, $\overrightarrow{s}_\mu \in L(\overrightarrow{s}_\nu)_{\nu=1}^3$ $(\mu = 1, 2, 3)$, since $\overrightarrow{s}_1 = 1\overrightarrow{s}_1 + 0\overrightarrow{s}_2 + 0\overrightarrow{s}_3$, $\overrightarrow{s}_2 = 0\overrightarrow{s}_1 + 1\overrightarrow{s}_2 + 0\overrightarrow{s}_3$, $\overrightarrow{s}_3 = 0\overrightarrow{s}_1 + 0\overrightarrow{s}_2 + 1\overrightarrow{s}_3$ (2 Pr 24, 2 Pr 26, Pr 10, Pr 9, Sgn 6). Let now (118) and

$$\lambda_1\overrightarrow{s}_1 + \lambda_2\overrightarrow{s}_2 + \lambda_3\overrightarrow{s}_3 = \overrightarrow{o} \tag{147}$$

hold. If (111), then (147) is equivalent to

$$(\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3, \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2 + \overline{\lambda}_3 m_3) = (o, o) \tag{148}$$

(2 Sgn 7, Pr 6, 2 Sgn 2, Sgn 5), whence

$$\lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3 = o. \tag{149}$$

Now (149) and (112) imply $\lambda_\nu = 0$ $(\nu = 1, 2, 3)$ (1 Pr 12), hence the linear independency of $\overrightarrow{s}_\nu$ $(\nu = 1, 2, 3)$.

On the other hand, let

$$\overrightarrow{\sigma}_\mu \in L(\overrightarrow{s}_\nu)_{\nu=1}^3 \qquad (\mu = 1, 2, 3, 4). \tag{150}$$

Now (150) and Sgn 6 imply that there exist

$$\lambda_{\mu\nu} \in S \qquad (\mu = 1, 2, 3, 4; \nu = 1, 2, 3) \tag{151}$$

with

$$\overrightarrow{\sigma}_\mu = \lambda_{\mu 1}\overrightarrow{s}_1 + \lambda_{\mu 2}\overrightarrow{s}_2 + \lambda_{\mu 3}\overrightarrow{s}_3 \qquad (\mu = 1, 2, 3, 4). \tag{152}$$

If (111), then (152) is equivalent to

$$\overrightarrow{\sigma}_\mu = (\lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3, \overline{\lambda}_{\mu 1} m_1 + \overline{\lambda}_{\mu 2} m_2 + \overline{\lambda}_{\mu 3} m_3) \tag{153}$$

$$(\mu = 1, 2, 3, 4).$$

Let

$$\alpha_\mu \in S \qquad (\mu = 1, 2, 3, 4) \tag{154}$$

be a non-zero solution of the system of equations

$$\sum_{\mu=1}^{4} \alpha_\mu \lambda_{\mu\nu} = 0 \qquad (\nu = 1, 2, 3), \tag{155}$$

i.e.

(156) $$\sum_{\mu=1}^{4} \alpha_\mu \overline{\alpha}_\mu \neq 0.$$

Now (155) imply

(157) $$\sum_{\mu=1}^{4} \overline{\alpha}_\mu \overline{\lambda}_\mu = 0 \qquad (\nu = 1, 2, 3)$$

and (153), (155), (157) imply

(158) $$\sum_{\mu=1}^{4} \alpha_\mu \overrightarrow{\sigma}_\mu = \overrightarrow{o}$$

with (156), i.e. the linear dependency of (150).

Pr 47. If

(159) $$\overrightarrow{s} \in L(\overrightarrow{s})_{\nu=1}^{3},$$

(160) $$\overrightarrow{s} \mid \overrightarrow{s}_\nu \qquad (\nu = 1, 2, 3),$$

then

(161) $$\overrightarrow{s} \wedge \overrightarrow{s}_\nu \qquad (\nu = 1, 2, 3).$$

Dm. (159), Sgn 6 imply: there exist (118) with

(162) $$\overrightarrow{s} = \lambda_1 \overrightarrow{s}_1 + \lambda_2 \overrightarrow{s}_2 + \lambda_3 \overrightarrow{s}_3.$$

If 2(2), 2(59), then (160), 3 Sgn 2 imply

(163) $$s \times s_\nu \neq o \qquad (\nu = 1, 2, 3).$$

Besides, (162), 2(2), 2(59), 2 Sgn 7, Pr 6 imply

(164) $$s = \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3, \qquad m = \overline{\lambda}_1 m_1 + \overline{\lambda}_2 m_2 + \overline{\lambda}_3 m_3.$$

Now (164), 2 Sgn 1, 1 Ax 8S, 1 Pr 7, Sgn 6, 5 Sgn 1 imply

(165) $$sm_1 + s_1 m = \lambda_2(s_1 m_2 + s_2 m_1) + \lambda_3(s_1 m_3 + s_3 m_1) = 0,$$

(166) $$sm_2 + s_2 m = \lambda_1(s_1 m_2 + s_2 m_1) + \lambda_3(s_2 m_3 + s_3 m_2) = 0,$$

(167) $$sm_3 + s_3 m = \lambda_1(s_1 m_3 + s_3 m_1) + \lambda_3(s_3 m_3 + s_3 m_2) = 0,$$

and(163), (165) — (167), 5 Sgn 1 imply (161).

Pr 48. If

(168) $$\overrightarrow{\sigma}_\mu \in L(\overrightarrow{s})_{\nu=1}^{3} \qquad (\mu = 1, 2, 3),$$

(169) $\quad \vec{\sigma}_\mu = (\bar{\sigma}_\mu, n_\mu) \qquad (\mu = 1, 2, 3),$

(170) $\quad \bar{\sigma}_1 \times \bar{\sigma}_2 \cdot \bar{\sigma}_3 \neq 0,$

then

(171) $\quad L(\vec{\sigma})_{\nu=1}^3 = L(\vec{s})_{\nu=1}^3.$

Dm. (168), Sgn 6 imply: there exist

(172) $\quad \lambda_{\mu\nu} \in S \qquad (\mu, \nu = 1, 2, 3)$

with

(173) $\quad \vec{\sigma}_\mu = \lambda_{\mu 1} \vec{s}_1 + \lambda_{\mu 2} \vec{s}_2 + \lambda_{\mu 3} \vec{s}_3 \qquad (\mu = 1, 2, 3).$

If (111), then (169) and (173) imply

(174) $\quad \bar{\sigma}_\mu = \lambda_{\mu 1} s_1 + \lambda_{\mu 2} s_2 + \lambda_{\mu 3} s_3 \qquad (\mu = 1, 2, 3),$

(175) $\quad n_\mu = \bar{\lambda}_{\mu 1} m_1 + \bar{\lambda}_{\mu 2} m_2 + \bar{\lambda}_{\mu 3} m_3 \qquad (\mu = 1, 2, 3)$

and (174), (175), 1 Ax 10S, 1 Ax 8S, 1 Pr 7, 2 Sgn 1, Sgn 6 imply

(176) $\quad \bar{\sigma}_\mu n_\nu + \bar{\sigma}_\nu n_\mu = 0 \qquad (\mu, \nu = 1, 2, 3).$

Now (170) and (176) imply that the left-hand side of (171) exists (Sgn 6). Then Pr 46.

Sgn 7. $\vec{s}_1 \vec{s}_2$ sgn: $s_1 s_2$ iff (20).

Df 8. $\vec{s}_1 \vec{s}_2$ is called the *scalar product* of $\vec{s}_1$ and $\vec{s}_2$.

Sgn 8. $\vec{s}^2$ sgn: $\vec{s}\,\vec{s}$ iff 2(1).

Df 9. $\vec{s}^2$ is called the *scalar square* of $\vec{s}$.

Pr 49. 2(53) imply $\vec{s}_1 \vec{s}_2 = \vec{s}_2 \vec{s}_1$.
Dm. Sgn 7, 1 Ax 8S.

Pr 50. 2(49), 2(53) imply $(\lambda \vec{s}_1)\vec{s}_2 = \lambda(\vec{s}_1 \vec{s}_2)$.
Dm. Sgn 7, 2 Sgn 7, 1 Ax 9S.

Pr 51. (31), (22) imply $(\vec{s}_1 + \vec{s}_2)\vec{s}_3 = \vec{s}_1 \vec{s}_3 + \vec{s}_2 \vec{s}_3$.
Dm. Sgn 7, Pr 6, 1 Ax 10S.

Pr 52. 2(1) implies $\vec{s}^2 \geqq 0$.
Dm. Sgn 8, Sgn 7, 1 Ax 11S.

Pr 53. 2(1), $\vec{s}^2 = 0$ imply $\vec{s} = \vec{o}$.
Dm. Sgn 8, Sgn 7, 1 Ax 12S, 2 Sgn 1.

Pr 54. $L(\vec{s})_{\nu=1}^3$ is a 3-dimensional Hermitean space over S with respect to the addition in W_S, to the multiplication 2 Sgn 7 of the elements of S and W_S, and to the scalar multiplication Sgn 7 of the elements of W_S.
Dm. Pr 45, Pr 46, Pr 49 — Pr 53.

Sch 3. The following considerations will be useful in the sequel.

Let the following problem be solved. If $\overrightarrow{s}_\nu = (s_\nu, m_\nu)$ $\nu = 1, 2)$ be given intersecting arrows, then find a third arrow $\overrightarrow{s} = (s_3, m_3)$ such that, first, $s_3 = s_1 \times s_2$; and, second, that there exists a $r \in V_S$ with r Z dir $\overrightarrow{s}_\nu$ $(\nu = 1, 2, 3)$, i.e. $r \times s_\nu = m_\nu$ $(\nu = 1, 2, 3)$. (The directrices of $\overrightarrow{s}_\nu$ exist, since $\overrightarrow{s}_\nu \neq \overrightarrow{o}$ $(\nu = 1, 2, 3)$, as 5 Sgn 1 and the definition of $\overrightarrow{s}_3$ imply.) in other words, conditions are sought for the consistency of the system of vector equations $r \times s_\nu = m_\nu$ $(\nu = 1, 2, 3)$, provided $s_3 = s_1 \times s_2$. According to 1 Pr 30, it is necessary and sufficient to this end that

(177) $$s_\mu m_\nu + s_\nu m_\mu = 0 \qquad (\mu, \nu = 1, 2, 3).$$

hold, i.e. that

(178) $$m_3 s_1 = -m_1 s_3, \qquad m_3 s_2 = -m_2 s_3, \qquad m_3 s_3 = 0$$

are satisfied (1 Ax 8S, 2 Sgn 1). The system (178) is equivalent to

(179) $$m_3 s_1 = m_1 \cdot s_2 \times s_1, \qquad m_3 s_2 = m_2 \cdot s_2 \times s_1, \qquad m_3 s \cdot s_1 \times s_2 = 0$$

(1 Pr 13). according to 1 Pr 26, the only solution m_3 of the system (179) of vector equations is

(180) $$m_3 = (m_1 \cdot s_2 \times s_1) s_1^{-1} + (m_2 \cdot s_2 \times s_1) s_2^{-1}.$$

i.e.

(181) $$m_3 = \frac{(m_1 \cdot s_2 \times s_1)(s_2 \times (s_1 \times s_2)) + (m_2 \cdot s_2 \times s_1)(s_1 \times (s_2) \times s_1)}{(s_1 \times s_2)^2}.$$

Now (181) and 1 Pr 14, 1 Pr 15, 1 Pr 8 imply

(182) $$m_3 = \frac{s_1 \times s_2 2}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2) s_1 + (s_1 \times s_2 \cdot m_1) s_2).$$

These conclusions give rise to the following definitions.

Sgn 9. $\overrightarrow{s}_1 \times \overrightarrow{s}_2$ sgn: $\overrightarrow{o}$ iff 2(53), 3(1), 3(8).

Sgn 10. $\overrightarrow{s}_1 \times \overrightarrow{s}_2$ sgn: (s, m) with

(183) $$s = s_1 \times s_2,$$

(184) $$m = \frac{s_1 \times s_2 2}{(s_1 \times s_2)^2} \times ((s_2 \times s_1 \cdot m_2) s_1 + (s_1 \times s_2 \cdot m_1) s_2)$$

iff (20), 5(1).

Df 10. $\overrightarrow{s}_1 \times \overrightarrow{s}_2$ is called the *vector product* of $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$.

Pr 55. 2(53) imply: $\overrightarrow{s}_1 \times \overrightarrow{s}_2$ exists iff $\overrightarrow{s}_1 + \overrightarrow{s}_2$ exists.

Dm. Sgn 9, Sgn 10, Pr 7.

Pr 56. (130) imply: $\overrightarrow{\sigma}_1 \times \overrightarrow{\sigma}_2$ exists.

Dm. Pr 55, Pr 44.

Pr 57. (31) imply

(185) $$\vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1,$$

provided the vector product exist.

Dm. Two cases are possible:

(186) $$\vec{s}_\nu = \vec{o} \qquad (1 \leqq \nu \leqq 3)$$

or

(187) $$\vec{s}_\nu \neq \vec{o} \qquad (\nu = 1, 2, 3).$$

If (186), then both sides of (185) are zeroes (2 Sgn 2, Sgn 9, 3 Pr 5, 3 Pr 26, Sgn 7).

If (187), then the following subcases are possible:

(188) $$\vec{s}_1 \mid \vec{s}_2$$

or

(189) $$\vec{s}_2 \mid \vec{s}_3$$

or

(190) $$\vec{s}_\nu \wedge \vec{s}_{\nu+1} \qquad (\nu = 1, 2).$$

Let (111) hold.

If (188), then $s_1 \times s_2 = o$ (3 Sgn 1) and (187) imply $s_1 = \lambda s_2$ $(\lambda \in S)$. Besides, the left-hand side of (185) is zero because of Sgn 9, Pr 54. Now if (189), then the right-hand side of (185) is zero too by the same reasons, hence (185) holds. If $\vec{s}_2 \nmid \vec{s}_3$, then $\vec{s}_2 \wedge \vec{s}_3$ by the assumption that $\vec{s}_2 \times \vec{s}_3$ exists (Sgn 9, Sgn 10), and Sgn 10 implies $\vec{s}_2 \times \vec{s}_3 = (s_2 \times s_3, m)$ with an appropriate $m \in V_S$. Hence $\vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = s_2 \times s_3 \cdot (\lambda s_2) = 0$ (Sgn 7, 1 Pr 7), i.e. (185) holds again.

In the same way it is proved that (185) holds in the subcase (189) of (187).

Let now (190) hold. Then Sgn 10 implies

(191) $$\vec{s}_1 \times \vec{s}_2 = (s_1 \times s_2, p), \qquad \vec{s}_2 \times \vec{s}_3 = (s_2 \times s_3, q)$$

with appropriate $p, q \in V_S$. Now (191) and Sgn 7 imply

(192) $$\vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = s_1 \times s_2 \cdot s_3, \qquad \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = s_2 \times s_3 \cdot s_1$$

and the validity of (185) is a direct corollary of (192) and 1 Ax 13S.

Pr 58. (111) — (113), 5(7), $s \in V_S$,

(193) $$\vec{s} = (s, r \times s)$$

imply

(194) $$\vec{s} \in L(\vec{s}_\nu)_{\nu=1}^3$$

Dm. (111) — (113), 5(7) imply (116) and $s \in V_S$, (112) imply

$$(195) \qquad s = \sum_{\nu=1}^{3} (s s_\nu^{-1}) s_\nu$$

(1 Pr 30). Now (195) implies

$$(196) \qquad r \times s = \sum_{\nu=1}^{3} (s_\nu^{-1} s) r \times s_\nu$$

(1 Pr 15, 1 Pr 13, 1 Pr 17, 1 Ax 8S) and (196), (116) imply

$$(197) \qquad r \times s = \sum_{\nu=1}^{3} (s_\nu^{-1} s) m_\nu .$$

Then (193), (195), (197), (111), 2 Sgn 7 imply (162) with

$$(198) \qquad \lambda_\nu = s s_\nu^{-1} \qquad (\nu = 1, 2, 3),$$

whence (194) (Sgn 6).

Pr 59. (111) — (113), 5(7), (194), 2(2) imply (193).

Dm. Sgn 6 implies: there exist (118) with (162), and (162), 2(1), (111) imply (164). The first relation (164) implies (198) (1 Pr 24). Now (198), (116), and the second relation (164) imply

$$(199) \qquad m = \sum_{\nu=1}^{3} (s_\nu^{-1} s) r \times s_\nu = r \times \sum_{\nu=1}^{3} (s s_\nu^{-1}) s_\nu = r \times s$$

in view of 1 Pr 15, 1 Pr 13, 1 Pr 17, 1 Ax 8S.

Pr 60. (111) — (113), (130) imply

$$(200) \qquad \vec{\sigma}_1 \times \vec{\sigma}_2 \in L(\vec{s}_\nu)_{\nu=1}^{3}.$$

Dm. The existence of $\vec{\sigma}_1 \times \vec{\sigma}_2$ is proved in Pr 56. Let

$$(201) \qquad \vec{\sigma}_\nu = (\bar{\sigma}_\nu, n_\nu) \qquad (\nu = 1, 2).$$

Then (201) and Pr 59 imply

$$(202) \qquad n_\nu = r \times \bar{\sigma}_\nu \qquad (\nu = 1, 2),$$

r being defined by 5(7).

Two cases are now possible:

$$(203) \qquad \bar{\sigma}_1 \times \bar{\sigma}_2 = \mathfrak{o}$$

or

$$(204) \qquad \bar{\sigma}_1 \times \bar{\sigma}_2 \neq \mathfrak{o}.$$

If (203), then $\vec{\sigma}_1 \mid \vec{\sigma}_2$ (3 Sgn 1). The supposition $\vec{\sigma}_1 \uparrow\downarrow \vec{\sigma}_2$ is wrong. Indeed, together with (201) it implies

(205) $\qquad \overline{\sigma}_1 + \overline{\sigma}_2 = \mathfrak{o}, \cdot \quad \mathfrak{n}_1 + \mathfrak{n}_2 \neq \mathfrak{o}$

(3 Sgn 7). Now the first relation (205) and (202) imply $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{r} \times (\overline{\sigma}_1 + \overline{\sigma}_2) = \mathfrak{r} \times \mathfrak{o} = \mathfrak{o}$, contrary to the second relation (205). In such a manner, $\vec{\sigma}_1 \overrightarrow{\uparrow\downarrow} \vec{\sigma}_2$, whence $\vec{\sigma}_1 \times \vec{\sigma}_2 = \vec{o}$ (Sgn 9), i.e. (200) holds good in the case (203) in view of Pr 45.

If (204), then $\vec{\sigma}_1 \wedge \vec{\sigma}_2$ (5 Sgn 1), since (202) imply

(206) $\qquad \overline{\sigma}_1 \mathfrak{n}_2 + \overline{\sigma}_2 \mathfrak{n}_1 = \overline{\sigma}_1 \cdot \mathfrak{r} \times \overline{\sigma}_2 + \overline{\sigma}_2 \cdot \mathfrak{r} \times \overline{\sigma}_1 = \overline{\mathfrak{r} \times \overline{\sigma}_2 \cdot \overline{\sigma}_1} + \overline{\mathfrak{r} \times \overline{\sigma}_1 \cdot \overline{\sigma}_2}$

$$= \mathfrak{r} \cdot \overline{\sigma}_2 \times \overline{\sigma}_1 + \mathfrak{r} \cdot \overline{\sigma}_1 \times \overline{\sigma}_2 = \mathfrak{r} \cdot (\overline{\sigma}_2 \times \overline{\sigma}_1 + \overline{\sigma}_1 \times \overline{\sigma}_2) = \mathfrak{r}\mathfrak{o} = 0$$

(1 Ax 8S, 1 Pr 8, 1 Pr 13). Now (201), (204), Sgn 10 imply

(207) $\qquad \vec{\sigma}_1 \times \vec{\sigma}_2 = (\overline{\sigma}_1 \times \overline{\sigma}_2, \mathfrak{n})$

with

(208) $$\mathfrak{n} = \frac{\overline{\sigma}_1 \times \overline{\sigma}_2}{(\overline{\sigma}_1 \times \overline{\sigma}_2)^2} \times ((\overline{\sigma}_2 \times \overline{\sigma}_1 \cdot \mathfrak{n}_2)\overline{\sigma}_1 + (\overline{\sigma}_1 \times \overline{\sigma}_2 \cdot \mathfrak{n}_1)\overline{\sigma}_2) .$$

On the other hand, (201), (202) and 2 Sgn 4 imply

(209) $\qquad \mathfrak{r}\ \mathrm{Z}\ \mathrm{dir}\ \vec{\sigma}_\nu \qquad (\nu = 1, 2)$

Now $\vec{\sigma}_1 \wedge \vec{\sigma}_2$, (209), 5 Pr 31 imply

(210) $$\mathfrak{r} = \frac{1}{2} \sum_{\nu=1}^{3} \overline{\sigma}_\nu^{-1} \times \mathfrak{n}_\nu$$

provided

(211) $\qquad \overline{\sigma}_3 = \overline{\sigma}_1 \times \overline{\sigma}_2,$

(212) $\qquad \mathfrak{n}_3 = (\mathfrak{n}_1 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)\overline{\sigma}_1^{-1} + (\mathfrak{n}_2 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)\overline{\sigma}_2^{-1}.$

At that, (210) satisfies

(213) $\qquad \mathfrak{r}\ \mathrm{Z}\ \mathrm{dir}(\overline{\sigma}_3, \mathfrak{n}_3),$

i.e.

(214) $\qquad \mathfrak{r} \times \overline{\sigma}_3 = \mathfrak{n}_3.$

Now (211), (212) imply

(215) $$\mathfrak{n}_3 = \frac{(\mathfrak{n}_1 \cdot \overline{\sigma}_2 \times \overline{\sigma}_3)(\overline{\sigma}_2 \times (\overline{\sigma}_1 \times \overline{\sigma}_2)) + (\mathfrak{n}_2 \cdot \overline{\sigma}_2 \times \overline{\sigma}_1)((\overline{\sigma}_1 \times \overline{\sigma}_2) \times \overline{\sigma}_1)}{(\overline{\sigma}_1 \times \overline{\sigma})^2},$$

i.e.

(216) $$\mathfrak{n}_3 = \frac{\overline{\sigma}_1 \times \overline{\sigma}_2}{(\overline{\sigma}_1 \times \overline{\sigma}_2)^2} \times ((\overline{\sigma}_2 \times \overline{\sigma}_1 \cdot \mathfrak{n}_2)\overline{\sigma}_1 + (\overline{\sigma}_1 \times \overline{\sigma}_2 \cdot \mathfrak{n}_1)\overline{\sigma}_2) ,$$

and (208), (216) imply

(217) $\qquad n = n_3.$

Now (211), (214), (217) imply

(218) $\qquad r \times (\bar{\sigma}_1 \times \bar{\sigma}_2) = n,$

and (207), (218), Pr 58 imply (200) in the case (204).

Sch 4. The following remark may be useful in the capacity of an economizer of technical and intellectual work.

As it is immediately seen, two different definitions are given of the vector product $\vec{s}_1 \times \vec{s}_2$ (when it exists) of two arrows $\vec{s}_1$ and $\vec{s}_2$, namely Sgn 9 and Sgn 10, in accordance with the mutual disposition of $\vec{s}_1$ and $\vec{s}_2$: whether $\vec{s}_1 \mid \vec{s}_2$ or $\vec{s}_1 \wedge \vec{s}_2$. This distinction may be avoided by virtue of Pr 58 and Pr 59. Indeed, if r denotes the intersecting point of the directrices of the arrows $\vec{s}_1$ and $\vec{s}_2$ in the case $\vec{s}_1 \wedge \vec{s}_2$ (i.e. r is defined by 5(7) provided 5(8), 5(9), see 5 Pr 31), then $\vec{s}_1 \times \vec{s}_2$ may be defined by means of the relation

(219) $\qquad \vec{s}_1 \times \vec{s}_2 \text{ sgn} : (s_1 \times s_2, r \times (s_1 \times s_2)),$

as it has been shown in the proof of Pr 60. Now the same relation (219) may be used in the case $\vec{s}_1 \mid \vec{s}_2$ too, r denoting in this latter case an arbitrary vector. Indeed, if $\vec{s}_1 \mid \vec{s}_2$, then $\vec{s}_1 \times \vec{s}_2 = \vec{o}$ according to Sgn 9; the right-hand side of (219) is, however, also equal to $\vec{o}$, since $s_1 \times s_2 = o$, in view of 3Sgn 1.

To summarize $\vec{s}_1 \times \vec{s}_2$ may be defined by means of (219) with 5(7) if $\vec{s}_1 \wedge \vec{s}_2$ and with any r if $\vec{s}_1 \mid \vec{s}_2$.

