ГОДИШНИК
на Софийския университет „Св. Климент Охридски“

Факултет по математика и информатика

ANNUAL
of Sofia University “St. Kliment Ohridski”

Faculty of Mathematics and Informatics

Том/Volume 112

УНИВЕРСИТЕТСКО ИЗДАТЕЛСТВО „СВ. КЛИМЕНТ ОХРИДСКИ“
ST. KLIMENT OHRIDSKI UNIVERSITY PRESS
СОФИЯ • 2025 • SOFIA

Годишник на Софийския университет „Св. Климент Охридски“
Факултет по математика и информатика

Annual of Sofia University "St. Kliment Ohridski"
Faculty of Mathematics and Informatics

Address for correspondence

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, J. Bourchier Blvd., POB 48
1164 Sofia, BULGARIA
email: annuaire@fmi.uni-sofia.bg

Aims and Scope. The *Annual* is the oldest Bulgarian journal, founded in 1904, devoted to pure and applied mathematics, mechanics and computer science. It is indexed in Scopus, Zentralblatt für Mathematik, Mathematical Reviews, Russian Referativnii Jurnal and EBSCO. The *Annual* publishes significant and original research papers of authors both from Bulgaria and abroad in some selected areas that comply with the traditional scientific interests of the Faculty of Mathematics and Informatics at the Sofia University "St. Kliment Ohridski", i.e., algebra, geometry and topology, analysis, probability and statistics, mathematical logic, theory of approximations, numerical methods, computer science, classical, fluid and solid mechanics, and their fundamental applications.

ISSN 1313–9215 (Print)
ISSN 2603–5529 (Online)

CONTENTS

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

NOTES ON TWO DEFINITE QUADRATURE FORMULAE OF ORDER THREE

ANA AVDZHIEVA and PETAR B. NIKOLOV

In two recent papers from 2017 and 2018 two definite quadrature formulae of order three were obtained which are modifications of the compound trapezium and midpoint quadrature rules, respectively. The criteria applied to the construction of these definite quadrature formulae is minimization of their error constants through the usage of appropriate formulae for numerical differentiation. In this note, we show that the aforementioned definite quadrature formulae also meet another criteria for optimality. For integrands having a continuous third derivative with a permanent sign, we present another error estimates, which are easier to evaluate and in many cases provide better error bounds.

Keywords: definite quadrature formulae, Peano kernel representation, Euler-Maclaurin summation formulae, a posteriori error estimates

2020 Mathematics Subject Classification: 41A55, 65D30, 65D32

1. Introduction and statement of the results

The definite integral

$$I[f] := \int_0^1 f(x)\,dx \tag{1.1}$$

is usually approximated by quadrature formulae, which are linear functionals of the form

$$Q[f] = \sum_{i=1}^{n} a_i f(x_i), \quad 0 \le x_1 < \cdots < x_n \le 1. \tag{1.2}$$

Throughout this paper, π_m stands for the set of real-valued algebraic polynomials of degree at most m. A quadrature formula Q has algebraic degree of precision

DOI: 10.60063/GSU.FMI.112.5-16

m (in short, ADP$(Q) = m$) if m is the largest non-negative integer such that its remainder functional

$$R[Q;f] := I[f] - Q[f]$$

vanishes whenever $f \in \pi_m$, and $R[Q;f] \neq 0$ when f is a polynomial of degree $m+1$. Let us recall that quadrature formula (1.2) is *definite of order* $r \in \mathbb{N}$, if there exists a real non-zero constant $c_r(Q)$ such that its remainder functional admits the representation

$$R[Q;f] = I[f] - Q[f] = c_r(Q) f^{(r)}(\xi)$$

for every real-valued function $f \in C^r[0,1]$ with some $\xi \in [0,1]$ depending on f. Furthermore, Q is called positive definite (resp., negative definite) of order r, if $c_r(Q) > 0$ ($c_r(Q) < 0$).

Definite quadrature formulae of order r provide one-sided approximation to $I[f]$ whenever $f^{(r)}$ has a permanent sign in the integration interval. For the sake of brevity, we introduce the following definition.

Definition 1.1. A real-valued function $f \in C^r[0,1]$ is called r-positive (resp., r-negative) if $f^{(r)}(x) \geq 0$ (resp. $f^{(r)}(x) \leq 0$) for every $x \in [0,1]$.

If $\{Q^+, Q^-\}$ is a pair of positive and negative definite quadrature formulae of order r, then $Q^+[f] \leq I[f] \leq Q^-[f]$ for every r-positive function f. This simple observation serves as a base for the derivation of a posteriori error estimates and rules for termination of calculations (stopping rules) in the algorithms for automatic numerical integration (see [3] for a survey). Most quadratures used in practice (e.g., quadrature formulae of Gauss, Radau, Lobatto, Newton-Cotes) are definite of certain order. The compound midpoint and trapezium rules

$$Q_n^{\mathrm{Mi}}[f] = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{2k-1}{2n}\right), \quad Q_{n+1}^{\mathrm{Tr}}[f] = \frac{1}{2n}\left(f(0)+f(1)\right) + \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) \quad (1.3)$$

are the best known definite quadratures, they are respectively positive and negative definite of order two with error constants $c_2(Q_n^{\mathrm{Mi}}) = \frac{1}{24n^2}$ and $c_2(Q_{n+1}^{\mathrm{Tr}}) = -\frac{1}{12n^2}$. Definite quadrature formulae of order three, which are appropriately modified compound trapezium and midpoint quadratures, were constructed in [1] and [2]. The optimal definite quadrature formulae of higher order are not known explicitly, although their existence and uniqueness is known, see [5–7]. Asymptotically optimal definite quadrature formulae of fourth order were constructed in [3].

Theorem A ([1, part of Theorem 1])**.** *For every $n \geq 8$, the quadrature formula*

$$Q_n[f] = \sum_{k=0}^{n-1} A_{k,n} f(x_{k,n}), \quad x_{k,n} = \frac{k}{n}$$

with coefficients $A_{k,n} = 1/n$, $3 \leq k \leq n-4$, and

$$A_{0,n} = \frac{81+\sqrt{3}}{216n}, \qquad A_{1,n} = \frac{126-\sqrt{3}}{108n}, \qquad A_{2,n} = \frac{207+\sqrt{3}}{216n},$$

$$A_{n-3,n} = \frac{297-\sqrt{3}}{216n}, \qquad A_{n-2,n} = \frac{\sqrt{3}-18}{108n}, \qquad A_{n-1,n} = \frac{495-\sqrt{3}}{216n},$$

is positive definite of order three with the error constant

$$c_3(Q_n) = \frac{\sqrt{3}}{216n^3} + \frac{27-\sqrt{3}}{72n^4}. \tag{1.4}$$

Theorem B ([2, part of Theorem 1])**.** *For every $n \in \mathbb{N}$, $n \geq 8$, the quadrature formula*

$$Q_n^*[f] = \sum_{k=0}^{n+1} A_{k,n} f(y_{k,n})$$

with nodes $y_{0,n} = 0$, $y_{k,n} = \dfrac{2k-1}{2n}$, $k = 1,\dots,n$, $y_{n+1,n} = 1$ and coefficients

$$\begin{aligned}
A_{0,n} &= \frac{-42+41\sqrt{3}}{162n}, & A_{1,n} &= \frac{678-203\sqrt{3}}{432n},\\
A_{2,n} &= \frac{357+199\sqrt{3}}{648n}, & A_{3,n} &= \frac{164-13\sqrt{3}}{144n},\\
A_{n-2,n} &= \frac{225-\sqrt{3}}{216n}, & A_{n-1,n} &= \frac{189+2\sqrt{3}}{216n},\\
A_{n,n} &= \frac{234-\sqrt{3}}{216n}, & A_{n+1,n} &= 0,
\end{aligned}$$

$A_{k,n} = 1/n$, $4 \leq k \leq n-3$, *is positive definite of order three with the error constant*

$$c_3(Q_n^*) = \frac{\sqrt{3}}{216n^3} + \frac{169\sqrt{3}-210}{2592n^4}. \tag{1.5}$$

We need one more definition.

Definition 1.2 ([2])**.** The quadrature formula Q in (1.2) is called symmetrical, if $a_k = a_{n+1-k}$ and $x_k = 1 - x_{n+1-k}$ for $k = 1,\dots,n$, and nodes-symmetrical, if only the requirement for the nodes is satisfied.

The quadrature formula $\widetilde{Q}[f] = \widetilde{Q}[Q;f] := \sum_{k=1}^{n} a_k f(x_{n+1-k})$ is called a reflected quadrature formula for (1.2).

Some advantages of the definite quadrature formulae in Theorems A and B become clear when taking into account the following proposition.

Proposition 1.3 ([2])**.**

(i) *If Q is a positive definite quadrature formula of order r (r is odd), then its reflected quadrature formula $\widetilde{Q}$ is negative definite of order r and vice versa. Moreover, $c_r(\widetilde{Q}) = -c_r(Q)$.*

(ii) *If quadrature formula Q in* (i) *is nodes-symmetrical and definite of order r (r is odd), and f is an r-positive function, then with Q^* standing for either Q or $\widetilde{Q}$ we have*

$$|R[Q^*;f]| \leq B[Q;f] := \left| \sum_{k=1}^{[n/2]} (a_k - a_{n+1-k})(f(x_{n+1-k}) - f(x_k)) \right|.$$

(iii) *Under the same assumptions for Q and f as in* (ii), *for $\hat{Q} = (Q + \widetilde{Q})/2$ we have*

$$\left|R[\hat{Q}; f]\right| \le \frac{1}{2} B[Q; f].$$

In view of Proposition 1.3(i), the reflected quadratures to the quadrature formulae of Theorems A and B are negative definite of order three. Since they (and their reflected) are nodes-symmetrical and use equispaced nodes, one can apply Proposition 1.3(ii), (iii) to derive simple error bounds in terms of only a few integrand's values provided the integrand is three-positive or three-negative function. Of course, at our disposal are also the error estimates

$$|R[Q_n; f]| \le c_3(Q_n)\|f'''\|_{C[0,1]} \tag{1.6}$$

with the error constants given in (1.4) or (1.5). However, this requires knowledge of $\|f'''\|_{C[0,1]}$, which may be unavailable or difficult to obtain. Here we propose alternative error estimates for the definite quadrature formulae in Theorems A and B, which require only knowledge of the values of the integrand's second derivative at the end-points.

Theorem 1.4. *Let Q_n be the quadrature formula in Theorem A. If f is three-positive or three-negative function, then*

$$|R[Q_n; f]| \le \frac{0.277223}{n^3}\left|f''(1) - f''(0)\right|. \tag{1.7}$$

Theorem 1.5. *Let Q_n be the quadrature formula in Theorem B. If f is three-positive or three-negative function, then*

$$|R[Q_n; f]| \le \frac{0.0369563}{n^3}\left|f''(1) - f''(0)\right|. \tag{1.8}$$

The rest of the paper is structured as follows. In Section 2 we provide a brief introduction to Peano's integral representation of linear functionals which vanish on π_{r-1}. The proofs of Theorem 1.4 and Theorem 1.5 are given in Section 3 and Section 4, respectively.

2. Peano kernel representation of linear functionals

By $W_1^r[0,1]$, $r \in \mathbb{N}$, we denote the Sobolev class of functions

$$W_1^r[0,1] := \left\{ f \in C^{r-1}[0,1] \colon f^{(r-1)} \text{ abs. continuous}, \int_0^1 |f^{(r)}(t)|\,dt < \infty \right\}.$$

In particular, $W_1^r[0,1]$ contains the class $C^r[0,1]$. If $\mathcal{L}$ is a linear functional defined in $W_1^r[0,1]$ that vanishes on π_{r-1}, then, by a classical result of Peano [9], $\mathcal{L}$ admits the integral representation

$$\mathcal{L}[f] = \int_0^1 K_r(t) f^{(r)}(t)\, dt, \quad K_r(t) = \mathcal{L}\left[\frac{(\cdot - t)_+^{r-1}}{(r-1)!}\right], \quad t \in [0,1],$$

where $u_+(t) = \max\{t, 0\}$, $t \in \mathbb{R}$.

In the case when $\mathcal{L}$ is the remainder $R[Q;\cdot]$ of a quadrature formula Q with $\mathrm{ADP}(Q) \ge r-1$, the function $K_r(t) = K_r(Q;t)$ is referred to as the r-th Peano kernel of Q. For Q as in (1.2), explicit representations for $K_r(Q;t)$, $t \in [0,1]$, are

$$K_r(Q;t) = \frac{(1-t)^r}{r!} - \frac{1}{(r-1)!}\sum_{i=1}^n a_i (x_i - t)_+^{r-1}, \tag{2.1}$$

$$K_r(Q;t) = (-1)^r\left[\frac{t^r}{r!} - \frac{1}{(r-1)!}\sum_{i=1}^n a_i (t - x_i)_+^{r-1}\right]. \tag{2.2}$$

Since for $f \in C^r[0,1]$ we have

$$R[Q;f] = \int_0^1 K_r(Q;t) f^{(r)}(t)\, dt, \tag{2.3}$$

it is clear that Q is a positive (negative) definite quadrature formula of order r if and only if $\mathrm{ADP}(Q) = r-1$ and $K_r(Q;t) \ge 0$ (resp. $K_r(Q;t) \le 0$) for all $t \in [0,1]$. The following lemma is an immediate consequence of this observation.

Lemma 2.1. *Assume that Q is a positive or negative definite quadrature formula of order r. If f is an r-positive or r-negative function, then*

$$|R[Q;f]| \le \max_{t\in[0,1]} |K_r(Q;t)| \cdot \left| f^{(r-1)}(1) - f^{(r-1)}(0) \right|. \tag{2.4}$$

The proofs of Theorems 1.4 and 1.5 make use of this lemma.

3. Proof of Theorem 1.4

We start with recalling the way the quadrature formula in Theorem A was obtained. Assuming $f \in W_1^3[0,1]$, then a particular case of the Euler-Maclaurin summation formula (c.f. [4, Satz 98]) yields

$$I[f] = Q_{n+1}^{\mathrm{Tr}}[f] - \frac{1}{12n^2}\left[f'(1) - f'(0)\right] - \frac{1}{n^3}\int_0^1 B_3(nx - \{nx\}) f'''(x)\, dx. \tag{3.1}$$

Here $\{\cdot\}$ is the fractional part function and B_3 is the third Bernoulli polynomial with leading coefficient $1/6$,

$$B_3(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}.$$

Writing (3.1) in the equivalent form

$$
\begin{aligned}
I[f] &= Q^{\mathrm{Tr}}_{n+1}[f] - \frac{1}{12\,n^2}\,[f'(1) - f'(0)] - \frac{\sqrt{3}}{216\,n^3}\,[f''(1) - f''(0)] \\
&\quad + \frac{1}{n^3}\int_0^1 \Big(\frac{\sqrt{3}}{216} - B_3(nx - \{nx\})\Big)\, f^{(3)}(x)\,dx \\
&=: \widetilde{Q}[f] + R[\widetilde{Q}; f],
\end{aligned}
\tag{3.2}
$$

we observe that the quadrature formula $\widetilde{Q}$ in the first line of (3.2) is positive definite of order three though not of the desired form as it involves values of the first and the second derivatives of the integrand. Therefore, the expressions with derivatives $f^{(j)}(0)$ resp. $f^{(j)}(1)$, $j = 1, 2$, are replaced by one-parametric linear combinations of $\{f(x_{i,n})\}_{i=0}^3$ resp. $\{f(x_{n-i,n})\}_{i=0}^3$ (i.e., numerical differentiation formulae) which evaluate these expressions to the exact value whenever $f \in \pi_2$. The resulting quadrature formula Q has nodes $x_{k,n} = k/n$, $k = 0, \dots, n$, and weights $A_{k,n} = 1/n$, $4 \le k \le n-4$, while $\{A_{k,n}\}_0^3$ depend on a single parameter, say θ, and $\{A_{k,n}\}_{n-3}^n$ depend on another single parameter, say ϱ. Specifically, we have

$$
\begin{aligned}
A_{0,n} &= \frac{108 - \theta}{216n}, & A_{1,n} &= \frac{171 + \sqrt{3} + 3\theta}{216n}, \\
A_{2,n} &= \frac{288 - 2\sqrt{3} - 3\theta}{216n}, & A_{3,n} &= \frac{189 + \sqrt{3} + \theta}{216n}, \\
A_{n-3,n} &= \frac{189 - \sqrt{3} + \varrho}{216n}, & A_{n-2,n} &= \frac{288 + 2\sqrt{3} - 3\varrho}{216n}, \\
A_{n-1,n} &= \frac{171 - \sqrt{3} + 3\varrho}{216n}, & A_{n,n} &= \frac{108 - \varrho}{216n}.
\end{aligned}
$$

On the interval $[x_{3,n}, x_{n-3,n}]$ the third Peano kernel of Q coincides with that of $\widetilde{Q}$ in (3.2), i.e.,

$$
K_3(Q;t) = \frac{1}{n^3}\left(\frac{\sqrt{3}}{216} - B_3(nt - \{nt\})\right) \ge 0, \qquad t \in [x_{3,n}, x_{n-3,n}]. \tag{3.3}
$$

It has been shown in [1, Section 3] that the requirement $K_3(Q;t) \ge 0$ on $(0, x_{3,n})$ and on $(x_{n-3,n}, 1)$ is equivalent respectively to $\theta \le \theta^* = 27 - \sqrt{3}$ and $\varrho \ge \varrho^* = 108$. In the limit case $(\theta, \varrho) = (\theta^*, \varrho^*)$, Q becomes the quadrature formula Q_n in Theorem 1.4 (note that $A_{n,n} = 0$ in that case). According to [1, Eqns. (3.12) and (3.14)], the integrals

$$
\int_0^{x_{3,n}} K_3(Q;t)\,dt \quad \text{and} \quad \int_{x_{n-3,n}}^1 K_3(Q;t)\,dt
$$

are resp. monotonically decreasing and monotonically increasing functions of θ (resp. ϱ), therefore the choice $(\theta, \varrho) = (\theta^*, \varrho^*)$ is optimal in the sense that it provides the smallest error constant (1.4).

We are going to show that this choice is also optimal in another sense: it minimizes $\|K_3(Q;\cdot)\|_{C[0,1]}$ while preserving non-negativity of $K_3(Q;t)$ on $[0,1]$. More precisely, we shall show that the choice $\theta=\theta^*$ minimizes $\max\limits_{t\in[0,x_{3,n}]} K_3(Q;t)$ while the choice $\varrho=\varrho^*$ minimizes $\max\limits_{t\in[x_{n-3,n},1]} K_3(Q;t)$.

Using formula (2.2) for Peano kernels with $r=3$, after the change of variable $t=u/n$ we arrive at the following representation of $K_3(Q;t)$ for $t\in[0,x_{3,n}]$

$$\begin{aligned} K_3(Q;t) &= -\frac{1}{6n^3}\left[u^3-\frac{108-\theta}{72}u^2-\frac{171+\sqrt{3}+3\theta}{72}(u-1)_+^2-\frac{288-2\sqrt{3}-3\theta}{72}(u-2)_+^2\right]\\ &=: -\frac{1}{6n^3}\varphi(\theta,u), \quad u\in[0,3]. \end{aligned}$$

It is easily verified that

$$\frac{\partial\varphi}{\partial\theta}=-\frac{1}{72n^3}\left[u^2-3(u-1)_+^2+3(u-2)_+^2\right]\le 0, \quad u\in[0,3],$$

hence φ is a decreasing function of θ for every fixed $u\in[0,3]$. Consequently,

$$\begin{aligned} &\max_{t\in[0,x_{3,n}]} K_3(Q;t) \ge \max_{u\in[0,3]}\frac{-\varphi(\theta^*,u)}{6n^3} = \max_{t\in[0,x_{3,n}]} K_3(Q_n;t)\\ &= \frac{1}{432n^3}\max_{u\in[0,3]}\left[-72u^3+(81+\sqrt{3})u^2+(252-2\sqrt{3})(u-1)_+^2+(207+\sqrt{3})(u-2)_+^2\right]. \end{aligned}$$

By a straightforward calculation we find that the latter maximum is attained at $u_0=(81+\sqrt{3})/108\in(0,1)$ and

$$\max_{t\in[0,x_{3,n}]} K_3(Q_n;t)=\frac{1}{12n^3}\left(\frac{81+\sqrt{3}}{108}\right)^3<\frac{0.0375}{n^3}. \tag{3.4}$$

Now we estimate $\max\limits_{t\in[x_{n-3,n},1]} K_3(Q;t)$. Using (2.1) with $r=3$, we obtain

$$\begin{aligned} K_3(Q;t) &= \frac{(1-t)^3}{6}-\frac{1}{2}\sum_{k=0}^{n}A_{k,n}(x_{k,n}-t)_+^2\\ &= \frac{(1-t)^3}{6}-\frac{1}{2}\sum_{k=0}^{n}A_{k,n}(1-t-x_{n-k,n})_+^2\\ &\overset{x=1-t}{=} \frac{x^3}{6}-\frac{1}{2}\sum_{k=0}^{n}A_{n-k,n}(x-x_{k,n})_+^2 =: \widetilde{K}_3(Q;x), \end{aligned}$$

where $x\in[0,x_{3,n}]$ when $t\in[x_{n-3,n},1]$. With the further change of variable $x=u/n$, $u\in[0,3]$, we get

$$\begin{aligned} \widetilde{K}_3(Q;x) &= \frac{1}{6n^3}\left[u^3-\frac{108-\varrho}{72}u^2-\frac{171-\sqrt{3}+3\varrho}{72}(u-1)_+^2-\frac{288+2\sqrt{3}-3\varrho}{72}(u-2)_+^2\right]\\ &=: \frac{1}{6n^3}\psi(\varrho,u). \end{aligned}$$

Since
$$\frac{\partial\psi}{\partial\varrho}=\frac{1}{72n^3}\Big[u^2-3(u-1)_+^2+3(u-2)_+^2\Big]\geq 0,\qquad u\in[0,3],$$
we conclude that ψ is an increasing function of ϱ for every fixed $u\in[0,3]$. Consequently,

$$\begin{aligned}\max_{t\in[x_{n-3,n},1]}K_3(Q;t)&=\max_{x\in[0,x_{3,n}]}\widetilde{K}_3(Q;x)\\&\geq\frac{1}{6n^3}\max_{u\in[0,1]}\psi(\varrho^*,u)=\max_{t\in[x_{n-3,n},1]}K_3(Q_n;t)\\&=\frac{1}{432n^3}\max_{u\in[0,3]}\Big[72u^3-(495-\sqrt{3})(u-1)_+^2+(36-2\sqrt{3})(u-2)_+^2\Big]\\&=:\frac{1}{432n^3}\max_{u\in[0,3]}g(u).\end{aligned}$$

The maximum of $g(u)$ in $[0,3]$ is attained at the point $u_1=1.4788175\ldots$, which is the smaller root of the quadratic equation
$$108u^2-(495-\sqrt{3})(u-1)=0,$$
and
$$\max_{t\in[x_{n-3,n},1]}K_3(Q_n;t)=\frac{1}{432n^3}g(u_1)=\frac{0.2772229\ldots}{n^3}<\frac{0.277223}{n^3}.\tag{3.5}$$

Finally, from (3.3) we have
$$\max_{t\in[x_{3,n},x_{n-3,n}]}K_3(Q_n;t)=\frac{2\|B_3\|_{C[0,1]}}{n^3}=\frac{\sqrt{3}}{108n^3}<\frac{0.0161}{n^3}.\tag{3.6}$$

From (3.4), (3.5) and (3.6) we conclude that
$$\max_{t\in[0,1]}K_3(Q_n;t)<\frac{0.277223}{n^3}.$$

This inequality and Lemma 2.1 applied with $r=3$ accomplish the proof of Theorem 1.4. Figure 1 depicts the Peano kernel $K_3(Q_8;t)$ and it shows that its greatest maximum in $[0,1]$ is the rightmost one.

4. Proof of Theorem 1.5

The starting point for derivation of the quadrature formula in Theorem B is the following identity, which is a consequence of another version of the Euler-Maclaurin summation formula (c.f. [4, Satz 98]):

$$\begin{aligned}I[f]=Q_n^{\rm Mi}[f]+\frac{1}{24n^2}\big[f'(1)-f'(0)\big]-\frac{\sqrt{3}}{216n^3}\big[f''(1)-f''(0)\big]\\+\frac{1}{n^3}\int_0^1\left(\frac{\sqrt{3}}{216}-B_3\big(\{nx+1/2\}\big)\right)f^{(3)}(x)\,dx=:\overline{Q}[f]+R[\overline{Q};f],\end{aligned}$$

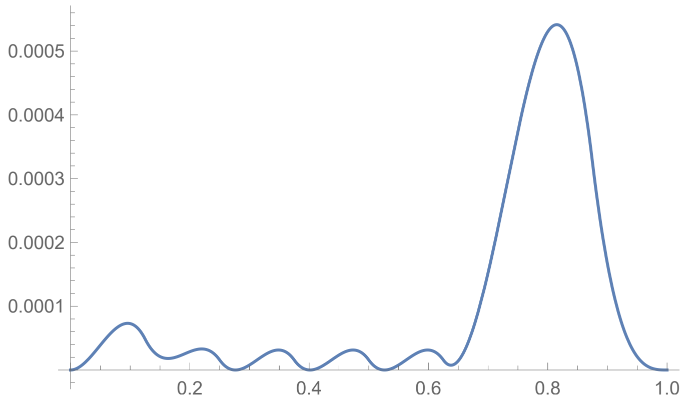


Figure 1. Graph of $K_3(Q_8;t)$, Theorem 1.4

where

$$\overline{Q}[f] = Q_n^{\mathrm{Mi}}[f] - \frac{1}{24n^2} f'(0) + \frac{\sqrt{3}}{216n^3} f''(0) + \frac{1}{24n^2} f'(1) - \frac{\sqrt{3}}{216n^3} f''(1). \tag{4.1}$$

Again, $\overline{Q}$ is a positive definite quadrature formula, though not of the desired type as it involves values of integrand's derivatives, therefore the expressions with derivatives values at the end-points are replaced by pairs of formulae for numerical differentiation involving only integrand's values at the few closest nodes. This yields a quadrature formula Q with $\mathrm{ADP}(Q) = 2$

$$Q[f] = \sum_{k=0}^{n+1} A_{k,n} f(y_{k,n}) \tag{4.2}$$

with nodes

$$y_{0,n} = 0, \quad y_{n+1,n} = 1, \quad y_{k,n} = \frac{2k-1}{2n}, \quad k = 1, \dots, n-1$$

and (for $n \geq 8$) weights $A_{k,n} = 1/n$, $4 \leq k \leq n-3$. The weights $\{A_{k,n}\}_{k=0}^{3}$ depend on a single parameter, say θ, while $\{A_{k,n}\}_{k=n-2}^{n+1}$ depend on another single parameter, say ϱ, these weights are given below

$$A_{0,n} = \frac{\theta}{81n}, \qquad A_{1,n} = \frac{234 + \sqrt{3} - 5\theta}{216n},$$

$$A_{2,n} = \frac{567 - 6\sqrt{3} + 10\theta}{648n}, \qquad A_{3,n} = \frac{225 + \sqrt{3} - \theta}{216n},$$

$$A_{n-2,n} = \frac{225 - \sqrt{3} - \varrho}{216n}, \qquad A_{n-1,n} = \frac{567 + 6\sqrt{3} + 10\varrho}{648n},$$

$$A_{n,n} = \frac{234 - \sqrt{3} - 5\varrho}{216n}, \qquad A_{n+1,n} = \frac{\varrho}{81n}.$$

On the interval $[y_{3,n}, y_{n-2,n}]$ the Peano kernel $K_3(Q;t)$ coincides with that of $\overline{Q}$

$$K_3(Q;t) = n^{-3}\left[\frac{\sqrt{3}}{216} - B_3\left(\{n\,t + 1/2\}\right)\right] \geq 0, \quad t \in [y_{3,n}, y_{n-2,n}], \tag{4.3}$$

thus care is to be taken only for finding the values of parameters θ and ϱ which ensure non-negativeness of $K_3(Q;t)$ on the intervals $[0, y_{3,n}]$ and $[y_{n-2,n}, 1]$. It has been shown in [2] that this is the case when $\theta \geq \theta^* = (41\sqrt{3} - 42)/2$ and $\varrho \leq \varrho^* = 0$. The quadrature formula Q_n in Theorem B is obtained from (4.2) with $(\theta, \varrho) = (\theta^*, \varrho^*)$ (and in this case $A_{n+1,n} = 0$). Since the integrals

$$\int_0^{y_{3,n}} K_3(Q;t)\,dt \quad \text{and} \quad \int_{y_{n-2,n}}^1 K_3(Q;t)\,dt$$

are respectively monotonically increasing function of θ and monotonically decreasing function of ϱ (see [2, equations (11) and (12)]), the choice $(\theta, \varrho) = (\theta^*, \varrho^*)$ is optimal in the sense that it provides the minimal error constant (1.5).

For the proof of Theorem 1.5 we need to find $\|K_3(Q_n;\cdot)\|_{C[0,1]}$, and this requires comparison of

$$\max_{t\in[0,y_{3,n}]} K_3(Q_n;t) \quad \text{and} \quad \max_{t\in[y_{n-2,n},1]} K_3(Q_n;t).$$

Similarly to the proof of Theorem 1.4, with a suitable change of the variable this task is reduced to comparison of the maxima of some spline functions independent of n on fixed intervals. Figure 2 depicts the Peano kernel $K_3(Q_8;t)$, showing that its greatest maximum in $[0,1]$ is the rightmost one. This situation persists for every $n \geq 8$ (only the number of equi-oscillations of $K_3(Q_8;t)$ in $[y_{3,n}, y_{n-2,n}]$ changes). Therefore,

$$\|K_3(Q_n;\cdot)\|_{C[0,1]} = \max_{t\in[y_{n-2,n},1]} K_3(Q_n;t).$$

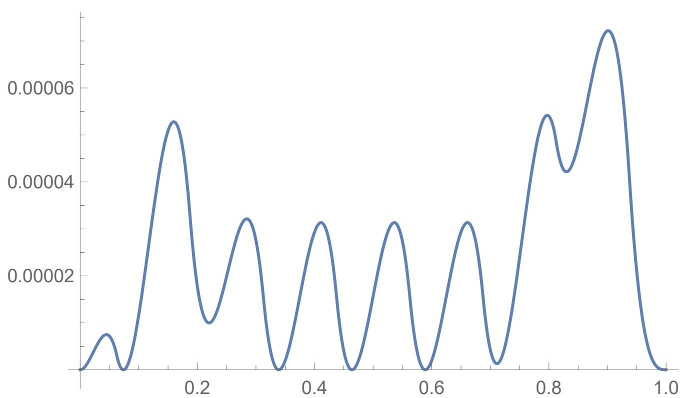


Figure 2. Graph of $K_3(Q_8;t)$, Theorem 1.5

From (2.2) with $r = 3$ we obtain

$$
\begin{aligned}
K_3(Q_n;t) &= \frac{(1-t)^3}{6} - \frac{1}{2}\sum_{k=0}^{n+1} A_{k,n}(y_{k,n}-t)_+^2 \\
&= \frac{(1-t)^3}{6} - \frac{1}{2}\sum_{k=0}^{n+1} A_{k,n}(1-t-y_{n+1-k,n})_+^2 \\
&\overset{x=1-t}{=} \frac{x^3}{6} - \frac{1}{2}\sum_{k=0}^{n+1} A_{n+1-k,n}(x-y_{k,n})_+^2 =: \widetilde{K}_3(Q_n;x).
\end{aligned}
$$

We observe that $x \in [0, y_{3,n}]$ when t varies in $[y_{n-2,n}, 1]$. By substituting further $x = u/n$, $u \in [0, 5/2]$, we get

$$
\widetilde{K}_3(Q;x) = \frac{1}{432n^3} h(u), \quad u \in [0, 5/2],
$$

where

$$
h(u) = 72u^3 - (234 - \sqrt{3})\left(u - \frac{1}{2}\right)_+^2 - (189 + 2\sqrt{3})\left(u - \frac{3}{2}\right)_+^2 .
$$

The maximum of $h(u)$ in $[0, 5/2]$ is attained at the point $u_1 = 0.79073\ldots$, which is the smaller root of the quadratic equation

$$
216u^2 - (234 - \sqrt{3})(2u - 1) = 0,
$$

and $h(u_1) < 15.9651065$, whence

$$
\|K_3(Q_n;\cdot)\|_{C[0,1]} \le \frac{1}{432n^3} h(u_1) < \frac{0.0369563}{n^3}.
$$

Theorem 1.5 now follows from the above inequality and Lemma 2.1.