Sch 5. In order to manifest the effectiveness of the "new definition" (219), let us prove, by its aid, the relation (185): we know, from the proof of Pr 57, that its direct deduction of the basis of Sgn 9 and Sgn 10 is a rather complicated one.

And so, let (111) hold. Then, according to (219),

(220) $\qquad \vec{s}_1 \times \vec{s}_2 = (s_1 \times s_2, r_1 \times (s_1 \times s_2)),$

(221) $\qquad \vec{s}_2 \times \vec{s}_3 = (s_2 \times s_3, r_2 \times (s_2 \times s_3)).$

At that, r_1 is any vector if $\vec{s}_1 \mid \vec{s}_2$ and it is defined by the conditions r_1 Z dir $\vec{s}_\nu$ ($\nu = 1, 2$) if $\vec{s}_1 \wedge \vec{s}_2$; similarly, r_2 is any vector if $\vec{s}_2 \mid \vec{s}_3$, and it is defined by the conditions r_2 Z dir $\vec{s}_\nu$ ($\nu = 2, 3$) if $\vec{s}_2 \wedge \vec{s}_3$.

Now (220), (221), (111), and Sgn 7 imply

(222) $\qquad \vec{s}_1 \times \vec{s}_2 \cdot \vec{s}_3 = (s_1 \times s_2, r_1 \times (s_1 \times s_2)) \cdot (s_3, m_3) = s_1 \times s_2 \cdot s_3,$

(223) $\qquad \vec{s}_2 \times \vec{s}_3 \cdot \vec{s}_1 = (s_2 \times s_3, r_2 \times (s_2 \times s_3)) \cdot (s_1, m_1) = s_2 \times s_3 \cdot s_1,$

and (185) is implied by (222), (223), and 1 Ax 13S.

Pr 57 has been proved directly above in order to afford an opportunity for a parallel between the two approaches. Some abridgements in the proofs of Pr 44, Pr 46, etc. are also posible, on the basis of Pr 58 and Pr 59.

Pr 61. (111) — (113), (168) imply

(224) $$(\vec{\sigma}_1 \times \vec{\sigma}_2) \times \vec{\sigma}_3 = (\vec{\sigma}_1 \vec{\sigma}_3)\vec{\sigma}_2 - (\vec{\sigma}_2 \vec{\sigma}_3)\vec{\sigma}_1.$$

Dm. Let

(225) $$\vec{\sigma}_\nu = (\bar{\sigma}_\nu, r \times \bar{\sigma}_\nu) \qquad (\nu = 1, 2, 3),$$

r being defined by 5(7) (Pr 59). Then, according to Sch 4, the relations

(226) $$\vec{\sigma}_1 \times \vec{\sigma}_2 = (\bar{\sigma}_1 \times \bar{\sigma}_2, r \times (\bar{\sigma}_1 \times \bar{\sigma}_2)),$$

(227) $$(\vec{\sigma}_1 \times \vec{\sigma}_2) \times \vec{\sigma}_3 = ((\bar{\sigma}_1 \times \bar{\sigma}_2) \times \bar{\sigma}_3, r \times ((\bar{\sigma}_1 \times \bar{\sigma}_2) \times \bar{\sigma}_3)),$$

hold good. At that, the left-hand sides of (226), (227) exist by virtue of Pr 56.

On the other hand, (225) and Sgn 7, 2 Sgn 7, 1 Pr 17, Pr 21 imply

(228)
$$\begin{aligned}(\vec{\sigma}_1 \vec{\sigma}_3)\vec{\sigma}_2 - (\vec{\sigma}_2 \vec{\sigma}_3)\vec{\sigma}_1 &= (\bar{\sigma}_1 \bar{\sigma}_3)(\bar{\sigma}_2, r \times \bar{\sigma}_2) - (\bar{\sigma}_2 \bar{\sigma}_3)(\bar{\sigma}_1, r \times \bar{\sigma}_1) \\ &= ((\bar{\sigma}_1 \bar{\sigma}_3)\bar{\sigma}_2, r \times ((\bar{\sigma}_1 \bar{\sigma}_3)\bar{\sigma}_2)) - ((\bar{\sigma}_2 \bar{\sigma}_3)\bar{\sigma}_1, r \times ((\bar{\sigma}_2 \bar{\sigma}_3)\bar{\sigma}_1)) \\ &= ((\bar{\sigma}_1 \bar{\sigma}_3)\bar{\sigma}_2 - (\bar{\sigma}_2 \bar{\sigma}_3)\bar{\sigma}_1, r \times ((\bar{\sigma}_1 \bar{\sigma}_3)\bar{\sigma}_2) - (\bar{\sigma}_2 \bar{\sigma}_3)\bar{\sigma}_1)).\end{aligned}$$

Now (224) is a direct corollary from (227), (228), and 1 Ax 14S.

Pr 62. $L(\vec{s}_\nu)_{\nu=1}^3$ is a standard vector space over S with respect to the addition in W_S, to the multiplication 2 Sgn 7 of the elements of S and W_S, to the scalar multiplication Sgn 7 of the elements of W_S, and to the vector multiplication Sgn 9, Sgn 10 in W_S.

Dm. Pr 45, Pr 46, Pr 54, Pr 56, Pr 57, Pr 61, and $\vec{s}_1 \times \vec{s}_2 \neq \vec{o}$ (Sgn 10).

Sch 6. A comparison of the present paper with the article [1] at once displays the complete analogy between the real and the complex algebras of arrows.

Further developments exposed in [1] will not be extended here for the complex case. They concern mainly the associativity of addition of arrows, as well as some facts about incidence of poles and arrows and metrical relations (distances, etc.). Some questions about incidence of arrows and planes are also omitted here, since they concern mainly systems of arrows.

In the third part of this series of articles dedicated to the algebraic theory of arrows finite systems of arrows will be discussed.

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ON THE COVERING RADIUS OF SOME BINARY CYCLIC CODES *

EVGUENIA VELIKOVA

Евгения Великова. РАДИУСЫ ПОКРЫТИЯ НЕКОТОРЫХ ДВОИЧНЫХ ЦИКЛИЧНЫХ КОДОВ. Рассматриваются двоичные цикличные коды длины $n = uv$, которые получаются следующим разбиением многочлена $x^{uv} - 1 = (x-1)\left(\frac{x^u-1}{x-1}\right) \times \left(\frac{x^v-1}{x-1}\right) f_1(x)$, где u и v — нечетные взаимно простые числа. Для этих кодов определены радиусы покрытия (для двух из них — только верхняя и нижняя границы радиуса).

Evguenia Velikova. ON THE COVERING RADIUS OF SOME BINARY CYCLIC CODES. The binary cyclic codes of length $u = uv$ obtained by the factorization $x^{uv} - 1 = (x-1) \times \left(\frac{x^u-1}{x-1}\right)\left(\frac{x^v-1}{x-1}\right) f_1(x)$, where u and v are odd relatively prime numbers are considered. For these codes we find the covering radius (for two codes only upper and lower bound on covering radius).

In this paper we study the problem of finding the covering radius of some binary cyclic codes. Let C be an $[n, k]$ binary linear code and with F denote the field with two elements $F = GF(2)$. The covering radius $R = R(C)$ of C is the smallest integer, such that any vector of the space F^n is within Hamming distance R from some code word. The covering radius of cyclic codes of length up to 31 are given in[1] and the covering radius of cyclic codes of length 33, 35 and 39, without 3 codes, are given in [2]. The codes, considered in this paper, are some binary cyclic codes of length which is a product of two relatively prime odd numbers.

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Let u and v be odd integers, such that $gcd(u, v) = 1$ and $u < v$. We consider the binary cyclic codes of length $n = uv$, obtained by the following factorization:

$$x^{uv} + 1 = f_0(x) \cdot f_1(x) \cdot f_u(x) \cdot f_v(x),$$

where

$$\begin{aligned} f_0(x) &= x + 1, \\ f_u(x) &= x^{u-1} + \cdots + x + 1 = \frac{x^u + 1}{x + 1}, \\ f_v(x) &= x^{v-1} + \cdots + x + 1 = \frac{x^v + 1}{x + 1}, \\ f_1(x) &= \frac{x^{uv} + 1}{f_0(x) \cdot f_u(x) \cdot f_v(x)}. \end{aligned}$$

Table 1

Code	Generator polynomial
C_1	$g_1(x)=f_1(x)f_u(x)$
C_2	$g_2(x)\equiv f_0(x)f_1(x)f_u(x)$
C_3	$g_3(x)\equiv f_1(x)f_v(x)$
C_4	$g_4(x)\equiv f_0(x)f_1(x)f_v(x)$
C_5	$g_5(x)\equiv f_0(x)f_v(x)$
C_6	$g_6(x)\equiv f_v(x)$
C_7	$g_7(x)\equiv f_0(x)f_u(x)$
C_8	$g_8(x)\equiv f_u(x)$
C_9	$g_9(x)\equiv f_1(x)$
C_{10}	$g_{10}(x)\equiv f_0(x)f_1(x)$
C_{11}	$g_{11}(x)\equiv f_0(x)f_u(x)f_v(x)$
C_{12}	$g_{12}(x)\equiv f_u(x)f_v(x)$

In other words we consider only the codes which generator polynomial is a product of some $f_0(x)$, $f_1(x)$, $f_u(x)$, $f_v(x)$. For each two numbers u and v there are 12 codes of this kind and let the codes be C_i and $g_i(x)$ be the generator polynomial of code C_i, as they are given in the Table 1 and R_i be the covering radius of code C_i.

Some of these codes have a parity polynomial which divides $x^s + 1$ ($s = u$ or $s = v$) and these codes are composed of some repetitions of (in our case) F^s or E_s (E_s is the $[s, s-1, 2]$ even weight code). For such codes we can calculate their covering radius using [3]. In this way we obtain the covering radius of the following codes:

— $g_1(x) = f_1(x)f_u(x)$ and C_1 is $[uv, v, u]$ $R_1 = \frac{v(u-1)}{2}$ code and C_1 is u repetitions of F^v,

— $g_2(x) = f_0(x)f_1(x)f_u(x)$ and C_2 is $[uv, v-1, 2u]$ $R_2 = 1 + \frac{v(u-1)}{2}$ code and C_2 is u repetitions of E_v,

— $g_3(x) = f_1(x)f_v(x)$ and C_3 is $[uv, u, v]$ $R_3 = \frac{u(v-1)}{2}$ code and C_3 is v repetitions of F^u,

— $g_4(x) = f_0(x)f_1(x)f_v(x)$ and C_4 is $[uv, u-1, 2v]$ $R_4 = 1 + \frac{u(v-1)}{2}$ code and C_4 is v repetitions of E_v.

We will use the following obvious proposition:

P r o p o s i t i o n 1. Let C be an $[n, k]$ binary linear code. If $H = (h_1 \dots h_n)$ is a parity check matrix of C, then $R(C)$ is the smallest integer, such that every nonzero vector of F^{n-k} is a sum of at most R columns of the matrix H, i.e.

$$x = h_{i_1} + \cdots + h_{i_t}, \qquad t \leqq R, \qquad \forall x \in F^{n-k} \setminus \{0\}.$$

If H has repeated columns, then only one of this columns can be taken for the calculation. Therefore, using this proposition we can calculate the covering radius of the codes dual to $C_1, \dots, C_4$ and namely the covering radius of the following codes:

— $g_5(x) = f_1(x)f_v(x)$ and C_5 is $[uv, uv - v, 2]$ $R_5 = v$ code and C_5 is the dual to C_1,

— $g_6(x) = f_v(x)$ and C_6 is $[uv, uv - v + 1, 2]$ $R_6 = \frac{v-1}{2}$ code and C_6 is the dual to C_2,

— $g_7(x) = f_0(x)f_u(x)$ and C_7 is $[uv, uv - u, 2]$ $R_7 = u$ code and C_7 is the dual to C_3,

— $g_8(x) = f_u(x)$ and C_8 is $[uv, uv - u + 1, 2]$ $R_8 = \frac{u-1}{2}$ code and C_8 is the dual to C_4.

The minimum distance of each of the rest four codes is given by the following theorem:

T h e o r e m 1. i) If $g_9(x) = f_1(x)$ then C_9 is a $[uv, u + v - 1, u]$ code and C_9 has a weight enumerator

$$A(z) = \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{2i} \left(z^{2i} + z^{u-2i}\right)^v,$$

ii) If $g_{10}(x) = f_0(x)f_1(x)$, then C_{10} is the $[uv, , u + v - 2, 2u]$ code,
iii) If $g_{11}(x) = f_0(x)f_u(x)f_v(x)$, then C_{11} is the $[uv, , uv - u - v + 1, 4]$ code,
iv) If $g_{12}(x) = f_u(x)f_v(x)$, then C_{12} is the $[uv, , uv - u - v + 2, 4]$ code.

Proof. i) The code C_9 contains the code C_1 which is u times repeated F^v, as well as the code C_3 which is v times repeated F^u. C_3 is generated by the words x_i, $i = 1, \dots, u$ with support $X_i = \{i, i + u, \dots, i + (v-1)u\}$ and C_1 is generated by words y_j, $j = 1, \dots, v$ with support $Y_j = \{j, j + v, \dots, j + (u-1)v\}$. We can arrange the coordinates $\{1, 2, \dots, n\}$ in a $u \times v$ matrix

$$\begin{pmatrix} i_{11} & \dots & i_{1v} \\ \dots\dots\dots\dots & & \\ i_{u1} & \dots & i_{uv} \end{pmatrix},$$

such that $i_{st} \equiv i_{s\ell} (\mathrm{mod} u)$ and $i_{st} \equiv i_{\ell t} (\mathrm{mod} v)$. Then the words x_i, $i = 1, \dots, u$

and y_j, $j = 1, \ldots, v$ are presented as

$$(1)\ x_i = \begin{pmatrix} 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 1 & \cdots & 1 \\ \ldots & \ldots & \ldots \\ 0 & \cdots & 0 \end{pmatrix} \leftarrow i\text{-th row}, \qquad y_j = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}.$$

j-th column

The word, which is a sum of t words of x_i, $i = 1, \ldots, u$ and s words of y_j, $j = 1, \ldots, v$ has a weight $t(v-s) + s(u-t)$ and in this way we obtain

$$A(z) = \sum_{c \in C_9} z^{wt(c)} = \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{2i} (z^{2i} + z^{u-2i})^v,$$

and the minimum distance of the C_9 is equal to u.

ii) The code C_{10} is the even weight subcode of C_9 and the minimum even distance in C_9 is $2u$.

iii) The generator polynomial of code C_{11} is $g_{11}(x) = \dfrac{(x^u+1)(x^v+1)}{x+1}$ and the word $(x+1)g_{11}(x) = x^{u+v} + x^v + x^u + 1$ belongs to the code C_{11}. Therefore $d_{11} \leqq 4$ but C_{11} is the even weight code and its dual code C_9 is not a repetition code, hence $d_{11} = 4$. We can compute the weight enumerator of C_{11} using the MacWilliams equations (see [4, p. 127]). The weight enumerator of C_{11} is

$$B(z) = \sum_{c \in C_{11}} z^{wt(c)}$$

$$= 2^{1-u-v} \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{2i} ((1-z)^{u-2i}(1+z)^{2i} + (1-z)^{2i}(1+z)^{u-2i})^v,$$

iv) The code C_{12} is a self-complementary and its even weight subcode is C_{11}. Hence the minimum distance of C_{12} is equal to $\min\{4, uv - t\}$,

where t is the maximal weight of C_{11}. But the weight enumerator $B(z)$ of code C_{11} has degree less than $uv - 3$. Hence $d_{12} = 4$.

The bounds on the covering radius of the codes C_9 and C_{10} are given by the following theorem:

T h e o r e m 2. i) Let $v_1 = v - \left\lfloor \frac{v}{2^{u-1}} \right\rfloor \cdot 2^{u-1}$ (v_1 is the residual of $v \bmod 2^{u-1}$) and $v_1 = \sum_{i=0}^{r} \binom{u}{i} + t$, where $0 \leqq t \leqq \binom{u}{r+1}$. Then the code C_9 with a generator polynomial $g_9(x) = f_1(x)$ has a covering radius R_9, where

$$\left\lfloor \frac{v}{2^{u-1}} \right\rfloor \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{i} i + \sum_{i=0}^{r} \binom{u}{i} i + t(r+1) \leqq R_9 \leqq \left\lfloor \frac{v}{2^{u-1}} \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{i} i \right\rfloor.$$

ii) The code C_{10} with a generator polynomial $g_{10}(x) = f_0(x)f_1(x)$ has a covering radius R_{10} where $R_9 + 1 \leqq R_{10} \leqq R_9 + u - 2$.

Proof. i) If we arrange the coordinates in a $u \times v$ matrix as in the proof of Theorem 1 i), then the words x_i, $i = 1, \ldots, u$ and y_j, $j = 1, \ldots, v$ from (1) generate the code C_9. Let I_j, $j = 1, \ldots, v$ be the block of coordinate places on the j-th column in that matrix. Then the word with support I_j belongs to C_9. From each block I_j we take away the element, which is on the last row and from the parity check matrix H of the code take away the columns with this numbers. In this way we obtain new blocks $\widetilde{I}_j$ and matrix H. The code $\widetilde{C}$ with the parity check matrix $\widetilde{H}$ is $[uv - v, u - 1]$ code and it is generated by the words

$$\left.\begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}\right\} u - 1$$

Then we can apply the upper bound on the covering radius of self-complementary code from [3] and obtain that $R_9 \leqq \left\lfloor \frac{v}{2^{u-1}} \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{i} i \right\rfloor$.

The lower bound on R_9 is constructive. Let us consider the word $a \in F^{uv}$. We take $\left\lfloor \frac{v}{2^{u-1}} \right\rfloor$ copies of each column of length u and weight no exceeding $\frac{u-1}{2}$ and the other $v_1 = v - \left\lfloor \frac{v}{2^{u-1}} \right\rfloor \cdot 2^{u-1}$ columns are distinct and have the minimum possible weight. The word a has a weight $w = \left\lfloor \frac{v}{2^{u-1}} \right\rfloor \sum_{i=0}^{\frac{1}{2}(u-1)} \binom{u}{i} i + \sum_{i=0}^{r} \binom{u}{i} i + t(r+1)$ and it is a leader of the coset $a + C_9$. Hence $R_9 \geqq w$.

ii) The code C_{10} is the even weight subcode of C_9 and then $R_{10} \geqq R_9 + 1$ (see [5]). Let x be a leader of coset of C_9 and

$$x = (x^1, \ldots, x^v) = \begin{pmatrix} x_{11} & \cdots & x_{1v} \\ \cdots & \cdots & \cdots \\ x_{u1} & \cdots & x_{uv} \end{pmatrix}$$

and let $x^i \neq 0$. The the words $(x^1, \ldots, x^i, \ldots, x^v)$ and $(x^1, \ldots, \overline{x}^i, \ldots, x^v)$ belong to the distinct cosets to C_{10} and they have weight $w = wt(x)$ and $w_1 \leqq w - 1 + u - 1 = w + u - 2$. Therefore, $R_{10} \leqq R_9 + u - 2$.

C o r o l l a r y 1. When $u = 3$ then $R_9 = \left\lfloor \frac{3v}{4} \right\rfloor$ and $R_{10} = 1 + \left\lfloor \frac{3v}{4} \right\rfloor$.

This corollary follows from the fact that the upper and lower bounds on R_9 and R_{10} from Theorem 2 are equal.

T h e o r e m 3. The code C_{11} with generator polynomial $g_{11}(x) = f_0(x)f_u(x)f_v(x)$ has a covering radius $R_{11} = v$.

Proof. The coefficients of the generator polynomial of this code are g_{11}: $\underbrace{11\ldots1}_{u}\underbrace{0\ldots0}_{v-u}\underbrace{1\ldots1}_{u}$ and then $R_{11} \leqq R' = v$, where R' is the covering radius of $[u + v, 1]$ code generated by the vector g_{11} (see [3]) a C_{11} is a subcode of code C_5

with $R_5 = v$. Then from the Supercode Lemma [5] follows that $R_{11} \geqq v$. Hence $R_{11} = v$.

The covering radius of the code C_{12} is obtained using the Proposition 1, as it is given in the following theorem:

T h e o r e m 4. The code C_{12} with a length $n = uv$, $u < v$ generator polynomial $g_{12}(x) = f_u(x)f_v(x)$ has a covering radius

$$R_{12} = \begin{cases} \frac{1}{2}(v-1) & \text{if} \quad u \leqq \frac{1}{2}(v-1), \\ u & \text{if} \quad \frac{1}{2}(v-1) < u < v. \end{cases}$$

Proof. This code is dual to code C_{10}. The code C_{10} contains the code C_2 which is u repetitions of E_v and code C_4 which is v repetitions of E_u. Therefore a parity check matrix of code C_{12} is equivalent to

$$H = \left(\begin{array}{c|c|c|c|c} 1\cdots1 & 0\cdots0 & & 0\cdots0 & 1\cdots1 \\ 0\cdots0 & 1\cdots1 & & \vdots & 1\cdots1 \\ \vdots & 0\cdots0 & & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & & 0\cdots0 & \vdots \\ 0\cdots0 & 0\cdots0 & & 1\cdots1 & 1\cdots1 \\ \hline L & L & \cdots & L & L \end{array}\right) = \left(\frac{h'_1 h'_2 \cdots h'_n}{h''_1 h''_2 \cdots h''_n}\right) = \left(\frac{H_1}{H_2}\right)$$

where L is the $(u-1) \times u$-matrix

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be an arbitrary syndrom of the code C_{12} and $y_1 \in F^{v-1}$, $y_2 \in F^{u-1}$ Let t_1 and t_2 be the smallest numbers such that y_i, $i = 1, 2$ is the sum of t_i columns of H_i. Then y_i can be represented as a sum of $t_i + 2m$ columns of H_i if we add m pairs of equal columns of H_i. There are two cases:

a) $t_1 \equiv t_2 (\mathrm{mod}2)$ and let $t = \max\{t_1, t_2\}$. Then y_1 and y_2 can be represented as a sum of t columns of H_1 or H_2 respectively: $y_1 = h'_{i_1} + \cdots + h'_{i_t}$ and $y_2 = h''_{j_1} + \cdots + h''_{j_t}$ and we obtain that:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h'_{i_1} \\ h''_{j_1} \end{pmatrix} + \cdots + \begin{pmatrix} h'_{i_t} \\ h''_{j_t} \end{pmatrix},$$

and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is a sum of t columns of H and $t \leqq \frac{v-1}{2}$;

b) $t_1 \not\equiv t_2 (\mathrm{mod}2)$. Then $t'_2 = u - t_2$ and if $y_2 = h''_{j_1} + \cdots + h''_{j_{t_2}}$, then y_2 is equal to the sum of the other t'_2 columns of L, because $\sum_{i=1}^{u} h''_i = 0$. Therefore $t''_2 \equiv t_1 (\mathrm{mod}2)$

and (as in the case a)) we can obtain that $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is the sum of $t = \max\{t_1, t_2'\}$ columns of H. We have $t_1 \leqq \frac{v-1}{2}$ and $t_2' \leqq u$, hence $t \leqq \max\{\frac{v-1}{2}, u\}$.

In this way we obtain that $R_{12} \leqq \max\{\frac{v-1}{2}, u\}$. On the other hand, $R_{12} \geqq \frac{v-1}{2}$ because the covering radius of the code with parity check matrix H_1 is equal to $\frac{v-1}{2}$ and $R_{12} \geqq u$ because the vector $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ with $y_1 = (1, 0, \ldots, 0)^T$ and $y_2 = (0, \ldots, 0)^T$ is the sum of u and no less than u columns of H. Therefore, we obtain that $R_{12} = \max\{\frac{v-1}{2}, u\}$ which is equal to

$$R_{12} = \begin{cases} \frac{1}{2}(v-1) & \text{if} \quad u \leqq \frac{1}{2}(v-1), \\ u & \text{if} \quad \frac{1}{2}(v-1) < u < v. \end{cases}$$

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INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, III

IVAN CHOBANOV

> ... mechanica rationalis erit scientia motuum, qui ex viribus quibuscunque resultant, et virium quae ad motus quoscunque requiruntur, accurate proposita ac demonstrata
>
> Isaac Newton: *Philosophiae Naturalis Principia Mathematica* (Auctoris praefacio ad lectorem)

Иван Чобанов. ВВЕДЕНИЕ В АЛГЕБРАИЧЕСКУЮ ТЕОРИЮ СКОЛЬЗЯЩИХ ВЕКТОРОВ, III. Эта работа является третьей частью серии статьей, посвещенных алгебраической теории скользящих векторов, первые две части [1, 2] которой опубликованы в этом *Ежегоднике*. Она занимается главным образом конечными системами скользящих векторов в комплексных стандартных векторных пространствах, в частности, или в стандартных векторных пространствах над комплексными расширениями произвольных упорядоченных полях, в общем. Определения этих пространств, а также некоторые основные моменты их алгебр, приведены в вводнуя часть статьи [2]; для более подробного ознакомления в этой связи читателю рекоммендуется статья [3] или книга [4]. Замечательно, что не наблюдаются никакие существенные расхождения между реальными и комплексными случаями алгебр скользящих векторов. Теорема о ранге играет ту же самую центральную роль в теории комплексных скользящих векторов, как и в теории реальных.