Remark 4.1. In [8] estimates similar to those in Theorems 1.4 and 1.5 are proved for the Radau-type quadrature formulae associated with the spaces of parabolic splines with double equidistant knots. These quadrature formulae are definite of order three, and under the assumptions of Theorems 1.4 and 1.5, the remainder $R[Q_{n+1}^R; f]$ of the $(n+1)$-point Radau quadrature formula Q_{n+1}^R admits the much better error bound

$$
\left|R[Q_{n+1}^R; f]\right| \le \frac{\sqrt{3}}{108n^3}\left|f''(1) - f''(0)\right| < \frac{0.01604}{n^3}\left|f''(1) - f''(0)\right|.
$$

However, since these quadrature formulae are not nodes-symmetrical, for them the error estimates implied by Proposition 1.3(ii),(iii) are not applicable.

Acknowledgements

The research of the second named author is supported by the Bulgarian National Science Fund under Contract KP-06-N62/4.

References

[1] A. Avdzhieva, V. Gushev and G. Nikolov, Definite quadrature formulae of order three with equidistant nodes, Ann. Univ. Sofia Fac. Math. and Inf. 104 (2017) 155–170.
[2] A. Avdzhieva, V. Gushev and G. Nikolov, Definite quadrature formulae of order three based on the compound midpoint rule, in: International Conference on Numerical Methods and Applications, LNCS, vol. 11189, Cham, Springer International Publishing, 2018, 227–234.
[3] A. Avdzhieva and G. Nikolov, Asymptotically optimal definite quadrature formulae of 4th order, J. Comput. Appl. Math. 311 (2017) 565–582.
[4] H. Braß, Quadraturverfahren, Vandenhoeck & Ruprecht, Göttingen, 1977.
[5] K. Jetter, Optimale Quadraturformeln mit semidefiniten Kernen, Numer. Math. 25 (1976) 239–249.
[6] G. Lange, Beste und optimale definite Quadraturformel, PhD Thesis, Technical University Clausthal, Germany, 1977.
[7] G. Lange, Optimale definite Quadraturformel, in: Numerische Integration, ed. by G. Hämmerlin, ISNM, vol. 45, Birkhäuser, Basel, Boston, Stuttgart, 1979, 187–197.
[8] G. Nikolov and P. Nikolov, On the error bounds of the Gauss-type quadrature formulae associated with spaces of parabolic and cubic spline functions with double equidistant knots, Ann. Univ. Sofia Fac. Math. and Inf. 112 (2025) 91–107.
[9] G. Peano, Resto nelle formule di quadratura espresso con un integrale definito, Atti della Reale Accademia dei Lincei: Rendiconti Ser. 5 22 (1913) 562–569.

Received on October 4, 2025
Revised on November 2, 2025
Accepted on November 3, 2025

Ana Avdzhieva

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: aavdzhieva@fmi.uni-sofia.bg

Petar B. Nikolov

Faculty of Pharmacy
Medical University of Sofia
2, Dunav Str.
1000 Sofia
BULGARIA

E-mail: p.nikolov@pharmfac.mu-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

A CHARACTERISATION OF THE RATE OF APPROXIMATION OF THE BASKAKOV-KANTOROVICH OPERATOR

Borislav R. Draganov

We characterise the K-functional that was previously shown to describe the rate of approximation of the Baskakov-Kantorovich operator in the L_p-spaces with the weight $(1+x)^\gamma$ with $\gamma \le 0$. The characterisation uses the Ditzian-Totik moduli of smoothness.

Keywords: Baskakov-Kantorovich operator, modulus of smoothness, K-functional

2020 Mathematics Subject Classification: 41A25, 41A35, 41A36, 41A81

1. Introduction and main results

Let $f(x)$ be Lebesgue integrable on any finite closed subinterval of $[0,\infty)$ and $n \in \mathbb{N}_+$, $n \ge 2$. We consider the Baskakov-Kantorovich operator defined by

$$\widetilde{V}_n f(x) := \sum_{k=0}^{\infty} v_{n,k}(x)\,(n-1)\int_{\frac{k}{n-1}}^{\frac{k+1}{n-1}} f(u)\,du, \quad x \ge 0,$$

where

$$v_{n,k}(x) := \binom{n+k-1}{k} x^k (1+x)^{-n-k}.$$

Let

$$w(x) := (1+x)^\gamma, \quad \gamma \in \mathbb{R}, \tag{1.1}$$

and

$$L_p(w)[0,\infty) := \{f \in L_{1,loc}(0,\infty) \colon wf \in L_p[0,\infty)\},$$

where $1 \le p \le \infty$. Here $L_{1,loc}(0,\infty)$ stands for the space of all functions which are Lebesgue integrable on any finite closed subinterval of $(0,\infty)$. Let $\|\cdot\|_p$ denote the standard norm in $L_p[0,\infty)$. The norm in $L_p(w)[0,\infty)$ is defined by $\|f\|_{w,p} := \|wf\|_p$.

DOI: 10.60063/GSU.FMI.112.17-29

The K-functional that turns out to naturally describe the approximation rate of $\widetilde{V}_n$ in $L_p(w)[0,\infty)$ is

$$\widetilde{K}(f,t)_{w,p} := \inf_{g\in\widetilde{W}_p(w)[0,\infty)} \{\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\},$$

where $\widehat{D}g(x) := (\varphi^2(x)g'(x))'$, $\varphi(x) := \sqrt{x(1+x)}$ and the space $\widetilde{W}_p(w)[0,\infty)$ is defined, in the case $\gamma \le 0$, by

$$\widetilde{W}_p(w)[0,\infty) := \Big\{g \in AC^1_{loc}(0,\infty)\colon g, \widehat{D}g \in L_p(w)[0,\infty),\ \lim_{x\to 0+0} \varphi^2(x)g'(x) = 0\Big\},$$

and, for $\gamma > 0$, the functions in $\widetilde{W}_p(w)[0,\infty)$ are, in addition, required to satisfy the condition $\lim_{x\to\infty} \varphi^2(x)g'(x) = 0$. As usually, we denote by $AC^m_{loc}(0,\infty)$, where $m \in \mathbb{N}_0$, the space of all functions on $(0,\infty)$, which possess absolute continuous derivatives up to order m on any finite closed subinterval of $(0,\infty)$.

Gadjev [9] proved the direct estimate

$$\|\widetilde{V}_n f - f\|_p \le c\widetilde{K}(f,n^{-1})_{1,p},$$

for all $f \in L_p[0,\infty)$, $1 \le p \le \infty$, and $n \in \mathbb{N}_+$, $n \ge 2$. Here c is a positive constant whose value is independent of f and n.

That estimate was generalised in $L_p(w)[0,\infty)$, $1 \le p \le \infty$, for any $\gamma \in \mathbb{R}$ by Parvanov [24]

$$\|w(\widetilde{V}_n f - f)\|_p \le c\widetilde{K}(f,n^{-1})_{w,p}, \quad f \in L_p(w)[0,\infty), \quad n > |\gamma| + 1. \tag{1.2}$$

Gadjev [9] (for $\gamma = 0$) and Gadjev and Uluchev [17] (for $\gamma < 0$) proved a two-term strong converse inequality when $1 < p \le \infty$, which shows that (1.2) cannot be improved for these p and γ.

We will characterise the K-functional $\widetilde{K}(f,t)_{w,p}$ for $\gamma \le 0$ by the weighted Ditzian-Totik modulus of continuity [4, (6.1.5) and (3.2.1)]

$$\bar{\omega}^1_{1+\chi}(f,t)_{w,p} := \sup_{0<h\le t} \|w\overrightarrow{\Delta}_{h(1+\chi)} f\|_p,$$

where $\chi(x) := x$ and

$$\overrightarrow{\Delta}_\tau f(x) := f(x+\tau) - f(x),$$

and the weighted Ditzian-Totik modulus of smoothness of a second order $\omega^2_\varphi(f,t)_{w,p}$, defined by [4, (6.1.5)]

$$\omega^2_\varphi(f,t)_{w,p} := \sup_{0<h\le t} \|w\Delta^2_{h\varphi} f\|_p,$$

where

$$\Delta^2_\tau f(x) := \begin{cases} f(x+\tau) - 2f(x) + f(x-\tau), & x-\tau \ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

We will show that the following equivalence relation holds. Before that we need to introduce a relation. We say that the real-valued functionals $A(f,t)$ and $B(f,t)$ are equivalent and write $A(f,t) \sim B(f,t)$ for f and t in specified domains if and only if there exists a positive constant c such that $c^{-1}B(f,t) \le A(f,t) \le cB(f,t)$ for all f and t in the specified domains.

Theorem 1.1. *Let* $1 < p \le \infty$ *and* $w(x)$ *be given by* (1.1) *with* $\gamma \le 0$. *Then there exists* $t_0 > 0$ *such that*

$$\widetilde{K}(f,t^2)_{w,p} \sim \omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p},$$

where $f \in L_p(w)[0,\infty)$ *and* $0 < t \le t_0$.

Remark 1.2. As we will show in the proof of the theorem, the inequality

$$\widetilde{K}(f,t^2)_{w,p} \le c\big(\omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p}\big),$$

where $f \in L_p(w)[0,\infty)$ and $0 < t \le t_0$ with some $t_0 > 0$, holds for $p = 1$ as well. Then, in the case $\gamma \le 0$, the direct estimate (1.2) can be stated in the form

$$\|w(\widetilde{V}_n f - f)\|_p \le c\big(\omega_\varphi^2(f,n^{-1/2})_{w,p} + \bar{\omega}_{1+\chi}^1(f,n^{-1})_{w,p}\big)$$

for all $f \in L_p(w)[0,\infty)$, $1 \le p \le \infty$, and $n \ge n_0$ with some constant $n_0 \ge 2$, which is independent of f.

Relations like the one in Theorem 1.1 are not new. The first one of this type known to the author was proved by Gonska and Zhou [18, Theorem 1.2 and Remark 1.3] (see also [1, Theorem B]) for the approximation by the Kantorovich operator in $L_p[0,1]$. It is for $1 < p \le \infty$, too. Later on, the author established a similar characterisation of the rate of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums [6]. About the case $p = 1$, Ivanov [22] introduced a different kind of a modulus of smoothness to characterise the K-functional associated to the Kantorovich and Durrmeyer operator.

A very closely related result like the one in Theorem 1.1 in the unweighted case was established in [10]. In the case $p = \infty$, the characterisation in Theorem 1.1 was essentially established in [7].

Let us recall that the weighted approximation by the Baskakov operator and the Meyer-König and Zeller operator are closely related [25] (see also [8] and [12, Section 2]). This relation might turn useful to transfer results about the rate of approximation by the Baskakov-Kantorovich operator to the Kantorovich form of the Meyer-König and Zeller operator. Direct estimates about the latter were established in [11, 13]. Also, it seems quite reasonable to expect that a similar characterisation of the rate of approximation can be established for the operators considered in [15, 16, 21] as well. In this regard, the K-functional used in [14], where such operators were considered, is equivalent to the one characterised in [5, Theorem 5.1] with $r = 2$. Finally, such an approach should be effective to characterise the K-functionals associated to the approximation rate of combinations of exponential-type operators of the type introduced in [2, 23] and [4, Section 9.2]. Approximation results about such operators were established, e.g., in [26–28] (see also [19]).

2. Preliminaries

We will relate the K-functional $\widetilde{K}(f,t)_{w,p}$ to two simpler ones. They are defined by

$$K_{2,\varphi}(f,t)_{w,p} := \inf\big\{\|w(f-g)\|_p + t\|w\varphi^2 g''\|_p : \\ g \in AC^1_{loc}(0,\infty),\ g, \varphi^2 g'' \in L_p(w)[0,\infty)\big\}$$

and

$$K_{1,1+\chi}(f,t)_{w,p} := \inf\big\{\|w(f-g)\|_p + t\|w(1+\chi)g'\|_p : \\ g \in AC_{loc}(0,\infty),\ g, (1+\chi)g' \in L_p(w)[0,\infty)\big\}.$$

To recall, we have set $\chi(x) := x$.

By [4, Theorem 6.1.1], there exists $t_0 > 0$ such that for all $f \in L_p(w)(0,\infty)$ and $t \in (0,t_0]$ there hold

$$K_{2,\varphi}(f,t^2)_{w,p} \sim \omega^2_\varphi(f,t)_{w,p} \tag{2.1}$$

and

$$K_{1,1+\chi}(f,t)_{w,p} \sim \bar{\omega}^1_{1+\chi}(f,t)_{w,p}. \tag{2.2}$$

We will also make use of the K-functional

$$K_{2,\phi}(f,t)_{p,[0,1]} := \inf\big\{\|f-g\|_{p,[0,1]} + t\|\phi^2 g''\|_{p,[0,1]} : \\ g \in AC^1_{loc}(0,1),\ g, \phi^2 g'' \in L_p[0,1]\big\},$$

where $\|\cdot\|_{p,J}$ stands for the standard L_p-norm on the interval J and $\phi(x) := \sqrt{x(1-x)}$. Similarly, we have (see [4, Theorem 2.1.1]) that there exists $t_0 > 0$ such that for all $f \in L_p[0,1]$ and $t \in (0,t_0]$ there holds

$$K_{2,\phi}(f,t^2)_{p,[0,1]} \sim \omega^2_\phi(f,t)_{p,[0,1]}. \tag{2.3}$$

Here $\omega^2_\phi(f,t)_{p,[0,1]}$ is the Ditzian-Totik modulus of smoothness of order 2, defined in [4, (2.1.2)] by

$$\omega^2_\phi(f,t)_{p,[0,1]} := \sup_{0<h\le t} \|\bar{\Delta}^2_{h\phi} f\|_{p,[0,1]},$$

where

$$\bar{\Delta}^2_\tau f(x) := \begin{cases} f(x+\tau) - 2f(x) + f(x-\tau), & x \pm \tau \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

The last two K-functionals which will play an auxiliary role in the proof of the main result are given by

$$K_{2,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \inf\big\{\|\chi^\gamma(f-g)\|_{p,[1/4,\infty)} + t\|\chi^{\gamma+2} g''\|_{p,[1/4,\infty)} : \\ g \in AC^1_{loc}(1/4,\infty),\ g, \chi^2 g'' \in L_p(\chi^\gamma)[1/4,\infty)\big\}$$

and

$$K_{1,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \inf\big\{\|\chi^\gamma(f-g)\|_{p,[1/4,\infty)} + t\|\chi^{\gamma+1}g'\|_{p,[1/4,\infty)} : \\ g \in AC_{loc}(1/4,\infty),\ g,\chi g' \in L_p(\chi^\gamma)[1/4,\infty)\big\}. \quad (2.4)$$

The first K-functional is related to the weighted Ditzian-Totik modulus of smoothness of second order, defined by [4, (6.1.5)]

$$\omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \sup_{0<h\le t}\|\chi^\gamma\Delta^2_{h\chi}f\|_{p,[1/4,\infty)},$$

and the second K-functional to the weighted Ditzian-Totik modulus of continuity, defined by [4, (6.1.5)]

$$\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} := \sup_{0<h\le t}\|\chi^\gamma\overrightarrow{\Delta}_{h\chi}f\|_{p,[1/4,\infty)},$$

where

$$\Delta^2_\tau f(x) := \begin{cases} f(x+\tau)-2f(x)+f(x-\tau), & x-\tau\ge 1/4,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\overrightarrow{\Delta}_\tau f(x) := f(x+\tau)-f(x).$$

We have, by [4, Theorem 6.1.1], that there exists $t_0>0$ such that for all $f \in L_p(w)(0,\infty)$ and $t\in(0,t_0]$ there hold

$$K_{2,\chi}(f,t^2)_{\chi^\gamma,p,[1/4,\infty)} \sim \omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} \quad (2.5)$$

and

$$K_{1,\chi}(f,t)_{\chi^\gamma,p,[1/4,\infty)} \sim \bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}. \quad (2.6)$$

In all the above instances, the modulus of continuity is defined by the forward finite difference rather than by the symmetric one as it was done in the cited definitions in [4]. We do that for technical convenience only. We still have (2.2) and (2.6) for the forward finite difference modulus of continuity. That was shown in [4, Theorem 3.2.1] for the case $\gamma=0$, but the proof can be readily extended to any real γ.

In order to estimate the K-functional $\widetilde{K}(f,t)_{w,p}$ from below by means of the K-functionals $K_{2,\varphi}(f,t)_{w,p}$ and $K_{1,1+\chi}(f,t)_{w,p}$, hence, in view of (2.1) and (2.2), by $\omega^2_\varphi(f,t)_{w,p}$ and $\bar{\omega}^1_{1+\chi}(f,t)_{w,p}$, we will use the embedding inequalities below. They are known (see [17, Lemma 4]), we include their short proof for the reader's convenience.

Proposition 2.1. *Let $1<p\le\infty$ and $w(x)$ be given by (1.1) with $\gamma\le 0$. Let $g\in\widetilde{W}_p(w)[0,\infty)$. Then*

$$\|w(1+\chi)g'\|_p \le \frac{p}{p-1}\big\|w\widehat{D}g\big\|_p \quad (2.7)$$

and

$$\|w\varphi^2 g''\|_p \le \frac{3p-1}{p-1}\big\|w\widehat{D}g\big\|_p. \quad (2.8)$$

For $p = \infty$, the expressions $p/(p-1)$ and $(3p-1)/(p-1)$ are to be interpreted as their limit at infinity.

Proof. Since $|w(x)\varphi^2(x)g''(x)| \le 2|w(x)(1+x)g'(x)| + |w(x)\widehat{D}g(x)|$, it is enough to show (2.7).

We let $\varepsilon \to 0+0$ in

$$\int_\varepsilon^x \widehat{D}g(u)\,du = \varphi^2(x)g'(x) - \varphi^2(\varepsilon)g'(\varepsilon), \quad x > 0,$$

to arrive at

$$x(1+x)g'(x) = \int_0^x \widehat{D}g(u)\,du, \quad x > 0. \tag{2.9}$$

We applied the Dominated Convergence Theorem and $\lim_{\varepsilon\to 0+0}\varphi^2(\varepsilon)g'(\varepsilon) = 0$. Regarding the former, we have $\widehat{D}g \in L_1[0,x]$ for any $x > 0$ by virtue of Hölder's inequality.

Next, (2.9) yields

$$|w(x)(1+x)g'(x)| \le \frac{1}{x}\int_0^x |w(u)\widehat{D}g(u)|\,du, \quad x > 0; \tag{2.10}$$

hence, for $1 < p < \infty$,

$$\left(\int_0^\infty |w(x)(1+x)g'(x)|^p\,dx\right)^{1/p} \le \left(\int_0^\infty \left(\frac{1}{x}\int_0^x |w(u)\widehat{D}g(u)|\,du\right)^p dx\right)^{1/p}.$$

Now, by virtue of Hardy's inequality (see [20, p. 245])

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x |F(u)|\,du\right)^p dx\right)^{1/p} \le \frac{p}{p-1}\,\|F\|_p,$$

with $F(u) = w(u)\widehat{D}g(u)$, we arrive at (2.7) for $1 < p < \infty$.

In the case $p = \infty$, we readily derive from (2.10) the inequality

$$\|w(1+\chi)g'\|_\infty \le \left\|w\widehat{D}g\right\|_\infty.$$

Thus (2.7) is established. □

3. Proof of Theorem 1.1

We denote by c and t_0 positive constants, whose value is independent of the functions involved, the function variable and t. Their value can vary at each occurrence.

First, we will show that there exist a positive constant t_0 such that for all $f \in L_p(w)[0,\infty)$ and $t \in (0, t_0]$, there holds

$$\omega_\varphi^2(f,t)_{w,p} + \bar{\omega}_{1+\chi}^1(f,t^2)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p}. \tag{3.1}$$

Let $g \in \widetilde{W}_p(w)[0,\infty)$. By Proposition 2.1, there hold

$$\|w(1+\chi)g'\|_p \le c\|w\widehat{D}g\|_p$$

and

$$\|w\varphi^2 g''\|_p \le c\|w\widehat{D}g\|_p.$$

Consequently, $(1+\chi)g', \varphi^2 g'' \in L_p(w)[0,\infty)$,

$$K_{1,1+\chi}(f,t)_{w,p} \le \|w(f-g)\|_p + t\|w(1+\chi)g'\|_p \le c\big(\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\big)$$

and

$$K_{2,\varphi}(f,t)_{w,p} \le \|w(f-g)\|_p + t\|w\varphi^2 g''\|_p \le c\big(\|w(f-g)\|_p + t\|w\widehat{D}g\|_p\big).$$

Next, we take the infimum on $g \in \widetilde{W}_p(w)[0,\infty)$ to arrive at

$$K_{1,1+\chi}(f,t)_{w,p} \le c\widetilde{K}(f,t)_{w,p}, \quad t>0,$$

and

$$K_{2,\varphi}(f,t)_{w,p} \le c\widetilde{K}(f,t)_{w,p}, \quad t>0.$$

Now, (2.1) and (2.2) imply

$$\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p},$$

and

$$\omega^2_{\varphi}(f,t)_{w,p} \le c\widetilde{K}(f,t^2)_{w,p},$$

where $0<t\le t_0$ with some t_0 independent of f; hence we get (3.1).

To establish the reverse relation, we use the same approach as in the proof of [7, Theorem 1.2], where the case $p=\infty$ was considered.

Let $1 \le p \le \infty$. For any $t \in (0,t_0]$ with some $t_0 \in (0,1]$ to be specified in the course of the proof, we will define a function $g_t \in \widetilde{W}_p(w)[0,\infty)$ such that

$$\|w(f-g_t)\|_p \le c\,\omega^2_{\varphi}(f,t)_{w,p}, \tag{3.2}$$

and

$$t^2\big\|w\widehat{D}g_t\big\|_p \le c\left(\omega^2_{\varphi}(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right). \tag{3.3}$$

Then we readily get

$$\widetilde{K}(f,t)_{w,p} \le c\left(\omega^2_{\varphi}(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0<t\le t_0.$$

To establish (3.3), we will show that

$$t^2\|w(1+\chi)g_t'\|_p \le c\left(\omega^2_{\varphi}(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right) \tag{3.4}$$

and

$$t^2\|w\varphi^2 g_t''\|_p \le c\left(\omega^2_{\varphi}(f,t)_{w,p} + \bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}\right). \tag{3.5}$$

We will define g_t by patching smoothly two approximants of f on $[0,3/4]$ and $[1/2,\infty)$, respectively, which satisfy (3.2), (3.4) and (3.5).

We begin with the approximant on $[0,3/4]$. We use the same argument as in [6, p. 132] and [7, pp. 25–26]. It is not long and we include it for the sake of completeness. Let $m\in\mathbb{N}_+$ be such that $t\in(1/(m+1),1/m]$ and let $Q_t(x):=Q_T(f)(x)$ be the quasi-interpolant spline operator of order $r+1$ with knots t_j, $j=-m+1,\dots,m-1$, used in the proof of [3, Chapter 6, Theorem 6.2] with $r=2$ for the interval $[0,1]$ instead of $[-1,1]$.

We have, by [3, Chapter 5, Proposition 4.6, and Chapter 6, Theorem 4.2, (6.22) and (6.24)]), that $Q_t\in AC^1[0,1]$ and

$$\begin{aligned}\|f-Q_t\|_{p,[0,1]}&\le c\,\omega_\phi^2(f,t)_{p,[0,1]},\\ t^2\|Q_t'\|_{p,[0,1]}&\le c\,\omega(f,t^2)_{p,[0,1]},\\ t^2\|\phi^2Q_t''\|_{p,[0,1]}&\le c\,\omega_\phi^2(f,t)_{p,[0,1]}\end{aligned}\tag{3.6}$$

for $0<t\le 1/4$. Here $\omega(f,t)_{p,[0,1]}$ denotes the classical modulus of continuity in $L_p[0,1]$.

Next, since $w(x)\ge c>0$ and $\phi(x)\le\varphi(x)$ on $[0,1]$, then

$$K_{2,\phi}(f,t)_{p,[0,1]}\le c\,K_{2,\varphi}(f,t)_{w,p};$$

hence, by virtue of and (2.1) and (2.3) (cf. [4, Theorem 4.1.1])

$$\omega_\phi^2(f,t)_{p,[0,1]}\le c\,\omega_\varphi^2(f,t)_{w,p}\tag{3.7}$$

for $0<t\le t_0$ with some $t_0\in(0,1)$. In addition, similarly,

$$\omega(f,t^2)_{p,[0,1]}\le c\,\bar\omega_{1+\chi}^1(f,t^2)_{w,p},\quad 0<t\le t_0.$$

We combine the last two estimates with (3.6) to deduce

$$\|f-Q_t\|_{p,[0,3/4]}\le c\,\omega_\varphi^2(f,t)_{w,p},\tag{3.8}$$

$$t^2\|Q_t'\|_{p,[0,3/4]}\le c\,\bar\omega_{1+\chi}^1(f,t^2)_{w,p},\tag{3.9}$$

$$t^2\|\varphi^2Q_t''\|_{p,[0,3/4]}\le c\,\omega_\varphi^2(f,t)_{w,p}\tag{3.10}$$

for $0<t\le t_0$.

We proceed to the definition of the approximant on $[1/2,\infty)$. We will use the same one as in the second half of the proof of [7, Theorem 1.2], where the case $p=\infty$ was only considered. The prove below include all $\gamma\in\mathbb{R}$ and $1\le p\le\infty$.

Let $1\le p<\infty$. Following [7, p. 27], we introduce the Steklov-type function

$$H_t(x):=\frac{2}{t^2}\int_0^{t/2}\int_0^{t/2}\left[f\big(x+x(u_1+u_2)\big)+f\big(x-x(u_1+u_2)\big)\right]du_1\,du_2,$$

where $0<t\le 1/2$ and $x\ge 1/2$.

By Minkowski's integral inequality, we arrive at

$$\begin{aligned}\|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)} &\le \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2}\|\chi^\gamma\Delta^2_{(u_1+u_2)\chi}f\|_{p,[1/2,\infty)}\,du_1\,du_2\\ &\le \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2}\omega^2_\chi(f,t)_{\chi^\gamma,[1/4,\infty)}\,du_1\,du_2\\ &\le \omega^2_\chi(f,t)_{\chi^\gamma,[1/4,\infty)}.\end{aligned}$$

Then similarly to (3.7), we use (2.1) and (2.5) to further deduce

$$\|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)} \le c\,\omega^2_\varphi(f,t)_{w,p}, \quad 0<t\le t_0. \tag{3.11}$$

We set

$$\widetilde{H}_t(x) := \frac{2}{t^2}\int_0^{t/2}\int_0^{t/2} f\big(x+x(u_1+u_2)\big)\,du_1\,du_2.$$

We have $H_t(x)=\widetilde{H}_t(x)+\widetilde{H}_{-t}(x)$.

We write $\widetilde{H}_t$ in the form

$$\widetilde{H}_t(x) = \frac{2}{x^2t^2}\int_x^{x(1+t/2)}(u-x)f(u)\,du + \frac{2}{x^2t^2}\int_{x(1+t/2)}^{x(1+t)}(x(1+t)-u)f(u)\,du.$$

Therefore, $\widetilde{H}_t \in AC^1[1/2,\infty)$ and straightforward calculations yield

$$\begin{aligned}\widetilde{H}'_t(x) &= \frac{2}{xt^2}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\\ &\qquad -\frac{4}{xt^2}\int_0^{t/2}\int_0^{t/2}\left[f\big(x+x(u_1+u_2)\big)-f\left(x+x\left(u_1+\frac{t}{2}\right)\right)\right]du_1\,du_2.\end{aligned}$$

To estimate the $L_p(\chi^{\gamma+1})[1/2,\infty)$-norm of each of the integrals above, we subtract and add $f(x)$, split the integral into two terms by the triangle inequality, and apply Minkowski's integral inequality. For the first integral, we have

$$\begin{aligned}&\left(\int_{1/2}^\infty\left|x^{\gamma+1}\frac{1}{x}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\right|^p dx\right)^{1/p}\\ &\quad\le\left(\int_{1/2}^\infty\left|x^\gamma\int_0^{t/2}\left|f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x)\right|du\right|^p dx\right)^{1/p}\\ &\qquad+\left(\int_{1/2}^\infty\left|x^\gamma\int_0^{t/2}|f(x+xu)-f(x)|\,du\right|^p dx\right)^{1/p}\\ &\quad\le\int_0^{t/2}\left\|\chi^\gamma\overrightarrow{\Delta}_{(u+t/2)\chi}f\right\|_{p,[1/2,\infty)}du+\int_0^{t/2}\left\|\chi^\gamma\overrightarrow{\Delta}_{u\chi}f\right\|_{p,[1/2,\infty)}du\\ &\quad\le t\,\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}.\end{aligned}$$

We estimate the $L_p(\chi^{\gamma+1})[1/2,\infty)$-norm of the second integral in the above expression of $\widetilde{H}'_t$ in a similar way. Thus, we arrive at

$$t^2\|\chi^{\gamma+1}\widetilde{H}'_t\|_{p,[1/2,\infty)} \le ct\,\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)} \le c\,\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}, \quad 0<t\le t_0,$$

as for the second estimate we took into account (2.2) and (2.6).

We prove the analogue of this estimate for $\widetilde{H}_{-t}$ in a similar way, as we take into account the fact that if we replace the forward finite difference in the definition of $\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}$ with the backward one, we get a modulus, which is still equivalent to the same K-functional; hence to $\bar{\omega}^1_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}$ (see [4, Section 3.2]).