Ivan Chobanov. INTRODUCTION TO AN ALGEBRAIC THEORY OF ARROWS, III. This article is the third part of a series of investigations on an algebraic theory of arrows or sliding vectors, the first two parts [1, 2] of which are published in this *Annual*. It is dedicated mainly to the finite systems of arrows in complex standard vector spaces or, more generally, in standard vector spaces over the complex extensions of arbitrary ordered fields. The definitions of these spaces, as well as some basic moments of their algebras, are given in the introductory part of the article [2]; for a more detailed exposition in this connection the reader is referred to the paper [3] or to the booklet [4]. It is remarkable that no essential divergences are observed between

the real and the complex cases of arrow algebras. The rank-theorem plays the same central role in the theory of complex arrows as in that of the real ones.

This article is the third part of a series of investigations on an algebraic theory of arrows or sliding vectors (*vecteurs glissants, gleitende Vektoren, скользящие векторы*), the first two parts [1, 2] of which are published in this *Annual*. The definitions and notations from [2] are systematically used in this third part, the following manner of quotation being adopted (the example is a ficticious one): notation 1, axiom 2, definition 3, proposition 4, scholium 5, and relation (6) of §7 of [2], for instance, are cited here by [2, 7 Sgn 1], [2, 7 Ax 2], [2, 7 Df 3], [2, 7 Pr 4], [2, 7 Sch 5], and [2, 7(6)] respectively. In general, the exposition of the present paper is an immediate continuation of the exposition of the second part [2] of the series in question.

This article is dedicated mainly to the theory of finite systems of arrows in real and complex standard vector spaces, in particular, and of standard vector spaces over ordered fields and over the complex extensions of such fields, in general. As it has been emphasized in [2], the definitions and the basic algebraic properties of the standard vector space V_F over the ordered field F and of the standard vector space $V_{C(F)}$ over the complex extension $C(F)$ of F may be found in the article [3], as well as in the booklet [4]. In any case, a brief account in this connection is given in the introductory paragraph (§1. Praeliminaria) of [2], where the notation V_S is used in order to reduce the real case $S = F$, as well as the complex one $S = C(F)$, to a common denominator.

In order to simplify references and to ease the exposition, some most fundamental notations from [2] are reproduced over again here. Such cases are, by the way, extremely rare: as a matter of fact, they are exceptions made in case of emergency.

Since the main, if not exclusive, domain of applications of arrows is proposed by analytical mechanics, kinematics as well as dynamics, and since the analytical mechanics deals predominantly, if not exceptionally, with finite systems of arrows, our interest will be focused on such systems in the main. In other words, we shall consider sets consisting of a finite number of arrows. Since the properties of such sets depend exclusively on the properties of the single arrows, it is quite natural to fix our attention on the latter. This has been done in [2], where the following basic definition has been adopted.

Sgn 1. W_S **sgn:** $\{(s, m) \in V_S^2 : s \neq \mathfrak{o},\ sm = 0 \text{ or } s = m = \mathfrak{o}\}$.

Df 1. The elements of W_S are called *S-arrows* or *arrows in* V_S.

Df 2. Any set

$$\underset{\rightarrow}{s} = \left\{\overrightarrow{s}_\nu\right\}_{\nu=1}^{n} \tag{1}$$

of arrows

$$\overrightarrow{s}_\nu \in W_S \qquad (\nu = 1, \ldots, n) \tag{2}$$

is called a (*finite*) *system of S-arrows* or *of arrows in* V_S.

Sch 1. For the sake of brevity (1) is usually called a *system of arrows*, S and V_S being implied by the context.

Sgn 2. Σ_S sgn: the set of all finite systems of S-arrows.

Df 3. The vector

$$s = \sum_{\nu=1}^{n} s_\nu \tag{3}$$

provided (1) and

$$\overrightarrow{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, \ldots, n) \tag{4}$$

is called the basis of $\underrightarrow{s}$.

Df 4. The vector

$$m = \sum_{\nu=1}^{n} m_\nu \tag{5}$$

provided (1), (4), is called the moment of $\underrightarrow{s}$.

Sch 2. In the light of Df 3, Df 4, as well as of [2, 2 Df 2, 2 Df 3], it is clear, that the basis and the moment of a system of arrows are the sums of the bases and of the moments respectively of the particular arrows composing the system in question.

Sch 3. Whereas there is a certain *dependence* between the basis s and the moment m of a single arrow $\overrightarrow{s}$, as it is seen from Sgn 1, there is no compulsory relation between the basis (3) and the moment (5) of a system of arrows (1). In other words, the vectors s and m, defined by (3) and (5) respectively, may be *completely arbitrary*. It may be proved that for any couple (s, m) of S-vectors s and m there exists one at least (infinitely many, as a matter of fact) systems (1) of arrows (4) for which the equalities (3) and (5) respectively take place. In particular, there exist systems (1) of arrows (4), for which (3) and (5) imply $sm = 0$, as in the case of a single arrow, but there exist also systems for which sm is equal to any value in S.

Sch 4. Similarly, whereas the basis s and the moment m of a single arrow $\overrightarrow{s}$ determine it completely by virtue of the definition

$$\overrightarrow{s} = (s, m), \tag{6}$$

the basis s and the moment m of a system of arrows $\underrightarrow{s}$ by no means determine $\underrightarrow{s}$ unanimously: as it has been mentioned above, *there are infinitely many systems of arrows with the same basis and the same moment.* In such sense, the notation

$$\underrightarrow{s} = (s, m), \tag{7}$$

similar to (6), is void of meaning. It is convenient, however, to introduce the following notation.

Sgn 3. $\underrightarrow{s}(s, m)$ sgn: s and m are the basis and the moment respectively of the system of arrows $\underrightarrow{s}$.

Sch 5. As it has been manifested in [2], an extremely important attribute of an arrow $\overrightarrow{s}$ is its r-moment, denoted by $\mathrm{mom}_r \overrightarrow{s}$ and defined in the following manner. If

$$r \in V_S, \tag{8}$$

$$\overrightarrow{s} = (s, m) \in W_S, \tag{9}$$

then by definition

$$\mathrm{mom}_r \overrightarrow{s} = m + s \times r. \tag{10}$$

In the same way, the notion of r-moment of a system of arrows $\underset{\rightarrow}{s}$, denoted by $\mathrm{mom}_r \underset{\rightarrow}{s}$, plays an extremely important role in the theory of the systems of arrows. It is introduced by the aid of the following definition, which is complete imitation of the definition (10).

Sgn 4. $\mathrm{mom}_r \underset{\rightarrow}{s}$ sgn: $m + s \times r$ iff (8) and

$$\underset{\rightarrow}{s}(s, m) \in \Sigma_S \tag{11}$$

Df 5. $\mathrm{mom}_r \underset{\rightarrow}{s}$ is called the *r-moment* of $\underset{\rightarrow}{s}$ or the *moment of* $\underset{\rightarrow}{s}$ *with respect to* r.

Df 6. r is called the *pole* of $\mathrm{mom}_r \underset{\rightarrow}{s}$.

Pr 1. (4), (1), (8) imply

$$\mathrm{mom}_r \underset{\rightarrow}{s} = \sum_{\nu=1}^{n} \mathrm{mom}_r \overrightarrow{s}_\nu. \tag{12}$$

Dm. By definition

$$\mathrm{mom}_r \overrightarrow{s}_\nu = m_\nu + s_\nu \times r \qquad (\nu = 1, \ldots, n). \tag{13}$$

Now (13), (3), (5), Sgn 4 imply

$$\sum_{\nu=1}^{n} \mathrm{mom}_r \overrightarrow{s}_\nu = \sum_{\nu=1}^{n} m_\nu + \sum_{\nu=1}^{n} s_\nu \times r = \dot{m} + s \times r = \mathrm{mom}_r \underset{\rightarrow}{s}. \tag{14}$$

Pr 2. (11) implies $m = \mathrm{mom}_o \underset{\rightarrow}{s}$.

Dm. Sgn 4.

Pr 3. (11) implies: $\mathrm{mom}_r \underset{\rightarrow}{s}$ is invariant with respect to r iff

$$s = o. \tag{15}$$

Dm. If (15), then Sgn 4 implies $\mathrm{mom}_r \underset{\rightarrow}{s} = m$ for any (8).

Let $\mathrm{mom}_r \underset{\rightarrow}{s}$ be invariant with respect to the poles (8), i.e. let

$$\mathrm{mom}_r \underset{\rightarrow}{s} = n \tag{16}$$

for any (8). Sgn 4 and (16) imply

$$m + s \times r = n \qquad (r \in V_S). \tag{17}$$

Let

$$r_\nu \in V_S \qquad (\nu = 1, 2, 3), \tag{18}$$

$$r_1 \times r_2 + r_2 \times r_3 + r_3 \times r_1 \neq o. \tag{19}$$

Then (17), (18) imply

$$m + s \times r_\nu = n \qquad (\nu = 1, 2, 3) \tag{20}$$

whence

$$s \times (r_\nu - r_3) = o \qquad (\nu = 1, 2). \tag{21}$$

The relation (19) is equivalent with

$$(r_1 - r_3) \times (r_2 - r_3) \neq o. \tag{22}$$

The system of vector equations (21) with (22), where s is unknown, has, according to [2, 1 Pr 29], the only solution (15), *q. e. d.*

Pr 4. (8), (11) imply

$$s \cdot \mathrm{mom}_r \underrightarrow{s} = sm. \tag{23}$$

Dm. Sgn 4.

Sch 6. The relation (23) displays that the scalar product $s \cdot \mathrm{mom}_r \underrightarrow{s}$ is *invariant* with respect to the pole r of $\mathrm{mom}_r \underrightarrow{s}$. This important fact is an analogue of [2, 2 Pr 5] and deserves a special attention.

Sgn 5. I($\underrightarrow{s}$) sgn: sm iff (11).

Df 7. I($\underrightarrow{s}$) is called the *first scalar invariant* of $\underrightarrow{s}$.

Sgn 6. II($\underrightarrow{s}$) sgn: $\frac{sm}{s^2}$ iff (11) and

$$s \neq o. \tag{24}$$

Df 8. II($\underrightarrow{s}$) is called the *second scalar invariant* of $\underrightarrow{s}$.

Sch 7. The analogue of the notion of *directrix* of a single arrow [2, 2 Sgn 4] is played, in the theory of systems of arrows, by the so-called *axis* of a system of arrows. This notion is generated by the following considerations.

Let a system of arrows (11) be given and let the following question be put: do there exist poles (8) for which $\mathrm{mom}_r \underrightarrow{s}$ are parallel to s? (Naturally, this problem is meaningless unless (24) is satisfied.) The affirmative answer of this question is equivalent with the assumption that the vector equation with respect to r:

$$s \times \mathrm{mom}_r \underrightarrow{s} = o \tag{25}$$

is consistent provided (11), (24). By virtue of Sgn 4, (25) is equivalent with

(26) $$s \times (m + s \times r) = o,$$

i.e. with

(27) $$(r \times s) \times s = m \times s$$

in view of [2, 1 Pr 13]. Now (27), (24), and [2, 1 Pr 32] imply: there exists $\alpha \in S$ with

(28) $$r \times s = \alpha s + \frac{s \times (m \times s)}{s^2}.$$

A scalar multiplication of (28) by s implies

(29) $$\alpha s^2 = 0,$$

and (29), (24), [2, 1 Ax 12S] imply $\alpha = 0$; therefore (28) implies

(30) $$r \times s = \frac{s \times (m \times s)}{s^2}$$

In such a way, if (8) satisfies (25) or, just the same, (26), then it must be sought among the solutions r of the vector equation (30). We shall establish now that any r satisfying (30) satisfies (25) too: any solution (8) of (30) is, at the same time, a solution of our initial problem (25).

Indeed, let (8) satisfy (30). Since [2, 1 Pr 16] implies

(31) $$s \times (m \times s) = s^2 m - (ms)s,$$

the relation (30) is equivalent with

(32) $$m + s \times r = \left(\frac{ms}{s^2}\right) s.$$

Now (32) and Sgn 4 imply

(33) $$\mathrm{mom}_r \underset{\rightarrow}{s} = \left(\frac{ms}{s^2}\right) s,$$

and (33) implies (25).

On the other hand, (30) is the equation of a line l, if r is regarded as a fluent radius-vector. Consequently, all poles (8), for which the moments of the system of arrows $\underset{\rightarrow}{s}$ are parallel to the basis s of $\underset{\rightarrow}{s}$, are located on the line (30). This line is called the *axis* of $\underset{\rightarrow}{s}$.

Let us now formalize this conclusion.

Sgn 7. ax $\underset{\rightarrow}{s}$ sgn: l with

(34) $$\left(s, \frac{s \times (m \times s)}{s^2}\right) \;\&\; l,$$

iff (11), (24).

Sch 8. The meaning of Sgn 7 is disclosed by the aid of Sgn 4 of §1 (p.86) of the article [5].

Df 9. ax $\underset{\rightarrow}{s}$ is called the *axis* of $\underset{\rightarrow}{s}$.

Pr 5. (11), (24), (8),

$$\mathfrak{r}\ \mathrm{Z}\ \mathrm{ax}\ \underset{\rightarrow}{s} \tag{35}$$

imply (33).

Dm. Sgn 7 and Sgn 1 of §4 (p. 117) of [5] imply that (35) is equivalent with (30), which on its part is equivalent with (32). Now (32) and Sgn 4 imply (33).

Sch 9. The following proposition discloses why ax $\underset{\rightarrow}{s}$ is an analogue of dir $\overrightarrow{s}$, as it has been mentioned in Sch 7.

Pr 6. If

$$\overrightarrow{o} \neq \overrightarrow{s} \in W_S, \tag{36}$$

$$\underset{\rightarrow}{s} = \left\{\overrightarrow{s}\right\}, \tag{37}$$

then

$$\mathrm{ax}\ \underset{\rightarrow}{s} = \mathrm{dir}\ \overrightarrow{s}. \tag{38}$$

Dm. (36), (6) imply

$$\mathfrak{s} \neq \mathfrak{o}, \qquad \mathfrak{sm} = 0. \tag{39}$$

On the other hand, (37) and Df 3, Df 4 imply (11). Now (11), (39), Sgn 7 imply

$$(\mathfrak{s}, \mathfrak{m})\ \&\ \mathrm{ax}\ \underset{\rightarrow}{s}. \tag{40}$$

On the other hand, [2, 2 Sgn 4] imply

$$(\mathfrak{s}, \mathfrak{m})\ \&\ \mathrm{dir}\ \overrightarrow{s}, \tag{41}$$

and (40), (41) imply (38) by virtue of Pr 18 of §1 (p. 86) of [5].

Sch 10. The following proposition displays that the moments of a system of arrows have a minimal property for the poles located on the axis of the system.

Pr 7. (8), (11), (24), (35),

$$\overline{\mathfrak{p}} \in V_S, \tag{42}$$

$$\overline{\mathfrak{p}}\ \overline{\mathrm{Z}}\ \mathrm{ax}\ \underset{\rightarrow}{s} \tag{43}$$

imply

$$\left(\mathrm{mom}_{\mathfrak{r}}\ \underset{\rightarrow}{s}\right)^2 < \left(\mathrm{mom}_{\overline{\mathfrak{p}}}\ \underset{\rightarrow}{s}\right)^2. \tag{44}$$

Dm. Sgn 7 implies that (35) and (43) are equivalent with (30) and

$$\overline{p} \times s \neq \frac{s \times (m \times s)}{s^2} \tag{45}$$

respectively, and Sgn 4 implies that (30) and (45) are equivalent with (33) and

$$\mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} \neq \left(\frac{ms}{s^2}\right) s \tag{46}$$

respectively. Let

$$\mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} = \left(\frac{ms}{s^2}\right) s + \overline{\sigma}, \tag{47}$$

where, in view of (46),

$$\overline{\sigma} \neq \mathrm{o}. \tag{48}$$

Now (47) and (33) imply

$$\mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s} = \mathrm{mom}_{\mathrm{r}} \underset{\rightarrow}{s} + \overline{\sigma}, \tag{49}$$

and (49), Pr 4 imply

$$s\overline{\sigma} = 0. \tag{50}$$

Then (33) and (50) imply

$$\overline{\sigma} \cdot \mathrm{mom}_{\mathrm{r}} \underset{\rightarrow}{s} = 0, \tag{51}$$

and (49), (51) imply

$$\left(\mathrm{mom}_{\overline{p}} \underset{\rightarrow}{s}\right)^2 = \left(\mathrm{mom}_{\mathrm{r}} \underset{\rightarrow}{s}\right)^2 + \overline{\sigma}^2. \tag{52}$$

At last, (52) and (48) imply (44).

Sch 11. The same minimal property characterizes $\mathrm{dir}\,\overrightarrow{s}$: let us remind [2, 2 Pr 8].

Sch 12. If l is a line parallel to $\mathrm{ax}\ \underset{\rightarrow}{s}$, then all moments of $\underset{\rightarrow}{s}$ with respect to poles incident with l are equal between themselves. Indeed, let

$$l \parallel \mathrm{ax}\ \underset{\rightarrow}{s}. \tag{53}$$

Then there exists a $n \in V_S$ with $sn = 0$ and

$$(s, n)\ \&\ l. \tag{54}$$

If (8) is incident with l, then

$$r \times s = n. \tag{55}$$

Now (55) and Sgn 4 imply

$$\text{mom}_{\mathfrak{r}} \underset{\rightarrow}{s} = m - n. \tag{56}$$

In such a way, the following proposition has been proved.

Pr 8. (11), (24), (54), (8), $\mathfrak{r}$ Z l imply (56).

Sch 13. Another important property of ax $\underset{\rightarrow}{s}$ is disclosed by the following considerations. Let

$$r_\nu \in V_S \qquad (\nu = 1, 2) \tag{57}$$

and let

$$\text{dist}(r_1, \text{ax } \underset{\rightarrow}{s}) = \text{dist}(r_2, \text{ax } \underset{\rightarrow}{s}), \tag{58}$$

i.e. the end-points of r_1 and r_2 are at equal distances from ax $\underset{\rightarrow}{s}$. Since

$$\text{dist}(r_\nu, \text{ax } \underset{\rightarrow}{s}) = \frac{1}{s}\left| r_\nu \times s - \frac{s \times (m \times s}{s^2} \right| \tag{59}$$

($\nu = 1, 2$), as Sgn 7 and Sgn 9 of §4 (p. 124) of [5] imply, (58) is equivalent to

$$\left(r_1 \times s - \frac{s \times (m \times s)}{s^2}\right)^2 = \left(r_2 \times s - \frac{s \times (m \times s)}{s^2}\right)^2. \tag{60}$$

Now (60) and Sgn 4 imply

$$\left(\text{mom}_{\mathfrak{r}_1} \underset{\rightarrow}{s} - \left(\frac{ms}{s^2}\right) s\right)^2 = \left(\text{mom}_{\mathfrak{r}_2} \underset{\rightarrow}{s} - \left(\frac{ms}{s^2}\right) s\right)^2, \tag{61}$$

and (61), Pr 4 imply

$$\left(\text{mom}_{\mathfrak{r}_1} \underset{\rightarrow}{s}\right)^2 = \left(\text{mom}_{\mathfrak{r}_2} \underset{\rightarrow}{s}\right)^2. \tag{62}$$

In such a manner, the following proposition has been proved.

Pr 9. (11), (24), (57), (58) imply (62).

Sch 14. The following proposition is an analogue of [2, 2 Pr 9].

Pr 10. (11), (24), (8), (42), (35) imply

$$\text{mom}_{\overline{p}} \underset{\rightarrow}{s} = (r - \overline{p}) \times s + \left(\frac{ms}{s^2}\right) s. \tag{63}$$

Dm. Sgn 4, (30).

Pr 11. (11), (57) imply

$$\text{mom}_{\mathfrak{r}_1} \underset{\rightarrow}{s} - \text{mom}_{\mathfrak{r}_2} \underset{\rightarrow}{s} = s \times (r_1 - r_2). \tag{64}$$

Dm. Sgn 4.

Sch 15. The relation (64) is called the *connection between the moments of a system of arrows with respect to two poles*, and it plays a fundamental role in the so-called *statical-kinematical analogy*, which will be discussed briefly below. The connection (64) is a complete analogue of the *connection between the moments of an arrow with respect to two poles* [2, 2(19)].

Sch 16. (64) implies

$$\boldsymbol{s} \cdot \mathrm{mom}_{\boldsymbol{r}_1} \underrightarrow{s} = \boldsymbol{s} \cdot \mathrm{mom}_{\boldsymbol{r}_2} \underrightarrow{s}. \tag{65}$$

The inference (65) from (64) is, however, trivial in the light of Pr 4.

Pr 12. (57),

$$\underrightarrow{s} \in \Sigma_S \tag{66}$$

imply

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\boldsymbol{r}_1} \underrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\boldsymbol{r}_2} \underrightarrow{s}. \tag{67}$$

Dm. Pr 11.

Pr 13. (66), (57),

$$\boldsymbol{r}_1 \neq \boldsymbol{r}_2, \tag{68}$$

$$S = P \quad \text{or} \quad S = C(P) \tag{69}$$

imply

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \cdot \mathrm{mom}_{\boldsymbol{r}_1} \underrightarrow{s} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)^0 \cdot \mathrm{mom}_{\boldsymbol{r}_2} \underrightarrow{s}. \tag{70}$$

Dm. Pr 12, [2, 1 Sgn 4].

Sch 17. The relation (70) gives expression of the fact that if (68) holds and if (57) are different poles, then the projections of $\mathrm{mom}_{\boldsymbol{r}_\nu} \underrightarrow{s}$ $(\nu = 1, 2)$ on the line l connecting them, i.e. on

$$(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{r}_2 \times \boldsymbol{r}_1) \,\&\, l, \tag{71}$$

are equal.

Pr 14. (11), (18), (19),

$$\mathrm{mom}_{\boldsymbol{r}_\nu} \underrightarrow{s} = \boldsymbol{n} \qquad (\nu = 1, 2, 3) \tag{72}$$

imply (15).

Dm. (72), Sgn 4 imply

$$\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{n} \qquad (\nu = 1, 2, 3) \tag{73}$$

and (73) imply (21). Since (19) is equivalent to (22), the system of equations (21) implies (15).

Sch 18. The condition (19) implies that (18) are non-colinear, i.e. there exists no line l with $r_\nu \; Z \; l$ ($\nu = 1,.2, 3$). In such a manner, the basis of the system $\underrightarrow{s}$ is certainly zero if there exist three non-colinear poles, the moments of $\underrightarrow{s}$ with respect to which are equal. The inverse statement is trivial in the light of Pr 3.

Sch 19. Before we proceed further, let us make a brief remark of an ideological character. The above exposition reveals the most important role the notion of moment of an arrow plays in the theory of arrows. It would not be far-fetched to state that both notions are genetically connected.

Indeed, an arrow $\overrightarrow{s}$ being given by (6), its r-moment is determined by (10). On the other hand, (10) implies $m = \mathrm{mom}_o \overrightarrow{s}$, so that (6) may be written in the form

$$\overrightarrow{s} = (s, \mathrm{mom}_o \overrightarrow{s}). \tag{74}$$

In other words, a special r-moment of $\overrightarrow{s}$, namely its o-moment, takes part in the very definition of $\overrightarrow{s}$ by means of (6). In other words, the arrow-concept is unthinkable at all without the moment-concept. On the other hand, not a step could be made in the theory of arrows without the intensive exploitation of moments. The rank-theorem, the discussion of which lies yet ahead, is a brilliant illustration of this assertion.

The general connection between arrow-concept and moment-concept being once comprehended, a problem of cardinal importance arises at once, and quite naturally at that. It concerns the logical possibilities for a formal, purely mathematical, generalization of the arrow theory.