Consequently,

$$t^2\|\chi^{\gamma+1}H'_t\|_{p,[1/2,\infty)} \le c\,\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}, \quad 0<t\le t_0. \tag{3.12}$$

By means of straightforward calculations, we arrive at

$$\begin{aligned}\widetilde{H}''_t(x) = &-\frac{8}{x^2t^2}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f(x+xu)\right]du\\ &-\frac{2}{x^2t}\int_0^{t/2}\left[f\left(x+x\left(u+\frac{t}{2}\right)\right)-f\left(x+x\frac{t}{2}\right)\right]du\\ &+\frac{12}{x^2t^2}\int_0^{t/2}\int_0^{t/2}\left[f(x+x(u_1+u_2))-f\left(x+x\left(u_1+\frac{t}{2}\right)\right)\right]du_1\,du_2\\ &+\frac{2(t+2)}{x^2t}\left[f(x+xt)-f\left(x+x\frac{t}{2}\right)\right]\\ &+\frac{2}{x^2t^2}\left[f(x+xt)-2f\left(x+x\frac{t}{2}\right)+f(x)\right].\end{aligned}$$

We estimate the $L_p(\chi^{\gamma+2})[1/2,\infty)$-norm, multiplied by t^2, of all the terms on the first four lines on the right-hand side above by $\bar{\omega}^1_{1+\chi}(f,t^2)_{w,p}$ as we did for $\widetilde{H}'_t$. Likewise, we estimate the norm of the corresponding terms in $\widetilde{H}''_{-t}$. As for the terms on the last line, we consider their sum with the corresponding terms in $\widetilde{H}''_{-t}$ and write this sum in the form

$$\begin{aligned}G_t(x) := &\frac{2}{x^2t^2}\left[f(x+xt)-2f\left(x+x\frac{t}{2}\right)+f(x)\right]\\ &+\frac{2}{x^2t^2}\left[f(x-xt)-2f\left(x-x\frac{t}{2}\right)+f(x)\right]\\ =&\frac{2}{x^2t^2}\,\Delta^2_{xt}f(x)-\frac{4}{x^2t^2}\,\Delta^2_{xt/2}f(x);\end{aligned}$$

hence

$$t^2\|\chi^{\gamma+2}G_t\|_{p,[1/2,\infty)} \le c\,\omega^2_\chi(f,t)_{\chi^\gamma,p,[1/4,\infty)}.$$

Further, we use (2.1) and (2.5) to get

$$t^2\|\chi^{\gamma+2}G_t\|_{p,[1/2,\infty)} \le c\,\omega^2_\varphi(f,t)_{w,p}.$$

Thus, we show that

$$t^2\|\chi^{\gamma+2}H_t''\|_{p,[1/2,\infty)} \le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0<t\le t_0. \tag{3.13}$$

We are ready to define $g_t(x)$ and derive that it satisfies (3.2), (3.4) and (3.5). We apply a standard argument (see, e.g., [3, p. 176]). Let $\psi \in C^\infty(\mathbb{R})$ be such that $\psi(x) = 0$ for $x \le 1/2$ and $\psi(x) = 1$ for $x \ge 3/4$. Clearly, there exists a positive constant c such that $|\psi^{(i)}(x)| \le c$, $x \in \mathbb{R}$, where $i = 0, 1, 2$. We set

$$g_t(x) := (1-\psi(x))Q_t(x) + \psi(x)H_t(x), \quad x \ge 0.$$

Then $g_t \in AC^1_{loc}(0,\infty)$ and $\lim_{x\to 0+0}\varphi^2(x)g_t'(x) = \lim_{x\to 0+0} x(1+x)Q_t'(x) = 0$.

Next, we take into account that $|\psi(x)| \le c$ for $x \in \mathbb{R}$, $\psi(x) = 0$ for $x \le 1/2$ and $\psi(x) = 1$ for $x \ge 3/4$, and (3.8) and (3.11) to get

$$\begin{aligned}\|w(f-g_t)\|_p &\le c\big(\|f-Q_t\|_{p,[0,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,\infty)}\big)\\ &\le c\,\omega_\varphi^2(f,t)_{w,p}, \quad 0<t\le t_0.\end{aligned}$$

Thus, (3.2) is established.

We represent the first derivative of g_t in the form

$$g_t'(x) = \psi'(x)[f(x)-Q_t(x)] - \psi'(x)[f(x)-H_t(x)] + (1-\psi(x))Q_t'(x) + \psi(x)H_t'(x).$$

Then, similarly to the last estimate, but using also (3.9) and (3.12), we get

$$\begin{aligned}t^2\|w(1+\chi)g_t'\|_p &\le c\big(\|f-Q_t\|_{p,[1/2,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,3/4]}\\ &\qquad + \|Q_t'\|_{p,[0,3/4]} + \|\chi^{\gamma+1}H_t'\|_{p,[1/2,\infty)}\big)\\ &\le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0<t\le t_0.\end{aligned}$$

Thus, (3.4) is established.

Finally, for the second derivative of g_t, we have

$$\begin{aligned}g_t''(x) &= \psi''(x)[f(x)-Q_t(x)] - \psi''(x)[f(x)-H_t(x)]\\ &\quad - 2\psi'(x)Q_t'(x) + 2\psi'(x)H_t'(x) + (1-\psi(x))Q_t''(x) + \psi(x)H_t''(x).\end{aligned}$$

Then, similarly to the last two estimates, but taking into account all relations (3.8)–(3.10) and (3.11)–(3.13), we arrive at

$$\begin{aligned}t^2\|w\varphi^2 g_t''\|_p &\le c\big(\|f-Q_t\|_{p,[1/2,3/4]} + \|\chi^\gamma(f-H_t)\|_{p,[1/2,3/4]}\\ &\qquad + \|Q_t'\|_{p,[1/2,3/4]} + \|\chi^{\gamma+1}H_t'\|_{p,[1/2,3/4]}\big)\\ &\qquad + \|Q_t''\|_{p,[0,3/4]} + \|\chi^{\gamma+2}H_t''\|_{p,[1/2,\infty)}\big)\\ &\le c\left(\omega_\varphi^2(f,t)_{w,p} + \bar\omega^1_{1+\chi}(f,t^2)_{w,p}\right), \quad 0<t\le t_0.\end{aligned}$$

Thus, we have shown (3.5) and completed the proof of $g_t \in \widetilde{W}_p(w)[0,\infty)$ and the second part of the proof of Theorem 1.1 for $1 \le p < \infty$. The validity of that part for $p = \infty$ was shown in [7, pp. 27–29].

Acknowledgements

The research is supported by the Bulgarian National Science Fund under Contract KP-06-N62/4 from 2022. I am thankful to the reviewers for their remarks which improved the exposition and corrected several mistakes.

References

[1] H. Berens and Y. Xu, On Bernstein-Durrmeyer polynomials with Jacobi weights: The cases $p = 1$ and $p = \infty$, in: Approximation, interpolation, and summation, ed. by S. Baron and D. Leviatan, Israel Mathematical Conference Proceedings, vol. 4, Jerusalem, Weizmann Science Press of Israel, 1991, 51–62.

[2] P. L. Butzer, Linear combinations of Bernstein polynomials, Canad. J. Math. 5 (1953) 559–567.

[3] R. A. DeVore and G. G. Lorentz, Constructive approximation, Springer-Verlag, Berlin, 1993.

[4] Z. Ditzian and V. Totik, Moduli of smoothness, Springer-Verlag, New York, 1987.

[5] B. R. Draganov, On simultaneous approximation by iterated Boolean sums of Bernstein operators, Results Math. 66 (2014) 21–41.

[6] B. R. Draganov, Strong estimates of the weighted simultaneous approximation by the Bernstein and Kantorovich operators and their iterated Boolean sums, J. Approx. Theory 200 (2015) 92–135.

[7] B. R. Draganov, Direct estimates of the weighted simultaneous approximation by the Baskakov operator, Results Math. 75 (2020) 156.

[8] I. Gadjev, Strong converse result for uniform approximation by Meyer-König and Zeller operator, J. Math. Anal. Appl. 428 (2015) 32–42.

[9] I. Gadjev, Approximation of functions by Baskakov-Kantorovich operator, Results Math. 70 (2016) 385–400.

[10] I. Gadjev, About characterization of one K-functional, J. Math. Anal. Appl. 450 (2017) 1076–1082.

[11] I. Gadjev, A direct theorem for MKZ-Kantorovich operator, Anal. Math. 45 (2019) 25–38.

[12] I. Gadjev and P. E. Parvanov, Weighted approximation in uniform norm by Meyer-König and Zeller operators, Ann. Sofia Univ. Fac. Math and Inf. 104 (2017) 77–87.

[13] I. Gadjev, P. E. Parvanov and R. Uluchev, Weighted approximation by Kantorovich type modification of Meyer-König and Zeller operator, Ann. Sofia Univ. Fac. Math and Inf. 105 (2018) 75–95.

[14] I. Gadjev, P. E. Parvanov and R. Uluchev, Higher order approximation of functions by modified Goodman-Sharma operators, Constr. Math. Anal. 7(4) (2024) 180–195.

[15] I. Gadjev, P. E. Parvanov and R. Uluchev, On a new modification of Baskakov operators with higher order of approximation, Mathematics 13 (2025) 64.

[16] I. Gadjev, P. E. Parvanov and R. Uluchev, Higher order approximation of continuous functions by a modified Meyer-König and Zeller-Type operator, arXiv preprint, arXiv:2506.14392, 2025.

[17] I. Gadjev and R. Uluchev, Strong converse result for weighted approximation by Baskakov-Kantorovich operator, in: Proc. Internat. Conf. "Constructive Theory of Functions", Sozopol, Bulgaria, June 2–8, 2019, ed. by B. Draganov, K. Ivanov, G. Nikolov and R. Uluchev, Bulgarian Acad. Sci., Sofia, 2020, 53–71.

[18] H. Gonska and X.-l. Zhou, The strong converse inequality for Bernstein-Kantorovich operators, Comput. Math. Appl. 30 (1995) 103–128.
[19] V. Gupta and G. Tachev, Approximation with positive linear operators and linear combinations, Springer, Cham, 2017.
[20] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, 1951.
[21] M. Heilmann and M. Wagner, The genuine Bernstein-Durrmeyer operators and quasi-interpolants, Results. Math. 62 (2012) 319–335.
[22] K. G. Ivanov, A characterization theorem for the K-functional for Kantorovich and Durrmeyer operators, in: Approximation theory, A volume dedicated to Borislav Bojanov, ed. by D. K. Dimitrov, G. Nikolov and R. Uluchev, Marin Drinov Academic Publishing House, Sofia, 2004, 112–125.
[23] C. P. May, Saturation and inverse theorems for combinations of a class of exponential-type operators, Canad. J. Math. 28 (1976) 1224–1250.
[24] P. E. Parvanov, Weighted approximation of functions in L_p-norm by Baskakov-Kantorovich operator, Anal. Math. 46 (2020) 821–842.
[25] V. Totik, Approximation by Meyer-König and Zeller type operators, Math. Z. 182 (1983) 425–446.
[26] S. Wang and L. Xie, Strong converse inequality for linear combinations of Szász-Mirakjan operators, J. Approx. Theory 273 (2022) 105651.
[27] S. Wang and L. Xie, The L_p-saturation of linear combinations of Szász-Mirakjan-Kantorovich operators, Int. J. Wavelets Multiresolut. Inf. Process. 23(1) (2025) 2450046.
[28] L. Xie, The saturation class for linear combinations of Bernstein operators, Arch. Math. 91 (2008) 86–96.

Received on October 11, 2025
Revised on November 6, 2025
Accepted on November 7, 2025

Borislav R. Draganov

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

and

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia
BULGARIA

E-mail: `bdraganov@fmi.uni-sofia.bg`

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

ON THE CONNECTION BETWEEN FIXED POINT THEOREMS ON METRIC SPACES WITH GRAPHS AND $\mathbb{P}$ SETS

VALENTIN GEORGIEV, ATANAS ILCHEV and BOYAN ZLATANOV

The Banach contraction principle is one of the most famous and applied results in recent mathematical history. Due to its utility, plenty of generalizations have been established. One of them considers a contraction principle on metric spaces with graphs, while another confines the contraction principle to pairs of elements inside a $\mathbb{P}$ set, a generalization of partial orders. In this work we examine the similarities of both approaches, establishing the connection between theorems of metric spaces with graphs and metric spaces with $\mathbb{P}$ sets and restating results from one approach to the other and vice versa.

Keywords: fixed point, metric space endowed with a graph, $\mathbb{P}$ sets

2020 Mathematics Subject Classification: 47H10, 05C40, 54H25

1. Introduction

The Banach Contraction Principle [1] has long served as a cornerstone in fixed point theory, guaranteeing the existence and uniqueness of fixed points for certain classes of self-maps defined on complete metric spaces. Over time, considerable effort has been invested in extending and generalizing this principle to encompass broader settings. Two notable extensions focus on restricting the contraction condition to certain elements of the metric space. One of these involves the study of mappings defined on metric spaces endowed with a directed graph, where the notion of a G-contraction is used to yield refined fixed point results. The other is concerned with $\mathbb{P}$ sets, a binary relation structure allowing the contraction condition to be imposed in a more flexible and generalized manner.

Studies of fixed points in graph-based frameworks demonstrate that adding a directed graph G to a metric space (X, ρ) enriches the classical theory. This approach

DOI: 10.60063/GSU.FMI.112.31-44

leads to fixed point theorems that unify and extend numerous known results obtained for mappings defined on partially ordered sets, cone metric spaces, or ordered normed spaces [7, 9, 11, 14]. Such graph-based results have broadened the applicability of the Banach Contraction Principle, enabling one to treat nonlinear integral equations, coupled fixed points, and iterative approximation in function spaces within a unified setting.

Simultaneously, the introduction of $\mathbb{P}$ sets has provided a versatile tool to handle fixed point problems where the contraction condition may vary depending on the points under consideration. These structures are inspired by results in partially ordered metric spaces, first introduced in [17] and later popularized in [2, 15]. By abstracting the partial order as a $\mathbb{P}$ set, researchers have proven natural formulation of contractive iterates, generalizing classical assumptions and linking them to iterative processes often seen in nonlinear analysis [3–5, 12, 13]. This setting encapsulates a wide spectrum of known contraction-type maps, from single-valued mappings on metric spaces to more sophisticated structures that underlie iterative approximation schemes. A deep observation in [12] makes a connection between fixed points and coupled fixed points, utilizing $\mathbb{P}$ sets.

Although these two directions (graph-based fixed point theory and $\mathbb{P}$-based approaches) originate from different motivations and employ distinct technical tools, they share a profound conceptual similarity. Both paradigms embed the classical contraction condition into a richer structural environment, capitalizing on additional relational properties to produce more general fixed point results. The key question this paper addresses is how these two seemingly different approaches relate to each other. We aim to demonstrate that many results obtained in the context of G-contractions have direct analogues within the $\mathbb{P}$ set framework, and vice versa.

By examining the conditions and conclusions of fixed point theorems in both settings, we establish a correspondence between the assumptions on the graph G and those on the $\mathbb{P}$ sets. This correspondence allows us to transfer results, insights, and techniques from one realm to the other, thereby yielding a unified perspective on fixed point theory that transcends the particularities of the chosen framework. Such a unification not only streamlines the existing theory but also opens new avenues for research, enabling known fixed point principles to be translated and applied in broader contexts.

In what follows, we present a reformulation of several theorems, translating from G-contractions to $\mathbb{P}$ sets and the other way around, highlighting their equivalences in terms of existence, uniqueness, and ordered structural properties of fixed points. The results herein show that the interplay between metric completeness, contractive behavior, and additional relational structures, be it a directed graph or a binary relation, gives rise to a more comprehensive and robust theory of fixed points.

2. Preliminaries

In what follows, we will use the notation $\mathbb{N}$ for the natural numbers ($\mathbb{N} = 1, 2, \dots$), $\mathbb{Z}$ for the integers, $\mathbb{Q}$ for the rational numbers, $\mathbb{R}$ for the reals, (X, ρ) for a metric space with a metric ρ, G for a graph a metric space is endowed with, f for a mapping from X to X when discussing metric spaces endowed with a graph, T for a mapping from X to X when discussing metric spaces with a $\mathbb{P}$ set and F for mappings from $X \times X$ to X for both metric spaces endowed with a graph and those with a $\mathbb{P}$ set.

2.1. Metric spaces endowed with a graph

In this subsection we follow the exposition in [7].

Definition 2.1 ([12])**.** Let (X, ρ) be a metric space. Two sequences $x_n, y_n \in X$, $n \in \mathbb{N}$ are said to be Cauchy equivalent if $\lim_{n\to\infty} \rho(x_n, y_n) = 0$.

Whenever we discuss a metric space (X, ρ) endowed with a graph, it will be understood that G is a weighted directed graph with a set of vertices $V(G) = X$ and an edge set $E(G) \subseteq X \times X$, where the weights of the edges will be calculated as the distance between their endpoints. We will also require some general notions from graph theory.

By G^{-1} we will denote the conversion of G, i.e., $V(G^{-1}) = V(G)$ and

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Further, $\widetilde{G}$ will be the undirected graph obtained from G, that is, $V(\widetilde{G}) = V(G)$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. A subgraph of G is called a graph (V', E') such that $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for each edge $(x, y) \in E'$, it holds that $x, y \in V'$.

If x and y are vertices of G, then a path of length n, $n \in \mathbb{N} \cup \{0\}$ is a sequence of vertices $\{x_i\}_{i=0}^{n}$ such that

$$x_0 = x,\ x_n = y,\ (x_{i-1}, x_i) \in E(G) \ \text{ for } \ i = 1, 2, \dots, n.$$

A graph is said to be connected if there is a path between any two vertices. Given that $\widetilde{G}$ is connected, G is weakly connected. If the edge set $E(G)$ of a graph G is symmetric, then the component of G containing a vertex x is defined as the subgraph G_x that includes all vertices and edges that lie on a path starting from x. By $[x]_G$ we will denote the equivalence class induced by the relation R defined on $V(G)$ as

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

It follows that $V(G_x) = [x]_G$. Let us also point out that $\widetilde{G}$ clearly has a symmetric edge set.

Definition 2.2 ([7])**.** Let (X, ρ) be a metric space endowed with a graph G. We say that a mapping $f : X \to X$ is a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,

$$\text{for all } x, y \in X((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way: there exists $\alpha \in (0, 1)$ such that for all $x, y \in X$

$$((x, y) \in E(G) \Rightarrow d(fx, fy) \le \alpha\rho(x, y)).$$

Let us recollect that the self-map f is said to be a Picard operator (PO) if for every $x \in X$, we have that $\lim_{n\to\infty} f^n(x) = x^*$, where $x^* \in X$ is the unique fixed point of the operator. A weaker notion is that of a weak Picard operator, that is, for the mapping f it holds that $\lim_{n\to\infty} f^n(x)$ is convergent for all $x \in X$ to a fixed point of f that may not be unique.

For the following theorem to hold, the assumption that $(x, x) \in E(G)$ for all $x \in X$ is made.

Theorem 2.3 ([7])**.** *Let (X, ρ) be a complete metric space and G be a directed graph on X. Assume that (X, ρ, G) satisfies the following property:*

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim\limits_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k.

Let $f\colon X \to X$ be a G-contraction and define

$$X_f := \{x \in X\colon (x, f(x)) \in E(G)\}.$$

Then the following hold:

(1) $\operatorname{card}(\operatorname{Fix}(f)) = \operatorname{card}\big(\{[x]_{\widetilde{G}}\colon x \in X_f\}\big)$.

(2) $\operatorname{Fix}(f) \neq \varnothing \iff X_f \neq \varnothing$.

(3) *f has a unique fixed point if and only if there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*

(4) *For every $x \in X_f$, the restriction $f|_{[x]_{\widetilde{G}}}$ is PO (Picard operator).*

(5) *If $X_f \neq \varnothing$ and G is weakly connected, then f is PO.*

(6) *If $X' = \bigcup_{x\in X_f} [x]_{\widetilde{G}}$, then $f|_{X'}$ is WPO (Weak Picard operator).*

(7) *If $f \subseteq E(G)$, then f is WPO.*

2.2. Metric spaces endowed with a $\mathbb{P}$ set

In order to state a part of the results, proven in the context of metric spaces with a $\mathbb{P}$ set, the following definitions are oftentimes used, following the exposition in [12, 13].

Definition 2.4 ([13])**.** Let X be a non-empty set, $\mathbb{P} \subseteq X\times X$ and $T\colon X \to X$ be a map. We say that $\mathbb{P}$ is T-closed if whenever $(x, y) \in \mathbb{P}$, it follows that $(Tx, Ty) \in \mathbb{P}$.

Next, we will present some examples of T-closed $\mathbb{P}$ sets.

Example 2.5 ([12])**.** Let $(X, \rho, \preccurlyeq)$ be a partially ordered metric space. Let the mapping $T\colon X \to X$ be an increasing function, i.e., $Tx \preccurlyeq Ty$, provided that $x \preccurlyeq y$. Then the set $\mathbb{P} = \{(x, y) \in X \times X\colon x \preccurlyeq y\}$ is T-closed.

Example 2.6 ([12])**.** Let $(X, \rho, \preccurlyeq)$ be a partially ordered metric space. For the mapping $T\colon X \to X$ let Tx be comparable with Ty, i.e., $Tx \asymp Ty$. Then the set $\mathbb{P} = \{(x, y) \in X \times X : x \asymp y\}$ is T-closed.

Example 2.7. Let us consider $\mathbb{R}$ with the usual metric. For $Tx = x^2$, we have that the set $\mathbb{P} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x, y \in \mathbb{Q}\}$ is T-closed.

Definition 2.8. ([13]) Let (X, ρ) be a metric space and $\mathbb{P} \subseteq X \times X$. The triple $(X, \rho, \mathbb{P})$ is said to be:

(a) i-$\mathbb{P}$-regular if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, such that for all n $(x_n, x_{n+1}) \in \mathbb{P}$, there holds $(x_n, x) \in \mathbb{P}$ for all n.

(b) d-$\mathbb{P}$-regular if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, such that for all n $(x_{n+1}, x_n) \in \mathbb{P}$, there holds $(x, x_n) \in \mathbb{P}$ for all n.

Definition 2.9 ([10])**.** Let X and Y be topological spaces. The graph of a map $T\colon X \to Y$ is the set $\{(x, y) \in X \times Y : y = T(x)\}$. It is said that T has a closed graph if its graph is a closed subset of $X \times Y$ endowed with the product topology.

We will also use the following notation – $T \subset \mathbb{P}$ if from $x \in X$ it follows that $(x, Tx) \in \mathbb{P}$.

Definition 2.10. ([5]) We say $\mathbb{P}$ has the transitive property on a set X if for any $x, y, z \in X$, whenever $(x, y) \in \mathbb{P}$ and $(y, z) \in \mathbb{P}$, it follows that $(x, z) \in \mathbb{P}$.

The next theorem is a generalization of the results for mappings with a contractive iterate at a point, first considered in [16] and later developed in [5, 6, 8].

Theorem 2.11 ([5])**.** *Let (X, ρ) be a complete metric space, $\mathbb{P} \subset X \times X$, $T\colon X \to X$ be a map and there hold*

(i) *$\mathbb{P}$ is T-closed and has the transitive property;*

(ii) *T either has a closed graph or the triple $(X, \rho, \mathbb{P})$ is* i-$\mathbb{P}$*-regular;*

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathbb{P}$;*

(iv) *there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \leq \alpha\rho(x, y).$$

Then

(a) $\mathrm{Fix}(f) \neq \varnothing$ *and for any arbitrarily chosen $x_0 \in X$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_n = T^n x_0$ converges to an element $x^* \in \mathrm{Fix}(T)$;*

(b) *For any $x \in X$ and x_0 so that $(x_0, Tx_0) \in \mathbb{P}$, satisfying $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent and hence u_n converges to $x^* \in \mathrm{Fix}(f)$, where $x^* = \lim_{n\to\infty} T^n x_0$;*

(c) *If* $y^* \in \mathrm{Fix}(T)$ *and either* $(x_0, y^*) \in \mathbb{P}$ *or* $(y^*, x_0) \in \mathbb{P}$ *or there is* $z \in X$ *so that either* $(x_0, z), (y^*, z) \in \mathbb{P}$ *or* $(z, x_0), (z, y^*) \in \mathbb{P}$*, then* $y^* = x^*$*;*

(d) *If in addition we suppose that for every* $x, y \in X$ *such that neither* $(x, y) \in \mathbb{P}$ *nor* $(y, x) \in \mathbb{P}$ *there is* $z \in X$ *so that* $(x, z), (y, z) \in \mathbb{P}$ *or* $(z, x), (z, y) \in \mathbb{P}$*, then* $\mathrm{Fix}(T) = \{x^*\}$*.*

Proposition 2.12 ([5])**.** *Given the conditions of Theorem 2.11, we have that any two sequences* $\{T^n u_0\}_{n=0}^{\infty}$ *and* $\{T^n v_0\}_{n=0}^{\infty}$ *are Cauchy equivalent, given that* $(u_0, v_0) \in \mathbb{P}$ *or* $(v_0, u_0) \in \mathbb{P}$*.*

3. Main result

3.1. From graphs to $\mathbb{P}$ sets

Let us point out that since the proof of Theorem 2.3 depends on $V(G) = X$ and for all $x \in X$ to hold that $(x, x) \in E(G)$, to restate the result in terms of $\mathbb{P}$ sets, we would require that for every $\mathbb{P}$ set of this subsection it holds that if $x \in X$, then $(x, x) \in \mathbb{P}$.

Lemma 3.1. *Let* (X, ρ) *be a metric space and* $f \colon X \to X$ *be a self-map. Then there exists a directed graph* G *with* $E(G) \subseteq X \times X$ *such that* f *is a Banach* G*-contraction if and only if there exists* $\mathbb{P} \subseteq X \times X$ *such that* $\mathbb{P}$ *is* f*-closed and* $\alpha \in (0, 1)$ *such that*

$$\rho(fx, fy) \leq \alpha\rho(x, y)$$

for all $(x, y) \in \mathbb{P}$*.*

Proof. Let G be such a directed graph with $E(G) \subseteq X \times X$ such that f is a Banach G-contraction. Let $\mathbb{P} = E(G)$. Then

$$(x, y) \in \mathbb{P} \Rightarrow (fx, fy) \in \mathbb{P},$$

or $\mathbb{P}$ is f-closed. Also, from f being a G-contraction, we get that there exists $\alpha \in (0, 1)$ such that

$$\rho(fx, fy) \leq \alpha\rho(x, y)$$

for all $(x, y) \in \mathbb{P}$.

Now let there exist $\mathbb{P} \subseteq X \times X$ such that $\mathbb{P}$ is f-closed and there exists $\alpha \in (0, 1)$ so that $\rho(fx, fy) \leq \alpha\rho(x, y)$ for all $(x, y) \in \mathbb{P}$. Let us construct a graph G such that $(x, y) \in E(G)$ if and only if $(x, y) \in \mathbb{P}$. Then clearly f is a G-contraction. □

We introduce the following notation:

$$\mathbb{P}^{-1} = \{(x, y) \in X \times X \colon (y, x) \in \mathbb{P}\},$$

$$\widetilde{\mathbb{P}} = \mathbb{P} \cup \mathbb{P}^{-1},$$

$$\widetilde{\mathbb{P}}(x) := \{y \in X \colon \text{there exists } z_i \in X \text{ such that } (x, z_1), (z_j, z_{j+1}), (z_n, y) \in \tilde{\mathbb{P}}, \\ i = 1, 2, \dots, n,\ j = 1, 2, \dots, n-1,\ n \in \mathbb{N}\}.$$

It is clear that these sets are analogues of the converse, the undirected graph and $V(G_x)$ notions in graphs.

Lemma 3.2. *Let (X, ρ) be a metric space, G be a directed graph with $E(G) \subseteq X \times X$ and $\mathbb{P} = E(G)$. Then $[x]_{\widetilde{G}} = \widetilde{\mathbb{P}}(x)$.*

Proof. Let us point out that $E(\widetilde{G}) = \widetilde{\mathbb{P}}$. Then $y \in X$ being in $[x]_{\widetilde{G}}$ means that there exists a path in $\widetilde{G}$ from x to y, or there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n-1$, $n \in \mathbb{N}$. This is equivalent to the existence of $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in \tilde{\mathbb{P}}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n-1$, $n \in \mathbb{N}$, which means that $y \in \widetilde{\mathbb{P}}(x)$. Therefore, $[x]_{\widetilde{G}} = \widetilde{\mathbb{P}}(x)$. □

Let us first state Theorem 2.3 in terms of $\mathbb{P}$ sets.

Theorem 3.3. *Let (X, ρ) be a complete metric space, $\mathbb{P} \subseteq X \times X$ and the triple $(X, \rho, \mathbb{P})$ have the following property:*

$$\begin{array}{l}\text{For any sequence } \{x_n\}_{n\in\mathbb{N}} \subseteq X, \text{ if } \lim\limits_{n\to\infty} x_n = x \text{ and } (x_n, x_{n+1}) \in \mathbb{P} \text{ for all} \\ n, \text{ then there exists a subsequence } \{x_{n_k}\}_{k\in\mathbb{N}} \text{ such that } (x_{n_k}, x) \in \mathbb{P} \text{ for all } k.\end{array} \tag{3.1}$$

Let $f\colon X \to X$, $\mathbb{P}$ be f-closed, $W = \{x \in X \colon (x, fx) \in \mathbb{P}\}$ and there exist $\alpha \in (0, 1)$ such that

$$\rho(fx, fy) \leq \alpha\rho(x, y)$$

for all $(x, y) \in \mathbb{P}$. Then:

1. $\operatorname{card}(\operatorname{Fix}(f)) = \operatorname{card}\left(\left\{\widetilde{\mathbb{P}}(x) \colon x \in W\right\}\right)$.
2. $\operatorname{Fix}(f) \neq \varnothing \iff W \neq \varnothing$.
3. *f has a unique fixed point if and only if there exists $x_0 \in W$ such that $W \subseteq \widetilde{\mathbb{P}}(x_0)$.*
4. *For every $x \in W$, the restriction $f|_{\widetilde{\mathbb{P}}(x)}$ is PO.*
5. *If $W \neq \varnothing$ and there exists $x \in X$ such that $X \subseteq \widetilde{\mathbb{P}}(x)$, then f is PO.*
6. *If $X' = \bigcup_{x\in W} \widetilde{\mathbb{P}}(x)$, then $f|_{X'}$ is WPO.*
7. *If $f \subseteq \mathbb{P}$, then f is WPO.*

In order to present a proof, we will use the following proposition.

Proposition 3.4. *Theorem 2.3 holds if and only if Theorem 3.3 holds.*

Proof. From Lemmas 3.1 and 3.2, we see that the statements of both theorems are equivalent. The only thing we need to show is that G being weakly connected is equivalent to $X \subseteq \widetilde{\mathbb{P}}(x)$ for some $x \in X$. Indeed, let G be weakly connected, i.e., for all $x, y \in X$ there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. But that is equivalent to the statement that for all $x, y \in X$ there exists $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in \widetilde{\mathbb{P}}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n-1$, $n \in \mathbb{N}$. This is true if and only if from $y \in X$ it follows that $y \in \widetilde{\mathbb{P}}(x)$, or $X \subseteq \widetilde{\mathbb{P}}(x)$. Thus, the proposition is proven. □

As a consequence of Proposition 3.4 and Theorem 2.3 having been proven, it follows that Theorem 3.3 holds as well.

Example 3.5. Let us consider $\mathbb{R}$ with the usual metric $\rho(x, y) = |x - y|$. Let $\mathbb{E} = \{x \in \mathbb{R} \colon |x| = 2^n, n \in \mathbb{Z}\} \cup \{0\}$ and let us have the map

$$Tx = \begin{cases} \frac{x}{2}, & x \in \mathbb{E}, \\ 3x + \sqrt{3}, & x \notin \mathbb{E} \end{cases}.$$

Let $\mathbb{P} = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon x, y \in \mathbb{E} \text{ or } x = y\}$. Then clearly condition (3.1) holds, $\mathbb{P}$ is T-closed and the contractive condition is fulfilled with $\alpha = \frac{1}{2}$. Therefore, we can apply Theorem 3.3. Clearly, $\widetilde{\mathbb{P}} = \mathbb{P}$. If $x \in \mathbb{E}$, then $\widetilde{\mathbb{P}}(x) = \mathbb{E}$, whereas if $x \notin \mathbb{E}$, then $\widetilde{\mathbb{P}}(x) = \{x\}$. Let us consider W. If $x \in \mathbb{E}$, it holds that $Tx \in \mathbb{E}$ and $(x, Tx) \in \mathbb{P}$. If however, $x \notin \mathbb{E}$, then we have that $\left(-\frac{\sqrt{3}}{2}, T\left(-\frac{\sqrt{3}}{2}\right)\right) \in \mathbb{P}$. Therefore, $W = \mathbb{E} \cup \left\{-\frac{\sqrt{3}}{2}\right\}$.

From conclusions (1) and (2), we know that $\operatorname{card}(\operatorname{Fix}(f)) = 2$. Conclusions (3), (5) and (7) do not hold, whereas (4) and (6) do hold. If $x \in \mathbb{E}$, then $\lim_{n\to\infty} T^n x = 0$ and if $x = -\frac{\sqrt{3}}{2}$, then $T^n\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$.

If we would prefer to express this example utilizing a graph, we could construct a graph G such that $V(G) = \mathbb{R}$ and $E(G) = \mathbb{P}$. Then the results from Theorem 2.3 can be applied. Due to Proposition 3.4, we will arrive at the same conclusions as we did using Theorem 3.3.

3.2. From $\mathbb{P}$ sets to graphs

Definition 3.6. We say that a mapping $f \colon X \to X$ is G-contraction with a contractive iterate at a point if f preserves edges of G, i.e,

$$\text{for all } x, y \in X((x, y) \in E(G) \Rightarrow (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way: there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in E(G)$ it holds

$$\rho(f^{n(x)}(x), f^{n(x)}(y)) \le \alpha\rho(x, y).$$

This definition is a clear analogue of Definition 2.2.

In order to produce a simpler proof of the graph version of 2.11, we will first generalize the result by replacing the i-$\mathbb{P}$-regularity with a weaker assumption via the next Lemma.

Lemma 3.7 ([7]). *Let (X, d) be a complete metric space, G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. Let $E(G)$ be transitive. Then for the triple (X, d, G) the following properties are equivalent:*

1. *for any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ so that $(x_{n_k}, x) \in E(G)$ for all k;*

2. *the i-$E(G)$-regular property.*

In view of this, we can restate Theorem 2.11 in the following way.

Theorem 3.8. *Let (X, ρ) be a complete metric space, $\mathbb{P} \subset X \times X$, $T\colon X \to X$ be a map and there hold*

(i) *$\mathbb{P}$ is T-closed and has the transitive property;*

(ii) *T either has a closed graph or the triple $(X, \rho, \mathbb{P})$ has the following property:*

For any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, if $\lim_{n\to\infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k;

(iii) *there exists $x_0 \in X$ such that $(x_0, Tx_0) \in \mathbb{P}$;*

(iv) *there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \leq \alpha\rho(x, y).$$

Then

(a) *$\mathrm{Fix}(f) \neq \varnothing$ and for any arbitrary chosen $x_0 \in X$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_n = T^n x_0$ converges to an element $x^* \in \mathrm{Fix}(T)$;*

(b) *For any $x \in X$ and x_0 so that $(x_0, Tx_0) \in \mathbb{P}$, satisfying $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent and hence u_n converges to $x^* \in \mathrm{Fix}(f)$, where $x^* = \lim_{n\to\infty} T^n x_0$;*

(c) *If $y^* \in \mathrm{Fix}(T)$ and either $(x_0, y^*) \in \mathbb{P}$ or $(y^*, x_0) \in \mathbb{P}$ or there is $z \in X$ so that either $(x_0, z), (y^*, z) \in \mathbb{P}$ or $(z, x_0), (z, y^*) \in \mathbb{P}$, then $y^* = x^*$;*

(d) *If in addition we suppose that for every $x, y \in X$ such that neither $(x, y) \in \mathbb{P}$ nor $(y, x) \in \mathbb{P}$ there is $z \in X$ so that $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (z, y) \in \mathbb{P}$, then $\mathrm{Fix}(T) = \{x^*\}$.*

Before we move on to the proof, let us prove the following lemmas.

Lemma 3.9. *The conditions of Theorem 2.11 and Theorem 3.8 are equivalent.*

Proof. Let us construct a graph G such that $V(G) = X$ and $E(G) = \mathbb{P}$. Then, from $\mathbb{P}$ having the transitive property, it is clear that $E(G)$ also has the transitive property. Therefore, the result quickly follows from Lemma 3.7. □

Lemma 3.10. *Given the conditions of Theorem 3.8, we have that any two sequences $\{T^n u_0\}_{n=0}^{\infty}$ and $\{T^n v_0\}_{n=0}^{\infty}$ are Cauchy equivalent, given that $(u_0, v_0) \in \mathbb{P}$ or $(v_0, u_0) \in \mathbb{P}$.*

Proof. This is a simple consequence of Lemma 3.9. □

Now we will prove Theorem 3.8.

Proof. The conditions of Theorems 2.11 and 3.8 are the same by Lemma 3.9. □

Lemma 3.11. *Let (X, d) be a metric space and $f: X \to X$ be a self-map. Then there exists a directed graph G with $E(G) \subseteq X \times X$ such that f is a G-contraction with a contractive iterate at a point if and only if there exists $\mathbb{P} \subseteq X \times X$ such that $\mathbb{P}$ is f-closed and there exists $\alpha \in [0, 1)$ so that for any $x \in X$ there is $n(x) \in \mathbb{N}$, such that for all $(x, y) \in \mathbb{P}$ it holds that*

$$\rho(T^{n(x)}(x), T^{n(x)}(y)) \leq \alpha \rho(x, y).$$

Proof. The proof is analogous to the proof of Lemma 3.1. □

We will now restate Theorem 3.8 in the context of a metric space with a graph. Let us note that in the following result, we do not require that $(x, x) \in E(G)$ for any $x \in X$.

Theorem 3.12. *Let (X, ρ) be a complete metric space and G be a directed graph on X with edge set $E(G)$ and $T: X \to X$ be a G contraction with a contractive iterate at a point. Let $W = \{x \in X : (x, Tx) \in E(G)\}$ and assume that*

(I) *$E(G)$ has the transitive property;*

(II) *T has a closed graph or $(X, \rho, E(G))$ satisfies the following property:*
For any sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, if $\lim_{n \to \infty} x_n = x$ and $(x_n, x_{n+1}) \in E(G)$ for all n, then there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(G)$ for all k;

(III) *$W \neq \varnothing$.*

Then

(A) *$\mathrm{Fix}(f) \neq \varnothing$. If $X' = \bigcup_{x \in W} [x]_{\widetilde{G}}$, then the restriction $T|_{X'}$ is a WPO;*

(B) *For any $x \in W$, $T|_{[x]_{\widetilde{G}}}$ is a PO;*

(C) *If G is weakly connected, then T is a PO.*

Proposition 3.13. *Theorem 3.8 holds if and only if Theorem 3.12 holds.*

Proof. Without loss of generality, let $E(G) = \mathbb{P}$. From Lemmas 3.11 and 3.9, we see that the statements I and II of Theorem 3.12 are equivalent to conditions i, ii and iv of Theorem 3.8. Condition III is clearly equivalent to condition iii of Theorem 3.8.

By definition, $x \in X'$ if and only if $(x, Tx) \in E(G) = \mathbb{P}$. Thus, for any $x \in X$ such that $(x, Tx) \in \mathbb{P}$ the sequence $T^n x$ converges to an element $x^* \in \mathrm{Fix}(T)$ if and only if $T|_{X'}$ is a WPO. Thus Theorem 3.8 a is equivalent to Theorem 3.12 A.

There exists an $x \in X$ such that $(x_0, x) \in \mathbb{P}$ or $(x, x_0) \in \mathbb{P}$, which is equivalent to $x \in \widetilde{\mathbb{P}}(x_0)$. Therefore by Lemma 3.2, $(x_0, Tx_0) \in \mathbb{P}$ and $x \in \widetilde{\mathbb{P}}(x_0)$, the sequences $x_n = T^n(x_0)$ and $u_n = T^n(x)$ are Cauchy equivalent if and only if for any $x \in V$, $T_{[x]_{\widetilde{G}}}$ is a PO. Thus Theorem 3.8 b and c are equivalent to Theorem 3.12 B.

Let us examine the last condition. The graph G being weakly connected means that for all $x, y \in X$ there exist $z_i \in X$ such that (x, z_1), (z_j, z_{j+1}), $(z_n, y) \in E(\widetilde{G})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n-1$, $n \in \mathbb{N}$. This is equivalent to the existence of $z \in X$ such that $(x, z), (z, y) \in E(\widetilde{G})$. Since $E(\widetilde{G}) = \widetilde{\mathbb{P}}$, the previous statement is true if and only if for all $x, y \in X$ there exists $z \in X$ such that $(x, z), (z, y) \in \widetilde{\mathbb{P}}$, which is equivalent to $(x, z), (z, y) \in \mathbb{P}$, $(z, x), (z, y) \in \mathbb{P}$, $(x, z), (y, z) \in \mathbb{P}$ or $(z, x), (y, z) \in \mathbb{P}$. If $(x, z), (z, y) \in \mathbb{P}$, then $(x, y) \in \mathbb{P}$. If $(z, x), (y, z) \in \mathbb{P}$, then $(y, x) \in \mathbb{P}$. Thus Theorem 3.8 b and d are equivalent to Theorem 3.12 C.

The proposition is proven. □

As a consequence of Proposition 3.13 and Theorem 3.8 having been proven, it follows that Theorem 3.12 holds as well.

3.3. Coupled fixed points

We will conclude by establishing a similar connection between metric spaces endowed with a graph and metric spaces with $\mathbb{P}$ sets by considering a result concerning coupled fixed points. Let us recall that for a mapping $F\colon X \times X \to X$ a coupled fixed point $(x, y) \in X \times X$ is called a point such that $(F(x, y), F(y, x)) = (x, y)$. The analogues to $\mathbb{P}$ sets in this context are often called $\mathbb{M}$ sets.

Definition 3.14 ([12])**.** Let (X, d) be a metric space and $F\colon X \times X \to X$ be an operator. A nonempty subset $\mathbb{M}$ of X^4 is said to be F-closed if for all $x, y, u, v \in X$ the following implication holds:

$$(x, y, u, v) \in \mathbb{M} \Rightarrow (F(x, y), F(y, x), F(u, v), F(v, u)) \in \mathbb{M}.$$

Theorem 3.15 ([12])**.** *Let (X, d) be a complete metric space, $\mathbb{M} \subseteq X^4$ and $F\colon X \times X \to X$ be an operator with a closed graph. Suppose*:

1. *$\mathbb{M}$ is F-closed*;

2. *there exists $(x_0, y_0) \in X \times X$ such that $(x_0, y_0, F(x_0, y_0), F(y_0, x_0)) \in \mathbb{M}$*;

3. *there is* $k \in [0,1]$ *such that if* $(x,y) \in X \times X$ *and* $(x,y,F(x,y),F(y,x)) \in \mathbb{M}$, *then*

$$d\left(F(x,y),F^2(x,y)\right)+d\left(F(y,x),F^2(y,x)\right)\leq k\left(d\left(x,F(x,y)\right)+d\left(y,F(y,x)\right)\right).$$

Then F *has at least one coupled fixed point* $(x^*,y^*) \in X \times X$ *and the sequences* $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ *and* $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ *converge to* x^* *and* y^* *respectively.*

For the restatement of this result, we will require the following definition.

Definition 3.16. Let (X,d) be a metric space and G be a directed graph on $X \times X$ with edge set $E(G)$. We say that a mapping $F\colon X \times X \to X$ is a coupled G-contraction if F preserves edges of G, i.e.,

$$\begin{gathered}\text{for all } x,y \in X(((x,y),(u,v)) \in E(G) \text{ it follows that}\\ \left(\left(F(x,y),F(y,x)\right),\left(F(u,v),F(v,u)\right)\right) \in E(G)),\end{gathered} \tag{3.2}$$

and F decreases weights of edges of G in the following way: there exists $k \in [0,1)$ so that for any $(x,y) \in X$ such that

$$((x,y),(F(x,y),F(y,x))) \in E(G),$$

it is true that

$$d\left(F(x,y),F^2(x,y)\right)+d\left(F(y,x),F^2(y,x)\right) \leq k\left(d\left(x,F(x,y)\right)+d\left(y,F(y,x)\right)\right). \tag{3.3}$$

The equivalent theorem in the context of metric spaces with a graph is:

Theorem 3.17. *Let* (X,d) *be a complete metric space,* G *be a directed graph on* $X \times X$ *with edge set* $E(G)$ *and* $F\colon X \times X \to X$ *be a coupled* G *contraction with a closed graph. Let*

$$W = \{(x,y) \in X \times X\colon\ ((x,y),(F(x,y),F(y,x))) \in E(G)\}$$

and assume that $W \neq \varnothing$. *Then* F *has at least one coupled fixed point* (x^*,y^*) *and the sequences* $(F^n(x_0,y_0))_{n\in\mathbb{N}}$ *and* $(F^n(y_0,x_0))_{n\in\mathbb{N}}$ *converge to* x^* *and* y^* *respectively.*

Proposition 3.18. *Theorem 3.15 holds if and only if Theorem 3.17 holds.*

Proof. The operator F being a coupled G contraction from Theorem 3.17 is equivalent to conditions (1) and (3) of Theorem 3.15. Furthermore, $W \neq \varnothing$ from Theorem 3.17 is equivalent to (2) of Theorem 3.15. Both theorems require that F has a closed graph. The conditions of Theorem 3.15 and Theorem 3.17 are equivalent. Therefore, Theorem 3.15 holds if and only if Theorem 3.17 holds. □

Since Theorem 3.15 is proven, by Proposition 3.18 we can conclude that Theorem 3.17 holds as well.

Acknowledgements

The European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project DUECOS BG-RRP-2.004-0001-C01, has partially supported the research.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133–181.

[2] T. Gnana Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65(7) (2006) 1379–1393.

[3] V. Georgiev, A. Ilchev and B. Zlatanov, A variational principle, fixed points and coupled fixed points on $\mathbb{P}$ sets, J. Fixed Point Theory Appl. 26(3) (2024) 33.

[4] V. Georgiev, V. Zhelinski and B. Zlatanov, Best proximity point theorem for mappings with a contractive iterate on $\mathbb{P}$ sets, MATTEX 2024, Conf. Proc. (2024) 79–90.

[5] V. Georgiev and B. Zlatanov, A generalization of fixed-point theorems for mappings with a contractive iterate, Mathematics 12(23) (2024) 3725.

[6] L. F. Guseman, Jr., Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 26(4) (1970) 615–618.

[7] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136(4) (2008) 1359–1373.

[8] S. Karaibryamov and B. Zlatanov, Fixed points for mappings with a contractive iterate at each point, Math. Slovaca 64(2) (2014) 455–468.

[9] S. K. Mohanta, Common fixed points in b-metric spaces endowed with a graph, Mat. Vesnik 68(2) (2016) 140–154.

[10] L. Narici and E. Beckenstein, Topological Vector Spaces, Second Edition, Pure and applied mathematics, Chapman & Hall/CRC, 2011.

[11] A. Özlem, A. Hassen and M. De la S. Manuel, New fixed point results via a graph structure, Mathematics 9(9) (2021) 1013.

[12] A. Petruşel, Fixed points vs. coupled fixed points, J. Fixed Point Theory Appl. 20(4) (2018) 150.

[13] A. Petruşel, G. Petruşel, Y.-B. Xiao and J.-C. Yao, Fixed point theorems for generalized contractions with applications to coupled fixed point theory, J. Nonlinear Convex Anal. 19(1) (2018) 71–88.

[14] M. Younis, D. Singh and A. Petruşel, Applications of graph Kannan mappings to the damped spring-mass system and deformation of an elastic beam, Discrete Dyn. Nat. Soc. 2019 (2019) 1315387.

[15] A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132(5) (2004) 1435–1443.

[16] V. M. Sehgal, A fixed point theorem for mappings with a contractive iterate, Proc. Amer. Math. Soc. 23(3) (1969) 631–634.

[17] M. Turinici, Abstract comparison principles and multivariable Gronwall-Bellman inequalities, J. Math. Anal. Appl. 117(1) (1986) 100–127.

Received on May 3, 2025
Revised on July 5, 2025
Accepted on July 23, 2025

Valentin Georgiev, Atanas Ilchev and Boyan Zlatanov

University of Plovdiv “Paisii Hilendarski“
Faculty of Mathematics and Informatics
24, Tsar Asen Str.
4000 Plovdiv
BULGARIA

E-mail: valentingeorgiev@uni-plovdiv.bg
atanasilchev@uni-plovdiv.bg
bobbyz@uni-plovdiv.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

A SHORT NOTE ON A WEIGHTED OPIAL-TYPE INEQUALITY

IRINA GEORGIEVA

This paper establishes new weighted Opial-type inequalities by employing recent advances in weighted Hardy inequalities. Our approach builds upon results of Nikolov and Uluchev and develops inequalities for both single and two functions.

Keywords: Opial's inequality, Opial-type inequalities, weighted Hardy inequality

2020 Mathematics Subject Classification: 26D10, 26D15, 33C45

1. Introduction and preliminaries

Opial's inequality, introduced in 1960, has played a central role in the theory of integral inequalities. The original result relates a function to its derivative in a sharp inequality with significant applications in analysis. Following Opial's work, many authors provided new proofs, refinements, and discrete analogues.

In this section we recall some classical results related to Opial's inequality, which provide the background and motivation for the results established later in this paper.

We begin with the original inequality due to Zdzisław Opial and then recall some of its later versions.

Theorem 1.1 ([9]). *If $f \in C^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0,h)$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{4}\int_0^h \left[f'(x)\right]^2 dx, \tag{1.1}$$

where the constant factor $h/4$ is best possible.

DOI: 10.60063/GSU.FMI.112.45-51

This inequality has found much attention among many mathematicians, and many different proofs, variants, extensions, and discrete analogues of Opial's theorem have been developed. We mention here only a few [2–4, 6, 8–10, 12]. For a detailed bibliography on the topic we refer to [1, 11].

Soon after the publication of Opial's paper, C. Olech [8], in a short note, showed that the condition $f(x) > 0$ is not necessary; however, without this assumption the constant on the right-hand side of the inequality becomes double. The result reads as follows.

Theorem 1.2 ([8]). *If f is absolutely continuous on $[0, h]$ with $f(0) = f(h) = 0$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h \left[f'(x)\right]^2 dx. \tag{1.2}$$

The sign of equality holds if and only if $f(x) = cx$, where c is a constant.

In 1965, C. Mallows [6], presented what is arguably the simplest and most elegant proof of Opial's inequality. Moreover, Mallows' version of Opial inequality does not require any condition on the function f at the right end of the interval $[0, h]$.

Theorem 1.3 ([6]). *If f is absolutely continuous on $[0, h]$ with $f(0) = 0$, then*

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h \left[f'(x)\right]^2 dx.$$

The sign of equality holds if and only if $f(x) = cx$, where c is a constant.

This is in fact the most well-known version of Opial inequality.

In 1962, Beesack [3], provided conditions under which a generalized weighted version of Opial inequality

$$\left(\int_a^b |g(x)|^q\,|g'(x)|^s v(x)\,dx\right)^{\frac{1}{q+s}} \le c\left(\int_a^b |g'(x)|^p u(x)\,dx\right)^{\frac{1}{p}}$$

holds for all $g(x)$, such that $g(a) = 0$, $p > q > 0$, $0 < s < p$, and $u(x)$, $v(x)$ are weight functions.

A key contribution was made by Sinnamon [12], who demonstrated that Hardy-type inequalities naturally yield Opial-type inequalities. Moreover, Sinnamon proposed a two-function generalization of Opial's inequality, where the inequality is formulated in terms of products of two functions and their derivatives. This result complemented Pachpatte's earlier contributions [10] and illustrated once again the link between Hardy- and Opial-type inequalities.

In 1997, exploiting Sinnamon's approach, Bloom [4] established some Opial-type inequalities involving generalized Hardy operators.

Research on Opial-type inequalities continues to develop rapidly, with ongoing work on refinements, generalizations, and discrete analogues, as well as important recent applications to fractional calculus [1].

The above short historical overview outlines some of the main steps in the evolution of Opial-type inequalities, without attempting to be complete. Given the extensive literature on the subject, we restrict ourselves to citing mainly works that are directly relevant to our approach.

In the next section we turn to recent advances in weighted Hardy inequalities.

2. Recent results on weighted Hardy inequalities

Recent work by Nikolov and Uluchev [7] studies a weighted Hardy-type inequality in the particular finite-dimensional space

$$\mathcal{H}_{n,\alpha} := \left\{ f\colon \int_0^x f(t)\,dt = x^{-\alpha/2} e^{-x/2} p(x),\ p \in \mathcal{P}_n,\ p(0) = 0 \right\}, \quad \alpha < 1,$$

where $\mathcal{P}_n$ is the set of algebraic polynomials of degree at most n with real coefficients. More precisely, the authors examine the sharp constant $C_{n,\alpha}$ in a Hardy inequality with weight $w(x) = x^\alpha$:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\,dt \right)^2 x^\alpha\,dx \le C_{n,\alpha} \int_0^\infty f^2(x)\,x^\alpha\,dx, \quad f \in \mathcal{H}_{n,\alpha}. \tag{2.1}$$

Lower and upper bounds for $C_{n,\alpha}$ of correct order in space dimension n are proved. The particular case $\alpha = 0$ was considered by Dimitrov, Gadjev, Nikolov and Uluchev in [5].

The following two-sided estimate for the sharp constant $C_{n,\alpha}$, $n \ge 3$ of (2.1) was proved in [7, Theorem 1.1]:

$$C_{n,\alpha} < \left(\frac{2}{1-\alpha} \right)^2 \left(1 - \frac{4\sqrt{6}}{(1-\alpha)^2 \left(\log\lfloor \frac{n+1}{2} \rfloor + 4 \right)^2 + 4\sqrt{6}} \right) =: \overline{C}_{n,\alpha}, \tag{2.2}$$

$$C_{n,\alpha} > \left(\frac{2}{1-\alpha} \right)^2 \left(1 - \frac{16}{(1-\alpha)^2 \left(\log\lfloor \frac{n+1}{2} \rfloor + \frac{8}{3} \right)^2 + 16} \right).$$

While the estimates in [5] for the lower and the upper bound of $C_{n,0}$ are of different order, in [7] the asymptotic rate of the lower estimate and the upper estimate is the same as $n \to \infty$ for any $\alpha < 1$, namely $O((\log n)^{-2})$.

3. A weighted Opial-type inequality

We now describe the framework in which a weighted Opial-type inequality will be derived. We begin with introducing some notations.

In what follows, $\|\cdot\|_{2,\alpha}$ denotes the weighted $L_2[0,\infty)$-norm with weight x^α,

$$\|f\|_{2,\alpha} := \left(\int_0^\infty f^2(x) x^\alpha\,dx \right)^{1/2}.$$

We will consider the class

$$\mathcal{G}_{n,\alpha} := \{g\colon g(x) = x^{-\alpha/2}e^{-x/2}\,p(x),\ p \in \mathcal{P}_n,\ p(0)=0\}, \quad \alpha < 1,$$

and the class used in [7]

$$\mathcal{H}_{n,\alpha} = \left\{ f\colon \int_0^x f(t)\,dt \in \mathcal{G}_{n,\alpha} \right\}.$$

Introducing the classes $\mathcal{G}_{n,\alpha}$ and $\mathcal{H}_{n,\alpha}$ allows us to translate the results on weighted Hardy-type estimates from [7] into Opial-type inequalities. The link between the two classes will be essential in the subsequent derivation.

Proposition 3.1. *If $g(x) \in \mathcal{G}_{n,\alpha}$, then $g'(x) \in \mathcal{H}_{n,\alpha}$.*

Proof. Indeed, any function $g(x) \in \mathcal{G}_{n,\alpha}$ has the form $x^{-\alpha/2}e^{-x/2}\,p(x)$, $p \in \mathcal{P}_n$ and $g(0) = 0$. Then

$$g(x) = g(x) - g(0) = \int_0^x g'(t)\,dt$$

is of the form $x^{-\alpha/2}e^{-x/2}\,p(x)$, where $p \in \mathcal{P}_n$, i.e., $g' \in \mathcal{H}_{n,\alpha}$. □

Theorem 3.2. *Let $g(x) \in \mathcal{G}_{n,\alpha}$, $q, s \geq 0$, $q + s = 2$ and $\alpha < 1$. Then the following weighted Opial-type inequality holds true*

$$\int_0^\infty |g(x)|^q\,|g'(x)|^s\,x^{\alpha-q}\,dx \leq (C_{n,\alpha})^{q/2} \int_0^\infty \big[g'(x)\big]^2 x^\alpha\,dx.$$

Proof. Since $g \in \mathcal{G}_{n,\alpha}$, we have that $g'(x) \in \mathcal{H}_{n,\alpha}$. Then using (2.1) for $g'(x) \in \mathcal{H}_{n,\alpha}$, we get that the following inequality holds true

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x g'(t)\,dt \right)^2 x^\alpha\,dx \right)^{1/2} \leq (C_{n,\alpha})^{1/2} \left(\int_0^\infty \big[g'(x)\big]^2 x^\alpha\,dx \right)^{1/2}.$$

Or, in other words, we get

$$\|g\|_{2,\alpha-2} \leq (C_{n,\alpha})^{1/2}\,\|g'\|_{2,\alpha}. \tag{3.1}$$

Denote

$$I := \int_0^\infty |g(x)|^q |g'(x)|^s x^{\alpha-q}\,dx = \int_0^\infty \big(|g(x)|^2 x^{\alpha-2}\big)^{q/2} \big(|g'(x)|^2 x^\alpha\big)^{s/2}\,dx.$$

Applying Hölder's inequality and (3.1), we obtain

$$\begin{aligned} I &\leq \left(\int_0^\infty |g(x)|^2 x^{\alpha-2}\,dx \right)^{q/2} \left(\int_0^\infty |g'(x)|^2 x^\alpha\,dx \right)^{s/2} \\ &\leq \|g\|_{2,\alpha-2}^q\,\|g'\|_{2,\alpha}^s \\ &\leq (C_{n,\alpha})^{q/2}\,\|g'\|_{2,\alpha}^q\,\|g'\|_{2,\alpha}^s \\ &= (C_{n,\alpha})^{q/2}\,\|g'\|_{2,\alpha}^2, \end{aligned}$$

which concludes the proof. □

To obtain an explicit, though non-sharp, version of the inequality in Theorem 3.2, we replace the sharp constant $C_{n,\alpha}$ by its computable upper bound $\overline{C}_{n,\alpha}$ from [7, Theorem 1.1], which yields the following result.

Corollary 3.3. *Let $g \in \mathcal{G}_{n,\alpha}$ with $\alpha < 1$, and let $q, s \geq 0$ satisfy $q + s = 2$. For $n \geq 3$,*

$$\int_0^\infty |g(x)|^q \, |g'(x)|^s \, x^{\alpha - q} \, dx \leq \big(\overline{C}_{n,\alpha}\big)^{q/2} \int_0^\infty |g'(x)|^2 \, x^\alpha \, dx, \tag{3.2}$$

where $\overline{C}_{n,\alpha}$, given in (2.2) is an explicit upper bound for the sharp Hardy constant $C_{n,\alpha}$.

Moreover, the constant satisfies the asymptotic relation

$$\big(\overline{C}_{n,\alpha}\big)^{q/2} = \left(\frac{2}{1-\alpha}\right)^q \Big(1 + O\big((\log n)^{-2}\big)\Big), \quad n \to \infty,$$

so that $\big(\overline{C}_{n,\alpha}\big)^{q/2}$ preserves the asymptotic rate of convergence of $(C_{n,\alpha})^{q/2}$.

Remark 3.4. The inequality (3.2) is non-sharp, since $C_{n,\alpha} < \overline{C}_{n,\alpha}$, and equality can occur only in the trivial case $g \equiv 0$.

4. An Opial-type inequality for two functions

Having derived the weighted Opial-type inequality for a single function, we next obtain the corresponding result for two functions.