As it has been emphasized at the very end of the introductory remarks of [2], discussing in his *Vortrag* [6] the problem of the axiomatical consolidation of the logical foundations of rational mechanics, Hilbert underlines that, in the course of this process, "auch wird der Mathematiker, wie er es in der Geometrie getan hat, nicht bloß die der Wirklichkeit nahe kommenden, sondern überhaupt alle logisch möglichen Theorien berücksichtigen zu haben". The real arrows (i.e. the elements of W_R) "sind der Wirklichkeit nahe kommende" mathematical entities; the complex ones are obviously not, to say nothing of the $C(F)$-arrows. And yet, maybe there are other generalizations of the arrow-concept, for instance, in multidimensional Hermitean or, at least, Euclidean spaces?

This question is answered in the negative, and the main reason for this state of affairs lies in the fact that the introduction of a fourth operation (vector multiplication) in an Euclidean or in a Hermitean space with the most economical requirement that it must satisfy the two only specific conditions [2, 1 Ax 13S, 1 Ax 14S] inevitably reduces the dimensions of the spaces in question to 3 (with the additional condition that this operation should not be trivial, i.e. that there must exist two at least Euclidean or Hermitean vectors, the vector product of which is different from zero). And this fourth operation is used most essentially in the definition (10) of the moment of an arrow.

It is true that there are made efforts to generalize the vector multiplication in multidimensional cases, but it is still a controversial point to what extent these generalizations may be rated as not far-fetched ones.

Sch 20. If $G \subset V_S$ and a mapping

$$\mu: \quad G \longrightarrow V_S \tag{75}$$

is defined, then it is said that a *vector field* over G is at hand. Now, a system of arrows (11) being given, a vector field

$$\mu: \quad V_S \longrightarrow V_S \tag{76}$$

is *eo ipso* defined, determined by the relation

$$\mu(r)\ \mathrm{sgn}: \ \mathrm{mom}_r \overrightarrow{s} \qquad (r \in V_S), \tag{77}$$

i.e.

$$\mu(r) = m + s \times r \qquad (r \in V_S), \tag{78}$$

by virtue of Sgn 4.

Df 10. The vector field (76) defined by (77) is called the *moment field* of $\underset{\rightarrow}{s}$.

Sch 21. Let us regard the image $\mu(V_S)$ of V_S through the mapping (78). By definition, it is a set consisting of S-vectors. Now any set of vectors has a natural characteristic, namely the maximal number of linearly independent elements of this set. Since any four standard vectors are linearly dependent, this maximal number may be 0, or 1, or 2, or 3 at most. After these explanations, the following definition may be accepted.

Sgn 8. rank $\underset{\rightarrow}{s}$ sgn: the maximal number of linearly independent elements of $\mu(V_S)$ iff (11), (77).

Df 11. rank $\underset{\rightarrow}{s}$ is called the *rank* of $\underset{\rightarrow}{s}$.

Sch 22. The rank of a system of arrows $\underset{\rightarrow}{s}$ is, as it will be seen in the sequel, a most important characteristic of $\underset{\rightarrow}{s}$. Therefore, a system of arrows $\underset{\rightarrow}{s}$ being given, the problem of the determination of rank $\underset{\rightarrow}{s}$ is a most actual one. The direct application to this end of the definition Sgn 8 is, however, a very clumsy approach to the goal.

Indeed, let us get aware of the nature of this direct application. What must one do in order to determinate rank $\underset{\rightarrow}{s}$?

To prove that rank $\underset{\rightarrow}{s} = 3$ means to find such three particular poles (18) that $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ $(\nu = 1, 2, 3)$ are linearly independent, i.e.

$$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} \cdot \mathrm{mom}_{r_3} \underset{\rightarrow}{s} \neq 0. \tag{79}$$

To prove that rank $\underset{\rightarrow}{s} = 2$ means, first, to establish that $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ $(\nu = 1, 2, 3)$ are linearly dependent for any three poles (18), i.e.

$$\mathrm{mom}_{r_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{r_2} \underset{\rightarrow}{s} \cdot \mathrm{mom}_{r_3} \underset{\rightarrow}{s} = 0; \tag{80}$$

and second, to find such two particular poles (57), that $\mathrm{mom}_{r_\nu} \underset{\rightarrow}{s}$ $(\nu = 1, 2)$ are linearly independent, i.e.

(81) $\mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} \neq \mathbf{o}.$

To prove that rank $\underset{\rightarrow}{s} = 1$ means, first, to demonstrate that $\mathrm{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s}$ ($\nu = 1, 2$) are linearly dependent for any two poles (57), i.e.

(82) $\mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{\mathbf{r}_2} \underset{\rightarrow}{s} = \mathbf{o};$

and second, to find such a particular pole (8), that $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}$ is linearly independent, i.e.

(83) $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} \neq 0.$

At last, to prove that rank $\underset{\rightarrow}{s} = 0$ means to establish that $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s}$ is linearly dependent for any pole (8), i.e.

(84) $\mathrm{mom}_{\mathbf{r}} \underset{\rightarrow}{s} = \mathbf{o}.$

It is obvious that all these procedures cannot be assessed as a very attractive mathematical task. They certainly require some amount of inventiveness. Therefore it is quite natural to seek a certain *Schablon* that would permit to determinate rank $\underset{\rightarrow}{s}$ automatically, as the saying goes.

This *Schablon* is proposed by the so-called *theorem of the rank of a system of arrows* (or, concisely, the *rank-theorem*). It consists in the following proposition.

Pr 15 (the *rank-theorem*). If (11), then

(85) $\mathrm{rank} \underset{\rightarrow}{s} = 0$

iff

(86) $\boldsymbol{s} = \mathbf{o}, \qquad \boldsymbol{m} = \mathbf{o};$

(87) $\mathrm{rank} \underset{\rightarrow}{s} = 1$

iff

(88) $\boldsymbol{s} = \mathbf{o}, \qquad \boldsymbol{m} \neq \mathbf{o};$

(89) $\mathrm{rank} \underset{\rightarrow}{s} = 2$

iff

(90) $\boldsymbol{s} \neq \mathbf{o}, \qquad \boldsymbol{sm} = \mathbf{o};$

(91) $\mathrm{rank} \underset{\rightarrow}{s} = 3$

iff

(92) $\boldsymbol{sm} = \mathbf{0}.$

Sch 23. The following scheme proposes an abbreviated formulation of the rank-theorem:

$$\text{(93)} \qquad \operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \text{iff} \quad \begin{cases} s = o, & m = o, \\ s = o, & m \neq o, \\ s \neq o, & sm = 0, \\ & sm \neq 0. \end{cases}$$

Sch 24. Two different proofs of the rank-theorem (93) will be given below. Before proceeding to these, let us get aware of the meaning of Pr 15.

Let a system (1) of arrows (4) be given. Then its basis s and its moment m are trivially determined by (3) and (5) respectively. Now, in order to determine rank $\underset{\rightarrow}{s}$, one must simply check up which of the cases (86), or (88), or (90), or (92) is at hand. This is why the rank-theorem has been called a *Schablon* in Sch 22. Where did the difficulties in the determination of rank $\underset{\rightarrow}{s}$, described in Sch 22, vanish? The answer is: these difficulties are overcome, once for ever, in the proof of Pr 15.

First proof of the rank-theorem. *Sufficiency.* Let (86) hold. Then (8) and Sgn 4 imply (84), whence (85) (Sgn 8).

Let (88) hold. Then (86) and Sgn 4 imply (83). On the other hand, (57) and Pr 3 imply (82), whence (87) (Sgn 8).

Let (90) hold. If (8), (35), then the second relation (90) and (30) (Sgn 7) imply

$$\text{(94)} \qquad r \times s = m.$$

Let

$$\text{(95)} \qquad s_\nu \in V_S \qquad (\nu = 1, 2),$$

$$\text{(96)} \qquad s_1 \times s_2 \cdot s_3 \neq 0,$$

and let by definition

$$\text{(97)} \qquad r_\nu = r + s_\nu \qquad (\nu = 1, 2, 3).$$

Then (97), Sgn 4, (94) imply

$$\text{(98)} \qquad \operatorname{mom}_{r_\nu} \underset{\rightarrow}{s} = m + s \times (r + s_\nu) = s \times s_\nu \qquad (\nu = 1, 2),$$

whence

$$\text{(99)} \qquad \operatorname{mom}_{r_1} \underset{\rightarrow}{s} \times \operatorname{mom}_{r_2} \underset{\rightarrow}{s} = (s \times s_1) \times (s \times s_2),$$

i.e.

$$\text{(100)} \qquad \operatorname{mom}_{r_1} \underset{\rightarrow}{s} \times \operatorname{mom}_{r_2} \underset{\rightarrow}{s} = (s \times r_1 \cdot s_2)s.$$

Now (100), (96), and the first relation (90) imply (81). On the other hand, if (18), then Sgn 4 and the second relation (90) imply

(101) $$\mathrm{mom}_{\mathfrak{r}_1 \underset{\rightarrow}{s}} \times \mathrm{mom}_{\mathfrak{r}_2 \underset{\rightarrow}{s}} \cdot \mathrm{mom}_{\mathfrak{r}_3 \underset{\rightarrow}{s}}$$

$$= ((\boldsymbol{r}_2\boldsymbol{m} - \boldsymbol{r}_1\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1 \cdot \boldsymbol{r}_2)\boldsymbol{s})(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_3).$$

Now (101) and the second relation (90) imply (80), whence (89) (Sgn 8).

At last, let (92) hold. If (8), (35), then (30) holds. Let (95), (96),

(102) $$\boldsymbol{s}_3 = \boldsymbol{s},$$

and let by definition

(103) $$\boldsymbol{r}_\nu = \boldsymbol{r} + \boldsymbol{s}_\nu \qquad (\nu = 1, 2, 3).$$

Then (103), Sgn 4, (30) imply

(104) $$\mathrm{mom}_{\mathfrak{r}_\nu \underset{\rightarrow}{s}} = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{m} + \boldsymbol{s} \times (\boldsymbol{r} \times \boldsymbol{s}_\nu) = \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_\nu$$

$$(\nu = 1, 2, 3),$$

and (104), (102) imply

(105) $$\mathrm{mom}_{\mathfrak{r}_1 \underset{\rightarrow}{s}} \times \mathrm{mom}_{\mathfrak{r}_2 \underset{\rightarrow}{s}} \cdot \mathrm{mom}_{\mathfrak{r}_3 \underset{\rightarrow}{s}}$$

$$= \left(\left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_1\right) \times \left(\left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right)\boldsymbol{s} + \boldsymbol{s} \times \boldsymbol{s}_2\right) \cdot \left(\left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right)\boldsymbol{s}\right)$$

$$= \left(\frac{\boldsymbol{ms}}{\boldsymbol{s}^2}\right)(\boldsymbol{s} \times \boldsymbol{s}_1 \cdot \boldsymbol{s}_2)\boldsymbol{s}^2.$$

Now (105), (92), (96) imply (79), whence (91) (Sgn 8).

Necessity. The sufficiency of the conditions (86), (88), (90), (92) for (85), (87), (89), (91) respectively once proved, the necessity of these conditions is trivial, since they are mutually inconsistent.

Second proof of the rank-theorem. *Necessity.* Let (85) hold. Then Sgn 8 implies (84) for any (8) whence

(106) $$\boldsymbol{m} = \mathbf{o}$$

(Pr 2) and (15) (Pr 3). Now (15), (106) imply (86).

Let (87) hold. Then Sgn 8 implies that there exists (42) with

(107) $$\mathrm{mom}_{\overline{\mathfrak{p}} \underset{\rightarrow}{s}} \neq \mathbf{o}$$

and that (57) imply (82). If

(108) $$\overline{\boldsymbol{\mu}} = \mathrm{mom}_{\overline{\mathfrak{p}} \underset{\rightarrow}{s}},$$

then (82) implies

(109) $$\mathrm{mom}_{\mathfrak{r} \underset{\rightarrow}{s}} \times \overline{\boldsymbol{\mu}} = \mathbf{o}$$

for any (8), and (109), Pr 2 imply

(110) $\quad \boldsymbol{m} \times \overline{\boldsymbol{\mu}} = \mathbf{o}.$

On the other hand, (109) and Sgn 4 imply

(111) $\quad (\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}) \times \overline{\boldsymbol{\mu}} = \mathbf{o},$

and (110), (111) imply

(112) $\quad (\boldsymbol{s} \times \boldsymbol{r}) \times \overline{\boldsymbol{\mu}} = \mathbf{o}$

for any (8). If (24), then there exist (57) with

(113) $\quad (\boldsymbol{s} \times \boldsymbol{r}_1) \times (\boldsymbol{s} \times \boldsymbol{r}_2) \neq \mathbf{o}.$

Since (112) holds for any (8),

(114) $\quad (\boldsymbol{s} \times \boldsymbol{r}_\nu) \times \overline{\boldsymbol{\mu}} = \mathbf{o} \qquad (\nu = 1, 2),$

and (113), (114) imply

(115) $\quad \overline{\boldsymbol{\mu}} = \mathbf{o},$

contrary to (107). This consideration is due to the supposition that (24) holds. Therefore (15). Now Pr 3, Pr 2 imply $\boldsymbol{m} = \overline{\boldsymbol{\mu}}$, whence

(116) $\quad \boldsymbol{m} \neq \mathbf{o},$

and (15), (116) imply (88).

Let (89) hold. Then Sgn 8 implies that there exist $\overline{\rho}_\nu \in V_S$ $(\nu = 1, 2)$ with

(117) $\quad \mathrm{mom}_{\overline{\rho}_1} \underset{\rightarrow}{s} \times \mathrm{mom}_{\overline{\rho}_2} \underset{\rightarrow}{s} \neq \mathbf{o},$

and that (18) imply (80). If (15), then (64) implies

(118) $\quad \mathrm{mom}_{\overline{\rho}_1} \underset{\rightarrow}{s} = \mathrm{mom}_{\overline{\rho}_2} \underset{\rightarrow}{s}$

contrary to (117), whence (24). On the other hand, if

(119) $\quad \boldsymbol{m}_\nu = \mathrm{mom}_{\overline{\rho}_\nu} \underset{\rightarrow}{s} \qquad (\nu = 1, 2).$

Then (80), Pr 2, (8) imply

(120) $\quad \boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \boldsymbol{m} = 0,$

(121) $\quad \boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot \mathrm{mom}_{\mathrm{r}} \underset{\rightarrow}{s} = 0.$

Now (121), Sgn 4 imply

(122) $\quad \boldsymbol{m}_1 \times \boldsymbol{m}_2 \cdot (\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}) = 0,$

and (120), (122) imply

(123) $\quad m_1 \times m_2 \cdot s \times r = 0$

for any (8). Let (18),

(124) $\quad r_1 \times r_2 \cdot r_3 \neq 0$

hold. Then (123) implies

(125) $\quad (m_1 \times m_2) \times s \cdot r_\nu = 0 \qquad (\nu = 1, 2, 3)$

and(124), (125) imply

(126) $\quad (m_1 \times m_2) \times s = o,$

i.e.

(127) $\quad (m_1 s) m_2 - (m_2 s) m_1 = 0.$

Now (127), (118), (119) imply

(128) $\quad m_1 s = 0,$

and (128), Pr 4 imply

(129) $\quad sm = 0.$

Then (24), (129) imply (90).

Let (91) hold. Then there exist (18) with (79). Let

(130) $\quad m_\nu = \mathrm{mom}_{r_\nu} \underset{\rightarrow}{s} \qquad (\nu = 1, 2, 3).$

Now (130), (79), Pr 2, Pr 4 imply

(131) $\quad m_1 \times m_2 \cdot m_3 \neq 0,$

(132) $\quad sm = sm_\nu \qquad (\nu = 1, 2, 3).$

If (129), then (131), (132) imply (15), and (15), (130), Sgn 4 imply

(133) $\quad m_\nu = m \qquad (\nu = 1, 2, 3),$

contrary to (131). Hence (92).

Sufficiency. The necessity of the conditions (86), (88), (90), (92) for (85), (87), (89), (91) respectively being, in such a manner, proved, the sufficiency of these conditions is verified trivially, since they are mutually inconsistent.

Sch 25. The first proof of the rank-theorem may be assessed as simpler than the second one.

Pr 16. (18), (66),

(134) $\quad m_\nu = \mathrm{mom}_{r_\nu} \underset{\rightarrow}{s} \qquad (\nu = 1, 2, 3)$

imply

(135) $$(\boldsymbol{r}_\nu - \boldsymbol{r}_{\nu+1})(\boldsymbol{m}_\nu - \boldsymbol{m}_{\nu+1}) = 0 \qquad (\nu = 1, 2, 3)$$

provided

(136) $$\boldsymbol{r}_4 = \boldsymbol{r}_1, \qquad \boldsymbol{m}_4 = \boldsymbol{m}_1.$$

Dm. (134), (11), Sgn 4 imply

(137) $$\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_\nu = \boldsymbol{m}_\nu \qquad (\nu = 1, 2, 3),$$

i.e.

(138) $$\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_{\nu+1} = \boldsymbol{m}_{\nu+1} \qquad (\nu = 1, 2, 3)$$

provided (136). Now (137), (138) imply

(139) $$\boldsymbol{s} \times (\boldsymbol{r}_\nu - \boldsymbol{r}_{\nu+1}) = \boldsymbol{m}_\nu - \boldsymbol{m}_{\nu+1} \qquad (\nu = 1, 2, 3)$$

and (139) imply (135).

Sch 26. In such a manner, the conditions (135) provided (136) are necessary for the consistency of the relations (134) or, just the same, of (137). As a matter of fact, (134) imply seemingly much more burdensome necessary conditions than (135), namely

(140) $$(\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu) = 0$$

$$(\mu, \nu, \sigma, \tau = 1, 2, 3).$$

Indeed, (137) imply

(141) $$\boldsymbol{s} \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) = \boldsymbol{m}_\mu - \boldsymbol{m}_\nu (\mu, \nu = 1, 2, 3),$$

(142) $$\boldsymbol{s} \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) = \boldsymbol{m}_\sigma - \boldsymbol{m}_\tau (\sigma, \tau = 1, 2, 3),$$

and (141), (142) imply

(143) $$\begin{aligned}(\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu) \\ = (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \cdot \boldsymbol{s} \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) \cdot \boldsymbol{s} \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \\ = \boldsymbol{s} \cdot (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau) \times (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) + \boldsymbol{s} \cdot (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu) \times (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)\end{aligned}$$

$(\mu, \nu, \sigma, \tau = 1, 2, 3)$, whence (140).

It will be seen, however, that all the relations (140) are simple corollaries from (135), rather than mutually independent.

As a matter of fact, the number of different among themselves relations (140) is 6, rather than 81, as one might think at a first glance. Indeed, let by definition

(144) $$L(\mu, \nu, \sigma, \tau) = (\boldsymbol{r}_\mu - \boldsymbol{r}_\nu)(\boldsymbol{m}_\sigma - \boldsymbol{m}_\tau) + (\boldsymbol{r}_\sigma - \boldsymbol{r}_\tau)(\boldsymbol{m}_\mu - \boldsymbol{m}_\nu)$$

$(\mu, \nu, \sigma, \tau = 1, 2, 3)$. Now obviously

$$L(\mu, \mu, \sigma, \tau) = 0, \tag{145}$$

$$L(\mu, \nu, \sigma, \sigma) = 0 \tag{146}$$

$(\mu, \nu, \sigma, \tau = 1, 2, 3)$. On the other hand,

$$L(\nu, \mu, \sigma, \tau) = -L(\mu, \nu, \sigma, \tau), \tag{147}$$

$$L(\mu, \nu, \tau, \sigma) = -L(\mu, \nu, \sigma, \tau), \tag{148}$$

$$L(\tau, \sigma, \mu, \nu) = L(\mu, \nu, \sigma, \tau) \tag{149}$$

$(\mu, \nu, \sigma, \tau = 1, 2, 3)$, etc. In such a manner, the relations (140) reduce to the following ones: (135) provided (136) and

$$(r_1 - r_2)(m_1 - m_3) + (r_1 - r_3)(m_1 - m_2) = 0, \tag{150}$$

$$(r_2 - r_3)(m_2 - m_1) + (r_2 - r_1)(m_2 - m_3) = 0, \tag{151}$$

$$(r_3 - r_1)(m_3 - m_2) + (r_3 - r_2)(m_3 - m_1) = 0. \tag{152}$$

Now (135) imply

$$\begin{aligned}(r_1 - r_2)(m_1 - m_3) + (r_1 - r_3)(m_1 - m_2) \\ = (r_1 - r_2)(m_1 - m_2 + m_2 - m_3) + (r_1 - r_3)(m_1 - m_3 + m_3 - m_2) \\ (r_1 - r_2)(m_1 - m_2) + (r_1 - r_2)(m_2 - m_3) \\ +(r_1 - r_3)(m_1 - m_3) + (r_1 - r_3)(m_3 - m_2) \\ = (r_3 - r_2)(m_2 - m_3) = 0,\end{aligned} \tag{153}$$

i.e. (150). Similarly, (135) imply

$$\begin{aligned}(r_2 - r_3)(m_2 - m_1) + (r_2 - r_1)(m_2 - m_3) \\ = (r_2 - r_3)(m_2 - m_3 + m_3 - m_1) + (r_2 - r_1)(m_2 - m_1 + m_1 - m_3) \\ = (r_2 - r_3)(m_2 - m_3) + (r_2 - r_3)(m_3 - m_1) \\ +(r_2 - r_1)(m_2 - m_1) + (r_2 - r_1)(m_1 - m_3) \\ = (r_3 - r_1)(m_1 - m_3) = 0,\end{aligned} \tag{154}$$

i.e. (151). At last, (135) imply

$$\begin{aligned}(r_3 - r_1)(m_3 - m_2) + (r_3 - r_2)(m_3 - m_1) \\ = (r_3 - r_1)(m_3 - m_1 + m_1 - m_2) + (r_3 - r_2)(m_3 - m_2 + m_2 - m_1) \\ (r_3 - r_1)(m_3 - m_1) + (r_3 - r_1)(m_1 - m_2) \\ +(r_3 - r_2)(m_3 - m_2) + (r_3 - r_2)(m_2 - m_1) \\ = (r_2 - r_1)(m_1 - m_2) = 0,\end{aligned} \tag{155}$$

i.e. (152).

Now the following question quite naturally arises: are the conditions (135) also sufficient for the consistency of the relations (134). In other words, (18) and

(156) $\quad m_\nu \in V_S \qquad (\nu = 1, 2, 3)$

with (135) being given, does there exist a system of arrows (66) satisfying (234)? Or otherwise, does there exist, under the same conditions, a solution s, m of the system of vector equations (137). As the following proposition displays, this question is answered in the affirmative, provided (19) holds.

Pr 17. (18), (19), (156), (135) provided (136) imply: there exist infinitely many systems of arrows (11) with (134), namely all these for which

$$(157) \qquad s = \frac{1}{2}\sum_{\nu=1}^{3} a_\nu^{-1} \times b_\nu$$

provided

$$(158) \qquad a_1 = r_1 - r_2, \qquad a_2 = r_2 - r_3,$$

$$(159) \qquad b_1 = m_1 - m_2, \qquad b_2 = m_2 - m_3,$$

$$(160) \qquad a_3 = r_1 \times r_2 + r_2 \times r_3 + r_3 \times r_1,$$

$$(161) \qquad b_3 = (b_1 \cdot a_2 \times a_1)a_1^{-1} + (b_2 \cdot a_2 \times a_1)a_2^{-1}.$$

and

$$(162) \qquad m = m_1 + r_1 \times s.$$

Dm. Since (134) is equivalent with (137), a solution s, m of this system of vector equations at the hypothesis (135) provided (136) must be found. The system (137) implies

$$(163) \qquad s \times a_\nu = b_\nu \qquad (\nu = 1, 2)$$

provided (158), (159). As it is well-known [2, 1 Pr 28], necessary conditions for the consistency of (163) are

$$(164) \qquad a_\mu b_\nu + a_\nu b_\mu = 0 \qquad (\mu, \nu = 1, 2).$$

Now the definitions (158), (159) imply that (164) are equivalent to

$$(165) \qquad (r_1 - r_2)(m_1 - m_2) = 0, \qquad (r_2 - r_3)(m_2 - m_3) = 0$$

and

$$(166) \qquad (r_1 - r_2)(m_2 - m_3) + (r_2 - r_3)(m_1 - m_2) = 0.$$

As regards the conditions (165), they are satisfied by virtue of the hypothesis (135); as regards the condition (166), it is satisfied by virtue of (151), which is a corollary from (135), as proved in Sch 26.

On the other hand, the conditions (164) are sufficient for the consistency of (163), provided

$$(167) \qquad a_1 \times a_2 \neq o$$

[2, 1 Pr 31]: if (167), (164) hold, then the system (163) has exactly one solution s, namely (157), provided

(168) $\qquad a_3 = a_1 \times a_2$

and (161). Now (158) imply that (167) is equivalent to

(169) $\qquad (r_1 - r_2) \times (r_2 - r_3) \neq o$

by virtue of (19), and that (168) is equivalent to (160).