Theorem 4.1. *Let $g(x) \in \mathcal{G}_{n,\alpha}$, $h(x) \in \mathcal{G}_{m,\beta}$, $m, n \in \mathbb{N}$, $\alpha, \beta < 1$, $q + s = 2$, $q = q_1 + q_2$, $s = s_1 + s_2$ and*

$$v(x) := x^{(\alpha-2)q_1/2} \, x^{\alpha s_1/2} \, x^{(\beta-2)q_2/2} \, x^{\beta s_2/2} = x^{\frac{\alpha(q_1+s_1)+\beta(q_2+s_2)}{2} - q}.$$

Then

$$\int_0^\infty |g(x)|^{q_1} \, |g'(x)|^{s_1} \, |h(x)|^{q_2} \, |h'(x)|^{s_2} \, v(x) \, dx \\ \leq \big(C_{n,\alpha}\big)^{q_1/2} \, (C_{m,\beta})^{q_2/2} \, \|g'\|_{2,\alpha}^{q_1+s_1} \, \|h'\|_{2,\beta}^{q_2+s_2}.$$

Proof. Denote

$$J := \int_0^\infty |g(x)|^{q_1} \, |g'(x)|^{s_1} \, |h(x)|^{q_2} \, |h'(x)|^{s_2} \, v(x) \, dx. \tag{4.1}$$

Since $g'(x) \in \mathcal{H}_{n,\alpha}$ and $h'(x) \in \mathcal{H}_{m,\beta}$, then from (3.1) it follows that

$$\|g\|_{2,\alpha-2} \leq \big(C_{n,\alpha}\big)^{1/2} \, \|g'\|_{2,\alpha}, \quad \|h\|_{2,\beta-2} \leq \big(C_{m,\beta}\big)^{1/2} \, \|h'\|_{2,\beta}.$$

Since $\frac{q_1}{2}+\frac{q_2}{2}+\frac{s_1}{2}+\frac{s_2}{2}=\frac{q+s}{2}=1$, the Hölder inequality applied to (4.1) yields

$$\begin{aligned} J &\le \left(\int_0^\infty |g(x)|^{q_1\frac{2}{q_1}}\, x^{\frac{\alpha-2}{2}q_1\frac{2}{q_1}}\,dx\right)^{q_1/2}\left(\int_0^\infty |g'(x)|^2\, x^\alpha\,dx\right)^{s_1/2} \\ &\quad\times\left(\int_0^\infty |h(x)|^{q_2\frac{2}{q_2}}\, x^{\frac{\beta-2}{2}q_2\frac{2}{q_2}}\,dx\right)^{q_2/2}\left(\int_0^\infty |h'(x)|^2 x^\beta\,dx\right)^{s_2/2} \\ &= \left(\|g\|_{2,\alpha-2}\right)^{q_1}\|g'\|_{2,\alpha}^{s_1}\left(\|h\|_{2,\beta-2}\right)^{q_2}\|h'\|_{2,\beta}^{s_2} \\ &\le \left(C_{n,\alpha}\right)^{q_1/2}\|g'\|_{2,\alpha}^{q_1+s_1}\left(C_{m,\beta}\right)^{q_2/2}\|h'\|_{2,\beta}^{q_2+s_2}. \end{aligned}$$

□

In the particular case $\alpha=\beta$, we have $v(x)=x^{\alpha-q}$ and the weighted Opial's-type inequality has a much simpler form.

Corollary 4.2. *Let $g(x)\in\mathcal{G}_{n,\alpha}$, $h(x)\in\mathcal{G}_{m,\alpha}$, $m,n\in\mathbb{N}$, $\alpha<1$, $q+s=2$, $q=q_1+q_2$, $s=s_1+s_2$. Then the following weighted Opial's type inequality holds true*

$$\int_0^\infty |g(x)|^{q_1}\,|g'(x)|^{s_1}\,|h(x)|^{q_2}\,|h'(x)|^{s_2}\,x^{\alpha-q}\,dx \le \left(C_{n,\alpha}\right)^{q_1/2}\left(C_{m,\alpha}\right)^{q_2/2}\|g'\|_{2,\alpha}^{q_1+s_1}\,\|h'\|_{2,\alpha}^{q_2+s_2}.$$

To obtain an explicit, though non-sharp, form of the inequality in Theorem 4.1, we replace the sharp constants $C_{n,\alpha}$ and $C_{m,\beta}$ by their computable upper bounds $\overline{C}_{n,\alpha}$ and $\overline{C}_{m,\beta}$ given in (2.2), which leads to the following result.

Corollary 4.3. *Let $g(x)\in G_{n,\alpha}$ and $h(x)\in G_{m,\beta}$ with $\alpha,\beta<1$ and $m,n\in\mathbb{N}$. Let $q+s=2$, $q=q_1+q_2$, $s=s_1+s_2$, and define*

$$v(x)=x^{\frac{\alpha(q_1+s_1)+\beta(q_2+s_2)}{2}-q}.$$

Then, for $n,m\ge 3$,

$$\int_0^\infty |g(x)|^{q_1}|g'(x)|^{s_1}|h(x)|^{q_2}|h'(x)|^{s_2}v(x)\,dx \le \left(\overline{C}_{n,\alpha}\right)^{q_1/2}\left(\overline{C}_{m,\beta}\right)^{q_2/2}\|g'\|_{2,\alpha}^{\frac{q_1+s_1}{2}}\,\|h'\|_{2,\beta}^{\frac{q_2+s_2}{2}}. \tag{4.2}$$

Moreover, the constant satisfies the asymptotic relation

$$\left(\overline{C}_{n,\alpha}\right)^{q_1/2}\left(\overline{C}_{m,\beta}\right)^{q_2/2}=\left(\frac{2}{1-\alpha}\right)^{q_1}\left(\frac{2}{1-\beta}\right)^{q_2}\left(1+O\big((\log n)^{-2}+(\log m)^{-2}\big)\right),$$

$n,m\to\infty$, so that the explicit bound preserves the asymptotic rate of convergence of $C_{n,\alpha}^{q_1/2}C_{m,\beta}^{q_2/2}$, $n,m\to\infty$.

Remark 4.4. The inequality in Corollary 4.2 is non-sharp, since $C_{n,\alpha}<\overline{C}_{n,\alpha}$ and $C_{m,\beta}<\overline{C}_{m,\beta}$, and equality can occur only in the trivial case $g\equiv h\equiv 0$.

The presented results demonstrate how recent advances in weighted Hardy inequalities can be employed to establish new weighted Opial-type inequalities with explicit constant estimates and clear asymptotic behavior.

Acknowledgements

The author is grateful to the anonymous referees for their careful reading and for their helpful comments and suggestions, which improved the paper.

This research is supported by the Bulgarian National Science Fund under Contract KP-06-N62/4.

References

[1] M. Andrić, J, Pečarić and I. Perić, Inequalities of Opial and Jensen, Monographs in Inequalities, vol. 10, Element, Zagreb, 2016, ISBN 978-953-197-598-8.

[2] R. P. Agarwal and P. Y. H. Pang, Opial Inequalities with Applications in Differential and Difference Equations, Mathematics and Its Applications, vol. 320, Kluwer, Dordrecht, 1995.

[3] P. R. Beesack, On an integral inequality of Z. Opial, Trans. Am. Math. Soc. 104(3) (1962) 470–475.

[4] S. Bloom, First and second order Opial inequalities, Studia Math. 126 (1997) 27–50.

[5] D. K. Dimitrov, I. Gadjev, G. Nikolov and R. Uluchev, Hardy's inequalities in finite dimensional Hilbert spaces, Proc. Am. Math. Soc. 149 (2021) 2515–2529.

[6] C. L. Mallows, An even simpler proof of Opial's inequality, Proc. Am. Math. Soc. 16 (1965) 173.

[7] G. Nikolov and R. Uluchev, On the sharp constant in a weighted Hardy inequality, Monatsh. Math. 206 (2025) 665–683.

[8] C. Olech, A simple proof of a certain result of Z. Opial, Ann. Polon. Math. 8 (1960) 61–63.

[9] Z. Opial, Sur une inégalité, Ann. Polon. Math. 8 (1960) 29–32.

[10] B. G. Pachpatte, On Opial-type integral inequalities, J. Math. Anal. Appl. 120 (1986) 541–556.

[11] B. G. Pachpatte, Mathematical Inequalities, North-Holland Mathematical Library, vol. 67, Elsevier, Amsterdam, 2005.

[12] G. J. Sinnamon, Weighted Hardy and Opial-type inequalities, J. Math. Anal. Appl. 160 (1991) 434–445.

Received on September 30, 2025
Revised on November 2, 2025
Accepted on November 3, 2025

Irina Georgieva
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia
BULGARIA
E-mail: irina@math.bas.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

SOME APPLICATIONS OF THE FUNCTIONS OF THE SYSTEM $\Gamma_{\mathcal{B}_s}$ TO THE THEORY OF THE UNIFORMLY DISTRIBUTED SEQUENCES

VASSIL GROZDANOV and ELMI SHABANI

The authors use the $\mathcal{B}_s$-adic functions constructed in Cantor systems to show some of their applications to the theory of the uniformly distributed sequences. The notions of multidimensional modified integrals from these functions are introduced. The following results – the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion, are presented in the terms of the introduced integrals.

Keywords: function system $\Gamma_{\mathcal{B}_s}$, multidimensional modified integrals, LeVeque's inequality, Koksma's formula, Erdős-Turán-Koksma's inequality, integral Weyl's criterion, uniformly distributed sequences

2020 Mathematics Subject Classification: Primary: 11K06; Secondary: 11K31, 11K36, 11K38, 11K45

1. Introduction

Let $s \geq 1$ be a fixed integer, which will denote the dimension through the paper. Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence in the unit cube $[0,1)^s$. For an arbitrary integer $N \geq 1$ and a subinterval $J \subseteq [0,1)^s$ let us denote $A(J;N) = \#\{n\colon 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$. Let λ_s denote the Lebesgue measure on $[0,1)^s$. Following Kuipers and Niederreiter [15] we will remind that the sequence ξ is called uniformly distributed in $[0,1)^s$ if the limit equality $\lim\limits_{N\to\infty} \dfrac{A(J;N)}{N} = \lambda_s(J)$ holds for each subinterval J of $[0,1)^s$.

To assess the quality of the distribution of the points of sequences and nets we use special quantitative measures, called discrepancy and diaphony. Thus, to present

DOI: 10.60063/GSU.FMI.112.53-75

the concepts of the extreme and the quadratic discrepancy, we will introduce some notations. For this purpose, let $\mathcal{J}$ denote a family of subintervals of $[0,1)^s$ of the form $J = \prod_{j=1}^{s}[u_j, v_j)$, where $0 \le u_j < v_j \le 1$ for $1 \le j \le s$. For an arbitrary vector $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$ let us denote $[0,\mathbf{x}) = [0,x_1) \times \cdots \times [0,x_s)$.

Let $\xi_N = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. The extreme discrepancy $D(\xi_N)$ and the quadratic discrepancy $T(\xi_N)$ of the net ξ_N are defined respectively as

$$D(\xi_N) = \sup_{J \in \mathcal{J}} \left| \frac{A(J;N)}{N} - \lambda_s(J) \right|$$

and

$$T(\xi_N) = \left(\int_{[0,1]^s} \left| \frac{A([0,\mathbf{x});N)}{N} - x_1 \dots x_s \right|^2 dx_1 \dots dx_s \right)^{\frac{1}{2}}.$$

For each integer $N \ge 1$ the extreme and the quadratic discrepancy of the sequence ξ are defined as the corresponding discrepancies of the first N its elements.

Some classes of orthonormal function systems with very big success are used to solve many problems of the theory of the uniformly distributed sequences. Such classes are the trigonometric function system and systems constructed in b-adic number system as the Walsh function system and the Haar function system.

Also, function systems constructed in the so-called *Cantor systems* are used as a tool for studying the uniformly distributed sequences. Let us remind the concept of the Cantor systems, which are natural generalizations of the ordinary b-adic number system. The algebraic basis of these systems is given by the following explanations: Let $B = \{b_0, b_1, \dots : b_i \ge 2 \text{ for } i \ge 0\}$ be an arbitrary sequence of integers, called bases. By using the sequence B, the so-called *generalized powers* are defined by the following recurrence manner: we put $B_0 = 1$ and for $i \ge 0$ define $B_{i+1} = B_i b_i$. The number system, which corresponds to the sequence B of bases and the sequence $\{B_0, B_1, \dots\}$ of generalized powers usually is called B-adic Cantor system or system with variable bases. We will stick to the terminology B-adic number system.

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An arbitrary integer $k \in \mathbb{N}_0$ and a real $x \in [0,1)$ in the B-adic system have representations of the form

$$k = \sum_{i=0}^{\nu} k_i B_i \quad \text{and} \quad x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}},$$

where $k_i, x_i \in \{0, 1, \dots, b_i - 1\}$ for $i \ge 0$ and $k_\nu \ne 0$. The representation of k is unique. Under the additional assumption that for infinitely many values i we have $x_i \ne b_i - 1$, than the B-adic representation of x is also unique.

The multidimensional version of the Cantor systems is given by the following manner: Let $\mathcal{B}_s = (B_1, \dots, B_s)$ be an arbitrary set of sequences of bases, where $B_j = \left\{b_0^{(j)}, b_1^{(j)}, \dots\right\}$ for $1 \le j \le s$ and $\left\{B_0^{(j)}, B_1^{(j)}, \dots\right\}$ be the corresponding to B_j

sequence of the generalized powers. The number system, which correspond to the set $\mathcal{B}_s$, we will call $\mathcal{B}_s$-adic Cantor system.

Everywhere in the article we strictly will stick to the above concepts for B-adic and $\mathcal{B}_s$-adic Cantor systems.

A set of functions constructed in Cantor systems was first of all proposed by Vilenkin [21] and independent of him was considered by Price [20]. The details are as follows: For an arbitrary $k \in \mathbb{N}_0$ and a real $x \in [0, 1)$ with the B-adic representations $k = \sum_{i=0}^{\nu} k_i B_i$ and $x = \sum_{i=0}^{\infty} \frac{x_i}{B_{i+1}}$, where $k_i, x_i \in \{0, 1, \ldots, b_i - 1\}$ for $i \geq 0$, $k_\nu \neq 0$ and for infinitely many values of i we have $x_i \neq b_i - 1$, the function ${}_B\mathrm{vil}_k \colon [0, 1) \to \mathbb{C}$ is defined as

$$ {}_B\mathrm{vil}_k(x) = \prod_{i=0}^{\nu} e^{2\pi \mathbf{i} \frac{k_i x_i}{b_i}}. $$

For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th multidimensional Vilenkin function is defined as

$$ {}_{\mathcal{B}_s}\mathrm{vil}_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\mathrm{vil}_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_j) \in [0, 1)^s. $$

The set $\mathrm{Vil}_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\mathrm{vil}_{\mathbf{k}}(\mathbf{x}) \colon \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0, 1)^s\}$ is called Vilenkin function system. For the system Vil_B usually the name multiplicative system is used, see Agaev et al. [1].

Quite recently, Hellekalek and Niederraiter [13] introduced the so-called **b**-adic function system $\Gamma_{\mathbf{b}}$, where $\mathbf{b} = (b_1, \ldots, b_s)$ is a vector of not necessary distinct integers $b_j \geq 2$. Some applications of the system $\Gamma_{\mathbf{b}}$ to the theory of the uniformly distributed sequences were presented. Petrova [19] generalized the construction of the functions of the system $\Gamma_{\mathbf{b}}$ to functions considered in $\mathcal{B}_s$-adic Cantor systems. We will recall this constructive principle.

Definition 1.1. For an arbitrary integer $k \geq 0$ and a real number $x \in [0, 1)$, which in B-adic system have the representations of the form

$$ k = \sum_{i=1}^{\nu} k_i B_i \quad \text{and} \quad x = \sum_{i=1}^{\infty} \frac{x_i}{B_{i+1}}, $$

where $k_i, x_i \in \{0, 1, \ldots, b_i - 1\}$ for $i \geq 0$, $k_\nu \neq 0$ and for infinitely many values of i we have that $x_i \neq b_i - 1$, the k-th B-adic function ${}_B\gamma_k \colon [0, 1) \to \mathbb{C}$ is defined as

$$ {}_B\gamma_k(x) = e^{2\pi \mathbf{i} \left(\frac{k_0}{B_1} + \frac{k_1}{B_2} + \cdots + \frac{k_\nu}{B_{\nu+1}}\right)(x_0 B_0 + x_1 B_1 + \cdots + x_\nu B_\nu)}. $$

The multidimensional functions constructed in $\mathcal{B}_s$-adic Cantor systems are defined in the following definition.

Definition 1.2. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}$-th multidimensional $\mathcal{B}_s$-adic function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}} \colon [0, 1)^s \to \mathbb{C}$ is defined as

$$ {}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{B_j}\gamma_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s. $$

The set $\Gamma_{\mathcal{B}_s} = \{{}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})\colon \mathbf{k} \in \mathbb{N}_0^s,\, \mathbf{x} \in [0,1)^s\}$ is called $\mathcal{B}_s$*-adic function system* and is proved that it is a complete orthonormal basis of the space $L_2([0,1)^s)$.

The inequality of LeVeque [17] is an upper bound of the extreme discrepancy of an arbitrary one-dimensional net in the terms of the functions of the trigonometric function system. This is one-dimensional result and to the authors is unknown multidimensional version of this inequality.

The Erdős-Turán-Koksma's inequality, which has a long history, gives upper bounds of the extreme discrepancy in the terms of the functions of some classes of orthonormal function systems. First, Erdős and Turán [3] and Koksma [14] presented this inequality in the terms of the functions of the trigonometric function system. Niederreiter [18] in its monograph "Random Number Generator and Quasi-Monte Carlo Methods" systematizes and generalizes the results related to the Erdős-Turán-Koksma's inequality, which are based on the trigonometric function system. The form of the Erdős-Turán-Koksma's inequality for the extreme discrepancy, which is based on the Walsh and the Haar functions in base b have been given by Hellekalek [10–12]. Grozdanov and Stoilova [8] presented the form of this inequality in the terms of the functions of an arbitrary complete orthonormal function system constructed in Cantor systems.

Grozdanov [6] presented the form of the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma' inequality and the integral Weyl criterion in the terms of the modified integrals from the Vilenkin functions. Also, the form of the LeVeque's inequality, the Koksma's formula and the integral Weyl criterion in the terms of the modified integrals from the Haar functions constructed in Cantor systems, were presented.

There are examples of sequences constructed in b-adic number system, see Faure [4], which are generalizations of the classical multidimensional Halton's sequence. The next step of a generalization of the Halton's sequence was developed by Bednařík, Lertchoosakul, Markes and Trojovský [2]. They consider sequences constructed in Cantor systems, which are natural generalization of the Halton's and Faure's sequences.

Bednařik et al. [2] studied the extreme discrepancy of this sequence. Also, Lertchoosakul and Nair [16] realized investigations related to the generalized sequence of Halton. Grozdanov and Sevdinova [7] studied the $(\Gamma_b; \alpha; \gamma)$-diaphony of the Van der Corput sequence constructed in Cantor systems.

In such a way, we have objects constructed in Cantor systems. To study of these sequences and nets, we need appropriate analytical tools, usually some complete orthonormal function systems constructed in the same Cantor systems. It is clear that two such systems are these of the Vilenkin and the Haar functions.

The purpose of our investigation is to show some applications of the functions of the system $\Gamma_{\mathcal{B}_s}$ to the theory of the uniformly distributed sequences. Some classical results of the quantitative theory of the uniformly distributed sequences, as the LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion in the terms of the modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$, will be shown.

The rest of the article is organized as following. In Section 2 the notions of multidimensional modified integrals from the functions of the system $\Gamma_{\mathcal{B}_s}$ are introduced. In Section 3 some useful preliminary statements are given. In Section 4 the main results of the article are presented and proved.

2. Multidimensional modified integrals of the functions of the system $\Gamma_{\mathcal{B}_s}$

We will introduce the following useful symbol. For arbitrary and fixed integers $b \geq 2$ and $\beta \in \{1, 2, \dots, b-1\}$ let us denote $\Delta(b;\beta) = \sum_{h=0}^{b-1} h e^{2\pi \mathbf{i} \frac{\beta}{b} h}$. Then, the following equality holds

$$\Delta(b;\beta) = -\frac{b}{2}\left(1 + \mathbf{i} \coth \frac{\pi}{b}\beta\right).$$

Following Fine [5], for an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ we will consider the integral from the function ${}_B\gamma_k(x)$, thus let us denote

$${}_BJ_k(x) = \int_0^x {}_B\gamma_k(t)\, dt.$$

We will introduce the concept of the multidimensional modified integrals from the functions of the $\mathcal{B}_s$-adic system. For this purpose, let us denote $S = \{1, 2, \dots, s\}$. For an arbitrary integer $0 \leq u \leq s$ let

$$A_u = \{\alpha_1, \dots, \alpha_u \colon 1 \leq \alpha_1 < \dots < \alpha_u \leq s\}$$

be an arbitrary subset of S. Obviously we have $C_s^u = \dfrac{s!}{u!(s-u)!}$ choices of the subsets A_u. Let us define $C_{s-u} = S \setminus A_u$ and to denote

$$C_{s-u} = \{\beta_1, \dots, \beta_{s-u} \colon 1 \leq \beta_1 < \dots < \beta_{s-u} \leq s\}.$$

In the case when $u = 0$ we will think that $A_0 = \varnothing$ and $C_s = S$. When $u = s$ we have that $A_s = S$ and $C_0 = \varnothing$.

Let us assume that $u = 0$. To the value $u = 0$ we can think that corresponds the s-dimensional vector $\mathbf{k} = \mathbf{0}$. Let us introduce the notion of *modified integral* from the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{0}}(\mathbf{x})$ as

$${}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}) = \prod_{j=1}^{s}(1 - x_j) - \frac{1}{2^s}, \quad \mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s.$$

Now, let us assume that $1 \leq u \leq s$ is a fixed index. For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ let us introduce the following assumption:

(C) *Exactly u of number coordinates of* $\mathbf{k}$ *are nonzero and let they be* $k_{\alpha_1}, \dots, k_{\alpha_u}$.

In addition, for $1 \le j \le u$ the coordinate $k_{\alpha_j} = k_{g_j}^{(\alpha_j)} B_{g_j}^{(\alpha_j)} + k'_{\alpha_j}$, where $g_j \ge 0$, $0 \le k'_{\alpha_j} \le B_{g_j}^{(\alpha_j)} - 1$ and $k_{g_j}^{(\alpha_j)} \in \{1, \dots, b_{g_j}^{(\alpha_j)} - 1\}$.

Now, we are able to introduce the notion of *modified integral of rang u* from the function ${}_{\mathcal{B}_s}\gamma_{\mathbf{k}}(\mathbf{x})$ as

$$
{}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}) = (-1)^u \prod_{j=1}^{u} {}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j}) \cdot \prod_{j=1}^{s-u}(1 - x_{\beta_j})
$$

$$
- \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}\right)} - 1} \cdot \Delta\left(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}\right) \right] \frac{1}{2^{s-u}},
$$

$\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$. In the case when $u = s$, we assume that $\prod_{j=1}^{s-u}(1 - x_{\beta_j}) = 1$.

3. Preliminary statements

To prove the main results of our investigation we need of some useful preliminary statements. For this purpose, in Lemmas 3.1, 3.2, 3.3 and 3.4 we will use an arbitrary one-dimensional B-adic Cantor system, thus let $B = \{b_0, b_1, \dots : b_i \ge 2 \text{ for } i \ge 0\}$ be an arbitrary sequence of bases and $\{B_0, B_1, \dots\}$ be the corresponding to B sequence of generalized powers. Let us start with the following lemma, where we will present some properties of the functions of the system Γ_B.

Lemma 3.1. *Let $k \ge 1$ be an arbitrary integer with the B-adic representation*

$$
k = k_g B_g + k_{g-1} B_{g-1} + \dots + k_1 B_1 + k_0,
$$

where $g \ge 0$ for $0 \le i \le g$, $k_i \in \{0, 1, \dots, b_i - 1\}$ and $k_g \ne 0$. Then the following holds:

(i) *For each integers h and μ, and a real x such that $0 \le h \le B_g - 1$, $0 \le \mu \le b_g - 1$ and $x \in \left[\frac{h}{B_g} + \frac{\mu}{B_{g+1}}, \frac{h}{B_g} + \frac{\mu+1}{B_{g+1}}\right)$ the equality holds*

$$
{}_B\gamma_k(x) = {}_B\gamma_k\left(\frac{h}{B_g}\right) \cdot e^{2\pi \mathbf{i} \frac{k_g}{b_g}\mu};
$$

(ii) *For each integer p such that $0 \le p \le B_g - 1$ the equality holds*

$$
\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\, dx = 0;
$$

(iii) *Additionally let us denote $k' = k_{g-1}B_{g-1} + \dots + k_1 B_1 + k_0$. Then, the equality holds*

$$
{}_B\gamma_k(x) = {}_B\gamma_{k_g B_g}(x) {}_B\gamma_{k'}(x) \text{ for all } x \in [0,1).
$$

Proof. (i) Let us use the B-adic representation $\dfrac{h}{B_g} = 0.h_0h_1\dots h_{g-1}$, where for $0 \le i \le g-1$, $h_i \in \{0,1,\dots,b_i-1\}$. Then, an arbitrary $x \in \left[\dfrac{h}{B_g} + \dfrac{\mu}{B_{g+1}}, \dfrac{h}{B_g} + \dfrac{\mu+1}{B_{g+1}}\right)$ has the B-adic representation of the form $x = 0.h_0h_1\dots h_{g-1}\mu x_{g+1}\dots$. According to Definition 1.1 we have that

$$\begin{aligned}{}_B\gamma_k(x) &= e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_g}{B_{g+1}}\right)(h_0+h_1B_1+\cdots+h_{g-1}B_{g-1}+\mu B_g)} \\ &= e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_g}{B_{g+1}}\right)(h_0+h_1B_1+\cdots+h_{g-1}B_{g-1})}\cdot e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} = {}_B\gamma_k\left(\frac{h}{B_g}\right)\cdot e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu}.\end{aligned}$$

(ii) If $p = 0$, then the statement obviously is true. Now, let us assume that $p \ge 1$. In this case, we will use the presentation

$$\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\,dx = \sum_{h=0}^{p-1}\sum_{\mu=0}^{b_g-1}\int_{\frac{h}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{h}{B_g}+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_k(x)\,dx.$$

We will use the part (i) of the Lemma and the above equality to obtain that

$$\int_0^{\frac{p}{B_g}} {}_B\gamma_k(x)\,dx = \frac{1}{B_{g+1}}\sum_{h=0}^{p-1} {}_B\gamma_k\left(\frac{h}{B_g}\right)\sum_{\mu=0}^{b_g-1} e^{2\pi\mathbf{i}\frac{k_g}{b_g}\mu} = 0.$$

(iii) For an arbitrary real $x \in [0,1)$ we will use the B-adic representation $x = 0.x_0x_1\dots$, where for infinitely many values of i we have that $x_i \ne b_i - 1$. Then, according to Definition 1.1, we consecutively obtain that

$$\begin{aligned}{}_B\gamma_k(x) &= e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_{g-1}}{B_g}+\frac{k_g}{B_{g+1}}\right)(x_0+x_1B_1+\cdots+x_gB_g)} \\ &= e^{2\pi\mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_gB_g)}\cdot e^{2\pi\mathbf{i}\left(\frac{k_0}{B_1}+\cdots+\frac{k_{g-1}}{B_g}\right)(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \\ &= {}_B\gamma_{k_gB_g}(x)\,{}_B\gamma_{k'}(x).\end{aligned}$$

The lemma is finally proved. □

Lemma 3.2. *For arbitrary integers $g \ge 0$ and $\kappa_g \in \{1,\dots,b_g-1\}$, and a real $x \in [0,1)$ with the B-adic representation $x = 0.x_0x_1x_2\dots$, where for infinitely many values of i we have that $x_i \ne b_i - 1$, let us define the function*

$$\delta_{\kappa_g}(x) = \begin{cases}\displaystyle\sum_{\mu=0}^{x_g-1} e^{2\pi\mathbf{i}\frac{\kappa_g}{b_g}\mu}, & \text{if } x_g \ne 0,\\ 0, & \text{if } x_g = 0.\end{cases}$$

Let $k \ge 1$ be an arbitrary integer of the form $k = k_gB_g + k'$, where $g \ge 0$, $0 \le k' \le B_g - 1$ and $k_g \in \{1,\dots,b_g-1\}$. Then, the modified integral ${}_BJ_k(x)$ from *the function* ${}_B\gamma_k(x)$ *satisfies the equality*

$$ {}_BJ_k(x) = \frac{1}{B_{g+1}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \delta_{k_g}(x) {}_B\gamma_{k'}(x) + {}_B\gamma_k(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}} $$

for all $x \in [0,1)$.

Proof. We will use the statements of Lemma 3.1 to obtain that for an arbitrary and fixed real $x \in [0,1)$ the presentations hold

$$ \begin{aligned} {}_BJ_k(x) &= \int_0^x {}_B\gamma_k(t)\,dt = \int_0^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}} {}_B\gamma_k(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_k(t)\,dt \\ &= \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t) {}_B\gamma_{k'}(t)\,dt = {}_B\gamma_{k'}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt \\ &= {}_B\gamma_{k'}(x) \left[\int_0^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}} {}_B\gamma_{k_gB_g}(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt \right] \\ &= {}_B\gamma_{k'}(x) \int_0^x {}_B\gamma_{k_gB_g}(t)\,dt = {}_B\gamma_{k'}(x) {}_BJ_{k_gB_g}(x). \end{aligned} $$

Hence, the equality holds

$$ {}_BJ_k(x) = {}_B\gamma_{k'}(x) {}_BJ_{k_gB_g}(x) \text{ for all } x \in [0,1). \tag{3.1} $$

Now, we will calculate the integral ${}_BJ_{k_gB_g}(x)$. Lemma 3.1 (ii) gives us that

$$ {}_BJ_{k_gB_g}(x) = \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}}^x {}_B\gamma_{k_gB_g}(t)\,dt. $$

First of all, let us assume that $x_g \neq 0$. Then, from the above equality we obtain that

$$ \begin{aligned} {}_BJ_{k_gB_g}(x) &= \sum_{\mu=0}^{x_g-1} \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu+1}{B_{g+1}}} {}_B\gamma_{k_gB_g}(t)\,dt + \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^x {}_B\gamma_{k_gB_g}(t)\,dt \\ &= e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \sum_{\mu=0}^{x_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g}\mu} \int_{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu}{B_{g+1}}}^{\frac{x_0}{B_1}+\cdots+\frac{x_{g-1}}{B_g}+\frac{\mu+1}{B_{g+1}}} dt \\ &\qquad + {}_B\gamma_{k_gB_g}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^x dt \\ &= \frac{1}{B_{g+1}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \sum_{\mu=0}^{x_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g}\mu} + {}_B\gamma_{k_gB_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \end{aligned} \tag{3.2} $$

Now, let us assume that $x_g = 0$. In this case, the equalities hold

$$ {}_{B}J_{k_g B_g}(x) = {}_{B}\gamma_{k_g B_g}(x) \int_{\frac{x_0}{B_1}+\cdots+\frac{x_g}{B_{g+1}}}^{x} dt = {}_{B}\gamma_{k_g B_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \tag{3.3} $$

By using the introduced function $\delta_{\kappa_g}(x)$, equalities (3.2) and (3.3) can be rewritten as

$$ {}_{B}J_{k_g B_g}(x) = \frac{1}{B_{g+1}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0 + x_1 B_1 + \cdots + x_{g-1} B_{g-1})} \delta_{k_g}(x) + {}_{B}\gamma_{k_g B_g}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}}. \tag{3.4} $$

Equalities (3.1) and (3.4) give us that the equality

$$ {}_{B}J_{k}(x) = \frac{1}{B_{g+1}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0 + x_1 B_1 + \cdots + x_{g-1} B_{g-1})} \delta_{k_g}(x) {}_{B}\gamma_{k'}(x) + {}_{B}\gamma_{k}(x) \sum_{h=g+1}^{\infty} \frac{x_h}{B_{h+1}} $$

holds for all $x \in [0, 1)$. The Lemma is finally proved. □

Lemma 3.3. *For each integer $k \geq 1$ with the B-adic representation $k = \sum_{i=0}^{g} k_i B_i$, where $g \geq 0$ for $0 \leq i \leq g$, $k_i \in \{0, \ldots, b_i - 1\}$ and $k_g \neq 0$, the equality holds*

$$ \int_0^1 {}_{B}J_{k}(x)\, dx = -\frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_g}{b_g}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0} + \cdots + \frac{k_g}{b_0 \cdots b_g}\right)} - 1} \cdot \Delta(b_g; k_g) $$

with the introduced in Section 2 symbol $\Delta(b_g; k_g)$.