The basis s of $\underrightarrow{s}$ being once determined in such a manner, the moment m is defined by (162). Let

(170) $\qquad p = m_2 + r_2 \times s.$

Now (162), (170) imply

(171) $\qquad m - p = m_1 - m_2 + (r_1 - r_2) \times s.$

Since s nullifies the right-hand side of (171), the last equation implies $p = m$ and (170) implies

(172) $\qquad m = m_2 + r_2 \times s.$

Similarly, let

(173) $\qquad q = m_3 + r_3 \times s.$

Now (170), (173) imply

(174) $\qquad p - q = m_2 - m_3 + (r_2 - r_3) \times s.$

Since s annulates the right-hand side of (174), the last equation implies $q = p = m$, and (173) implies

(175) $\qquad m = m_3 + r_3 \times s.$

In such a manner, the proposition is proved.

Sch 27. Pr 17 being proved, the following problem arises: determine the rank of the systems of arrows $\underrightarrow{s}$, satisfying (137), (18) with (19) and (156) with (135) being given.

According to Pr 3, Pr 15 and Pr 17, the following relations hold

$$(176) \qquad \operatorname{rank} \underrightarrow{s} = \begin{cases} 0 & \text{iff} \quad m_\nu = o \qquad (\nu = 1, 2, 3), \\ 1 & \text{iff} \quad m_1 = m_2 = m_3 \neq o. \end{cases}$$

as regards the other cases, they demand a more detailed discussion.

According to Pr 4, Pr 15 and Pr 17,

$$(177) \qquad \operatorname{rank} \underrightarrow{s} = \begin{cases} 2 \\ 3 \end{cases} \text{ iff } \begin{cases} s \neq o, \quad sm_1 = 0, \\ sm_1 \neq 0. \end{cases}$$

Now (157) and the definition [2, 1(23) provided 1(24)] of the reciprocal vectors a_ν^{-1} ($\nu = 1, 2, 3$) imply

(178) $$2(\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3)\boldsymbol{s} = \sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1} \times \boldsymbol{a}_{\nu+2}) \times \boldsymbol{b}_\nu.$$

provided

(179) $$\boldsymbol{a}_{\nu+3} = \boldsymbol{a}_\nu \qquad (\nu = 1, 2),$$

and (178) implies (24) iff

(180) $$\sum_{\nu=1}^{3}((\boldsymbol{a}_{\nu+1}\boldsymbol{b}_\nu)\boldsymbol{a}_{\nu+2} - (\boldsymbol{a}_{\nu+2}\boldsymbol{b}_\nu)\boldsymbol{a}_{\nu+1}) \neq \mathfrak{o}.$$

On the other hand, by virtue of (179) the relation (180) may be written in the form

(181) $$\sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+2}\boldsymbol{b}_{\nu+1} - \boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})\boldsymbol{a}_\nu \neq \mathfrak{o}$$

provided

(182) $$\boldsymbol{b}_{\nu+3} = \boldsymbol{b}_\nu \qquad (\nu = 1, 2),$$

and (181), in view of

(183) $$\boldsymbol{a}_\mu\boldsymbol{b}_\nu + \boldsymbol{a}_\nu\boldsymbol{b}_\mu = 0 \qquad (\mu, \nu = 1, 2, 3)$$

is equivalent to

(184) $$\sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})\boldsymbol{a}_\nu \neq \mathfrak{o}.$$

Because of

(185) $$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 \neq 0,$$

the condition (184) is equivalent to

(186) $$\sum_{\nu=1}^{3}(\boldsymbol{a}_\nu\boldsymbol{b}_{\nu+1})(\boldsymbol{b}_{\nu+1}\boldsymbol{a}_\nu) \neq 0.$$

As regards the condition

(187) $$\boldsymbol{s}\boldsymbol{m}_1 = 0,$$

the above considerations display that it is equivalent to

(188) $$\sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})(\boldsymbol{a}_\nu\boldsymbol{m}_1) = 0,$$

and

(189) $$\boldsymbol{s}\boldsymbol{m}_1 \neq 0$$

is equivalent to

(190) $$\sum_{\nu=1}^{3}(\boldsymbol{a}_{\nu+1}\boldsymbol{b}_{\nu+2})(\boldsymbol{a}_\nu \boldsymbol{m}_1) \neq 0.$$

Summing up, we may formulate the following proposition.

Pr 18. (18), (19), (156), (135) provided (136) imply: if (66) satisfies (134), then

(191) $$\operatorname{rank} \underset{\rightarrow}{s} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \quad \text{iff} \quad \begin{cases} \boldsymbol{m}_\nu = \mathbf{o} \quad (\nu = 1, 2, 3), \\ \boldsymbol{m}_1 = \boldsymbol{m}_2 = \boldsymbol{m}_3 \neq \mathbf{o}, \\ (184),\ (188), \\ (190) \end{cases}$$

provided (158) — (161), (179), (182).

Sch 28. As it has been emphasized in Sch 4, although the basis $\boldsymbol{s}$ and the moment $\boldsymbol{m}$ of a system of arrows $\underset{\rightarrow}{s}$ are extremely important characteristics of $\underset{\rightarrow}{s}$ (see, for instance, the role of $\boldsymbol{s}$ and $\boldsymbol{m}$ in the rank-theorem Pr 15), they do not determine $\underset{\rightarrow}{s}$ completely, so that the notation (7) is void of sense: there are infinitely many systems of arrows that have the same basis $\boldsymbol{s}$ and the same moment $\boldsymbol{m}$ for any $(\boldsymbol{s}, \boldsymbol{m}) \in V_S^2$. And yet, in the applications of the theory of arrows to analytical mechanics at least, dynamics as well as statics, the role of these characteristics of the systems of arrows is, to the highest degree, a predestinating one. The reason is concealed in the fact that, both in statics and dynamics, the systems of arrows available in the mechanical problem in the capacity of forces (both active and passive) acting on mass-points and rigid bodies, are authoritative by means of their bases and moments rather than by themselves as individual mechanical entities. Putting it more specific, a system of arrows (66) is of interest to statics and dynamics inasmuch as its basis $\boldsymbol{s}$ and its moment $\boldsymbol{m}$ are concerned, rather than the particular arrows entering into its composition. Figuratively speaking, one may say that for analytical statics and analytical dynamics the definition of (66) by (11) is sufficient, and the definition of (66) by (1) provided (2) is not necessary (inasmuch as the determination of the particular reactions of the geometrical constraints imposed on the mass-points or on the rigid bodies is neglected). To put it another way, both the statical and the dynamical effects of a system of forces (arrows) $\underset{\rightarrow}{s}$ remain unaffected if $\underset{\rightarrow}{s}$ is replaced by a system of forces $\underset{\rightarrow}{\sigma}$ having the same basis and the same moment as $\underset{\rightarrow}{s}$.

The causality of all these circumstances is rooted in the fact that a system of forces acting on a mass-point or a rigid body is presented in the basis equations of analytical statics and analytical dynamics by its basis and its moment namely, and not by the particular arrow-components which this system involves. This question is discussed in some details in the following three scholiums.

Sch 29. Analytical statics and analytical dynamics are concerned with the mechanical behaviour of mass-points and rigid bodies subjected to certain *geometrical constraints* and to the action of *active forces* (wholly determined in the conditions of the statical or dynamical problem in question) and of *passive forces* or *reactions of the constraints*. More precisely, analytical statics is concerned with the *equilibrium* of the said mass-points and rigid bodies, and analytical dynamics with their *motion*.

In order to fix the ideas, let a rigid body B be given, subjected to certain geometrical constraints, restricting to l its degrees of freedom, and let

$$q_\lambda \in R \qquad (\lambda = 1, \ldots, l) \tag{192}$$

be mutually independent parameters of B. Let B be under the action of the active forces (arrows)

$$\vec{F}_\mu = (\boldsymbol{F}_\mu, \boldsymbol{M}_\mu) \qquad (\mu = 1, \ldots, m), \tag{193}$$

and let the geometrical constraints generate (by virtue of special statical or dynamical axioms) the passive forces (reactions of the constraints)

$$\vec{R}_\nu = (\boldsymbol{R}_\nu, \boldsymbol{N}_\nu) \qquad (\nu = 1, \ldots, n). \tag{194}$$

For the sake of brevity let by definition

$$\vec{F}_{m+\nu} = \vec{R}_\nu \qquad (\nu = 1, \ldots, n), \tag{195}$$

i.e.

$$\boldsymbol{F}_{m+\nu} = \boldsymbol{R}_\nu, \qquad \boldsymbol{M}_{m+\nu} = \boldsymbol{N}_\nu \qquad (\nu = 1, \ldots, n) \tag{196}$$

provided

$$\vec{F}_{m+\nu} = (\boldsymbol{F}_{m+\nu}, \boldsymbol{M}_{m+\nu}) \qquad (\nu = 1, \ldots, n), \tag{197}$$

and let

$$\boldsymbol{F} = \sum_{\nu=1}^{m+n} \boldsymbol{F}_\nu, \qquad \boldsymbol{M} = \sum_{\nu=1}^{m+n} \boldsymbol{M}_\nu. \tag{198}$$

If

$$\underset{\rightarrow}{F} = \left\{ \vec{F}_\nu \right\}_{\nu=1}^{m+n} \tag{199}$$

denotes the system of all forces (both active and passive) acting on B, then obviously $\boldsymbol{F}$ and $\boldsymbol{M}$ are the basis and the moment of $\underset{\rightarrow}{F}$ respectively.

In any statical and dynamical problem all active forces (193) are completely determined functions

$$\vec{F}_\mu = \vec{F}_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{200}$$

$(\mu = 1, \ldots, m)$ of the parameters (192) of B, of their velocities (derivatives with respect to the time t)

$$\dot{q}_\lambda \in R \qquad (\lambda = 1, \ldots, l), \tag{201}$$

and possibly of the time t itself. (As a matter of fact, tradionally in statics the active forces (193) do not depend explicitely on the velocities (201) and on t.) In other words, the bases and the moments of (193) are completely determined functions

$$\boldsymbol{F}_\mu = \boldsymbol{F}_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{202}$$

and

$$\boldsymbol{M}_\mu = \boldsymbol{M}_\mu(q_1, \ldots, q_l; \dot{q}_1, \ldots, \dot{q}_l; t) \tag{203}$$

($\mu = 1, \ldots, m$) respectively of (192), (201) and t.

As regards the reactions of the geometrical constraints (194), the situation is a quite different one: they are unknown quantities the determination of which is in store for statics and dynamics, in the process of the solving of the corresponding mechanical problem. Now, any of the reactions (194) is generated by a single geometrical constraint imposed on B, so that the number of the passive forces is equal to the number of these geometrical constraints, i.e. to n. Let

$$\boldsymbol{c}_\nu \in V \qquad (\nu = 1, \ldots, n) \tag{204}$$

be the radius-vector of the corresponding point of contact of the rigid body B with the ν-th geometrical constraint. Then (204) are wholly determined functions

$$\boldsymbol{c}_\nu = \boldsymbol{c}_\nu(q_1, \ldots, q_l; t) \qquad (\nu = 1, \ldots, n) \tag{205}$$

of the parameters (192) of B and possibly of t. According to a kinetical (statical as well as dynamical) axiom, the directrix of the ν-th reaction $\vec{R}_\nu$ is incident with the point of contact $\boldsymbol{c}_\nu$ of the geometrical constraint generating $\vec{R}_\nu$ ($\nu = 1, \ldots, n$), i.e.

$$\boldsymbol{c}_\nu \times \boldsymbol{R}_\nu = \boldsymbol{N}_\nu \qquad (\nu = 1, \ldots, n). \tag{206}$$

In such a manner, it is enough to know the bases $\boldsymbol{R}_\nu$ of the reactions $\vec{R}_\nu$ in order to know $\vec{R}_\nu$ themselves ($\nu = 1, \ldots, n$). In other words, these namely bases $\boldsymbol{R}_\nu$ ($\nu = 1, \ldots, n$) are unknown quantities in the statical or dynamical problem under consideration, as well as the parameters (192) of B. In the case of a dynamical problem, initial values

$$q_{\lambda_0} = q_\lambda(0) \qquad (\lambda = 1, \ldots, l) \tag{207}$$

and

$$\dot{q}_{\lambda_0} = \dot{q}_\lambda(0) \qquad (\lambda = 1, \ldots, l) \tag{208}$$

are prescribed to (192) and (201) respectively, and functions

$$q_\lambda = q_\lambda(t) \qquad (\lambda = 1, \ldots, l) \tag{209}$$

of the time t are sought.

Sch 30. This information once grasped, let us proceed to the formulation of a statical problem concerning a rigid body B.

The fundamental problem of rigid body statics may be described in the following manner.

Let a rigid body B be given, subjected to n geometrical constraints with points of contact (204) which generate the reactions (194), and let B be under the action of the active forces (193). Find the equilibrium of B and the reactions of the constraints.

In this formulation the term *equilibrium* has been used. It is a fundamental statical notion ant it requires a special definition. This definition is given in the following manner.

It is said that a rigid body is *in equilibrium* under the action of a system of forces when the rank of this system iz zero.

Under the notations introduced in Sch 29, the rigid body will be in equilibrium under the action of the active forces (193) and of the reactions of the constraints (194) if, and only if, the following conditions are satisfied:

$$\boldsymbol{F} = \boldsymbol{O}, \qquad \boldsymbol{M} = \boldsymbol{O}. \tag{210}$$

In such a manner, it is seen, that the necessary and sufficient condition (210) for the equilibrium of the rigid body is expressed exclusively by the aid of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system of forces (199) acting on the rigid body. Let us, however, discuss this formulation somewhat closer.

The equations (210) may be written in the form

$$\sum_{\mu=1}^{m} \boldsymbol{F}_\mu + \sum_{\nu=1}^{n} \boldsymbol{R}_\nu = \boldsymbol{O} \tag{211}$$

and

$$\sum_{\mu=1}^{m} \boldsymbol{M}_\mu + \sum_{\nu=1}^{n} \boldsymbol{c}_\nu \times \boldsymbol{R}_\nu = \boldsymbol{O} \tag{212}$$

respectively, by virtue of (198), (196), and (206). In such a manner, the fundamental problem of rigid body statics is reduced mathematically to the vector equations (211), (212), where the unknown quantities are the parameters (192) of the rigid body B and the reactions $\boldsymbol{R}_\nu$ ($\nu = 1, \ldots, n$) of the constraints. In this general formulation the statical problem is indeterminate: as a rule, the number of the unknown quantities is much greater than the number of the equations (211), (212) (two vectors equations, equivalent to six scalar ones) available for their determination. In order to make this problem mathematically wholly determinate one, various additional hypotheses about the mechanical nature of the geometrical constraints are made, which reduce the number of the unknown quantities to six. (A classical for analytical statics hypothesis of this kind, possessing a sound physical motivation at that, consists in the assumption that the geometrical constraints are *smooth*, i.e. that they generate reactions perpendicular to the corresponding constraints.)

It is clear that the equations (211), (212) include unknown quantities of two kinds: first, the parameters (192) of B; and, second, the reactions $\boldsymbol{R}_\nu$ $(\nu = 1, \ldots, n)$.

With a view to a better understanding of the essence of the problem, let us suppose that $Oxyz$ is an orthonormal right-hand orientated Cartesian system of reference, the origin O of which coincides with the zero-pole (i.e. the pole, with respect to which the moments $\boldsymbol{M}_\mu$ and $\boldsymbol{N}_\nu$ $(\mu = 1, \ldots, m;\ \nu = 1, \ldots, n)$ of (193) and (194) respectively are taken). If $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the vectores of the axes Ox, Oy, Oz respectively, then these conditions imply

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1, \qquad \boldsymbol{ij} = \boldsymbol{jk} = \boldsymbol{ki} = 0 \tag{213}$$

and

$$\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}. \tag{214}$$

Let by definition

$$\boldsymbol{F}_\mu = F_{\mu x}\boldsymbol{i} + F_{\mu y}\boldsymbol{j} + F_{\mu z}\boldsymbol{k} \qquad (\mu = 1, \ldots, m), \tag{215}$$

$$\boldsymbol{M}_\mu = M_{\mu x}\boldsymbol{i} + M_{\mu y}\boldsymbol{j} + M_{\mu z}\boldsymbol{k} \qquad (\mu = 1, \ldots, m), \tag{216}$$

$$\boldsymbol{R}_\nu = R_{\nu x}\boldsymbol{i} + R_{\nu y}\boldsymbol{j} + R_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n), \tag{217}$$

$$\boldsymbol{c}_\nu = c_{\nu x}\boldsymbol{i} + c_{\nu y}\boldsymbol{j} + c_{\nu z}\boldsymbol{k} \qquad (\nu = 1, \ldots, n). \tag{218}$$

At these notations the vector equation (211) is equivalent to the following three scalar equations

$$\sum_{\mu=1}^{m} F_{\mu x} + \sum_{\nu=1}^{n} R_{\nu x} = 0, \tag{219}$$

$$\sum_{\mu=1}^{m} F_{\mu y} + \sum_{\nu=1}^{n} R_{\nu y} = 0, \tag{220}$$

$$\sum_{\mu=1}^{m} F_{\mu z} + \sum_{\nu=1}^{n} R_{\nu z} = 0, \tag{221}$$

and the vector equation (212) is equivalent to the following three scalar equations

$$\sum_{\mu=1}^{m} M_{\mu x} + \sum_{\nu=1}^{n} (c_{\nu y} R_{\nu z} - c_{\nu z} R_{\nu y}) = 0, \tag{222}$$

$$\sum_{\mu=1}^{m} M_{\mu y} + \sum_{\nu=1}^{n} (c_{\nu z} R_{\nu x} - c_{\nu x} R_{\nu z}) = 0, \tag{223}$$

$$\sum_{\mu=1}^{m} M_{\mu z} + \sum_{\nu=1}^{n} (c_{\nu x} R_{\nu y} - c_{\nu y} R_{\nu x}) = 0. \tag{224}$$

In such a way, the statical problem is reduced to the determination, by means of the equations (219) — (224), of the equilibrium values (if any) of the parameters (192) of B and of the equilibrium reactions (if any) of the constraints

$$(225) \qquad R_{\nu x}, R_{\nu y}, R_{\nu z} \qquad (\nu = 1, \ldots, n).$$

The expression "if any" used above means "if such exist". Now some explanations in this connection are more than indispensable.

Any statical problem consists, as a matter of fact, of three subproblems:

1. Does equilibrium exist?

2. If yes, then which is the equilibrium position of the rigid body?

3. If yes, then which are the reactions that realize it?

The first question is equivalent with the problem of existence of a solution of the statical problem. As in any mathematical problem, in a statical problem the existence of a solution is a fact that must be proved *ad hoc*, rather than a presumptive postulate.

As a matter of fact, the statical problem under consideration has a solution if, and only if, the system of equations (219) — (224) is consistent. And it is consistent if, and only if, there exist such quantities (192) and (225) that satisfy it. Tautological though these statements seem, they emphasize the importance of the existence problem in statics, which in mechanics in general not infrequent is found to be neglected.

The second question is equivalent with the problem of determination of the equilibrium values (i.e. those satisfying the system of equations (219) — (224)) of the parameters (192) of the rigid body.

The third problem is equivalent to the problem of determination of the equilibrium values (i.e. those satisfying the system of equations (219) — (224)) of the components (225) of the reactions of the geometrical constraints.

And that is that. In such, and in such only, sense the statement in Sch 28 must be understood, namely that "a system of arrows (66) is of interest to statics and dynamics, inasmuch as its basis $\boldsymbol{s}$ and its moment $\boldsymbol{m}$ are concerned, rather than the particular arrows entering into its composition": the conditions of equilibrium (210) are expressed by means of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system (199) of all forces (passive as well as active) acting on the rigid body, and not by means of the particular arrows this system contains.

Sch 31. An analogous situation is observed when dealing with a dynamical problem concerning a rigid body.

The fundamental problem of rigid body dynamics may be formulated in the following manner.

Let a rigid body B be given, subjected to n geometrical constraints with points of contact (204) which generate the reactions (194), and let B be under the action of the active forces (193). Initial conditions (207), (208) being prescribed to the parameters (192) of B and to their velocities (201), find the motion of B and the reactions of the constraints.

In this formulation the term *motion* has been used. It is a fundamental mechanical notion and it requires a special definition. Since this definition is a rather

intricate one, we shall not reproduce it here and, making a long story short, we shall substitute a mathematical surrogate for it.

To this end, let us first introduce two fundamental dynamical attributes for any rigid body.

If P is an arbitrary point of the rigid body B, $\boldsymbol{r} = \boldsymbol{OP}$, $v = dr/dt$ (the derivative being taken with respect to the system of reference $Oxyz$), and dm denotes a "mass-element" of B (all these terms being duly defined), then

$$\boldsymbol{K} = \int v dm \tag{226}$$

and

$$\boldsymbol{L} = \int \boldsymbol{r} \times v dm \tag{227}$$

are by definition the *momentum* and the *moment of momentum* (*kinetical moment*) of B respectively with regard to $Oxyz$, the integrals in (226) and (227) being taken over the part of the space occupied by the rigid body.

One of the greatest discoveries in all the history not only of rational mechanics, but of the whole mathematical physics as well, was made by Euler in 1775, and is reflected in his article [7]. In accordance with it, there exists one at least system of reference (*inertial system of reference*) that, all derivatives being taken with respect to it, for any rigid body B and for any system of forces $\underset{\rightarrow}{F}$ (passive as well as active) acting on B, the derivative with respect to the time t of the momentum and of the moment of momentum of B are equal to the basis and the moment of F respectively, both moments being taken with respect to the origin of $Oxyz$.

These laws or principles of Euler are called the *first* and the *second Eulerian dynamical axioms* respectively (or *Euler's laws* or *principles of momentum* and *of moment of momentum* of a rigid body respectively).

The mathematical formulation of both Eulerian dynamical axioms read

$$\frac{d}{dt} \int v dm = \boldsymbol{F} \tag{228}$$

and

$$\frac{d}{dt} \int \boldsymbol{r} \times v dm = \boldsymbol{M} \tag{229}$$

respectively provided (193) — (199).

These preliminaries settled, it is said that a rigid body is *moving* or is *in motion* under the action of the system of forces $\underset{\rightarrow}{F}(\boldsymbol{F}, \boldsymbol{M})$ when (228) and (229) hold good.

In such a manner, it is seen that the necessary and sufficient conditions (228), (229) for the motion of the rigid body are expressed exclusively by the aid of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system of forces (199) acting on the rigid body. Let us, however, discuss this formulation somewhat closer.

The equations (228) and (229) may be written in the form

$$\frac{d}{dt}\int v\,dm = \sum_{\mu=1}^{m} F_\mu + \sum_{\nu=1}^{n} R_\nu \tag{230}$$

and

$$\frac{d}{dt}\int r \times v\,dm = \sum_{\mu=1}^{m} M_\mu + \sum_{\nu=1}^{n} c_\nu \times R_\nu \tag{231}$$

respectively, by virtue of (198), (196), and (206). In such a manner, the fundamental problem of rigid body dynamics is reduced mathematically to the vector equations (230), (231), where the unknown quantities are the parameters (192) of the rigid body B as functions (209) of the time t and the reactions R_ν $(\nu = 1, \ldots, n)$ of the constraints. As in the statical case, in this general formulation the dynamical problem is indeterminate: as a rule, the number of the unknown quantities is much greater than the number of the equations (230), (231) (two vector equations equivalent to six scalar ones) available for their determination. In order to make this problem mathematically a whole determinate one, in dynamics, as well as in statics, various additional hypotheses about the mechanical nature of the geometrical constraints are made, which reduce the number of the unknown quantities to six. (As in statics, the most popular, traditional, and even classical hypothesis of this kind is the *postulate of smooth geometrical constraints.*)

It is clear that the equations (230), (231) include unknown quantities of two kinds: first, the parameters (192) of B, their velocities (201), and their accelerations

$$\ddot{q}_\lambda \in R \qquad (\lambda = 1, \ldots, l) \tag{232}$$

and, second, the reactions R_ν $(\nu = 1, \ldots, n)$. In other words, the equations (230), (231) are, as the saying goes, of a heterogeneous type: being differential equations of second order with respect to the time t with regard to the unknown functions (209), they are, in the same time, linear algebraic equations with regard to the unknown reactions of the costraints R_ν $(\nu = 1, \ldots, n)$.

As in the statical case, $Oxyz$ being an inertial system of reference with (213), (214), let by definition the relations (215) — (218) hold. Then the vector equation (230) is equivalent to the following three scalar equations:

$$\frac{d}{dt}\int \dot{x}\,dm = \sum_{\mu=1}^{m} F_{\mu x} + \sum_{\nu=1}^{n} R_{\nu x}, \tag{233}$$

$$\frac{d}{dt}\int \dot{y}\,dm = \sum_{\mu=1}^{m} F_{\mu y} + \sum_{\nu=1}^{n} R_{\nu y}, \tag{234}$$

$$\frac{d}{dt}\int \dot{z}\,dm = \sum_{\mu=1}^{m} F_{\mu z} + \sum_{\nu=1}^{n} R_{\nu z} \tag{235}$$

provided

$$r = xi + yj + zk \tag{236}$$

and the vector equation (231) is equivalent with the following three scalar equations:

$$(237) \qquad \frac{d}{dt}\int(y\dot{z}-z\dot{y})dm = \sum_{\mu=1}^{m} M_{\mu x} + \sum_{\nu=1}^{n}(c_{\nu y}R_{\nu z} - c_{\nu z}R_{\nu y}),$$

$$(238) \qquad \frac{d}{dt}\int(z\dot{x}-x\dot{z})dm = \sum_{\mu=1}^{m} M_{\mu y} + \sum_{\nu=1}^{n}(c_{\nu z}R_{\nu x} - c_{\nu x}R_{\nu z}),$$

$$(239) \qquad \frac{d}{dt}\int(x\dot{y}-y\dot{x})dm = \sum_{\mu=1}^{m} M_{\mu z} + \sum_{\nu=1}^{n}(c_{\nu x}R_{\nu y} - c_{\nu y}R_{\nu x}).$$

In such a way, the dynamical problem is reduced to the determination, by means of the equations (233) — (235) and (237) — (239) (or by the aid of other systems of differential equations, mathematically equivalent to (233) — (235) and (237) — (239), but technically much more convenient; dynamical equations, containing such characteristic for a rigid body quantities, as its moments of inertia and moments of deviation, have been proposed by Euler and are called today the *Eulerian dynamical equations*), of those functions (209) (if any) that govern the motion of rigid body and of the reactions of the constraints (225) (if any) that produce this motion (along with the active forces (193) acting on the rigid body).

The expression "if any" used above means "if such exists". As in statics, some explanations in this connection are more than unavoidable.

As a matter of fact, any dynamical problem consists of three subproblems:

1. Does motion exist?
2. If yes, then which is the motion of the rigid body?
3. If yes, then which are the reactions that realize it?

The first question is equivalent to the problem of existence of a solution of the dynamical problem. As in any mathematical problem, *in a dynamical problem the existence of a solution is not an a priori datum*, but a fact that needs an *ad hoc* demonstration. As a matter of fact, the dynamical problem under consideration has a solution if, and only if, the system of equations (233) — (235) and (237) — (239) is consistent.

This point needs a closer elucidation. Most of the authors of mechanical writings do not investigate at all the problem of existence of solutions of statical and dynamical problems. One of the reasons for such a mathematical behaviour is purely psychological. Solution of a statical or dynamical problem means, for these authors, rest or motion of the rigid body involved, and the existence of rest and motion is regarded by them as something self-apparent, self-evident, and self-explanatory.

This is not so. As Brelot [8] notes:

"On sait que lorsqu'on schématise selon l'habitude des problèmes méchaniques concrete physiquement possibles, on obtient parfois des problèmes mathématiques impossibles ... Exemples avec le frottement (Painlevé), exemple connu de la barre pesante diamétrale glissant sans frottement dans une sphère" (p. 7).

Writing "des problèmes mathèmatiques impossibles" this author has in mind mechanical problems without solutions. In this connection the reader may see, for instance, our article [9].

Another reason that explains the traditional carelessness of most authors of mechanical writings as regards the existence of dynamical solutions is rooted in the fact that, in their majority, they solve dynamical problems by the aid of the *Lagrangean dynamical equations*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} - \frac{\partial T}{\partial q_\lambda} - Q_\lambda = 0 \qquad (\lambda = 1, \ldots, l), \tag{240}$$

rather than by means of the Eulerian dynamical axioms (228), (229). Now the equations (240) do not contain the unknown reactions of the costraints (194): these reactions are expelled from the mechanical paradise by the aid of the *postulate of ideal constraints*

$$\sum_{\nu=1}^{n} \boldsymbol{R}_\nu d\boldsymbol{c}_\nu = 0, \tag{241}$$

and this act of heroism is regarded by the adherents of Lagrangean dynamical tradition as the chief achievement of their Teacher. The reactions of the constraints are, however, the mathematical factor that may deprive a dynamical problem of its solution. In such a manner, the Lagrangean dynamical equations (240), faced with the existence problem, may be compared with daltonists faced with a colour-table. Moreover, they are pointblank blind when confronted with this problem. It is needless to underline that the Lagrangeanists are as helpless in front of the third of the above questions (which are the reactions of the constraints?) as they are in front of the first of them: the only thing the Lagrangean dynamical equations (240) can do is to describe mathematically the motion *if any*.

The second question is equivalent to the solving of the equations (240) which are neither more nor less than the projections of the Eulerian dynamical axioms (228), (229) on appropriate axes: these axes are chosen in such a manner that the unknown reactions of the constraints (194) vanish when (228), (229) are projected on them. In such a way the functions (209) are determined, satisfying the initial conditions (207), (208).

The third question is equivalent to the problem of determination of the reactions (194) of the constraints. Since they take part linearly in the equations (230), (231), this determination, provided the functions (209) are known, offers no difficulties.

And that is that. In such, and in such only, sense the statement in Sch 28 must be understood, namely that "the systems of arrows available in the mechanical problem in the capacity of forces ... are authoritative by means of their bases and moments rather than by themselves as individual mechanical entities": the conditions of motion (228), (229) are expressed by means of the basis $\boldsymbol{F}$ and the moment $\boldsymbol{M}$ of the system (199) of all forces (passive as well as active) acting on the rigid body, and not by means of the particular arrows this system contains.

Sch 32. Analytical mechanics is sometimes described as the mathematics of the equilibria and motions of mass-points and rigid bodies, and of the forces that generate these equilibria and motions and are generated by them.

This description is clear in the light of the above explanations in connection with the fundamental problems of statics and dynamics. Not less clear is also that

the theory of arrows is predestinated to play an important role in the problems of analytical mechanics.

Still the first generations of mechanicians have become well aware of this fact. Not a few experimental and intellectual work has been dedicated to the doctrine of forces, aiming at the elucidation of various aspects of the force-concept from all angles. In order to realize that it is sufficient to remember the amount of painstaking work accomplished by mathematicians and physicists in connection with the problem of the *parallelogram of forces.*

This problem is a most remarkable one. Let B be a *physical* rigid body and let P_1 and P_2 be its points. Let Q_1 and Q_2 be small fixed pulleys and let w_1 and w_2 be weights fastened to the ends of two inextensible flexible strings, the other ends of which are fastened to P_1 and P_2 respectively, and which are passed through Q_1 and Q_2 respectively. If B is in equilibrium under the action of its weight w_3 and of the weights w_1 and w_2, then Q_1, Q_2, and the mass-center G of B lie in a vertical plane, as a simple experiment displays.

Let e_ν be the unit vectors of the vectors $P_\nu Q_\nu$ $(\nu = 1, 2)$ respectively and let e_3 be the unit vector of the downwards directed vertical. Let by definition $s_\nu = w_\nu e_\nu$ $(\nu = 1, 2, 3)$. If O is any fixed point, chosen for the zero-pole, let by definition $m_\nu = OP_\nu \times s_\nu$ $(\nu = 1, 2)$, $m_3 = OG \times s_3$. Then obviously

$$\overrightarrow{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2, 3), \tag{242}$$

and the rigid body B is in equilibrium under the action of the forces (242) iff

$$\sum_{\nu=1}^{3} s_\nu = 0, \qquad \sum_{\nu=1}^{3} m_\nu = 0. \tag{243}$$

Only a genius could guess that the action of the forces $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$ could be replaced, without disturbing the equilibrium of B, by a single force $\overrightarrow{s}$, the basis s of which is equal to the sum $s_1 + s_2$ of the bases of $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$, and the directrix of which is passing through G. The name of this genius is unknown. It is also an enigma where, when, and who was the first to confirm this brilliant conjecture by an experiment fated to become classical even in the primary course in physics today. It is easily seen that

$$\overrightarrow{s} = (s_1 + s_2, m_1 + m_2), \tag{244}$$

i.e. that

$$\overrightarrow{s} = \overrightarrow{s}_1 + \overrightarrow{s}_2. \tag{245}$$

This story is instructive to the highest degree.

In the first place, here lies the germ of the idea of *reduction of a system of forces.*

In the second place, here lies the germ of the notion of *equivalent systems of forces.*

In the third place, here lies the germ of the concept of *sum of two forces.*

In the fourth place, here lies the germ of the idea of *elementary statical operations.*

In the fifth place, this proces has provoked the mathematicians to solve the problem of the *parallelogram of forces* and to initiate, in such a manner, a scientific politics in mechanics aimed at its mathematification — a tendency half-finished as yet.

For what reason did the ancient mechanician invent the intricate mechanism with the pulleys described above? The answer is obvious: simplicity. Two arrows are something more complex than a single one. Now this simple discovery may be generalized. Anyone knows from his school-days how this device is accomodated to the addition of two arrows with parallel directrices, etc. The common of all these processes is the simplification of a system of arrows by diminishing the number of its elements.

It must not be left unnoticed that replacing two arrows by their sum in a system of arrows $\underset{\rightarrow}{s}$ one does not change the basis and the moment of $\underset{\rightarrow}{s}$. In the due course of time the mechanicians became concious of the fact that there exist four elementary operations, by means of which any system of forces may be transformed into the simplest possible form; at that it preserves unchanged its basis and its moment. This mathematical process has been called the reduction of the system of forces in question.

All these facts, initially established in the statical case, have been inductively transferred from it to the dynamical one. The fact, that a reduced system of forces has the same, dynamical as well as statical, effect as the starting system, has been realized long before the definition (210) of equilibrium has been formulated in its complete generality, and out and away earlier than the Eulerian dynamical axioms (228), (229) have been proclaimed.

After these explications the aim of which is to give a physical motivation and a heuristic background for the following considerations, we shall proceed now to the mathematical formalization of the circumstances described,

Sgn 9. $\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_2}$ sgn:

$$s_1 = s_2, \qquad m_1 = m_2 \tag{246}$$

iff

$$\underset{\rightarrow}{s_\nu}(s_\nu, m_\nu) \in \Sigma_S \qquad (\nu = 1, 2). \tag{247}$$

Df 12. $\underset{\rightarrow}{s_1}$ is called equivalent to $\underset{\rightarrow}{s_2}$ if

$$\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_2}. \tag{248}$$

Pr 19. (66) implies $\underset{\rightarrow}{s} \sim \underset{\rightarrow}{s}$.

Dm. Sgn 9.

Pr 20. If

$$\underset{\rightarrow}{s_\nu} \in \Sigma_S \qquad (\nu = 1, 2), \tag{249}$$

(248), then $\underset{\rightarrow}{s_2} \sim \underset{\rightarrow}{s_1}$.

Dm. Sgn 9.

Pr 21. If

$$\underset{\rightarrow}{s_\nu} \in \Sigma_S \qquad (\nu = 1, 2, 3), \tag{250}$$

(248), $\underset{\rightarrow}{s_2} \sim \underset{\rightarrow}{s_3}$, then $\underset{\rightarrow}{s_1} \sim \underset{\rightarrow}{s_3}$.

Dm. Sgn 9.

Pr 22. The relation $\sim$ in Σ_S defined by Sgn 9 is an equivalence relation in Σ_S.

Dm. Pr 19 — Pr 21.

Df 13. Any equivalence class in Σ_S, generated by the relation $\sim$ in Σ_S, is called an S-action or an action in V_S.

Sgn 10. A_S sgn: the set of all actions in V_S.

Sch 33. In the light of the explanation given in Sch 29 — Sch 31 one could say that namely the actions in V, alias the elements of A_R, rather than the particular systems of arrows in V, alias the elements of Σ_R, are that determine the mechanical (dynamical as well as statical) behaviour of a rigid body in analytical mechanics, the term behaviour meaning equilibrium or motion.

Sch 34. $\underset{\rightarrow}{s}$ being a system of arrows, the qualification of $\underset{\rightarrow}{s}$ as simple or as complicated is deprived of the possibilities of an objective judgement, save, maybe, by means of the number of its elements. And yet, immediately below four types of systems of arrows are proposed, any of them with its particular designation, in the capacity of the simplest possible kinds of systems.

Sgn 11. $\underset{\rightarrow}{o}$ sgn: $\{\overrightarrow{o}\}$.

Df 14. $\underset{\rightarrow}{o}$ is called the *zero-system*.

Df 15. $\{\overrightarrow{s}\}$ is called a *monosystem* iff $\overrightarrow{o} \neq \overrightarrow{s} \in W_S$.

Df 16. $\{\overrightarrow{s}_\nu\}_{\nu=1}^2$ is a called a *dipole* iff

$$\overrightarrow{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2), \tag{251}$$

$$s_1 + s_2 = o, \qquad m_1 + m_2 \neq \mathfrak{o}. \tag{252}$$

Df 17. $\{\overrightarrow{s}_\nu\}_{\nu=1}^2$ is called a *bisystem* iff (251),

$$s_1 \times s_2 \neq \mathfrak{o}, \qquad s_1 m_2 + s_2 m_1 \neq 0. \tag{253}$$

Sgn 12. $\Sigma_\nu (\nu = 0, 1, 2, 3)$ sgn: $\underset{\rightarrow}{o}$ iff $\nu = 0$; the set of all dipoles iff $\nu = 1$; the set of all monosystems iff $\nu = 2$; the set of all bisystems iff $\nu = 3$.

Sgn 13. Σ sgn: the union of $\Sigma_\nu (\nu = 0, 1, 2, 3)$.

Pr 23. $\{\overrightarrow{s}_\nu\}_{\nu=1}^2 \in \Sigma_1$ implies $\overrightarrow{s}_1 \uparrow\downarrow \overrightarrow{s}_2$.

Dm. Sgn 12, Df 16, [2, 3 Sgn 7].

Pr 24. $\{\overrightarrow{s}_\nu\}_{\nu=1}^2 \in \Sigma_3$ implies $\overrightarrow{s}_1 \otimes \overrightarrow{s}_2$.

Dm. Sgn 12, Df 17 [2, 5 Sgn 3].

Pr 25. (66) implies: there exists $\underset{\rightarrow}{\sigma} \in \Sigma$ with

$$\underset{\rightarrow}{s} \sim \underset{\rightarrow}{\sigma}. \tag{254}$$

Dm. If (11), then one exactly of the cases (86), (88), (90), and (92) is possible.

If (86), then $\underset{\rightarrow}{\sigma} = \underset{\rightarrow}{0}$ satisfies (254) (Sgn 11, [2, 2 Sgn 2], Sgn 9, Sgn 12, Sgn 13).

If (88), let $\overline{\rho}, \overline{\sigma} \in V_S$, $\overline{\rho} \times \overline{\sigma} = m$, $\overrightarrow{s}_1 = (-\overline{\sigma}, \mathfrak{o})$, $\overrightarrow{s}_2 = (\overline{\sigma}, m)$. Then $m \neq \mathfrak{o}$ implies $\overline{\sigma} \neq \mathfrak{o}$. Besides, obviously $\overline{\sigma} m = 0$. Hence

(255) $\overrightarrow{s}_\nu \in W_S$ $(\nu = 1, 2)$

(Sgn 1). If

(256) $\underset{\rightarrow}{\sigma} = \{\overrightarrow{s}_\nu\}_{\nu=1}^{2}$,

then $\underset{\rightarrow}{\sigma} \in \Sigma_1$ (Df 16, Sgn 12) and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sgn 9, Sgn 13).

If (90), let by definition (9) hold (Sgn 1). If $\underset{\rightarrow}{\sigma} = \{\overrightarrow{s}\}$, then $\underset{\rightarrow}{\sigma} \in \Sigma_2$ (Df 15, Sgn 12) and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sgn 9, Sgn 13).

If (92), let $s_1 \in V_S$,

(257) $s_1 \times s \neq o, \qquad s_1 m = sm$

and let by definition

(258) $\overrightarrow{s}_\nu = (s_\nu, m_\nu) \qquad (\nu = 1, 2)$

provided

(259) $m_1 = o, \qquad s_2 = s - s_1, \qquad m_2 = m.$

Then (257) — (259) imply (255) (Sgn 1). If (256), then $\underset{\rightarrow}{\sigma} \in \Sigma_3$ (Df 17, Sgn 12), since (257), (259), (92) imply

(260) $s_1 \times s_2 = s_1 \times s \neq o, \qquad s_1 m_2 + s_2 m_1 = s_1 m \neq 0,$

and $\underset{\rightarrow}{\sigma}$ satisfies (254) (Sign 9, Sgn 13).

The following proposition is an immediate corollary from Pr 25 and from the rank-theorem.

Pr 26. (66) implies: a necessary and sufficient condition for the existence of a

(261) $\underset{\rightarrow}{\sigma} \in \Sigma_\nu$ $(\nu = 0, 1, 2, 3)$

with (254) is

(262) $\operatorname{rank} \underset{\rightarrow}{s} = \nu$ $(\nu = 0, 1, 2, 3)$

respectively.

Sch 35. Naturally, the ancient mechanicians worked, technically at least, in a quite different way. Not disposing with an algebraic definition of the arrow concept and with all the technical facilitations this definition proposes, they were compelled to work synthetic-geometrically and invented, with an eye on the reduction of the systems of forces, the so-called *elementary statical operations*. We give here a non-formal description of these operations.

Let $\underset{\rightarrow}{s}$ be a system of forces and let two of its elements $\overrightarrow{s}_\nu \in \underset{\rightarrow}{s} (\nu = 1, 2)$ possess a sum $\overrightarrow{s}_1 + \overrightarrow{s}_2$. Let $\underset{\rightarrow}{\sigma}$ be a system of forces, defined in the following manner: $\underset{\rightarrow}{\sigma}$ consists of all elements of $\underset{\rightarrow}{s}$ except for $\overrightarrow{s}_1$ and $\overrightarrow{s}_2$; besides, $\overrightarrow{s}_1 + \overrightarrow{s}_2 \in \underset{\rightarrow}{\sigma}$. Then it is said that $\underset{\rightarrow}{\sigma}$ is obtained from $\underset{\rightarrow}{s}$ by means of the *first elementary statical operation*, and the transition from $\underset{\rightarrow}{s}$ to $\underset{\rightarrow}{\sigma}$ or, alias, the substitution of

$\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *first elementary statical operation.*

Let $\underrightarrow{s}$ be a system of forces and let one of its elements $\overrightarrow{s} \in \underrightarrow{s}$ be decomposed in two addends $\overrightarrow{s}_\nu (\nu = 1, 2)$, i.e. $\overrightarrow{s} = \overrightarrow{s}_1 + \overrightarrow{s}_2$. Let $\underrightarrow{\sigma}$ be a system of forces defined in the following manner: $\underrightarrow{\sigma}$ consists of all elements of $\underrightarrow{s}$ except for $\overrightarrow{s}$; besides, $\overrightarrow{s}_\nu \in \underrightarrow{\sigma} (\nu = 1, 2)$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *second elementary statical operation*, and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *second elementary statical operation.*

Let $\underrightarrow{s}$ be a system of forces including the zero-arrow, i.e. $\overrightarrow{o} \in \underrightarrow{s}$. Let $\underrightarrow{\sigma}$ be a system of forces defined in the following manner: $\underrightarrow{\sigma}$ consists of all elements of $\underrightarrow{s}$ except for $\overrightarrow{o}$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *third elementary statical operation*, and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *third elementary statical operation.*

Let $\underrightarrow{s}$ be any system of forces and let the system of forces $\underrightarrow{\sigma}$ be defined in the following manner: $\underrightarrow{\sigma}$ consists of all the elements of $\underrightarrow{s}$ and, moreover, of the zero-arrow $\overrightarrow{o}$. Then it is said that $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of the *fourth elementary statical operation* and the transition from $\underrightarrow{s}$ to $\underrightarrow{\sigma}$ or, alias, the substitution of $\underrightarrow{\sigma}$ for $\underrightarrow{s}$ in forthcoming mechanical considerations is called the *fourth elemehtary statical operation.*

It is trivially seen that, if $\underrightarrow{\sigma}$ is obtained from $\underrightarrow{s}$ by means of any of the described four elementary statical operations, then $\underrightarrow{s} \sim \underrightarrow{\sigma}$. In other words, *any of the four elementary statical operations preserves the basis and the moment of the initial system of forces.*

It is self-evident that the desultory application of the four elementary statical operations on a particular system of forces $\underrightarrow{s}$ leads to a dead-end. It turn out, however, that by the aid of purposeful applications of these operations on any system of forces $\underrightarrow{s}$ the latter may be reduced to a zero-system, or to a dipole, or to a mono-system, or, at last, to a bi-system, in other words, to some of the systems of $\Sigma_\nu (0 \leqq \nu \leqq 3)$ [10, p. 115—122]. This fact reveals the meaning of the term reduction of a system of arrows.

A system of arrows $\underrightarrow{s}$ being given, the process of discovering, by means of the elementary statical operations, a system $\underrightarrow{\sigma} \in \Sigma_\nu (0 \leqq \nu \leqq 3)$ with $\underrightarrow{s} \sim \underrightarrow{\sigma}$, is called a *reduction* of $\underrightarrow{s}$.

N. B. 1. This is obviously a non-mathematical description of the term reduction.

N. B. 2. In the above description it is said "a reduction" and not "the reduction". The reason for that is the fact that a reduction of a system of arrows $\underrightarrow{s}$ is not an univocally determined mathematical phenomenon: if (11) and $s = 0$, $m \neq 0$, then there exist two at least different $\underrightarrow{\sigma_\nu} \in \Sigma_1$ with $\underrightarrow{s} \sim \underrightarrow{\sigma_\nu} (\nu = 1, 2)$. Similarly, if (11) and $sm \neq 0$, then there exist two at least different $\underrightarrow{\sigma_\nu} \in \Sigma_3$ with

$\underset{\rightarrow}{s} \sim \underset{\rightarrow}{\sigma_\nu} (\nu = 1, 2)$.

Sch 36. The indefiniteness of the simplest dipoles or bisystems equivalent with a given system of arrows $\underset{\rightarrow}{s}$ in the cases rank $\underset{\rightarrow}{s} = 1$ and rank $\underset{\rightarrow}{s} = 3$ respectively leads to the formulation of various problems aimed at the clarification of the nature and degree of this indefiniteness. Some of these problems are discussed immediately below.

Sch 37. The following question quite naturally arises. A system of arrows (66) with (91), as well as (57), being given, do there exist (255) with

$$r_\nu \; Z \; \mathrm{dis} \, \overrightarrow{s}_\nu \qquad (\nu = 1, 2) \tag{263}$$

and

$$\underset{\rightarrow}{s} \sim \{\overrightarrow{s}_\nu\}_{\nu=1}^2. \tag{264}$$

Pr 26 implies that a necessary and sufficient condition for (264) is

$$\{\overrightarrow{s}_\nu\}_{\nu=1}^2 \in \Sigma_3, \tag{265}$$

and Sgn 12 implies that (265) is equivalent with (253) provided (258). In other words, the problem is equivalent with the following one.

If (57) and

$$(s, m) \in V_S^2, \tag{266}$$

(92), then do there exist

$$(s_\nu, m_\nu) \in V_S^2 \qquad (\nu = 1, 2) \tag{267}$$

with (253),

$$s_\nu m_\nu = 0 \qquad (\nu = 1, 2), \tag{268}$$

$$s_1 + s_2 = s, \qquad m_1 + m_2 = m, \tag{269}$$

$$r_\nu \times s_\nu = m_\nu \qquad (\nu = 1, 2). \tag{270}$$

In order to answer this question, let us first note that necessarily (68) must hold. Indeed, the supposition $r_1 = r_2 = r$, along with (270), implies

$$r \times s_\nu = m_\nu \qquad (\nu = 1, 2), \tag{271}$$

contrary to the second relation (253) [2, 1 Pr 28].

Let us suppose that the above question is answered in the affirmative, and let by definition

$$m_1 = r_1 \times s_1, \tag{272}$$

$$s_2 = s - s_1, \tag{273}$$

$$m_2 = r_2 \times (s - s_1). \tag{274}$$

In other words, the left-hand sides of (272) — (274) are certainly known if s_1 is known, and the problem is reduced to the determination of s_1 namely.