Proof. By using the result of Lemma 3.2 we obtain that

$$ \int_0^1 {}_{B}J_{k}(x)\, dx = \frac{1}{B_{g+1}} \int_0^1 e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0 + x_1 B_1 + \cdots + x_{g-1} B_{g-1})} \delta_{k_g}(x) {}_{B}\gamma_{k'}(x)\, dx + \sum_{h=g+1}^{\infty} \frac{1}{B_{h+1}} \int_0^1 x_h {}_{B}\gamma_{k}(x)\, dx. \tag{3.5} $$

We will calculate the integrals in the right side of equality (3.5). For this purpose, let us denote $k' = \sum_{i=0}^{g-1} k_i B_i$. Hence, we consecutively have that

$$ \int_0^1 e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0 + x_1 B_1 + \cdots + x_{g-1} B_{g-1})} \delta_{k_g}(x) {}_{B}\gamma_{k'}(x)\, dx $$

$$= \sum_{t_0=0}^{b_0-1} \cdots \sum_{t_g=0}^{b_g-1} \int_{\frac{t_0}{B_1}+\cdots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\cdots+\frac{t_g+1}{B_{g+1}}} e^{2\pi \mathbf{i} \frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \delta_{k_g}(x)_B\gamma_{k'}(x)\,dx$$

$$= \frac{1}{B_{g+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \ldots$$

$$\sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1}b_g}\right)t_{g-1}} \sum_{t_g=0}^{b_g-1} \delta_{k_g}(t_g). \quad (3.6)$$

It is easy to calculate that

$$\sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \ldots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1}b_g}\right)t_{g-1}}$$

$$= \frac{e^{2\pi \mathbf{i}\frac{k_g}{b_g}}-1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)}-1}. \quad (3.7)$$

By using the definition of the function $\delta_{k_g}(x)$, we consecutively have that

$$\sum_{t_g=0}^{b_g-1} \delta_{k_g}(t_g) = \sum_{t_g=1}^{b_g-1}\sum_{\mu=0}^{t_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}\mu} = \sum_{h=0}^{b_g-1}(b_g-1-h)e^{2\pi \mathbf{i}\frac{k_g}{b_g}h}$$

$$= (b_g-1)\sum_{h=0}^{b_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}h} - \sum_{h=0}^{b_g-1} he^{2\pi \mathbf{i}\frac{k_g}{b_g}h} = -\sum_{h=0}^{b_g-1} he^{2\pi \mathbf{i}\frac{k_g}{b_g}h} = -\Delta(b_g;k_g). \quad (3.8)$$

From equalities (3.6), (3.7) and (3.8) we obtain that

$$\int_0^1 e^{2\pi \mathbf{i}\frac{k_g}{B_{g+1}}(x_0+x_1B_1+\cdots+x_{g-1}B_{g-1})} \delta_{k_g}(x)_B\gamma_{k'}(x)\,dx$$

$$= -\frac{1}{B_{g+1}} \cdot \frac{e^{2\pi \mathbf{i}\frac{k_g}{b_g}}-1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)}-1} \cdot \Delta(b_g;k_g). \quad (3.9)$$

Now, we will calculate the second integral in the right hand side of equality (3.5). Thus, let $h \geq g+1$ be a fixed index. We will use the presentations

$$\int_0^1 x_{h\,B}\gamma_k(x)\,dx = \sum_{t_0=0}^{b_0-1} \cdots \sum_{t_g=0}^{b_g-1} \cdots \sum_{t_h=0}^{b_h-1} \int_{\frac{t_0}{B_1}+\cdots+\frac{t_h}{B_{h+1}}}^{\frac{t_0}{B_1}+\cdots+\frac{t_h+1}{B_{h+1}}} x_{h\,B}\gamma_k(x)\,dx$$

$$= \frac{1}{B_{h+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\cdots+\frac{k_g}{b_0\ldots b_g}\right)t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\cdots+\frac{k_g}{b_1\ldots b_g}\right)t_1} \ldots$$

$$\sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i}\frac{k_g}{b_g}t_g} \ldots \sum_{t_h=0}^{b_h-1} t_h = 0. \quad (3.10)$$

The equalities (3.5), (3.9) and (3.10) finally prove the statement of the lemma. □

Lemma 3.4. *For each integer $k \geq 1$ with the B-adic representation $k = \sum_{i=0}^{g} k_i B_i$, where $g \geq 0$ for $0 \leq i \leq g$, $k_i \in \{0, 1, \dots, b_i - 1\}$ and $k_g \neq 0$, the equality holds*

$$\int_0^1 x_B \gamma_k(x)\, dx = \frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_g}{b_g}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0} + \dots + \frac{k_g}{b_0 \dots b_g}\right)} - 1} \cdot \Delta(b_g; k_g)$$

with the introduced in Section 2 symbol $\Delta(b_g; k_g)$.

Proof. We will use the following presentations:

$$\int_0^1 x_B \gamma_k(x)\, dx = \sum_{t_0=0}^{b_0-1} \sum_{t_1=0}^{b_1-1} \cdots \sum_{t_{g-1}=0}^{b_{g-1}-1} \sum_{t_g=0}^{b_g-1} \int_{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\dots+\frac{t_g+1}{B_{g+1}}} x_B \gamma_k(x)\, dx$$

$$= \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\dots+\frac{k_g}{b_1 \dots b_g}\right) t_1} \dots$$

$$\sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \int_{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}}^{\frac{t_0}{B_1}+\dots+\frac{t_g}{B_{g+1}}+\frac{1}{B_{g+1}}} x\, dx$$

$$= \sum_{t_0=1}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \sum_{t_1=0}^{b_1-1} e^{2\pi \mathbf{i}\left(\frac{k_1}{b_1}+\dots+\frac{k_g}{b_1 \dots b_g}\right) t_1} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}}$$

$$\times \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \left\{ \frac{1}{B_{g+1}} \left(\frac{t_0}{B_1} + \frac{t_1}{B_2} + \dots + \frac{t_g}{B_{g+1}} \right) + \frac{1}{2B_{g+1}^2} \right\}$$

$$= \frac{1}{B_{g+1}} \left[\frac{1}{B_1} \sum_{t_0=0}^{b_0-1} t_0 \cdot e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \right.$$

$$+ \dots + \frac{1}{B_g} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} t_{g-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$\left. + \frac{1}{B_{g+1}} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} t_g e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g} \right]$$

$$+ \frac{1}{2B_{g+1}^2} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$= \frac{1}{B_{g+1}^2} \sum_{t_0=0}^{b_0-1} e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0}+\dots+\frac{k_g}{b_0 b_1 \dots b_g}\right) t_0} \dots \sum_{t_{g-1}=0}^{b_{g-1}-1} e^{2\pi \mathbf{i}\left(\frac{k_{g-1}}{b_{g-1}}+\frac{k_g}{b_{g-1} b_g}\right) t_{g-1}} \sum_{t_g=0}^{b_g-1} t_g e^{2\pi \mathbf{i} \frac{k_g}{b_g} t_g}$$

$$= \frac{1}{B_{g+1}^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_g}{b_g}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0}{b_0} + \dots + \frac{k_g}{b_0 \dots b_g}\right)} - 1} \cdot \Delta(b_g; k_g).$$

The lemma is finally proved. □

In the next lemma we will use an arbitrary $\mathcal{B}_s$-adic Cantor system. Thus, let $\mathcal{B}_s = (B_1, \dots, B_s)$, where for $1 \le j \le s$ $B_j = \{b_0^{(j)}, b_1^{(j)}, \dots : b_i^{(j)} \ge 2 \text{ for } i \ge 0\}$ be a sequence of bases and $\{B_0^{(j)}, B_1^{(j)}, \dots\}$ be the corresponding to B_j sequence of generalized powers.

Lemma 3.5. *Let $\xi_N = \{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by $N \ge 1$ points in $[0,1)^s$ and for $0 \le n \le N-1$ let us denote $\mathbf{x}_n = (x_n^{(1)}, \dots, x_n^{(s)})$. For an arbitrary vector $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$ let $g(\xi_N; \mathbf{x}) = \dfrac{1}{N} A([0,\mathbf{x}); \xi_N) - x_1 \dots x_s$ be the so-called local discrepancy of the net ξ_N.*

Then, for an arbitrary vector $\mathbf{k} \in \mathbb{N}_0^s$ the Fourier's coefficient

$$\widehat{g}(\xi_N; \mathbf{k}) = \int_{[0,1]^s} g(\xi_N; \mathbf{x})_{\mathcal{B}_s}\overline{\gamma}_{\mathbf{k}}(\mathbf{x})\, d\mathbf{x}$$

of the function $g(\xi_N; \mathbf{x})$ with respect to the functions of the system $\Gamma_{\mathcal{B}_s}$ satisfies the following presentations:

(i) *If $\mathbf{k} = \mathbf{0}$, then the equality $\widehat{g}(\xi_N; \mathbf{0}) = \dfrac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} (1 - x_n{}^{(j)}) - \dfrac{1}{2^s} \right]$ holds;*

(ii) *Let $\mathbf{k} \ne \mathbf{0}$ and for some u, such that $1 \le u \le s$, the vector $\mathbf{k}$ satisfies the presented in Section 2 assumption (C). Then, the equality holds*

$$\overline{\widehat{g}}(\xi_N; \mathbf{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \left\{ (-1)^u \prod_{j=1}^{u} B_{\alpha_j} J_{k_{\alpha_j}}(x_n^{(\alpha_j)}) \prod_{j=1}^{s-u} (1 - x_n^{(\beta_j)}) \right.$$
$$\left. - \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \dots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \frac{1}{2^{s-u}} \right\}$$

with the additional condition, that in the case when $u = s$, we have

$$\prod_{j=1}^{s-u} (1 - x_n^{(\beta_j)}) = 1.$$

Proof. For an arbitrary vector $\mathbf{x} = (x_1, \dots, x_s) \in [0,1)^s$ the local discrepancy $g(\xi_N; \mathbf{x})$ of the net ξ_N satisfies the presentation

$$g(\xi_N; \mathbf{x}) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} 1_{[x_n^{(j)}, 1)}(x_j) - x_1 \dots x_s \right],$$

where $1_{[x_n^{(j)},1)}(x_j)$, $0 \le n \le N-1$, $1 \le j \le s$, is the characteristic function over the interval $[{x_n}^{(j)}, 1)$.

For an arbitrary vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}^s$ the Fourier's coefficient of the function $g(\xi_N; \mathbf{x})$ satisfies the equality

$$\overline{\widehat{g}}(\xi_N; \mathbf{k}) \\ = \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} \int_0^1 1_{[{x_n}^{(j)},1)}(x_j) {}_{B_j}\gamma_{k_j}(x_j)\, dx_j - \prod_{j=1}^{s} \int_0^1 x_j\, {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \right]. \quad (3.11)$$

(i) Let us assume that $\mathbf{k} = \mathbf{0}$. Then, by using equality (3.11) we obtain that

$$\begin{aligned} \widehat{g}(\xi_N; \mathbf{0}) &= \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} \int_0^1 1_{[x_n^{(j)},1)}(x_j)\, dx_j - \prod_{j=1}^{s} \int_0^1 x_j\, dx_j \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \left[\prod_{j=1}^{s} (1 - {x_n}^{(j)}) - \frac{1}{2^s} \right]. \end{aligned}$$

(ii) Let us assume that $u = s$, thus for each $1 \le j \le s$ we have that $k_j \ne 0$. Lemma 3.1 (ii) gives us the following

$$\begin{aligned} \int_0^1 1_{[x_n^{(j)},1)}(x_j) {}_{B_j}\gamma_{k_j}(x_j)\, dx_j &= \int_{x_n^{(j)}}^1 {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \\ &= \int_0^1 {}_{B_j}\gamma_{k_j}(x_j)\, dx_j - \int_0^{x_n^{(j)}} {}_{B_j}\gamma_{k_j}(x_j)\, dx_j \\ &= -\int_0^{x_n^{(j)}} {}_{B_j}\gamma_{k_j}(x_j)\, dx_j = -{}_{B_j}J_{k_j}(x_n^{(j)}). \end{aligned} \quad (3.12)$$

By using the statement of Lemma 3.4, for $1 \le j \le s$ we have that

$$\int_0^1 x_j\, {}_{B_j}\gamma_{k_j}(x_j)\, dx_j = \frac{1}{\left[B_{g_j+1}^{(j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(j)}}{b_{g_j}^{(j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(j)}}{b_0^{(j)}} + \dots + \frac{k_{g_j}^{(j)}}{b_0^{(j)} \dots b_{g_j}^{(j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(j)}; k_{g_j}^{(j)}). \quad (3.13)$$

From equalities (3.11), (3.12) and (3.13) we obtain the presentation

$$\overline{\widehat{g}}(\xi_N; \mathbf{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \left[(-1)^s \prod_{j=1}^{s} {}_{B_j}J_{k_j}(x_n^{(j)}) \right. \\ \left. - \prod_{j=1}^{s} \frac{1}{\left[B_{g_j+1}^{(j)}\right]^2} \cdot \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(j)}}{b_{g_j}^{(j)}}} - 1}{e^{2\pi \mathbf{i} \left(\frac{k_0^{(j)}}{b_0^{(j)}} + \dots + \frac{k_{g_j}^{(j)}}{b_0^{(j)} \dots b_{g_j}^{(j)}} \right)} - 1} \cdot \Delta(b_{g_j}^{(j)}; k_{g_j}^{(j)}) \right].$$

Let us assume that $1 \le u \le s-1$. By using equality (3.11) and the statements of Lemma 3.1 (ii) and Lemma 3.4 we obtain that

$$\overline{\overline{g}}(\xi_N;\mathbf{k}) = \frac{1}{N}\sum_{n=0}^{N-1}\left[\prod_{j=1}^{u}\int_0^1 1_{[x_n^{(\alpha_j)},1)}(x_{\alpha_j})\,{}_{B_{\alpha_j}}\gamma_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j}\prod_{j=1}^{s-u}\int_0^1 1_{[x_n^{(\beta_j)},1)}(x_{\beta_j})\,dx_{\beta_j}\right.$$
$$\left. - \prod_{j=1}^{u}\int_0^1 x_{\alpha_j}\,{}_{B_{\alpha_j}}\gamma_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j}\prod_{j=1}^{s-u}\int_0^1 x_{\beta_j}\,dx_{\beta_j}\right]$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\left\{(-1)^u\prod_{j=1}^{u}{}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_n^{(\alpha_j)})\prod_{j=1}^{s-u}(1-x_n^{(\beta_j)})\right.$$
$$\left. - \prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\frac{e^{2\pi \mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\dots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\dots b_{g_j}^{(\alpha_j)}}\right)}-1}\cdot\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right]\frac{1}{2^{s-u}}\right\}.$$

The last finally proves the lemma. □

4. The main results

Now, we are able to present the main results of our article. In Theorem 4.1 we will use an arbitrary B-adic Cantor system. Thus, let $B = \{b_0, b_1, \dots : b_i \ge 2 \text{ for } i \ge 0\}$ be a sequence of bases and $\{B_0, B_1, \dots\}$ be the corresponding to B sequence of generalized powers. The following theorem holds.

Theorem 4.1 (The LeVeque's inequality). *Let $\xi_N = \{x_0, \dots, x_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)$. Then the extreme discrepancy $D(\xi_N)$ of the net ξ_N satisfies the inequality*

$$D^3(\xi_N) \le 12\sum_{g=0}^{\infty}\sum_{k_g=1}^{b_g-1}\sum_{k=k_gB_g}^{(k_g+1)B_g-1}\left|\frac{1}{N}\sum_{k=0}^{N-1}{}_BJ_{g,k_g,k}(x_n)\right|^2.$$

Proof. Following Kuipers and Niederreiter [15, Ch. 2, Theorem 2.4], let us denote $S(\xi_N) = \sum_{n=0}^{N-1}\left(x_n - \frac{1}{2}\right)$. The following inequality

$$D^3(\xi_N) \le \frac{12}{N^2}\int_0^1 [R(\xi_N;x) + S(\xi_N)]^2\,dx \tag{4.1}$$

is proved, where $R(\xi_N;x) = \sum_{n=0}^{N-1}\left[1_{[x_n,1)} - x\right]$.

For the integral in inequality (4.1) we will use the following presentations

$$\int_0^1 \left[R(\xi_N;x)+S(\xi_N)\right]^2\,dx = \int_0^1 R^2(\xi_N;x)\,dx + 2S(\xi_N)\sum_{n=0}^{N-1}\int_0^1 \left[1_{[0,1)}(x)-x\right]\,dx$$
$$+\,S^2(\xi_N) = \int_0^1 R^2(\xi_N;x)\,dx - S^2(\xi_N). \quad (4.2)$$

From (4.1) and (4.2) we obtain the inequality

$$D^3(\xi_N) \le \frac{12}{N^2}\left[\int_0^1 R^2(\xi_N;x)\,dx - S^2(\xi_N)\right].$$

To the integral of the above inequality, we will apply the Parseval's formula. For this purpose, the Fourier's coefficients of the function $R(\xi_N;x)$ were calculated in Lemma 3.5. In this way, we obtain that

$$D^3(\xi_N) \le 12\sum_{g=0}^{\infty}\sum_{k_g=1}^{b_g-1}\sum_{k=k_gB_g}^{(k_g+1)B_g-1}\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{B}J_{g,k_g,k}(x_n)\right|^2.$$

Theorem 4.1 is finally proved. □

In Theorems 4.2, 4.3 and 4.4 we will use an arbitrary $\mathcal{B}_s$-adic Cantor system. Thus, let $\mathcal{B}_s=(B_1,\dots,B_s)$, where for $1\le j\le s$ $B_j=\{b_0^{(j)},b_1^{(j)},\dots : b_i^{(j)}\ge 2 \text{ for } i\ge 0\}$ be a sequence of bases and $\{B_0^{(j)},B_1^{(j)},\dots\}$ be the corresponding to B_j sequence of generalized powers.

Theorem 4.2 (The Koksma's formula). *Let $\xi_N=\{\mathbf{x}_0,\dots,\mathbf{x}_{N-1}\}$ be an arbitrary net composed by $N\ge 1$ points in $[0,1)^s$. Then, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the equality*

$$T^2(\xi_N) = \left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)\right|^2 + \sum_{u=1}^{s}\sum_{\{\alpha_1,\dots,\alpha_u\}\subseteq S}\prod_{j=1}^{u}\sum_{g_j=0}^{\infty}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$
$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2.$$

Proof. By using the Parseval's formula, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the presentation

$$T^2(\xi_N) = \int_{[0,1]^s} g^2(\xi_N;\mathbf{x})\,d\mathbf{x} = \left|\widehat{g}(\xi_N;\mathbf{0})\right|^2 + \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}}\left|\widehat{g}(\xi_N;\mathbf{k})\right|^2.$$

In Lemma 3.5 the Fourier's coefficients of the local discrepancy $g(\xi_N;\mathbf{x})$ were calculated. We put the obtained results in the above equality and obtain that

$$T^2(\xi_N) = \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s} J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) \right|^2 + \sum_{u=1}^{s} \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{\infty} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)} B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s} J_{(g_1,\ldots,g_n),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

Theorem 4.2 is finally proved. □

Theorem 4.3 (The Erdős-Turán-Koksma's inequality)**.** *Let us assume that the coordinate sequences of the $\mathcal{B}_s$-adic system are bounded from above, i.e., there exists an absolute integer constant C, such that for $1 \le j \le s$ and each $i \ge 0$, the inequality $b_i^{(j)} \le C$ holds. Let us denote $b = \min_{1\le j\le s} \min_{i\ge 0} b_i^{(j)}$, and let us define the constant*

$$K(C;s) = \begin{cases} 9/4, & \text{if } C = 2, \\ \left[1 + \left(C^2/8\right)^s\right]^2, & \text{if } C \ge 3. \end{cases}$$

Let $M > 1$ be an arbitrary integer. Let $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$. Then, the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the inequality

$$T^2(\xi_N) \le K(C;s) \left[1 + (C-1)\frac{b}{b-1}\right]^s \cdot \frac{1}{b^M} + \left(\frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s} J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) \right)^2$$

$$+ \sum_{u=1}^{s} \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S} \prod_{j=1}^{u} \sum_{g_j=0}^{M-1} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)} \cdot B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)\cdot B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s} J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

Proof. Let M be as in the condition of the theorem and let us denote

$$A(M;s;\xi_N) = \sum_{u=1}^{s} \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S} \sum_{\substack{(g_1,\ldots,g_u)\in\mathbb{N}_0^u \\ \text{there is at least} \\ \text{one index } \delta, \\ 1\le\delta\le u, \\ \text{such that } g_\delta \ge M}} \prod_{j=1}^{u} \sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1} \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)} B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$

$$\times \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_{\mathcal{B}_s} J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n) \right|^2 .$$

The result of Theorem 4.2 gives us that the quadratic discrepancy $T(\xi_N)$ of the net ξ_N satisfies the presentation

$$T^2(\xi_N) = \left(\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n)\right)^2 + \sum_{u=1}^{s}\sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}\prod_{j=1}^{u}\sum_{g_j=0}^{M-1}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$
$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2 + A(M;s;\xi_N). \quad (4.3)$$

We will obtain un upper bound of the quantity $A(M;s;\xi_N)$. Without loss of the community of our consideration, let us assume that $g_1 \geq M$, $g_2 \geq 0, \ldots, g_u \geq 0$. In this way, we obtain that

$$A(M;s;\xi_N) \leq \sum_{u=1}^{s}\sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\sum_{g_2=0}^{\infty}\cdots\sum_{g_u=0}^{\infty}\prod_{j=1}^{u}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1}$$
$$\times\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x}_n)\right|^2. \quad (4.4)$$

We will prove that for each parameters $(g_1,\ldots,g_u)$, $(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)})$ and $(k_{\alpha_1},\ldots,k_{\alpha_u})$ the inequality holds

$$\left|{}_{\mathcal{B}_s}J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x})\right| \leq \left[1+\left(\frac{C^2}{8}\right)^u\right]\prod_{j=1}^{u}\frac{1}{B_{g_j}^{(\alpha_j)}}. \quad (4.5)$$

Really, by using the definition of the notion of modified integral, we obtain that

$$\left|{}_{\mathcal{B}_s}J_{(g_1,\ldots,g_u),(k_{g_1}^{(\alpha_1)},\ldots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\ldots,k_{\alpha_u})}(\mathbf{x})\right| \leq \prod_{j=1}^{u}\left|{}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j})\right| \times \prod_{j=1}^{s-u}(1-x_{\beta_j})$$
$$+\prod_{j=1}^{u}\left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2}\cdot\left|\frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}}-1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\ldots b_{g_j}^{(\alpha_j)}}\right)}-1}\right|\cdot\left|\Delta(b_{g_j}^{(\alpha_j)};k_{g_j}^{(\alpha_j)})\right|\right]\cdot\frac{1}{2^{s-u}}. \quad (4.6)$$

We will estimate the expressions in the above inequality. For this purpose, let $k \geq 1$ be an arbitrary integer and $B_g \leq k \leq B_{g+1}-1$ for some $g \geq 0$. Let $x \in [0,1)$ be an arbitrary and fixed real and let q be the integer such that $0 \leq q \leq B_g - 1$ and $\frac{q}{B_g} \leq x < \frac{q+1}{B_g}$. Then, by using Lemma 3.1 (ii) we obtain that

$$|{}_BJ_k(x)| = \left|\int_0^x {}_B\gamma_k(t)\,dt\right| = \left|\int_0^{\frac{q}{B_g}} {}_B\gamma_k(t)\,dt + \int_{\frac{q}{B_g}}^x {}_B\gamma_k(t)\,dt\right| = \left|\int_{\frac{q}{B_g}}^x {}_B\gamma_k(t)\,dt\right|$$
$$\leq \int_{\frac{q}{B_g}}^x |{}_B\gamma_k(t)|\,dt = \int_{\frac{q}{B_g}}^x dt = x - \frac{q}{B_g} \leq \frac{1}{B_g},$$

i.e., the inequality holds

$$|_{B}J_k(x)| \leq \frac{1}{B_g}, \quad x \in [0,1). \tag{4.7}$$

For arbitrary integers $b \geq 2$ and $\beta \in \{1,\dots,b-1\}$ we have that $|\Delta(b;\beta)| = \frac{b}{2} \cdot \frac{1}{\sin \pi \frac{\beta}{b}}$. We will use the facts that for $t \in \left[0, \frac{\pi}{2}\right]$ and $t \in \left[\frac{\pi}{2}, \pi\right]$, respectively the inequalities $\sin t \geq \frac{2}{\pi} t$ and $\sin t \geq 2 - \frac{2}{\pi} t$ hold. In this way, we obtain that $\sin \pi \frac{\beta}{b} \geq \frac{2}{b}$ and hence,

$$|\Delta(b;\beta)| \leq \frac{b^2}{4}. \tag{4.8}$$

For an arbitrary real t the equality $|e^{2\pi \mathbf{i} t} - 1| = 2|\sin \pi t|$ holds. Hence, for each $1 \leq j \leq u$ we have that

$$\left| \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}\right)} - 1} \right| \leq \frac{1}{\sin \pi \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right)}.$$

The inequalities

$$\frac{1}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \leq \frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \leq 1 - \frac{1}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}$$

give us that the lower bound

$$\sin \pi \left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}} \right) \geq \frac{2}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}$$

holds. In this way, we obtain that

$$\left| \frac{e^{2\pi \mathbf{i} \frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi \mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}} + \cdots + \frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}\right)} - 1} \right| \leq \frac{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}{2}. \tag{4.9}$$

From (4.6)–(4.9) we obtain that

$$\left| _{\mathcal{B}_s} J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}) \right|$$
$$\leq \prod_{j=1}^{u} \frac{1}{B_{g_j}^{(\alpha_j)}} + \prod_{j=1}^{u} \frac{1}{\left[B_{g_j}^{(\alpha_j)}\right]^2} \cdot \frac{b_0^{(\alpha_j)} \dots b_{g_j}^{(\alpha_j)}}{2} \cdot \frac{\left[b_{g_j}^{(\alpha_j)}\right]^2}{4} \leq \left[1 + \left(\frac{C^2}{8}\right)^u\right] \prod_{j=1}^{u} \frac{1}{B_{g_j}^{(\alpha_j)}}.$$

In this way, inequality (4.5) is finally proved.

From (4.4) and (4.5) we obtain that

$$
\begin{aligned}
&A(M;s;\xi_N)\\
&\le \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{1}{\left[B_{g_1}^{(\alpha_1)}\right]^2}\sum_{g_2=0}^{\infty}\frac{1}{\left[B_{g_2}^{(\alpha_2)}\right]^2}\cdots\sum_{g_u=0}^{\infty}\frac{1}{\left[B_{g_u}^{(\alpha_u)}\right]^2}\\
&\qquad\times\prod_{j=1}^{u}\sum_{k_{g_j}^{(\alpha_j)}=1}^{b_{g_j}^{(\alpha_j)}-1}\ \sum_{k_{\alpha_j}=k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)}}^{(k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)}-1} 1\\
&= \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{b_{g_1}^{(\alpha_1)}-1}{B_{g_1}^{(\alpha_1)}}\sum_{g_2=0}^{\infty}\frac{b_{g_2}^{(\alpha_2)}-1}{B_{g_2}^{(\alpha_2)}}\cdots\sum_{g_u=0}^{\infty}\frac{b_{g_u}^{(\alpha_u)}-1}{B_{g_u}^{(\alpha_u)}}\\
&\le \sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2 (C-1)^u \sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}\sum_{g_1=M}^{\infty}\frac{1}{B_{g_1}^{(\alpha_1)}}\sum_{g_2=0}^{\infty}\frac{1}{B_{g_2}^{(\alpha_2)}}\cdots\sum_{g_u=0}^{\infty}\frac{1}{B_{g_u}^{(\alpha_u)}}.
\end{aligned}
\tag{4.10}
$$

We will use the following estimations

$$
\sum_{g_1=M}^{\infty}\frac{1}{B_{g_1}^{(\alpha_1)}}\le\sum_{g=M}^{\infty}\frac{1}{b^g}=\frac{b}{b-1}\cdot\frac{1}{b^M}\ \text{and for } 2\le j\le u\ \sum_{g_j=0}^{\infty}\frac{1}{B_{g_j}^{(\alpha_j)}}\le\sum_{g=0}^{\infty}\frac{1}{b^g}=\frac{b}{b-1}.
$$

Then, from inequality (4.10) we obtain that

$$
\begin{aligned}
A(M;s;\xi_N) &\le \frac{1}{b^M}\sum_{u=1}^{s}\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\left[(C-1)\frac{b}{b-1}\right]^u\sum_{\{\alpha_1,\ldots,\alpha_u\}\subseteq S}1\\
&= \left[\sum_{u=0}^{s}C_s^u\cdot\left[(C-1)\frac{b}{b-1}\right]^u\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\right]\frac{1}{b^M}.
\end{aligned}
\tag{4.11}
$$

Let us assume that $C=2$. Then, for $1\le u\le s$ we have that $\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\le\frac{9}{4}$. In the case, when $C\ge 3$ we have that $\left[1+\left(\frac{C^2}{8}\right)^u\right]^2\le\left[1+\left(\frac{C^2}{8}\right)^s\right]^2$. In this way, from inequality (4.11) we obtain that

$$
\begin{aligned}
&A(M;s;\xi_N)\\
&\quad\le K(C;s)\left[\sum_{u=0}^{s}C_s^u\left[(C-1)\frac{b}{b-1}\right]^u\right]\frac{1}{b^M}=K(C;s)\left[1+(C-1)\frac{b}{b-1}\right]^s\frac{1}{b^M}
\end{aligned}
$$

with the defined in the condition of the theorem constant $K(C;s)$. Equality (4.3) and the above estimation finally prove the statement of the theorem. □

Theorem 4.4 (The Integral Weyl's criterion). *Let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. The sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the following conditions hold:*

(i) *The limit equality* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x}_n) = 0$ *holds;*

(ii) *The limit equality*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x}_n) = 0$$

holds for each choice of the parameters $1 \leq u \leq s$, $\{\alpha_1,\dots,\alpha_u\} \subseteq S$, $(g_1,\dots,g_u) \in \mathbb{N}_0^s$, for $1 \leq j \leq u$ $k_{g_j}^{(\alpha_j)} \in \{1,\dots,b_{g_j}^{(\alpha_j)}-1\}$ and $k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)} \leq k_{\alpha_j} \leq (k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)} - 1$.