Such a s_1 does not necessarily exist. Indeed, (272), (274), and the second relation (269) imply

(275) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}_1 = \boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2,$$

and (275) implies that the condition

(276) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) = 0$$

is necessary for the affirmative answer of the above question. Let us discuss this problem somewhat closer.

Sgn 4 implies that (275) is equivalent with

(277) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}_1 = \mathrm{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s},$$

and (276) is equivalent to

(278) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\boldsymbol{r}_2} \underset{\rightarrow}{s} = 0.$$

On the other hand , (278) and (67) imply

(279) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \mathrm{mom}_{\mathbf{r}_1} \underset{\rightarrow}{s} = 0,$$

and (279), Sgn 4 imply

(280) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1) = 0.$$

The relations (276) and (280) are, however, not mutually independent. Indeed, the supposition

(281) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2)(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_1) \neq 0$$

leads, along with (275), to the contradiction

(282) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot \boldsymbol{s} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2) \neq 0.$$

The conditions (68) and (276) display that the equation (275) with respect to $\boldsymbol{s}_1$ is consistent. As it is well known, (275) implies

(283) $$\boldsymbol{s}_1 = \lambda(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \frac{(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} \qquad (\lambda \in S),$$

and (283) implies

(284) $$\boldsymbol{s}_1 \times \boldsymbol{s} = \overline{\lambda}((\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s}) + \frac{(\boldsymbol{m} + \boldsymbol{s} \times \boldsymbol{r}_2) \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{(\boldsymbol{r}_1 - \boldsymbol{r}_2)^2} \times \boldsymbol{s}.$$

Two cases are possible:

(285) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s} = \mathbf{o}.$$

or

(286) $$(\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{s} \neq \mathbf{o}.$$

If (285), then (68), (92) imply

(287) $$\boldsymbol{s} = \mu(\boldsymbol{r}_1 - \boldsymbol{r}_2) \qquad (0 \neq \mu \in S),$$

and (284), (285), (287), (276) imply

(288) $$s_1 \times s = \overline{\mu}(m + s \times r_2).$$

The supposition

(289) $$m + s \times r_2 = o$$

is wrong, since it implies $sm = 0$ contrary to (92). In such a manner, if (185), then (287), (288) imply

(290) $$s_1 \times s \neq o.$$

If (286), then (284) and

(291) $$s_1 \times s = o$$

imply

(292) $$\lambda((r_1 - r_2) \times s)^2 = (r_1 - r_2) \times s \cdot s \times \frac{(m + s \times r_2) \times (r_1 - r_2)}{(r_1 - r_2)^2}.$$

Let by definition

(293) $$A = (r_1 - r_2) \times s \cdot s \times ((m + s \times r_2) \times (r_1 - r_2)).$$

Then

(294) $$\begin{aligned} A &= (s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s) \\ &- ((m + s \times r_2) \times (r_1 - r_2) \cdot (r_1 - r_2))s^2, \end{aligned}$$

i.e.

(295) $$A = (s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)$$

and (292), (293), (295) imply

(296) $$\lambda = \frac{(s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)}{(r_1 - r_2)^2((r_1 - r_2) \times s)^2}.$$

In other words, (291) is equivalent to (296).

In such a manner, (290) holds good for any $\lambda \in S$ if (285) and for any

(297) $$\lambda \neq \frac{(s(r_1 - r_2))((m + s \times r_2) \times (r_1 - r_2) \cdot s)}{(r_1 - r_2)^2((r_1 - r_2) \times s)^2}$$

if (286).

It will be now proved that (283) with any $\lambda \in S$ if (285) and with any $\lambda \in S$ satisfying (297) if (286), along with (272) — (274), propose a solution of the problem under consideration.

Indeed, (273) implies the first relation (269). On the other hand, (272) and (274) imply

(298) $$m_1 + m_2 = (r_1 - r_2) \times s + r_2 \times s.$$

Now (298) and (275) imply the second relation (269).

Besides, (272) and (274), (273) imply (270).

As regards the first relation (253), the definition (273) implies that it is equivalent to (290) and, by hypothesis, is satisfied for any $\lambda \in S$ in the case (285) and for any $\lambda \in S$ satisfying (287) in the case (286).

At last, the left-hand side of the relation (253) is equal to

$$(299) \qquad s_1 \cdot r_2 \times (s - s_1) + (s - s_1) \cdot r_1 \times s_1$$
$$= s_1 \cdot r_2 \times s + s \cdot r_1 \times s_1 = s_1 \cdot s \times (r_1 - r_2).$$

On the other hand, (283) implies

$$(300) \qquad s_1 \cdot s \times (r_1 - r_2) = \frac{(m + s \times r_2) \times (r_1 - r_2) \cdot s \times (r_1 - r_2)}{(r_1 - r_2)^2} = sm \neq 0$$

by virtue of (276) and (92), i.e. the second relation (253) is satisfied for any $\lambda \in S$. *Q. e. d.*

There do not exist solutions of the problem under consideration other than the ones described above, since the latter are consistent with all the necessary conditions for the existence of a solution, being their corollaries, as a matter of fact.

In such a way, the following proposition has been proved.

Pr 27. If $r_\nu \in V_S (\nu = 1, 2)$ and $\underset{\rightarrow}{s}(s, m)$ is a given system of arrows with rank $\underset{\rightarrow}{s} = 3$, then (68) and (276) are necessary conditions for the existence of

$$(301) \qquad \overrightarrow{s}_\nu = (s_\nu, m_\nu) \in W_S \qquad (\nu = 1, 2)$$

satisfying (263) and (264). These conditions are sufficient at the same time: if they are satisfied, then (301) defined by (283) with an arbitrary $\lambda \in S$ in the case (285) or with any $\lambda \in S$ satisfying (297) in the case (286) and by (272) — (274) are the only arrows satisfying (263) and (264).

Sch 38. The second problem may by formulated in the following manner. A system of arrows (66) with (91) being given, let (42) be any point and l with

$$(302) \qquad (a, b) \& l \qquad (a \neq o, ab = 0)$$

a line. The question now arises whether there exist arrows (255) with (264) and

$$(303) \qquad \overline{p} \; Z \; \mathrm{dir} \; \overrightarrow{s}_1,$$

$$(304) \qquad l = \mathrm{dir} \overrightarrow{s}_2.$$

The relations (303), (304) are equvalent to

$$(305) \qquad \overline{p} \times s_1 = m_1,$$

$$(306) \qquad s_2 = \lambda a, \qquad m_2 = \overline{\lambda} b \qquad (\lambda \in S)$$

respectively, by virtue of (302); and (264) is equivalent to (269). Now (305), (306), (269) imply

$$(307) \qquad s_1 + \lambda a = s, \qquad \overline{p} \times s_1 + \overline{\lambda} b = m,$$

and (307) imply

(308) $\qquad \overline{\rho} \times \boldsymbol{s} - \overline{\lambda}(\overline{\rho} \times \boldsymbol{a}) = \boldsymbol{m} - \overline{\lambda}\boldsymbol{b}.$

Then (308), (302) imply

(309) $\qquad (\overline{\rho} \times \boldsymbol{s} - \boldsymbol{m})\boldsymbol{a} = 0.$

In other words, (309) is a necessary condition for the affirmative answer of the problem under consideration.

Another necessary condition is that

(310) $\qquad \overline{\rho} \times \boldsymbol{s} \neq \boldsymbol{m}.$

Indeed,

(311) $\qquad \overline{\rho} \times \boldsymbol{s} = \boldsymbol{m}$

and Sgn 4 imply

(312) $\qquad \mathrm{mom}_{\overline{\rho}} \underset{\rightarrow}{s} = \mathrm{o}.$

Now (312) and Pr 4 imply $\boldsymbol{sm} = 0$ contrary to (92) (Pr 15).

A third necessary condition for the affirmative answer of the question under consideration reads

(313) $\qquad \overline{\rho} \times \boldsymbol{a} - \boldsymbol{b} \neq \mathrm{o}.$

Indeed, otherwise (302), (304) imply

(314) $\qquad \overline{\rho}\ \mathrm{Z}\ \mathrm{dir}\ \overrightarrow{s}_2,$

i.e.

(315) $\qquad \overline{\rho} \times \boldsymbol{s}_2 = \boldsymbol{m}_2.$

Now (305) and (315) contradict (253).

Let now (309), (310), and (313) be satisfied. The relations (307) imply

(316) $\qquad \overline{\rho} \times (\boldsymbol{s} - \lambda\boldsymbol{a}) + \overline{\lambda}\boldsymbol{b} = \boldsymbol{m},$

i.e.

(317) $\qquad \overline{\lambda}(\boldsymbol{b} - \overline{\rho} \times \boldsymbol{a}) = \boldsymbol{m} + \boldsymbol{s} \times \overline{\rho},$

and (317) implies

(318) $\qquad \lambda\,(\boldsymbol{s}(\boldsymbol{b} - \overline{\rho} \times \boldsymbol{a})) = \boldsymbol{sm}.$

The right-hand side of (318) is different from zero by virtue of (92). Therefore necessarily

(319) $\qquad (\overline{\rho} \times \boldsymbol{a} - \boldsymbol{b})\boldsymbol{s} \neq 0,$

and (318), (319) imply

$$\lambda = \frac{sm}{s(a \times \bar{\rho} + b)}. \tag{320}$$

Let us note that the necessary condition (319) is obviously stronger than the condition (313). Now (320) and the first relation (307) imply

$$s_1 = s - \left(\frac{sm}{s(a \times \bar{\rho} + b)}\right) a. \tag{321}$$

In such a manner, all unknown quantities are determined, provided the conditions of the problem are consistent. Now it remains to be proved that these quantities propose a solution of the problem.

First of all, (321) and the first relation (306) with (320) imply the first relation (269).

In the second place, (305) and the second relation (306) with (320), along with (308) and (321) imply

$$\begin{aligned} m_1 + m_2 &= \bar{\rho} \times s_1 + \bar{\lambda} b = \bar{\rho} \times s - \bar{\lambda}(\bar{\rho} \times a) + \bar{\lambda} b \\ &= m - \bar{\lambda} b + \bar{\lambda}(\bar{\rho} \times a) - \bar{\lambda}(\bar{\rho} \times a) + \bar{\lambda} b = m, \end{aligned} \tag{322}$$

i.e. the second relation (269).

In the third place, (321) with (320) and the first relation (306) imply

$$s_1 \times s_2 = s \times (\lambda a) = \bar{\lambda}(s \times a). \tag{323}$$

Now (320) and (92) imply $\bar{\lambda} \neq 0$. In other words, according to (323), the first relation (253) is equivalent to

$$s \times a \neq o, \tag{324}$$

and it is clear that the necessary condition (324) must be also hypothesized.

At last, (321) with (320) and (305), (306), (302) imply

$$\begin{aligned} s_1 m_2 + s_2 m_1 &= (s - \lambda a)(\lambda b) + (\lambda a)(\bar{\rho} \times (s - \lambda a)) \\ &= \lambda(sb) + \lambda(a \cdot \bar{\rho} \times s) = \lambda(sb + s \cdot a \times \bar{\rho}) = \lambda(s(b + a \times \bar{\rho})) \neq 0 \end{aligned} \tag{325}$$

by virtue of $\lambda \neq 0$ and (319).

In such a way, the following proposition is proved.

Pr 28. If $\underset{\rightarrow}{s}(s, m)$ is a given system of arrows with rank $\underset{\rightarrow}{s} = 3$, $\bar{\rho} \in V_S$ and the line l is defined by means of (302), then (309), (310), (319), and (324) are necessary conditions for the existence of arrows (301) with (264) and (303), (304). These conditions are sufficient too: if they are satisfied, then the only solution is proposed by (321) and (305), (306) with (320).

Sch 39. The last problem of this kind we shall now discuss is the following one. Let a system of arrows (11) with (91) be given and let

$$(a_\nu, b_\nu) \,\&\, l_\nu \qquad (\nu = 1, 2) \tag{326}$$

be two lines, i.e.

(327) $\qquad \boldsymbol{a}_\nu \neq \mathbf{o}, \qquad \boldsymbol{a}_\nu \boldsymbol{b}_\nu = 0 \qquad (\nu = 1, 2).$

Now we ask: do there exist two arrows (301) with (264) and

(328) $\qquad \operatorname{dir} \overrightarrow{s}_\nu = l_\nu \qquad (\nu = 1, 2).$

By virtue of (326), the relations (328) are equivalent to

(329) $\qquad \boldsymbol{s}_\nu = \lambda_\nu \boldsymbol{a}_\nu, \qquad \boldsymbol{m}_\nu = \overline{\lambda}_\nu \boldsymbol{b}_\nu \qquad (\nu = 1, 2)$

with appropriate

(330) $\qquad \lambda_\nu \in S \qquad (\nu = 1, 2).$

In view of (329), the relation (264) is equivalent to

(331) $\qquad \lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 = \boldsymbol{s}, \qquad \overline{\lambda}_1 \boldsymbol{b}_1 + \overline{\lambda}_2 \boldsymbol{b}_2 = \boldsymbol{m}.$

Pr 26, Sgn 12, Df 17, and (264) imply (253). Hence (329) imply

(332) $\qquad \lambda_\nu \neq 0 \qquad (\nu = 1, 2),$

(333) $\qquad \boldsymbol{a}_1 \times \boldsymbol{a}_2 \neq \mathbf{o}, \qquad \boldsymbol{a}_1 \boldsymbol{b}_2 + \boldsymbol{a}_2 \boldsymbol{b}_1 \neq 0.$

The first relation (331) implies

(334) $\qquad \lambda_1 \boldsymbol{a}_1^2 + \lambda_2 (\boldsymbol{a}_2 \boldsymbol{a}_1) = \boldsymbol{s}\boldsymbol{a}_1, \qquad \lambda_1 (\boldsymbol{a}_1 \boldsymbol{a}_2) + \lambda_2 \boldsymbol{a}_2^2 = \boldsymbol{s}\boldsymbol{a}_2,$

and the first relation (333) implies

(335) $\qquad \boldsymbol{a}_1^2 \boldsymbol{a}_2^2 - (\boldsymbol{a}_1 \boldsymbol{a}_2)(\boldsymbol{a}_2 \boldsymbol{a}_1) \neq 0.$

Now (334), (335) imply

(336) $$\lambda_1 = \frac{\begin{vmatrix} \boldsymbol{s}\boldsymbol{a}_1 & \boldsymbol{a}_2 \boldsymbol{a}_1 \\ \boldsymbol{s}\boldsymbol{a}_2 & \boldsymbol{a}_2^2 \end{vmatrix}}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}, \qquad \lambda_2 = \frac{\begin{vmatrix} \boldsymbol{a}_1^2 & \boldsymbol{s}\boldsymbol{a}_1 \\ \boldsymbol{a}_1 \boldsymbol{a}_2 & \boldsymbol{s}\boldsymbol{a}_2 \end{vmatrix}}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}$$

On the other hand, (331) implies that the condition

(337) $\qquad \boldsymbol{s} \cdot \boldsymbol{a}_1 \times \boldsymbol{a}_2 = 0$

is necessary for the consistency of the problem under consideration. Now (337) and the first relation (333) imply

(338) $\qquad \boldsymbol{s} = (\boldsymbol{s}\boldsymbol{a}_1^{-1})\boldsymbol{a}_1 + (\boldsymbol{s}\boldsymbol{a}_2^{-1})\boldsymbol{a}_2,$

where

(339) $$\boldsymbol{a}_1^{-1} = \frac{\boldsymbol{a}_2 \times (\boldsymbol{a}_1 \times \boldsymbol{a}_2)}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2}, \qquad \boldsymbol{a}_2^{-1} = \frac{(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \times \boldsymbol{a}_1}{(\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2},$$

and (339) imply

$$(340)\qquad (\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2 \boldsymbol{s}\boldsymbol{a}_1^{-1} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{s} \times \boldsymbol{a}_2 = (\boldsymbol{s}\boldsymbol{a}_1)\boldsymbol{a}_2^2 - (\boldsymbol{a}_2\boldsymbol{a}_1)(\boldsymbol{s}\boldsymbol{a}_2),$$

$$(341)\qquad (\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2 \boldsymbol{s}\boldsymbol{a}_2^{-1} = \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_1 \times \boldsymbol{s} = \boldsymbol{a}_1^2(\boldsymbol{s}\boldsymbol{a}_2) - (\boldsymbol{s}\boldsymbol{a}_1)(\boldsymbol{a}_1\boldsymbol{a}_2).$$

Now (340), (341), and (336) imply

$$(342)\qquad \lambda_\nu = \boldsymbol{s}\boldsymbol{a}_\nu^{-1} \qquad (\nu = 1, 2)$$

and (342), (331), (338), and the first relation (329) imply the first relation (269).

The second relation (269) is equivalent to the second relation (331), i.e. with

$$(343)\qquad \begin{vmatrix} \boldsymbol{a}_1\boldsymbol{s} & \boldsymbol{a}_1\boldsymbol{a}_2 \\ \boldsymbol{a}_2\boldsymbol{s} & \boldsymbol{a}_2^2 \end{vmatrix} \boldsymbol{b}_1 + \begin{vmatrix} \boldsymbol{a}_1^2 & \boldsymbol{a}_1\boldsymbol{s} \\ \boldsymbol{a}_2\boldsymbol{a}_1 & \boldsymbol{a}_2\boldsymbol{s} \end{vmatrix} \boldsymbol{b}_2 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2)^2 \boldsymbol{m}$$

by virtue of (336). In other words, the condition (343) is necessary for the affirmative answer of the question under consideration. As regards the conditions (253); they are satisfied in view of (329), (332), (333).

In such a manner, the following proposition is proved.

Pr 29. If $\underset{\rightarrow}{s}(\boldsymbol{s}, \boldsymbol{m})$ is a given system of arrows with rank = 3 and the lines l_ν $(\nu = 1, 2)$ are defined by means of (326) with (327), then the conditions (333), (337), and (343) are necessary for the existence of arrows (301) with (264) and (328). These conditions are sufficient too: if they are satisfied, then the only solution of the problem is proposed by (329) with (336).

Sch 40. As it has been promised, we shall conclude our exposition with some remarks in connection with the *statical-kinematical analogy*. Under this term the following mathematical phenomenon is understood.

A rigid body B being given and $\boldsymbol{a}$ denoting any of its points, there exists exactly one function $\overline{\omega}$ of the time t with the following property: a point $\boldsymbol{r}$ belongs to B if, and only if, the condition

$$(344)\qquad \frac{d}{dt}(\boldsymbol{r} - \boldsymbol{a}) = \overline{\omega} \times (\boldsymbol{r} - \boldsymbol{a})$$

is satisfied for any t.

This discovery is due to Euler, and $\overline{\omega}$ is called the *instantaneous angular velocity* of B. Euler's theorem (344) implies

$$(345)\qquad \frac{d\boldsymbol{r}_1}{dt} - \frac{d\boldsymbol{r}_2}{dt} = \overline{\omega} \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

for any two points $\boldsymbol{r}_\nu$ $(\nu = 1, 2)$ of B.

There is a formal analogy between (345) and the connection (64) between the moments of a system of arrows with respect to two poles $\boldsymbol{r}_\nu$ $(\nu = 1, 2)$. Indeed, it is sufficient to substitute $\mathrm{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s}$ for $d\boldsymbol{r}_\nu/dt$ $(\nu = 1, 2)$ respectively and $\boldsymbol{s}$ for $\overline{\omega}$ in (345) in order to obtain (64). Inversely, it is sufficient to substitute $d\boldsymbol{r}_\nu/dt$ for $\mathrm{mom}_{\mathbf{r}_\nu} \underset{\rightarrow}{s}$ $(\nu = 1, 2)$ respectively and $\overline{\omega}$ for $\boldsymbol{s}$ in (64) in order to obtain (345). This fact has far reaching consequences.

It turns out that for any theorem concerning systems of arrows there corresponds a true theorem concerning rigid body kinematics. At that, there is no need

to prove the kinematical theorem: it is certainly true, provided the arrow-theorem is proved. Furthermore, to any construction in the theory of the systems of arrows there corresponds a meaningful kinematical construction concerning rigid bodies. We shall enter in no details in this connection, and we shall confine our exposition to two examples only.

Let $\underset{\rightarrow}{\omega}(\overline{\omega}, \frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega})$ be a system of arrows *intrinsic* to the rigid body B by virtue of the statical-kinematical analogy. The kinematical analogue of the axis of a system of arrows (11) with (24) (Sgn 7) is the *helicoidal axis* of the rigid body B provided $\overline{\omega} \neq \mathbf{o}$, i.e. $\operatorname{ax} \underset{\rightarrow}{\omega}$, in other words the line l defined by

$$\left(\overline{\omega}, \frac{\overline{\omega} \times \left(\frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega}\right) \times \overline{\omega}}{\overline{\omega}^2}\right) \ \& \ l. \tag{346}$$

The helicoidal axis (346) of B has important kinematical interpretations.

The second example is connected with the rank-theorem (Pr 15). Its kinematical analogue reads:

Pr 30. $\underset{\rightarrow}{\omega}(\overline{\omega}, \frac{d\boldsymbol{a}}{dt} + \boldsymbol{a} \times \overline{\omega})$ denoting a system of arrows intrinsic for the rigid body B by virtue of the statical-kinematical analogy, the motion of B in the moment of time t is a rest, a translation, a rotation, or a most general helicoidal movement according to the cases rank $\underset{\rightarrow}{\omega} = 0$, 1, 2, or 3 respectively.

As mentioned, we shall enter in no more details here in connection with the statical-kinematical analogy. A special study will be dedicated to this theme in due time.

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ON THE REDUCTION OF POLYADIC RECURSIVE PROGRAMS TO MONADIC ONES*

DIMITER SKORDEV

Димитр Скордев. О СВЕДЕНИИ ПОЛИАДИЧЕСКИХ РЕКУРСИВНЫХ ПРОГРАММ К МОНАДИЧЕСКИМ. В работе рассматривается один метод сведения полиадических рекурсивных программ к монадическим. Он основан на идеях, использованных Л. Ивановым при некоторых применениях его алгебраической теории рекурсии. Обсуждается применимость этого метода для реализации полиадических рекурсивных программ в языках программирования, допускающих только рекурсивные процедуры без параметров.

Dimiter Skordev. ON THE REDUCTION OF POLYADIC RECURSIVE PROGRAMS TO MONADIC ONES. In the paper, a certain method is considered for the reduction of polyadic recursive programs to monadic ones. The method is based on ideas used before by L. Ivanov in some applications of his algebraic recursion theory. The applicability of this method is discussed for the implementation of polyadic recursive programs in programming languages admitting only recursive procedures without parameters.

The recursive programs considered in this paper are, roughly speaking, interpreted recursive program schemes (for the last notion, cf. e.g. [1, 2]). The monadic recursive programs can be described as recursive programs which contain only unary functional and predicate symbols and satisfy the condition that branching in them is controlled only by primitive predicates (compare with [1] pp. 7–4, 8–1). The polyadic recursive programs are those recursive programs which are not monadic. Their reduction to monadic ones can be used for their implementation in programming languages admitting only recursive procedures without parameters and hence

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not permitting to implement polyadic recursive programs in a straightforward manner (such languages are, for example, Basic and Forth, where in the case of the first of them its GOSUB-feature can be used). In the present paper, a method of reduction will be exposed which is based on ideas used before by L. Ivanov in some applications of his algebraic recursion theory [3, 4]. In a slightly different form, and without considerations about the practical implementation, this method has been exposed in [5].

Let $\mathcal{A} = \langle A;\ F_1,\ F_2,\ \ldots;\ P_1,\ P_2,\ \ldots\rangle$ be a (possibly partial) algebraic system.

D e f i n i t i o n 1 (inductive definition of the notion of an *$\mathcal{A}$-quasi-termal partial mapping* of A^m into A^n):

(1.1) $F_1,\ F_2,\ \ldots$ and the projection functions $\mathbf{pr}_{m,j}(a_1,\ \ldots,\ a_m) = a_j$, $m = 1,\ 2, \ldots,\ j = 1,\ \ldots,\ m$, are $\mathcal{A}$-quasi-termal.

(1.2) If $F : A^m - \to A^n$ and $G : A^l - \to A^m$ are $\mathcal{A}$-quasi-termal, then so is $F \circ G : A^l - \to A^n$ defined by

$$F \circ G(\bar{a}) \simeq F(G(\bar{a})).$$

(1.3) If $F : A^m - \to A^n$ and $G : A^m - \to A^p$ are $\mathcal{A}$-quasi-termal, then so is $F \times G : A^m \longrightarrow A^{n+p}$ defined by

$$F \times G(\bar{a}) \simeq F(\bar{a}).G(\bar{a})$$

(the multiplication sign in the right-hand side denotes concatenation).

(1.4) If P_i is l-ary, and $F : A^m - \to A^l$, $G,\ H : A^m - \to A^n$ are $\mathcal{A}$-quasi-termal, then so is $(P_i \circ F \Longrightarrow G,\ H) : A^m - \to A^n$ defined by

$$(P_i \circ F \Longrightarrow G,\ H)(\bar{a}) \simeq \begin{cases} G(\bar{a}) & \text{if} \quad P_i(F(\bar{a})) = \text{true}, \\ H(\bar{a}) & \text{if} \quad P_i(F(\bar{a})) = \text{false}. \end{cases}$$

D e f i n i t i o n 1′ (inductive definition of the notion of an *$\mathcal{A}$-quasi-termal operator*): a uniform relativized version of Definition 1.

In the case when $F_1,\ F_2,\ \ldots,\ P_1,\ P_2,\ \ldots$ are unary, we adopt also

D e f i n i t i o n 2 (inductive definition of the notion of a *monadic $\mathcal{A}$-quasi-termal mapping* of A into A):

(2.1) $F_1,\ F_2,\ \ldots$ and $\mathbf{id}_A = \mathbf{pr}_{1,1}$ are monadic $\mathcal{A}$-quasi-termal mappings.

(2.2) If F and G are monadic $\mathcal{A}$-quasi-termal mappings, then so is $F \circ G$.

(2.3) If G and H are monadic $\mathcal{A}$-quasi-termal mappings, then so is

$$(P_i \Longrightarrow G,\ H) = (P_i \circ \mathbf{id}_A \Longrightarrow G,\ H).$$

D e f i n i t i o n 2′ (inductive definition of the notion of a *monadic $\mathcal{A}$-quasi-termal operator*): a uniform relativized version of Definition 1′.

R e m a r k 1. The assumption that $F_1, F_2, \ldots,\ P_1,\ P_2,\ \ldots$ are unary does not always assure that all $\mathcal{A}$-quasi-termal mappings of A into A are monadic. Let

$\mathcal{A} = \langle \mathbf{N}; \mathbf{predec}; \mathbf{eqzero} \rangle$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$, $\mathbf{predec}(a) = a - 1$, and $\mathbf{eqzero}(a)$ is equivalent to $(a = 0)$. Then the mapping $(\mathbf{eqzero} \circ \mathbf{predec} \Longrightarrow \mathbf{id_N}, \mathbf{predec})$ is not monadic (only the mappings $\lambda a.\ a - i$, $i = 0, 1, 2, \ldots$, are monadic).

We return again to the general case (i. e. $F_1, F_2, \ldots, P_1, P_2, \ldots$ are not necessarily unary).

D e f i n i t i o n 3. The set of all finite sequences of elements of A will be denoted by A^*.

D e f i n i t i o n 4. For any $F : A^m - \to A^n$, we define $F^* : A^* - \to A^*$ as follows:

$$F^*(\langle a_1, \ldots, a_s \rangle) \simeq \begin{cases} F(a_1, \ldots, a_m) \cdot \langle a_{m+1}, \ldots, a_s \rangle \text{ if } s \geq m, \\ \bot \quad \text{otherwise} \end{cases}$$

(this construction has been used in [6]).

D e f i n i t i o n 5. For any partial m-ary predicate P on A, we define a partial unary predicate P^* on A^* as follows:

$$P^*(\langle a_1, \ldots, a_s \rangle) \simeq \begin{cases} P(a_1, \ldots, a_m) \text{ if } s \geq m, \\ \bot \quad \text{otherwise.} \end{cases}$$

D e f i n i t i o n 6. The partial mappings $\mathbf{drop}$, $\mathbf{dup}$, $\mathbf{roll}_1$, $\mathbf{roll}_2$, ... of A^* into A^* are defined as follows:

$$\mathbf{drop}(\langle a_1, \ldots, a_s \rangle) \simeq \begin{cases} \langle a_2, \ldots, a_s \rangle \text{ if } s > 0, \\ \bot \quad \text{otherwise;} \end{cases}$$

$$\mathbf{dup}(\langle a_1, \ldots, a_s \rangle) \simeq \begin{cases} \langle a_1, a_1, \ldots, a_s \rangle \text{ if } s > 0, \\ \bot \quad \text{otherwise;} \end{cases}$$

$$\mathbf{roll}_n(\langle a_1, \ldots, a_s \rangle) \simeq \begin{cases} \langle a_{n+1}, a_1, \ldots, a_n, a_{n+2}, \ldots, a_s \rangle \text{ if } s > n, \\ \bot \quad \text{otherwise} \end{cases}$$

(these denotations are borrowed, mutatis mutandis, from the programming language Forth).

D e f i n i t i o n 7. Let $\mathcal{A}^*$ be the unary partial structure defined by

$$\mathcal{A}^* = \langle A^*; F_1^*, F_2^*, \ldots, \mathbf{drop}, \mathbf{dup}, \mathbf{roll}_1, \mathbf{roll}_2, \ldots; P_1^*, P_2^*, \ldots \rangle.$$

T h e o r e m 1. Whenever F is an $\mathcal{A}$-quasi-termal partial mapping of A^m into A^n, then F^* is a monadic $\mathcal{A}^*$-quasi-termal partial mapping of A^* into A^*.

T h e o r e m 1′. If Γ is an $\mathcal{A}$-quasi-termal operator then there is a monadic $\mathcal{A}^*$-quasi-termal operator Γ^* having the same number of arguments and satisfying the equality

$$\Gamma^*(G_1^*, \ldots, G_k^*) = (\Gamma(G_1, \ldots, G_k))^*$$

for all sequences $G_1, \ldots, G_k$ belonging to $\mathbf{dom}\ \Gamma$.

Proof (of both theorems). We use the equalities

$$\begin{aligned}(\mathbf{pr}_{m,j})^{\bullet} &= (\mathbf{drop} \circ \mathbf{roll}_1)^{m-j} \circ (\mathbf{drop})^{j-1},\\ (\mathbf{pr}_{1,1})^{\bullet} &= \mathbf{drop} \circ \mathbf{dup},\\ (F \circ G)^{\bullet} &= F^{\bullet} \circ G^{\bullet},\end{aligned}$$

the fact that, whenever $F : A^m \dot{-}\!\!\rightarrow A^n$ and $G : A^m - \!\!\rightarrow A^p$, then

$$(F \times G)^{\bullet} = \begin{cases} F^{\bullet} \circ G^{\bullet} & \text{if } m = 0,\\ F^{\bullet} \circ (\mathbf{roll}_{p+m-1})^m \circ G^{\bullet} \circ \mathbf{copy}_m & \text{otherwise,}\end{cases}$$

where $\mathbf{copy}_m = (\mathbf{pick}_{m-1})^m$ if $m > 1$, $\mathbf{copy}_1 = \mathbf{dup}$, $\mathbf{copy}_0 = \mathbf{id}_{A^*}$, and $\mathbf{pick}_i = \mathbf{roll}_i \circ (\mathbf{roll}_{i+1})^i \circ \mathbf{dup} \circ \mathbf{roll}_i$, as well as the fact that, whenever P_i is l-ary, and $F : A^m - \!\!\rightarrow A^l$, $G, H : A^m - \!\!\rightarrow A^n$, then

$$(P_i \circ F \Longrightarrow G,\ H)^{\bullet} = (P_i^{\bullet} \Longrightarrow G^{\bullet} \circ (\mathbf{drop})^l,\ H^{\bullet} \circ (\mathbf{drop})^l) \circ F^{\bullet} \circ \mathbf{copy}_m.$$

If we modify the definition of $\mathcal{A}^{\bullet}$ by adding suitable other primitive operations, then the above proof may be replaced by a simpler one. For example, in the case of Theorem 1′ we could add all monadic mappings quasi-termal with respect to the original $\mathcal{A}^{\bullet}$ as additional operations, and we would have yet a non-tautological result. When applying the construction from the proof to concrete operators, sometimes also the addition of certain predicates of the form $P_i \circ \Phi$ with monadic Φ could happen to be convenient.

A certain kind of calculus can be developed for the construction of the operators $\Gamma^{\bullet}$. Let us adopt the convention that, when given an expression intended to denote some element of some set A^n, the same expression will denote also the corresponding mapping of A^0 into A^n. Then, given an $\mathcal{A}$-quasi-termal operator Γ having k arguments and producing m-ary functions, we form the corresponding expression $U^{\bullet}$, where U is an expression for $\Gamma(g_1, \ldots, g_k)(x_1, \ldots, x_m)$, assuming $g_1, \ldots, g_k$ to be variables for functions of the suitable types, and $x_1, \ldots, x_m$ to be variables for elements of A. The next step is to transform $U^{\bullet}$ into an expression of the form $V \circ x_1^{\bullet} \circ \ldots \circ x_m^{\bullet}$, where V does not contain $x_1^{\bullet}, \ldots, x_m^{\bullet}$, and V is obtained from $g_1^{\bullet}, \ldots, g_k^{\bullet}$, from denotations of monadic mappings and from predicates $P_i \circ \Phi$ by means of composition and branching (this transformation can be done by using equalities from the proof of Theorems 1, 1′ and some other equalities, e. g. $\mathbf{dup} \circ F^{\bullet} = F^{\bullet} \circ F^{\bullet}$ for each $F : A^0 - \!\!\rightarrow A$). Taking the expression V and replacing the occurrences of $g_1^{\bullet}, \ldots, g_k^{\bullet}$ in it by variables $\gamma a_1, \ldots, \gamma a_k$ for unary operations in A^*, we obtain an expression for $\Gamma^{\bullet}(\gamma a_1, \ldots, \gamma a_k)$.

E x a m p l e 1. Let $\mathcal{A}$ be the same as in Remark 1. Let U be the expression

$$g(g(\mathbf{predec}(y),\ z),\ g(\mathbf{predec}(x),\ z)),$$

where g is a variable for functions of type $\mathbf{N}^2 - \!\!\rightarrow \mathbf{N}$, and x, y, z are variables for elements of $\mathbf{N}$. Then

$$\begin{aligned}U^{\bullet} &= g^{\bullet} \circ (g(\mathbf{predec}(y),\ z))^{\bullet} \circ (g(\mathbf{predec}(x),\ z))^{\bullet}\\ &= g^{\bullet} \circ g^{\bullet} \circ (\mathbf{predec}(y))^{\bullet} \circ z^{\bullet} \circ (g(\mathbf{predec}(x),\ z))^{\bullet}\end{aligned}$$

$$= g^* \circ g^* \circ \mathbf{roll}_2^{-1} \circ (g(\mathbf{predec}(x),\ z))^* \circ (\mathbf{predec}(y))^* \circ z^*$$

$$= g^* \circ g^* \circ \mathbf{roll}_2^{-1} \circ g^* \circ (\mathbf{predec}(x))^* \circ z^* \circ (\mathbf{predec}(y))^* \circ z^*$$

$$= g^* \circ g^* \circ \mathbf{roll}_2^{-1} \circ g^* \circ F^* \circ x^* \circ y^* \circ z^*,$$

where $\mathbf{roll}_2^{-1}$ is the inverse mapping of $\mathbf{roll}_2$, and the mapping F of type $\mathbf{N}^3 \to \mathbf{N}^4$ is defined by

$$F(x,\ y,\ z) = \langle \mathbf{predec}(x),\ z,\ \mathbf{predec}(y),\ z \rangle.$$

Hence, taking γ to be a variable for unary operations in A^*, we may set

$$\Gamma^*(\gamma) = \gamma \circ \gamma \circ \mathbf{roll}_2^{-1} \circ \gamma \circ F^*.$$

Definition 8. A *recursive program over* $\mathcal{A}$ is a system of equations of the form

$$g_i = \Gamma a_i(g_1,\ \ldots,\ g_k), \qquad i = 1,\ \ldots,\ k,$$

where $\Gamma a_1,\ \ldots,\ \Gamma a_k$ are $\mathcal{A}$-quasi-termal operators. The above program is called *monadic* if $g_1,\ \ldots,\ g_k$ are variables for unary functions, and the operators $\Gamma a_1,\ \ldots,\ \Gamma a_k$ are monadic. *The functions computed by the program* are the components of the least solution of the system.

Remark 2. Some authors use the term "monadic program" in a sense which is not the same as in the above definition. Cf., for example, [7, p. 544].

Definition 9. Given a recursive program over $\mathcal{A}$ having the form from Definition 8, *the monadic transform* of this program is the monadic recursive program

$$\gamma a_i = \Gamma a_i^*(\gamma a_1,\ \ldots,\ \gamma a_k), \qquad i = 1,\ \ldots,\ k,$$

over $\mathcal{A}^*$, where $\gamma a_1,\ \ldots,\ \gamma a_k$ are variables for unary operations in A^*.

Using the continuity of the $\mathcal{A}$-quasi-termal operators and of the operation $\lambda F,\ F^*$, one easily proves.

Theorem 2. If $G_1,\ \ldots,\ G_k$ are the functions computed by a given recursive program over $\mathcal{A}$, then $G_1^*,\ \ldots,\ G_k^*$ are the functions computed by its monadic transform.

Since

$$G_i(a_1,\ \ldots,\ a_{m_i}) = G_i^*(\langle a_1,\ \ldots,\ a_{m_i} \rangle),$$

where m_i is the arity of G_i, Theorem 2 shows a way for the reduction of recursive programs over $\mathcal{A}$ to monadic recursive programs over $\mathcal{A}^*$.

Example 2. Let $\mathcal{A} = \langle \mathbf{N};\ \mathbf{predec},\ C,\ L, R;\ \mathbf{eqzero} \rangle$, where

$$C(x,\ y) = \frac{1}{2}(x+y)(x+y+1) + x,$$

$$L(C(x,\ y)) = x, \qquad R(C(x,\ y)) = y$$

for all x, y in $\mathbf{N}$. Consider the two-argument partial recursive function ω defined recursively by

$$\omega(C(x,\ y),\ z) \simeq \begin{cases} C(x,\ z) & \text{if } y = 0, \\ y - 1 & \text{if } y > 0,\ x = 0, \\ \omega(\omega(y-1,\ z),\ \omega(x-1,\ z)) & \text{otherwise} \end{cases}$$

(according to [8], ω is an universal function for the unary partial recursive functions). The above recursive definition of ω can be represented by a recursive program in the sense of Definition 8 such that ω will be the function computed by that recursive program. We could, for example, take the recursive program $g = \Delta(g) \circ G$, where g is a variable for two-argument partial functions in $\mathbf{N}$, the operator Δ transforms such functions into three-argument partial functions in $\mathbf{N}$, G is a mapping of $\mathbf{N}^2$ into $\mathbf{N}^3$, and the following defining equalities hold:

$$\Delta(g)(x,\ y,\ z) \simeq \begin{cases} C(x,\ z) & \text{if } y = 0, \\ y - 1 & \text{if } y > 0, \quad x = 0, \\ g(g(y-1,\ z),\ g(x-1,\ z)) & \text{otherwise,} \end{cases}$$

$$G(t,\ z) = \langle L(t),\ R(t),\ z \rangle.$$

Since $(\Delta(g) \circ G)^\bullet = (\Delta(g))^\bullet \circ G^\bullet = \Delta^\bullet(g^\bullet) \circ G^\bullet$, the monadic transform of the above recursive program can be written as $\gamma = \Delta^\bullet(\gamma) \circ G^\bullet$. For constructing an explicit expression for $\Delta^\bullet(\gamma)$, we consider the expression $U^\bullet$, where U is the expression $\Delta(g)(x,\ y,\ z)$. We have the equality

$$\begin{aligned} U^\bullet \;=\; (\mathtt{eqzero}^\bullet \circ y^\bullet \Longrightarrow\ & (C(x,\ z))^\bullet,\ (\mathtt{eqzero}^\bullet \circ x^\bullet \Longrightarrow (\mathtt{predec}(y))^\bullet, \\ & (g(g(\mathtt{predec}(y),\ z),\ g(\mathtt{predec}(x),\ z)))^\bullet)). \end{aligned}$$

Making use of Example 1, we get

$$\begin{aligned} U^\bullet \;&=\; (\mathtt{eqzero}^\bullet \circ y^\bullet \Longrightarrow\ C^\bullet \circ x^\bullet \circ z^\bullet,\ (\mathtt{eqzero}^\bullet \circ x^\bullet \Longrightarrow \mathtt{predec}^\bullet \circ y^\bullet, \\ &\qquad g^\bullet \circ g^\bullet \circ \mathtt{roll}_2^{-1} \circ g^\bullet \circ F^\bullet \circ x^\bullet \circ y^\bullet \circ z^\bullet)) \\ &=\; (\mathtt{eqzero}^\bullet \circ \mathtt{drop} \Longrightarrow C^\bullet \circ \mathtt{drop} \circ \mathtt{roll}_1, \mathtt{eqzero}^\bullet \Longrightarrow \mathtt{predec}^\bullet \circ \mathtt{drop}, \\ &\qquad g^\bullet \circ g^\bullet \circ \mathtt{roll}_2^{-1} \circ g^\bullet \circ F^\bullet)) \circ x^\bullet \circ y^\bullet \circ z^\bullet. \end{aligned}$$

Hence

$$\begin{aligned} \Delta^\bullet(\gamma) \;=\; (\mathtt{eqzero}^\bullet \circ \mathtt{drop} \Longrightarrow\ & C^\bullet \circ \mathtt{drop} \circ \mathtt{roll}_1, (\mathtt{eqzero}^\bullet \Longrightarrow \\ & \mathtt{predec}^\bullet \circ \mathtt{drop}, \gamma \circ \gamma \circ \mathtt{roll}_2^{-1} \circ \gamma \circ F^\bullet)). \end{aligned}$$

So we have found the explicit form of the monadic transform of the original recursive program computing ω. According to Theorem 2, this monadic transform computes the function $\omega^\bullet$, and therefore could be used for computing values of ω.

Of course, a recursive program in Pascal for computing values of ω can be written in a straightforward manner. But the monadic transform constructed above gives us the possibility to construct relatively simple programs computing values of ω in programming languages holding out much more restricted possibilities for a straightforward recursive programming. To demonstrate this, we wrote a program in Applesoft.Basic for the some purpose, using the obtained monadic transform and the GOSUB-feature of Basic. The length of the program is less than 1000 bytes. Sequences from $\mathbf{N}^*$ are represented on a stack principle, the members of the sequence following each other in direction from top to bottom. The stack is realized by means of an array **A** supplied with a counter **D**, and the dimension of **A** is chosen in concordance with the restriction in Applesoft.Basic about the number of the nested subroutine calls. The flow diagram of the program and the way of

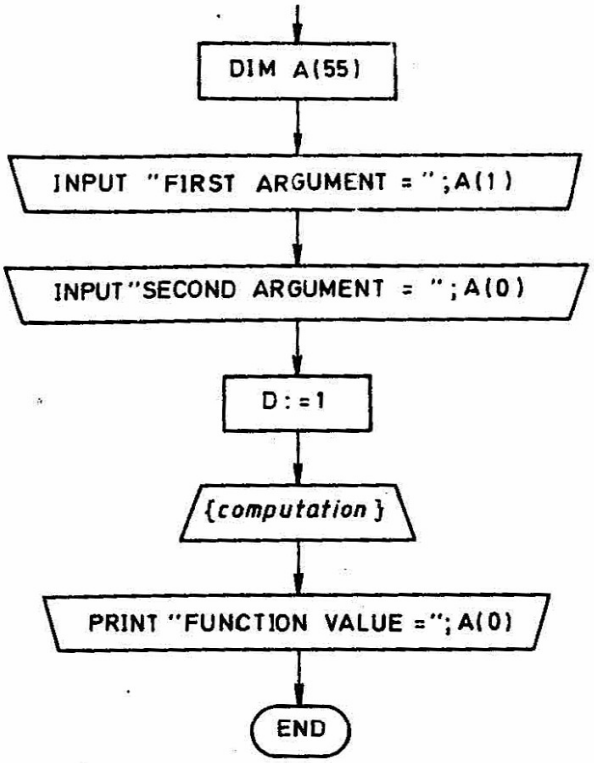

Fig. 1. Flow diagram of the program in Basic for computing ω

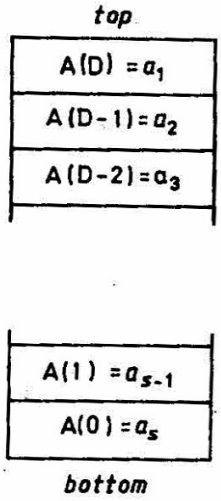

Fig. 2. The stack representation of the sequence $\langle a_1, \ldots a_s \rangle$

using the array A for the stack representation of elements of $\mathbf{N}^*$ are shown on Fig. 1, 2. An essential part of the program is a subroutine called "*computation*" which contains three calls of itself. The flow diagram of this subroutine is shown on Fig. 3. As to the efficiency of the considered program, the situation is not so bad as somebody could suspect. Namely, some experiments were made by the author on a "Pravetz-8M" microcomputer, and they showed the compiled variant of the program running faster then the directly written recursive Pascal program processed in the Apple UCSD Pascal system. For obtaining more information about the efficiency of using monadic transforms, the author wrote also a monadic recursive and a non-recursive program in Apple Pascal for computing values of ω

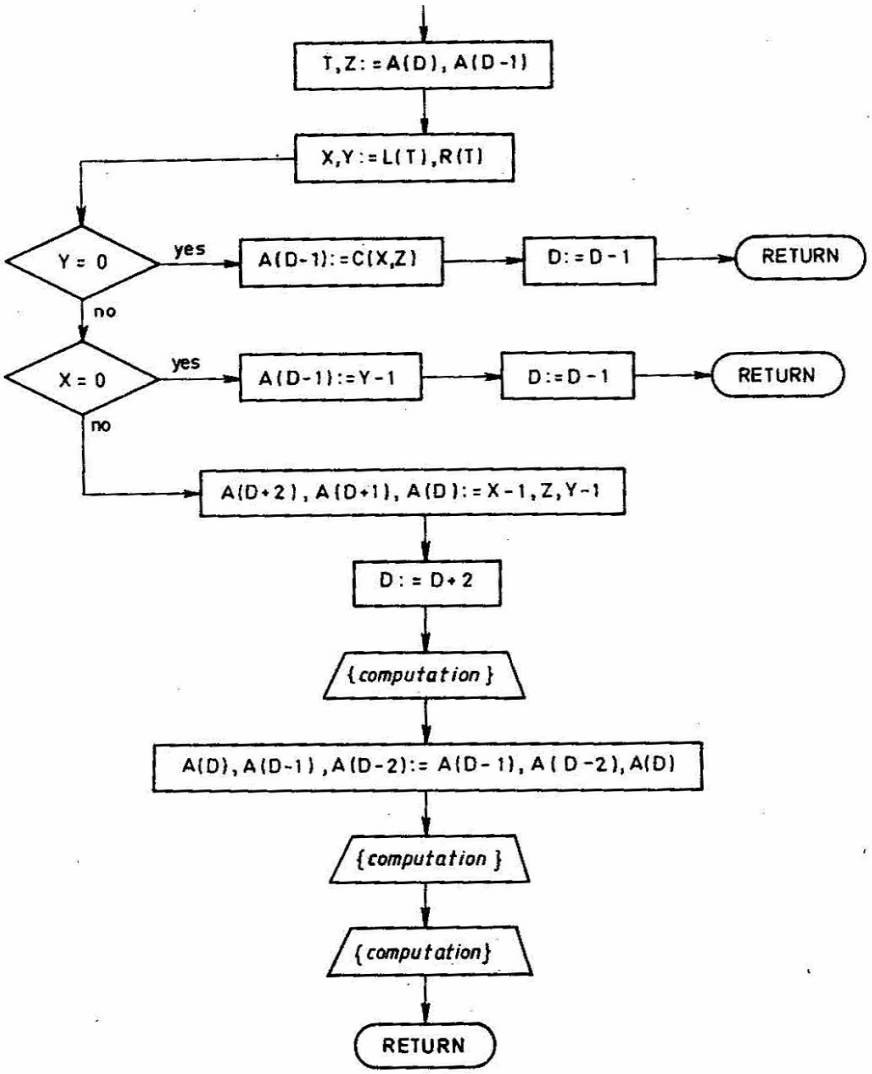

Fig. 3. Flow diagram of the subroutine "*computation*"

using again the monadic transform of the original recursive program. The table below contains some average ratios of run-times (the timing-experiments were based on computing $\omega(4,\ 0)$, $\omega(11,\ 1)$, $\omega(11,\ 3)$, $\omega(41,\ 50)$, $\omega(60,\ 61)$ and $\omega(312,\ 3)$):

Applesoft.Basic		Apple UCSD Pascal		
interpreted	compiled	polyadic recursive	monadic recursive	non-recursive
1.94	0.86	1	1.06	1.10

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