Proof. To prove the theorem we will use the so-called integral criterion of Weyl [22]. Thus, the sequence $\xi = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is uniformly distributed if and only if for each Riemann integrable over $[0,1]^s$ function, the limit equality holds

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n) = \int_{[0,1]^s} f(\mathbf{x})\,d\mathbf{x}.$$

In this way, we must prove that the equality $\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})\,d\mathbf{x} = 0$ holds, and for each choice of the parameters $1 \leq u \leq s$, $\{\alpha_1,\dots,\alpha_u\} \subseteq S$, $(g_1,\dots,g_u) \in \mathbb{N}_0^u$, $1 \leq j \leq u$, $k_{g_j}^{(\alpha_j)} \in \{1,\dots,b_{g_j}^{(\alpha_j)}-1\}$ and $k_{g_j}^{(\alpha_j)}B_{g_j}^{(\alpha_j)} \leq k_{\alpha_j} \leq (k_{g_j}^{(\alpha_j)}+1)B_{g_j}^{(\alpha_j)} - 1$, the equality $\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\,d\mathbf{x} = 0$ holds.

By using the definition of the modified integral ${}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})$, we directly obtain that

$$\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{\mathrm{mod},\mathbf{0}}(\mathbf{x})\,d\mathbf{x} = 0.$$

Now, let the above parameters be fixed. By using the definition of the modified integral ${}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})$ and the statement of Lemma 3.3 we consecutively obtain that

$$\int_{[0,1]^s} {}_{\mathcal{B}_s}J_{(g_1,\dots,g_u),(k_{g_1}^{(\alpha_1)},\dots,k_{g_u}^{(\alpha_u)}),(k_{\alpha_1},\dots,k_{\alpha_u})}(\mathbf{x})\,d\mathbf{x}$$

$$= (-1)^u \prod_{j=1}^{u} \int_0^1 {}_{B_{\alpha_j}}J_{k_{\alpha_j}}(x_{\alpha_j})\,dx_{\alpha_j} \prod_{j=1}^{s-u} \int_0^1 (1-x_{\beta_j})\,dx_{\beta_j}$$

$$- \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\cdots b_{g_j}^{(\alpha_j)}}\right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \times \frac{1}{2^{s-u}}$$

$$= \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\ldots b_{g_j}^{(\alpha_j)}}\right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \times \frac{1}{2^{s-u}}$$

$$- \prod_{j=1}^{u} \left[\frac{1}{\left[B_{g_j+1}^{(\alpha_j)}\right]^2} \cdot \frac{e^{2\pi\mathbf{i}\frac{k_{g_j}^{(\alpha_j)}}{b_{g_j}^{(\alpha_j)}}} - 1}{e^{2\pi\mathbf{i}\left(\frac{k_0^{(\alpha_j)}}{b_0^{(\alpha_j)}}+\cdots+\frac{k_{g_j}^{(\alpha_j)}}{b_0^{(\alpha_j)}\ldots b_{g_j}^{(\alpha_j)}}\right)} - 1} \cdot \Delta(b_{g_j}^{(\alpha_j)}; k_{g_j}^{(\alpha_j)}) \right] \times \frac{1}{2^{s-u}} = 0.$$

In this way, the proof of the theorem is finally completed. □

5. Conclusion

The application of the functions of some classes of complete orthonormal function systems to the theory of the uniformly distributed sequences is realized as two main approaches.

The first approach is their direct applications. The main results in this direction are the exponential Weyl's criterion, which is a necessary and sufficient condition that a sequence to be uniformly distributed and the Erdős-Turán-Koksma's inequality, which gives an upper bound of the extreme discrepancy. Both results are presented in the terms of the trigonometric sum of these functions. Also, an important result here, is to define the notion of diaphony, which is based on using some classes of complete orthonormal function systems.

The second approach is to introduce the notions of modified integrals from these functions and to show some their applications for an investigation of the uniformly distributed sequences. Grozdanov and Stoilova [8] realized this approach with respect to the functions of the system $\mathrm{Vil}_{\mathcal{B}_s}$ of the Vilenkin functions.

As it was shown, as an appropriate tool for investigation of sequences and nets constructed in Cantor systems, the functions of some complete orthonormal systems constructed over these number systems, are used.

In this article, we realized our main purpose – to use the function system $\Gamma_{\mathcal{B}_s}$ and to show some of its applications to the quantitative and qualitative theory of the uniformly distributed sequences. The LeVeque's inequality, the Koksma's formula, the Erdős-Turán-Koksma's inequality and the integral Weyl's criterion were presented in the terms of these integrals. We consider that the obtained results are related to a wide aspect of the quantitative and the qualitative theory of the uniformly distributed sequences.

References

[1] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzafarli and A. I. Rubinstain, Multiplicative system and harmonic analysis on zero-dimensional groups, ELM, Baku, 1981.

[2] D. Bednařík, P. Lertchoosakul, D. Marques and P. Trojovský, The generalized and modified Halton sequences in Cantor bases, Mon. Math. 188 (2019) 1–29.

[3] P. Erdős and P. Turán, On a problem in the theory of uniform distribution I, II, Indag. Math. 10 (1948) 370–378; 406–413.

[4] H. Faure, Discrépances de suites associées à un système de numération (en dimension un), Bull. Soc. Math. France 109 (1981) 143–182.

[5] N. J. Fine, On the Walsh functions, J. Trans. Amer. Math. Soc. 65 (1949) 372–414.

[6] V. Grozdanov, Quantitative measures for the irregularity of the distribution of sequences and nets, Dissertation for degree Doctor of Sciences, University of Shumen "Bishop Konstantin Preslavski", 2024.

[7] V. Grozdanov and M. Sevdinova, On the $(Vil_B;\alpha;\gamma)$-diaphony of the Van der Corput sequences constructed in Cantor systems, Ann. Sofia Univ. Fac. Math. and Inf. 109 (2023) 71–90, https://doi.org/10.60063/GSU.FMI.109.71--90.

[8] V. Grozdanov and S. Stoilova, The inequality of Erdős-Turán-Koksma. An arbitrary complete orthonormal functional system in $\mathcal{B}$-adic base, C. R. Acad. Bulg. Sci. 57(5) (2004) 23–28.

[9] V. Grozdanov and S. Stoilova, Price and Haar type functions and uniform distribution of sequences, JIPAM 5(2) (2004) 1–17.

[10] P. Hellekalek, General discrepancy estimates: The Walsh function system, Acta Arith. LXVII(3) (1994) 209–218.

[11] P. Hellekalek, General discrepancy estimates II: The Haar function system, Acta Arith. LXVII(4) (1994) 313–322.

[12] P. Hellekalek, General discrepancy estimates III: The Erdős-Turán-Koksma inequality for the Haar function system, Mon. Math. 120 (1995) 25–45.

[13] P. Hellekalek and H. Niederreiter, Constructions of uniformly distributed sequences using the b-adic method, Uniform Distribution Theory 6(1) (2011) 185–200.

[14] J. F. Koksma, Some theorems on diophantine inequalities, Math. Centrum, Amsterdam, Scriptum No. 5, 1950.

[15] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, John Wiley and Sons, New York, 1974.

[16] P. Lertchoosakul and R. Nair, Distribution functions for subsequences of the Van der Corput sequence, Indag. Math. N. S. 24(3) (2013) 593–601.

[17] W. J. LeVeque, An inequality connected with Weyl's criterion for uniform distribution, Proc. Symp. Pure Math., Vol. VIII, AMS, 1965, 20–30.

[18] H. Niederreiter, Random number generator and quasi-Monte Carlo methods, SIAM, Philadelphia, 1992.

[19] Ts. N. Petrova, Quasi-Monte Carlo integration in hybrid Korobov and Sobolev spaces, PhD Dissertation, South West University "Neofit Rilski", Blagoevgrad, Bulgaria, 2021.

[20] J. J. Price, Certain groups of orthonormal step functions, Canad. J. Math. 9 (1957) 413–425.

[21] N. Ya. Vilenkin, On a class of complete orthonormal systems, Izv. Akad. Nauk. SSSR Ser. Math. 11 (1947) 363–400 (in Russian).

[22] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann. 77(3) (1916) 313–352.

Received on November 17, 2024
Revised on February 10, 2025
Accepted on February 13, 2025

Vassil Grozdanov

Department of Mathematics
Faculty of Natural Sciences and Mathematics
South West University "Neofit Rilski"
66, Ivan Michailov Str.
2700 Blagoevgrad
BULGARIA

E-mail: vassgrozdanov@yahoo.com

Elmi Shabani

Department of Business Management
Faculty of Business
University "Haxhi Zeka"
30000 Pejë
REPUBLIC OF KOSOVO

E-mail: elmishabani1@gmail.com

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

TWO PROPERTIES OF THE PARTIAL THETA FUNCTION

VLADIMIR PETROV KOSTOV

For the partial theta function $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, q, $z \in \mathbb{C}$, $|q| < 1$, we prove that its zero locus is connected. This set is smooth at every point $(q^\flat, z^\flat)$ such that $z^\flat$ is a simple or double zero of $\theta(q^\flat, .)$. For $q \in (0,1)$, $q \to 1^-$ and $a \geq e^\pi$, there are $o(1/(1-q))$ and $(\ln(a/e^\pi))/(1-q) + o(1/(1-q))$ real zeros of $\theta(q,.)$ in the intervals $[-e^\pi, 0)$ and $[-a, -e^{-\pi}]$ respectively (and none in $[0,\infty)$). For $q \in (-1,0)$, $q \to -1^+$ and $a \geq e^{\pi/2}$, there are $o(1/(1+q))$ real zeros of $\theta(q,.)$ in the interval $[-e^{\pi/2}, e^{\pi/2}]$ and $(\ln(a/e^{\pi/2})/2)/(1+q) + o(1/(1+q))$ in each of the intervals $[-a, -e^{\pi/2}]$ and $[e^{\pi/2}, a]$.

Keywords: partial theta function, separation in modulus, limit density of the real zeros

2020 Mathematics Subject Classification: 26A06

1. Introduction

1.1. Definition of the partial theta function

For $q \in \mathbb{D}_1$, $z \in \mathbb{C}$, where $\mathbb{D}_r$ stands for the open disk of radius r centered at $0 \in \mathbb{C}$, one defines the *partial theta function* by the formula

$$\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j. \tag{1.1}$$

This terminology is explained by the resemblance of formula (1.1) with the one defining the *Jacobi theta function* $\Theta(q,z) := \sum_{j=-\infty}^{\infty} q^{j^2} z^j$; the word “partial” refers to the summation in the case of θ taking place only over the nonnegative values of j. One has $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$. We consider q as a parameter and z as a variable. For each q fixed, $\theta(q,.)$ is an entire function.

DOI: 10.60063/GSU.FMI.112.77-91

The function θ has been studied as Ramanujan-type series in [21]. Its applications in statistical physics and combinatorics are explained in [20]. Other fields, where θ is used, are the theory of (mock) modular forms (see [4]) and asymptotic analysis (see [2]). Asymptotics, modularity and other properties of partial and false theta functions are considered in [5] with regard to conformal field theory and representation theory, and in [3] when asymptotic expansions of regularized characters and quantum dimensions of the $(1,p)$-singlet algebra modules are studied.

A recent impetus for the interest in θ (in the case when the parameter q is real) was given by the theory of *section-hyperbolic polynomials*, i.e., real univariate polynomials of degree ≥ 2 with all roots real negative and such that, when their highest-degree monomial is deleted, this gives again a polynomial having only real negative roots. The classical results of Hardy, Petrovitch and Hutchinson in this direction (see [6,7,19]) have been continued in [8,17,18]. Various analytic properties of θ are studied in [9–11,13–15]. See more about θ in [1].

1.2. The zero locus and the spectrum of the partial theta function

In the present paper we consider the zero locus of θ, i.e., the set $S := \{(q,z) \in \mathbb{D}_1 \times \mathbb{C}, \theta(q,z) = 0\}$. In Section 2 we prove the following theorem.

Theorem 1.1. *In the space $\mathbb{D}_1 \times \mathbb{C}$, the zero locus S is an irreducible hence connected analytic curve. It is smooth at every point $(q^\flat, z^\flat)$ such that $z^\flat$ is a simple or double zero of $\theta(q^\flat, .)$.*

Remarks 1.2. (1) B. Z. Shapiro has introduced the notion of *spectrum* of θ as the set of values of q for which $\theta(q,.)$ has a multiple zero, see [17]. Suppose that q is real, i.e., $q \in (-1,0) \cup (0,1)$ (the case $q = 0$ is of little interest since $\theta(0,z) \equiv 1$). If $q \in (0,1)$, then $\theta(q,.)$ has infinitely-many real zeros and they are all negative. There are also infinitely-many spectral numbers $0 < \tilde{q}_1 < \tilde{q}_2 < \cdots < \tilde{q}_k < \cdots < 1$, $\lim_{k\to\infty} \tilde{q}_k = 1^-$, see [9].

(2) For $q \in (0, \tilde{q}_1)$ (where $\tilde{q}_1 = 0.3092\ldots$), all zeros of $\theta(q,.)$ are real, negative and distinct: $\cdots < \xi_2 < \xi_1 < 0$; one has $\theta(q,x) > 0$ for $x \in (\xi_{2j+1}, \xi_{2j})$ and $\theta(q,x) < 0$ for $x \in (\xi_{2j}, \xi_{2j-1})$. For $q \in (\tilde{q}_k, \tilde{q}_{k+1}) \subset (0,1)$, $k \in \mathbb{N}$, $\tilde{q}_0 := 0$, the function $\theta(q,.)$ has exactly k pairs of complex conjugate zeros (counted with multiplicity). When $q \in (0,1)$ increases and passes through the spectral value $\tilde{q}_k$, the two zeros ξ_{2k-1} and ξ_{2k} coalesce and then form a complex conjugate pair, see [9]. The index j of the zero ξ_j is meaningful as long as ξ_j is real, i.e., for $q \in (0, \tilde{q}_{[(j+1)/2]}]$, where $[.]$ stands for "the integer part of".

(3) Asymptotic expansions of the numbers $\tilde{q}_k$ are obtained in [10] and [14]. The formula of [14] reads:

$$\tilde{q}_k = 1 - \pi/2k + (\ln k)/8k^2 + O(1/k^2), \quad \tilde{y}_k = -e^{\pi} e^{-(\ln k)/4k + O(1/k)} \tag{1.2}$$

where $e^{\pi} = 23.1407\ldots$ and $\tilde{y}_k < 0$ is the double zero of $\theta(\tilde{q}_k,.)$. It is the rightmost of its real zeros and $\theta(\tilde{q}_k,.)$ has a minimum at $\tilde{y}_k$.

(4) For $k \in \mathbb{N}^*$, one has $\theta(q, -q^{-k}) \in (0, q^k)$, see [9, Proposition 9]. For $q > 0$ small enough, one has $\operatorname{sgn}(\theta(q, -q^{-k-1/2})) = (-1)^k$ and $|\theta(q, -q^{-k-1/2})| > 1$, see [9, Proposition 12].

Remarks 1.3. (1) If $q \in (-1, 0)$ is sufficiently small, then $\theta(q, .)$ has infinitely-many real negative and infinitely-many real positive zeros: $\cdots < \xi_4 < \xi_2 < 0 < \xi_1 < \xi_3 < \cdots$; one has $\theta(q, x) < 0$ for $x \in (\xi_{4j+4}, \xi_{4j+2})$ and $x \in (\xi_{4j+1}, \xi_{4j+3})$, and $\theta(q, x) > 0$ for $x \in (\xi_{4j+2}, \xi_{4j})$, $x \in (\xi_{4j+3}, \xi_{4j+5})$ and $x \in (\xi_2, \xi_1)$. For $q \in (-1, 0)$, there are also infinitely-many spectral numbers, see [13]. We denote them by $\bar{q}_k$, where $-1 < \bar{q}_k < 0$.

(2) For $s \geq 1$, one has $-1 < \bar{q}_{2s+1} < \bar{q}_{2s-1} < 0$, see [13, Lemma 4.11]. For k sufficiently large, one has $\bar{q}_{k+1} < \bar{q}_k$, see [13, Lemmas 4.10, 4.11 and 4.17]. The inequality $\bar{q}_{k+1} < \bar{q}_k$ being proved only for k sufficiently large we admit the possibility finitely-many equalities of the form $\bar{q}_i = \bar{q}_j$ to hold true, where at least one of the numbers i and j is even.

(3) When $q \in (-1, 0)$ decreases and passes through a spectral value $\bar{q}_k$, then for $k = 2s - 1$ (resp. $k = 2s$), $s \in \mathbb{N}^*$, the zeros ξ_{4s-2} and ξ_{4s} (resp. ξ_{4s-1} and ξ_{4s+1}) coalesce. Thus for $q \in (\bar{q}_{k+1}, \bar{q}_k) \subset (-1, 0)$ and k sufficiently large, the function $\theta(q, .)$ has exactly k pairs of complex conjugate zeros (counted with multiplicity). The zero ξ_1 remains real and simple for any $q \in (-1, 0)$.

(4) Asymptotic expansions of the numbers $\bar{q}_k$ are found in [13]:

$$\bar{q}_k = 1 - (\pi/8k) + o(1/k), \quad |\bar{y}_k| = e^{\pi/2} + o(1), \tag{1.3}$$

where $\bar{y}_k$ is the double zero of $\theta(\bar{q}_k, .)$ and $e^{\pi/2} = 4.81477382\ldots$. For k odd (respectively, k even) $\theta(\bar{q}_k, .)$ has a local minimum (respectively, maximum) at $\bar{y}_k$, and $\bar{y}_k$ is the rightmost of the real negative zeros of $\theta(\bar{q}_k, .)$ (respectively, for k sufficiently large, $\bar{y}_k$ is the second from the left of the real positive zeros of $\theta(\bar{q}_k, .)$).

Remarks 1.4. (1) All coefficients in the series (1.1) are real. Hence a priori spectral numbers are either real or they form complex conjugate pairs. It is proved in [15] that there exists at least one such pair which equals $0.4353184958\cdots \pm i0.1230440086\ldots$. Numerical results suggest that one should expect there to be infinitely-many such pairs.

(2) In any set of the form $\mathbb{D}_r \setminus \{0\}$, $r \in (0, 1)$, the number of spectral values of θ is finite (because the spectrum is locally a codimension 1 analytic subset in $\mathbb{D}_1 \setminus \{0\}$). For any spectral number q, the function $\theta(q, .)$ has finitely-many multiple zeros, see [11]. The number $\tilde{q}_1 = 0.3092\ldots$ is the only spectral number of θ in the disk $\overline{\mathbb{D}_{0.31}}$.

(3) For all spectral numbers $\tilde{q}_j \in (0, 1)$ and, for k sufficiently large, for all spectral numbers $\bar{q}_k \in (-1, 0)$, it is true that the function $\theta(\tilde{q}_j, .)$, resp. $\theta(\bar{q}_k, .)$, has exactly one double real zero while all its other real zeros are simple (see [9] and [13]). It would be interesting to prove (or disprove) that for any spectral value (real or complex) the partial theta function has just one double zero all its other zeros being simple. If true, this would mean in particular (see Theorem 1.1) that S is globally smooth and connected. If false, it would be of interest to describe the eventual singularities of S.

1.3. The limit distribution of the real zeros

In the present subsection we consider the case $q \in \mathbb{R}$, i.e., $q \in (-1,0) \cup (0,1)$.

Notation 1.5. For $q \in (-1,0) \cup (0,1)$, given a finite interval $J \subset \mathbb{R}$, we denote by $Z_J(q)$ the number of zeros of $\theta(q,.)$ (counted with multiplicity) belonging to J. For $q \in (0,1)$ and $a \geq e^{\pi}$, we set $\ell_a(q) := Z_{[-a,-e^{\pi}]}(q)$. For $q \in (-1,0)$ and $a \geq e^{\pi/2}$, we set $n_a(q) := Z_{[-a,-e^{\pi/2}]}(q)$ and $p_a(q) := Z_{[e^{\pi/2},a]}(q)$.

Theorem 1.6. (1) *For $q \in (0,1)$, one has $Z_{[-e^{\pi},0)}(q) = o(1/(1-q))$.*

(2) *The set of zeros of $\theta(\tilde{q}_k,.)$ (over all $k \in \mathbb{N}^*$) is everywhere dense in $(-\infty, -e^{\pi}]$. One has $\lim_{q\to 1^-} \ell_a(q)(1-q) = \ln(a/e^{\pi})$.*

(3) *For $q \in (-1,0)$, one has $Z_{[-e^{\pi/2},0)}(q) = o(1/(1+q))$.*

(4) *The set of zeros of $\theta(\bar{q}_{2s-1},.)$ (over all $s \in \mathbb{N}^*$) is everywhere dense in $(-\infty, -e^{\pi/2}]$. One has $\lim_{q\to -1^+} n_a(q)(1+q) = \ln(a/e^{\pi/2})/2$.*

(5) *For $q \in (-1,0)$, one has $Z_{(0,e^{\pi/2}]}(q) = o(1/(1+q))$.*

(6) *The set of zeros of $\theta(\bar{q}_{2s},.)$ (over all $s \in \mathbb{N}^*$) is everywhere dense in $[e^{\pi/2}, \infty)$. One has $\lim_{q\to -1^+} p_a(q)(1+q) = \ln(a/e^{\pi/2})/2$.*

The theorem is proved in Section 3.

Remark 1.7. The quantity $1/a = \lim_{\varepsilon\to 0^+}((\ln((a+\varepsilon)/e^{\pi}) - \ln(a/e^{\pi})/\varepsilon)$ can be interpreted as limit density of the real zeros of $\theta(q,.)$ as $q \to 1^-$ at the point $-a \leq -e^{\pi}$. Similarly for the quantity $1/(2a)$ at $\pm a$, $a \geq e^{\pi/2}$, as $q \to -1^+$. For the rest of the real line the limit density is 0. Indeed, for $q \in (0,1)$, there are no nonnegative zeros of $\theta(q,.)$; for $0 < a < e^{\pi}$, see part (1) of Theorem 1.6. For $q \in (-1,0)$, see parts (3) and (5) of Theorem 1.6.

2. Proof of Theorem 1.1

2.1. Smoothness

We prove the smoothness first. If $z^\flat$ is a simple zero of $\theta(q^\flat,.)$, then $(\partial\theta/\partial z)(q^\flat, z^\flat) \neq 0$ hence

$$\mathrm{Grad}(\theta)(q^\flat, z^\flat) \neq 0 \tag{2.1}$$

and S is smooth at $(q^\flat, z^\flat)$. The function θ satisfies the following differential equation (see (1.1)):

$$2q(\partial\theta/\partial q) = z(\partial^2/\partial z^2)(z\theta). \tag{2.2}$$

The right-hand side equals $2z(\partial\theta/\partial z) + z^2(\partial^2\theta/\partial z^2)$. If $z^\flat$ is a double zero of $\theta(q^\flat,.)$, then

$$\theta(q^\flat, z^\flat) = (\partial\theta/\partial z)(q^\flat, z^\flat) = 0 \neq (\partial^2\theta/\partial z^2)(q^\flat, z^\flat).$$

One has neither $q^\flat = 0$, because $\theta(0,.) \equiv 1 \neq 0$, nor $z^\flat = 0$, because $\theta(q,0) \equiv 1$. Hence

$$(\partial\theta/\partial q)(q^\flat, z^\flat) = ((z^\flat)^2/2q^\flat)(\partial^2\theta/\partial z^2)(q^\flat, z^\flat) \neq 0.$$

Therefore one has (2.1), so S is smooth at $(q^\flat, z^\flat)$.

2.2. Separation in modulus

For fixed $q \in \mathbb{D}_1 \setminus \{0\}$, we denote by $\mathcal{C}_k$, $k \in \mathbb{N}^*$, the circumference in the z-space $|z| = |q|^{-k-1/2}$. When q is close to 0, one can enumerate the zeros of θ, because there exists exactly one zero such that $\xi_k \sim -q^{-k}$ (see [9, Proposition 10]). For $0 < |q| \leq c_0 := 0.2078750206\ldots$, one has

$$|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2}, \tag{2.3}$$

see [15, Lemma 1]. In this sense we say that for $q \in \mathbb{D}_{c_0} \setminus \{0\}$, the zeros of θ are separated in modulus (that is, their moduli are separated by the circumferences $\mathcal{C}_k$). We say that, for given q, *strong separation* of the zeros of θ takes place for $k \geq k_0$, if for any $k \geq k_0$, there exists exactly one zero ξ_k of θ satisfying conditions (2.3).

Set $\alpha_0 := \sqrt{3}/2\pi = 0.2756644477\ldots$. The following result can be found in [15].

Theorem 2.1. *For $n \geq 5$ and for $|q| \leq 1 - 1/(\alpha_0 n)$, strong separation of the zeros of θ takes place for $k \geq n$.*

Theorem 2.1 has several important corollaries:

i) For each path $\gamma \subset \mathbb{D}_1 \setminus \{0\}$ in the q-space which avoids the spectral numbers of θ, one can define by continuity the zeros of θ as functions of q as q varies along γ. One can find $k \in \mathbb{N}$ such that $\gamma \subset \mathbb{D}_{1-1/(\alpha_0 k)}$. For $n \geq k$, the zero ξ_n is an analytic function in $q \in \mathbb{D}_{1-1/(\alpha_0 k)}$. Thus the indices of the zeros ξ_n are meaningful for $n \geq k$ and $q \in \mathbb{D}_{1-1/(\alpha_0 k)}$.

ii) Denote by Γ the spectrum of θ. If $\gamma \subset D := \mathbb{D}_{1-1/(\alpha_0 k)} \setminus \{\Gamma \cup \{0\}\}$ is a loop, then the zeros of θ lying inside $\mathcal{C}_k$ might undergo a *monodromy* as q varies along γ, i.e., a permutation which depends on the class of homotopy equivalence of γ in D. Therefore it might not be possible to correctly define the indices of these zeros for $q \in D$.

iii) For no $q_* \in \mathbb{D}_1 \setminus \{0\}$ does a zero of θ go to infinity as $q \to q_*$. That is, zeros are not born and do not disappear at infinity.

iv) For $(0,1) \ni q = \tilde{q}_j \in \Gamma$, the function $\theta(q,.)$ has one double zero and infinitely-many simple zeros, see part (3) of Remarks 1.4 and part (3) of Remarks 1.2. The double zero is a Morse critical point for θ. Suppose that γ is a small loop in $\mathbb{D}_1 \setminus \{0\}$ circumventing $\tilde{q}_j$. Then the two zeros ξ_{2j-1} and ξ_{2j} of $\theta(q,.)$ which coalesce for $q = \tilde{q}_j$ are exchanged as q varies along γ. For $(-1,0) \ni q = \bar{q}_k \in \Gamma$, $k = 2s-1$ or $2s$, where $s \geq 1$ is sufficiently large, the same remark applies to the zeros ξ_{4s-2} and ξ_{4s} or ξ_{4s-1} and

ξ_{4s+1}, see part (3) of Remarks 1.3. For the remaining values of k, if, say, p spectral values $\bar{q}_i$ coincide, then the function $\theta(\bar{q}_i, .)$ has p double real zeros (its other real zeros are simple) and the monodromy defined by the class of homotopy equivalence of γ exchanges the zeros in p non-intersecting couples of zeros (which are close to the double zeros of $\theta(\bar{q}_i, .)$).

v) Theorem 2.1 implies that the monodromy around $0 \in \mathbb{D}_1$ is trivial.

2.3. Connectedness of S

When $q \in \mathbb{D}_{c_0} \setminus \{0\}$, the zeros ξ_j can be considered as analytic functions in q. We discuss the possible monodromies which they can undergo when the parameter q runs along certain loops in $\mathbb{D}_1 \setminus \{0\}$. First of all we recall that for $q \in (0, \tilde{q}_j)$, the zeros $0 > \xi_{2j-1} > \xi_{2j} > \xi_{2j+1} > \cdots$ are simple, real negative and continuously depending on q, see part (2) of Remarks 1.2; for $q = \tilde{q}_j$, the zeros ξ_{2j-1} and ξ_{2j} coalesce.

Suppose that $a \in (0, c_0)$ and that $\mathcal{C}^\sharp \subset \mathbb{D}_1 \setminus \{0\}$ is a small circumference of radius ε centered at the spectral number $\tilde{q}_j$, see parts (1) and (2) of Remarks 1.2; no spectral number other than $\tilde{q}_j$ belongs to the circumference $\mathcal{C}^\sharp$ or to its interior. Define $\gamma_j \subset \mathbb{D}_1 \setminus \{0\}$ as the path consisting of the segment $\sigma_+ := [a, \tilde{q}_j - \varepsilon] \subset \mathbb{R}$, the circumference $\mathcal{C}^\sharp$ (which is run, say, counterclockwise) and the segment $\sigma_- := [\tilde{q}_j - \varepsilon, a]$. Hence if one considers the analytic continuation of the function ξ_{2j-1} (resp. ξ_{2j}) along the loop γ_j, the result will be the function ξ_{2j} (resp. ξ_{2j-1}), see *iv)* in Subsection 2.2. We denote this symbolically by $\gamma_j : \xi_{2j-1} \leftrightarrow \xi_{2j}$. If we need to indicate only the image of ξ_{2j-1} we might write $\gamma_j : \xi_{2j-1} \mapsto \xi_{2j}$.

Remark 2.2. For $j > 1$, the two segments $\sigma_\pm$ of the path γ_j pass through the spectral numbers $\tilde{q}_1, \ldots, \tilde{q}_{j-1}$. If one insists the path γ_j to bypass all spectral numbers $\tilde{q}_j$, then one should modify γ_j. Namely, parts of the two segments $\sigma_\pm$ which are segments of the form $\sigma_s := [\tilde{q}_s - \varepsilon', \tilde{q}_s + \varepsilon']$, $0 < \varepsilon' \ll \varepsilon$, $1 \leq s \leq j-1$, should be replaced by small half-circumferences with diameters σ_s which bypass the spectral numbers $\tilde{q}_s$ from above or below.

Suppose that $q \in (-1, 0)$. We will make use of Remarks 1.3. We construct a path δ_j consisting of a segment $\tau_- := [-a, \bar{q}_j + \varepsilon] \subset \mathbb{R}$, $-c_0 < -a < 0$ (c_0 is defined at the beginning of Subsection 2.2), a circumference $\mathcal{C}^\triangle \subset \mathbb{D}_1 \setminus \{0\}$ of radius ε centered at $\bar{q}_j$ (and run, say, counterclockwise) and the segment $\tau_+ := [\bar{q}_j + \varepsilon, -a]$. If $\bar{q}_{j_1} \neq \bar{q}_j$, then the spectral number $\bar{q}_{j_1}$ does not belong to $\mathcal{C}^\triangle$ or to its interior.

Suppose that $j = 2s-1$ (resp. $j = 2s$). If one considers the analytic continuation of the functions ξ_{4s-2} and ξ_{4s} (resp. of ξ_{4s-1} and ξ_{4s+1}) along the loop δ_j, the result will be that the functions ξ_{4s-2} and ξ_{4s} (resp. ξ_{4s-1} and ξ_{4s+1}) exchange their values, see iv) in Subsection 2.2. We denote this symbolically by $\delta_{2s-1} : \xi_{4s-2} \leftrightarrow \xi_{4s}$ or $\delta_{2s} : \xi_{4s-1} \leftrightarrow \xi_{4s+1}$.

Remark 2.3. Similarly to what was done with the path γ_j, see Remark 2.2, one can modify the path δ_j so that it should pass through no spectral value of θ. We do not claim, however, that an equality of the form $\bar{q}_{j_1} = \bar{q}_{j_2}$, $j_1 \neq j_2$, does not take place (this is not proved in [13]; see part (2) of Remarks 1.3). Nevertheless, even if such an equality holds true, then it does not affect our reasoning, because when q runs along $\mathcal{C}^{\triangle}$ close to the coinciding spectral numbers $\bar{q}_{j_1}$ and $\bar{q}_{j_2}$, the exchange of zeros ξ_i which occurs concerns two couples of zeros with no zero in common.

By combining the monodromies defined by the paths γ_j and δ_j one can obtain any monodromy $\xi_k \mapsto \xi_m$. Indeed, denote by η_+ a half-circumference centered at 0, of radius a, belonging to the upper half-plane (hence the segment $[-a, a]$ is its diameter) and run counterclockwise, by η_- the same half-circumference run clockwise, by $\gamma_j\gamma_\ell$ the concatenation of the paths γ_j and γ_ℓ (defined for one and the same value of a, γ_j is followed by γ_ℓ) and similarly for the loops (all with base point a) $\gamma_j\eta_+\delta_s\eta_-$, $\eta_+\delta_s\eta_-\gamma_j$, etc. Thus for $s \geq 1$, one obtains the monodromies

$$\begin{aligned}
\gamma_{2s-1} &: \xi_{4s-3} \leftrightarrow \xi_{4s-2},\\
\gamma_{2s} &: \xi_{4s-1} \leftrightarrow \xi_{4s},\\
\delta_{2s-1} &: \xi_{4s-2} \leftrightarrow \xi_{4s},\\
\delta_{2s} &: \xi_{4s-1} \leftrightarrow \xi_{4s+1},\\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_- &: \xi_{4s-3} \mapsto \xi_{4s},\\
\eta_+\delta_{2s-1}\eta_-\gamma_{2s-1} &: \xi_{4s} \mapsto \xi_{4s-3},\\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_-\gamma_{2s} &: \xi_{4s-3} \mapsto \xi_{4s-1},\\
\gamma_{2s-1}\eta_+\delta_{2s-1}\eta_-\gamma_{2s}\eta_+\delta_{2s}\eta_- &: \xi_{4s-3} \mapsto \xi_{4s+1}, \text{ etc.}
\end{aligned}$$

This means that, for suitably chosen loops, the root ξ_{4s-3} can be mapped by the corresponding monodromies into any of the roots ξ_{4s-2}, ξ_{4s-1}, ξ_{4s} or ξ_{4s+1}. After this one can repeat the reasoning with $\xi_{4s+1} = \xi_{4(s+1)-3}$ (i.e., one can shift the value of s by 1) and so on.

Thus the subset S^0 of S on which all zeros of θ are simple is connected. The set $S \setminus S^0$ belongs to the topological closure of S (because the zeros of θ depend continuously on q), so S is connected. The theorem is proved.

3. Proof of Theorem 1.6

Part (1). In the proof of parts (1) and (2) of the theorem, when considering the values of q from an interval of the form $(\tilde{q}_k, \tilde{q}_{k+1})$, we take into account the first of formulae (1.2), so as q tends to 1^- (hence k tends to ∞) one has $1 - q = O(1/k)$. We prove first the following lemma.

Lemma 3.1.

(1) *For every* $r \in (0, 1)$, *there exists* $K_r \in \mathbb{N}$ *such that for every* $q \in (0, r]$, *one has* $Z_{[-e^\pi, 0)}(q) \leq K_r$.

(2) *When the zeros* ξ_{2s-1} *and* ξ_{2s} *are real (see part (2) of Remarks 1.2), they belong to the interval* $(-q^{-2s}, -q^{-2s+1})$.

(3) *For* $q \in [\tilde{q}_k, \tilde{q}_{k+1})$, *one has* $Z_{[-e^\pi,0)}(q) = o(k)$.

Proof. Part (1). It follows from part (4) of Remarks 1.2 that for $q > 0$ small enough, all zeros ξ_j of $\theta(q,.)$ are real and the zeros ξ_{2s-1} and ξ_{2s} belong to the interval $(-q^{-2s}, -q^{-2s+1})$, so they are smaller than $-r^{-2s+1}$. And in the same way, for any $q \in (0,1)$, the zeros ξ_{2s-1} and ξ_{2s}, when they are real, belong to the interval $(-q^{-2s}, -q^{-2s+1})$ (which proves part (2)).

When q increases and becomes equal to $\tilde{q}_s$, the zeros ξ_{2s-1} and ξ_{2s} coalesce. For $q > \tilde{q}_s$, they form a complex conjugate pair, see part (2) of Remarks 1.2. For $q \in (0, r]$ and $2s - 1 > \pi/\ln(1/r)$, i.e., $q^{-2s+1} \geq r^{-2s+1} > e^\pi$ hence $-q^{-2s+1} < -e^\pi$, the zero ξ_j, $j \geq 2s-1$, is either smaller than $-e^\pi$ or it has given birth (together with ξ_{j-1} or ξ_{j+1} depending on the parity of j) to a complex conjugate pair. Therefore $Z_{[-e^\pi,0)}(q) \leq [\pi/\ln(1/r)] + 1$ and one can set $K_r := [\pi/\ln(1/r)] + 1$.

Part (3). Suppose first that $q = \tilde{q}_k$. The interval $I := [-e^\pi, \tilde{y}_k]$ contains all real zeros of $\theta(\tilde{q}_k, .)$ belonging to the interval $J := [-e^\pi, 0)$. The rightmost of these zeros which is in I is the double zero $\tilde{y}_k$ which is the result of the confluence of ξ_{2k-1} and ξ_{2k}, see parts (2) and (3) of Remarks 1.2. Denote by s_0 the smallest of the numbers s for which $-(\tilde{q}_k)^{-2s+1} < -e^\pi$. Hence there are not more than

$$t_0 := 2(s_0 - 1) - 2(k-1) + 1 = 2(s_0 - k) + 1$$

real zeros of $\theta(\tilde{q}_k, .)$ in I (counted with multiplicity), see the proof of part (1) of the present lemma. One has $\tilde{q}_k = 1 - \pi/2k + o(1/k)$, see (1.2). Therefore

$$-(\tilde{q}_k)^{-2s_0+1} < -e^\pi \;\Leftrightarrow\; (-2s_0+1)\ln(\tilde{q}_k) > \pi \;\Leftrightarrow\; 2s_0 - 1 > \pi/(\ln(1/\tilde{q}_k)) = 2k + o(k).$$

On the other hand, it follows from the definition of s_0 that $2s_0 - 3 \leq \pi/(\ln(1/\tilde{q}_k)) = 2k + o(k)$. Thus $s_0 = k + o(k)$ and $t_0 = o(k)$. Suppose now that $q \in (\tilde{q}_k, \tilde{q}_{k+1})$. Hence when one counts the real zeros of $\theta(q,.)$ in J, one should take into account that:

1) The double root $\tilde{y}_k$ gives birth to a complex conjugate pair of zeros, i.e., two real zeros are lost; for $q \in (\tilde{q}_k, \tilde{q}_{k+1})$, these are the only real zeros that are lost, see part (2) of Remarks 1.2;

2) Denote by $s_*(q)$ the smallest of the numbers s for which one has $-q^{-2s+1} < -e^\pi$ (hence $s_*(\tilde{q}_k) = s_0$). For fixed s, the number $-q^{-2s+1}$ increases with q, so $s_*(q)$ also increases, i.e., new real zeros might enter the interval J from the left.

Thus for $q \in (\tilde{q}_k, \tilde{q}_{k+1})$, one has $Z_J(q) \leq t_1 + 2$, where t_1 is the quantity t_0 defined for $k+1$ instead of k, hence $Z_J(q) = o(k)$. Indeed, the numbers $-q^{-2s+1}$ increase with q. We cannot claim that if for $s = s_*(q) - 1$, one has $-q^{-2s+1} \geq -e^\pi$, then the zeros ξ_{2s-1} and ξ_{2s} are larger or smaller than $-e^\pi$; this is why 2 is added to t_1. □

The proof of part (1) of Theorem 1.6 results from part (3) of Lemma 3.1. Indeed, one has $k = O(1/(1-\tilde{q}_k))$, see (1.2). □

Part (2). The function θ satisfies the following functional equation

$$\theta(q,x) = 1 + qx\theta(q,qx). \tag{3.1}$$

For $q = \tilde{q}_k \in \Gamma$, we denote by $\cdots < x_2 < x_1 < x_0 < 0$ the numbers $x_0 = \tilde{y}_k$, $x_s = x_{s-1}/\tilde{q}_k$, $s \in \mathbb{N}$ (i.e., $x_s = \tilde{y}_k/(\tilde{q}_k)^s$). Hence $\theta(\tilde{q}_k, x_0) = 0$, $\theta(\tilde{q}_k, x_1) = 1 + x_0\theta(\tilde{q}_k, x_0) = 1$ (see (3.1)), and for $s > 1$,

(i) if $\theta(\tilde{q}_k, x_s) < 0$, then $\theta(\tilde{q}_k, x_{s+1}) = 1 + x_s\theta(\tilde{q}_k, x_s) > 1$;

(ii) if $\theta(\tilde{q}_k, x_s) \geq 1$ (this is the case for $s = 1$), then for k sufficiently large, one has $x_s < -e^\pi/2$ (see (1.2)), $\tilde{q}_k \in (0.3, 1)$ (see parts (1) and (2) of Remarks 1.2) hence $\tilde{q}_k x_s < -0.3 \times e^\pi/2 < -3$ and

$$\theta(\tilde{q}_k, x_{s+1}) = 1 + \tilde{q}_k x_s \theta(\tilde{q}_k, x_s) < 1 - 3 = -2 < 0.$$

Thus for k sufficiently large, we have $\theta(\tilde{q}_k, x_s) < 0$ for $s \geq 2$ even and $\theta(\tilde{q}_k, x_s) > 0$ for $s \geq 3$ odd. Hence each interval (x_{s+1}, x_s) contains a zero of θ. For a fixed interval $[-a, -e^\pi]$, consider the intervals (x_{s+1}, x_s) which are its subintervals. As $k \to \infty$ (hence $\tilde{q}_k \to 1^-$) the lengths of these intervals tend uniformly to 0. Indeed, the largest of them is the last one and its length is $\leq (a - aq) = (1-q)a$. Therefore for any $a > e^\pi$, the set of zeros of $\theta(\tilde{q}_k, .)$ (over all k sufficiently large) is everywhere dense in the interval $[-a, -e^\pi]$. This proves the first claim of part (2) of the theorem. To prove the second one we first consider the case $q = \tilde{q}_k \in \Gamma$. We define the quantities u_0, $u_1 \in \mathbb{N}$ by the conditions

$$|\tilde{y}_k|/q^{u_0} = |x_{u_0}| \leq e^\pi < |x_{u_0+1}| = |\tilde{y}_k|/q^{u_0+1},$$

and

$$|\tilde{y}_k|/q^{u_1} = |x_{u_1}| \leq a < |x_{u_1+1}| = |\tilde{y}_k|/q^{u_1+1}.$$

Hence (remember that $\ln q < 0$)

$$(u_0+1)\ln q < \ln(|\tilde{y}_k|/e^\pi) \leq u_0 \ln q \quad \text{and} \quad (u_1+1)\ln q < \ln(|\tilde{y}_k|/a) \leq u_1 \ln q$$

which, taking into account that as $q \to 1^-$, implies $\ln q = \ln(1 + (q-1)) = (q-1) + o(q-1)$, one has

$$u_0(q-1) = \ln(|\tilde{y}_k|/e^\pi) + o(q-1) \quad \text{and} \quad u_1(q-1) = \ln(|\tilde{y}_k|/a) + o(q-1).$$

It is clear that $\ell_a(q) = u_1 - u_0 + O(1)$. Thus

$$\ell_a(q)(1-q) = (u_1 - u_0)(1-q) + O(1)(1-q) = \ln(a/e^\pi) + O(1-q).$$

Now suppose that $q \in (\tilde{q}_k, \tilde{q}_{k+1})$. Our reasoning is similar to the one in the proof of Lemma 3.1. The double zero $\tilde{y}_k$ gives birth to a complex conjugate pair, so two real zeros are lost. If for $q = q_* \in (\tilde{q}_k, \tilde{q}_{k+1})$, the interval $(-q_*^{-2s}, -q_*^{-2s+1})$ is a subset of the interval $[-a, 0)$, then the same is true for $q = \tilde{q}_{k+1}$. Thus

$$\ell_a(q_*) \leq Z_{[-a,0)}(\tilde{q}_{k+1}) + 2. \tag{3.2}$$

One adds 2 in order to take into account the two zeros of $\theta(\tilde{q}_{k+1},.)$ of the not more than one interval $(-q_*^{-2s},-q_*^{-2s+1})$ which belongs partially, but not completely, to $[-a,0)$. The number 2 of lost zeros and the number 2 in (3.2) are $o(1/(1-q_*))$. According to part (1) of the theorem

$$Z_{[-a,0)}(\tilde{q}_{k+1})=\ell_a(\tilde{q}_{k+1})+o(1/(1-q_*)),$$

and for $q=\tilde{q}_{k+1}$, it was shown that $(1-\tilde{q}_{k+1})\ell_a(\tilde{q}_{k+1})=\ln(a/e^{\pi})+o(1)$, so $\ell_a(q_*)=\ell_a(\tilde{q}_{k+1})+o(1/(1-q_*))$ and $(1-q_*)\ell_a(q_*)=\ln(a/e^{\pi})+o(1)$ which proves part (2) of the theorem. □

Part (3). We need the following lemma.

Lemma 3.2. *Suppose that $q\in(-1,0)$ and set $\rho:=|q|$. Then:*

(1) *For $\rho>0$ small enough, one has*

$$\begin{array}{lll}\xi_{4s}\in(-\rho^{-4s-1},-\rho^{-4s+1}), & & \xi_{4s+2}\in(-\rho^{-4s-3},-\rho^{-4s-1}),\\ \xi_{4s-1}\in(\rho^{-4s+2},\rho^{-4s}), & \textit{and} & \xi_{4s+1}\in(\rho^{-4s},\rho^{-4s-2}).\end{array}\tag{3.3}$$

Moreover, the mentioned zeros ξ_j are the only zeros of $\theta(q,.)$ in the indicated intervals.

(2) *For $\rho\in(0,1)$, one has $\theta(q,-q^{-2k})=\theta(q,-\rho^{-2k})\in(0,\rho^{2k}+\rho^{4k+1})$.*

(3) *For $q\in[\bar{q}_{2s-1},0)$, the zeros ξ_{4s-2} and ξ_{4s} belong to the interval $I^{\bullet}:=(-\rho^{-4s},-\rho^{-4s+2})$.*

Proof. Part (1). We consider the following four series:

$$\theta^{\diamond}:=\theta(-\rho,-\rho^{-4s+1})=\sum_{j=0}^{\infty}d_j,\quad d_j:=(-1)^{j(j+3)/2}\rho^{-(4s-1)j+j(j+1)/2},$$

$$\theta^{\nabla}:=\theta(-\rho,-\rho^{-4s-1})=\sum_{j=0}^{\infty}h_j,\quad h_j:=(-1)^{j(j+3)/2}\rho^{-(4s+1)j+j(j+1)/2},$$

$$\theta^{\heartsuit}:=\theta(-\rho,\rho^{-4s})=\sum_{j=0}^{\infty}r_j,\qquad r_j:=(-1)^{j(j+1)/2}\rho^{-4sj+j(j+1)/2},\quad\text{and}$$

$$\theta^{\star}:=\theta(-\rho,\rho^{-4s+2})=\sum_{j=0}^{\infty}\lambda_j,\qquad \lambda_j:=(-1)^{j(j+1)/2}\rho^{-(4s-2)j+j(j+1)/2}.$$

For the first series, its terms of largest modulus are d_{4s-1} and d_{4s-2}; one has $d_{4s-1}=d_{4s-2}=-\rho^{-8s^2+6s-1}$. The moduli of the terms decrease rapidly as $j>4s-1$ increases or as $j<4s-2$ decreases. In this series the sign $(-1)^{j(j+3)/2}$ is positive for $j=4\nu$ and $j=4\nu+1$ and negative for $j=4\nu+2$ and $j=4\nu+3$. Hence for ρ small enough, the sign of $\theta^{\diamond}$ is the same as the one of $d_{4s-1}+d_{4s-2}$, i.e., one has $\theta^{\diamond}<0$.

For the other three series the largest modulus terms are respectively $h_{4s} = h_{4s+1} = \rho^{-8s^2-2s} > 0$, $r_{4s-1} = r_{4s} = \rho^{-8s^2+2s} > 0$ and $\lambda_{4s-3} = \lambda_{4s-2} = -\rho^{-8s^2+10s-3} < 0$, so in the same way $\theta^\nabla > 0$, $\theta^\heartsuit > 0$ and $\theta^\star < 0$. Hence there is at least one zero of θ in the interval $(-\rho^{-4s-1}, -\rho^{-4s+1})$. In fact, there is exactly one zero, and this is ξ_{4s}. Indeed, for ρ small enough this is true, because one has $\xi_m \sim -q^{-m}$, see [12] (the zeros ξ_{4s-1} and ξ_{4s+1} are positive, so only ξ_{4s} belongs to $(-\rho^{-4s-1}, -\rho^{-4s+1})$). For any $\rho \in (0, 1)$, this follows from the fact that as ρ increases, new complex conjugate pairs are born, but the inverse does not take place, see part (2) of Remarks 1.2. In the same way one proves the rest of part (1) of the lemma.

Part (2). One checks directly that

$$\begin{aligned}\theta(q, -\rho^{-2k}) &= \sum_{j=0}^{\infty} q^{j(j+1)/2}(-\rho^{-2k})^j = \sum_{j=0}^{\infty}(-1)^{j(j+3)/2}\rho^{-2kj+j(j+1)/2} \\ &= \sum_{j=4k}^{\infty}(-1)^{j(j+3)/2}\rho^{-2kj+j(j+1)/2}.\end{aligned}$$

The last of these equalities follows from the fact that the first $4k$ terms of the series cancel (the first with the $(4k)$th, the second with the $(4k-1)$st, etc.). The signs of the terms of the last of these series are $+, +, -, -, +, +, -, -, \cdots$ and the exponents $-2kj + j(j+1)/2$ are increasing for $j \geq 4k$. Hence the series is the sum of two Leibniz series with positive first terms, so its sum is positive and not larger than the sum of the first terms of these two series. The latter sum is $\rho^{2k} + \rho^{4k+1}$ which proves part (2).

Part (3). For ρ sufficiently small, the zeros ξ_{4s-2} and ξ_{4s} belong to $I^\bullet$. Indeed, by part (2) of the present lemma, at the endpoints of $I^\bullet$ the function $\theta(q, .)$ is positive while it is negative at $-\rho^{-4s+1}$ (we showed already that $\theta^\diamond < 0$). As $\theta(q, .)$ is positive at the endpoints for any $q \in (-1, 0)$, the zeros ξ_{4s-2} and ξ_{4s} belong to $I^\bullet$ exactly for $q \in [\bar{q}_{2s-1}, 0)$, see part (3) of Remarks 1.3. This proves part (3) of Lemma 3.2. □

Suppose first that $q = \bar{q}_{2\nu-1}$, $\nu \in \mathbb{N}$. The rightmost of the negative zeros of $\theta(\bar{q}_{2\nu-1}, .)$ is the double zero $\bar{y}_{2\nu-1} = \xi_{4\nu-2} = \xi_{4\nu}$, see part (3) of Remarks 1.3. Denote by $s^\dagger = s^\dagger(\bar{q}_{2s-1})$ the largest of the numbers $s \in \mathbb{N}$ for which one has $-(\bar{q}_{2\nu-1})^{-4s} \geq -e^{\pi/2}$. Hence the zero $\xi_{4s^\dagger}$ is in the interval $[-e^{\pi/2}, 0)$ and the zero $\xi_{4(s^\dagger+1)}$ is to its left, i.e., outside it. Thus the number $\tilde{N}(\bar{q}_{2\nu-1}) := Z_{[-e^{\pi/2},0)}(\bar{q}_{2\nu-1})$ (the zeros in $[-e^{\pi/2}, 0)$ have only even indices i, see Remarks 1.3) is

$$\tilde{N}(\bar{q}_{2\nu-1}) = (4s^\dagger - 4\nu + 2)/2 + u = 2(s^\dagger - \nu) + 1 + u,$$

where $u \leq 1$ (the presence of the number u reflects the fact that we do not say whether the zero $\xi_{4s^\dagger+2}$ belongs or not to the interval $[-e^{\pi/2}, 0)$). The conditions

$$-(\bar{q}_{2\nu-1})^{-4(s^\dagger+1)} < -e^{\pi/2} \leq -(\bar{q}_{2\nu-1})^{-4(s^\dagger)}$$

are equivalent to $-4(s^\dagger+1)\ln|\bar{q}_{2\nu-1}|>\pi/2\geq -4s^\dagger\ln|\bar{q}_{2\nu-1}|$ or to

$$\begin{cases} 4(s^\dagger+1)>(\pi/2)/(\ln(1/|\bar{q}_{2\nu-1}|))=(\pi/2)/(\ln(1+\pi/(8(2\nu-1))+o(1/\nu))) \\ \qquad\qquad =4\nu+O(1), \\ \qquad 4s^\dagger\leq(\pi/2)/(\ln(1/|\bar{q}_{2\nu-1}|)), \end{cases}$$

see the first of formulae (1.3). Thus

$$s^\dagger=\nu+O(1) \quad \text{and} \quad \tilde{N}(\bar{q}_{2\nu-1})=O(1). \tag{3.4}$$

One can also write $\tilde{N}(\bar{q}_{2\nu-1})=o(\nu)=o(1/(1+\bar{q}_{2\nu-1}))$. Hence $\tilde{N}(\bar{q}_{2\nu+1})=o(\nu)$.

Now suppose that $q\in(\bar{q}_{2\nu+1},\bar{q}_{2\nu-1})$. When counting the zeros ξ_i in the interval $[-e^{\pi/2},0)$ one takes into account that the double zero $\xi_{4\nu-2}=\xi_{4\nu}$ is lost (it gives birth to a complex conjugate pair). The numbers $-\rho^{-4s}$ (which are left endpoints of intervals $I^\bullet$) increase, so new zeros ξ_i might enter the interval $[-e^{\pi/2},0)$ from the left. The number of such intervals $I^\bullet$ which belong entirely to $[-e^{\pi/2},0)$ is not greater than their number for $q=\bar{q}_{2\nu+1}$. There is at most one interval $I^\bullet$ which belongs only partially to $[-e^{\pi/2},0)$, so ignoring it means not counting at most 2 zeros $\xi_i\in[-e^{\pi/2},0)$. Therefore $\tilde{N}(q)=\tilde{N}(\bar{q}_{2\nu+1})+O(1)=o(\nu)=o(1/(1+q))$. Part (3) of Theorem 1.6 is proved. □

Part (4). Consider an interval of the form $[-a,-e^{\pi/2}]$ and its subinterval $(-a^*,-a^\triangle)$, $e^{\pi/2}<a^\triangle<a^*<a$. For $\nu\in\mathbb{N}$ sufficiently large, the double zero $\bar{y}_{2\nu-1}=\xi_{4\nu-2}=\xi_{4\nu}$ of $\theta(\bar{q}_{2\nu-1},.)$ is to the right of $-a^\triangle$ (see the second of formulae (1.3)) and there exists an interval of the form $I^\bullet$ (see Lemma 3.2) such that $I^\bullet\subset(-a^*,-a^\triangle)$. Indeed, the length of $I^\bullet$ equals $\rho^{-4s}(1-\rho^2)$. For each s sufficiently large, one can choose $\rho\in(0,1)$ such that

$$-\rho^{-4s}\in(-a^*,(-a^*-a^\triangle)/2). \tag{3.5}$$

If one chooses a larger s, then one can achieve condition 3.5 by choosing ρ closer to 1. This means that, as ρ^{-4s} remains bounded, the length of $I^\bullet$ tends to 0 and one can attain both conditions (3.5) and $-\rho^{-4s+2}\in(-a^*,-a^\triangle)$. Thus $\xi_{4s-2},\xi_{4s}\in(-a^*,-a^\triangle)$, see part (3) of Lemma 3.2. This proves the first claim of part (4) of Theorem 1.6.

To prove the second claim, for $q^*\in(-1,0)$, we denote by $s^\sharp(q^*)$ the value of $s\in\mathbb{N}$ corresponding to the leftmost of the numbers $-(q^*)^{-4s}$ belonging to the interval $[-a,0)$. In the proof of part (4) of Theorem 1.6 we set $\rho:=|q^*|$, so $-(q^*)^{-4s}=-\rho^{-4s}$. Hence

$$\lim_{\rho\to1^-}(-\rho^{-4s^\sharp(q^*)})=-a, \quad -\rho^{-4s^\sharp(q^*)}>-a \quad \text{and} \quad -\rho^{-4(s^\sharp(q^*)+1)}<-a.$$

From the latter two inequalities, having in mind that $\ln(1/\rho)=(1-\rho)+o(1-\rho)$, one gets

$$s^\sharp(q^*)\sim(\ln a)/(4(1-\rho)). \tag{3.6}$$

Now we partition the zeros of $\theta(q^*,.)$ with negative real parts in several sets (we remind that there are no zeros of $\theta(q^*,.)$ on the imaginary axis for any $q^*\in(-1,0)$, see [16]):

1) The set S_∞ of zeros ξ_j belonging to the intervals $I^\bullet$ with $s \geq s^\sharp(q^*)+2$. These zeros (when considered as depending continuously on $q \in [q^*, 0)$) are real and do not belong to the interval $[-a, 0)$ for any $q \in [q^*, 0)$.

2) The set S_0 of the two zeros of the interval $I^\bullet$ with $s = s^\sharp(q^*)+1$.

3) The set S_R of the other real negative zeros of $\theta(q^*, .)$. We subdivide this set into $S_R([-a, -e^{\pi/2}])$ and $S_R((-e^{\pi/2}, 0))$ of zeros belonging to the respective intervals.

4) The set S_I of the complex conjugate pairs of zeros of $\theta(q^*, .)$ which have negative real parts. For $q^* \in (\bar{q}_{2\nu+1}, \bar{q}_{2\nu-1})$, their number is ν. For $q^* < 0$ close to zero, the zeros of the set S_I are real and belong to intervals $I^\bullet$, and as q^* decreases, they form complex conjugate pairs, see Remarks 1.3.

By abuse of notation we denote by the same symbols sets (e.g. S_I, S_R, etc.) and the number of zeros of θ which they contain. We remind that the numbers $n_a(q^*)$ and $s^\dagger(q^*)$ are defined in Notation 1.5 and in the proof of part (3) of the present theorem respectively; the number $s^\dagger(q^*)$ satisfies the first of conditions (3.4). Hence for $q^* \in (\bar{q}_{2\nu+1}, \bar{q}_{2\nu-1})$, one has

$$n_a(q^*) = S_R([-a, -e^{\pi/2}]) + A, \tag{3.7}$$

where $A = 0$, 1 or 2 is the number of zeros of the set S_0 which belong to the interval $[-a, -e^{\pi/2}]$. On the other hand,

$$S_R([-a, -e^{\pi/2}]) = 2s^\sharp(q^*) - S_R((-e^{\pi/2}, 0)) - S_I. \tag{3.8}$$

Recall that $S_I = 2\nu$. By the first of equations (3.4) one has $\nu = s^\dagger(q^*) + O(1)$, and by part (3) of the present theorem one has $S_R((-e^{\pi/2}, 0)) = o(\nu)$. That's why equations (3.7) and (3.8) imply

$$n_a(q^*) = 2s^\sharp(q^*) - 2s^\dagger(q^*) + o(\nu). \tag{3.9}$$

The factor 2 corresponds to the fact that there are two zeros ξ_i in the interval $I^\bullet$. One can apply formula (3.6) with $a = e^{\pi/2}$ to obtain $s^\dagger(q^*) \sim \ln(e^{\pi/2})/(4(1-\rho))$ and from (3.9) one concludes that $n_a(q^*) = (\ln(a/e^{\pi/2}))/(2(1-\rho)) + o(1/(1-\rho))$ from which part (4) of the theorem follows. □

Parts (5) *and* (6). We begin by proving the first claim of part (6); in this part of the proof we write q instead of $\bar{q}_{2s}$. For any $\varepsilon > 0$, there exists $s^\nabla \in \mathbb{N}$ such that for $s \geq s^\nabla$, one has $\bar{y}_{2s} \in (e^{\pi/2} - \varepsilon, e^{\pi/2} + \varepsilon)$, see formulae (1.3). We assume that $\varepsilon < 1/2$, so $\bar{y}_{2s} > 3$. For $s \geq s^\nabla$, we set $x_j := \bar{y}_{2s}/q^j$, $j \in \mathbb{N}$. One has $\theta(q, x_0) = 0$, $x_{2m} > 0$, $x_{2m+1} < 0$ and $|x_j| > 3$. Therefore

$$\begin{aligned}
&\theta(q, x_1) = 1 + x_0\theta(q, x_0) = 1 > 0,\\
&\theta(q, x_2) = 1 + x_1\theta(q, x_1) = 1 + x_1(1 + x_0\theta(q, x_0)) < 1 - 3 = -2 < 0,\\
&\theta(q, x_3) = 1 + x_2\theta(q, x_2) < -2 < 0 \quad \text{and}\\
&\theta(q, x_4) = 1 + x_3\theta(q, x_3) > 2 > 0.
\end{aligned}$$

In the same way one shows that $\theta(q, x_{4m+2}) < -2 < 0$ and $\theta(q, x_{4m+4}) > 2 > 0$. Hence at least one zero of $\theta(q, .)$ belongs to the interval (x_{4m+2}, x_{4m+4}). The longest of these intervals for which $x_{4m+2} \in [e^{\pi/2} + \varepsilon, a]$ is the last one, i.e., the one with largest value of m. Its length is $\leq a((1/q^2) - 1)$ which quantity tends to 0 as $q \to -1^+$ (i.e., as $s \to \infty$). Hence the zeros of $\theta(q, .)$ are everywhere dense in the interval $[e^{\pi/2} + \varepsilon, a]$, and as $\varepsilon > 0$ is arbitrary, they are everywhere dense in $[e^{\pi/2}, a]$. This proves the first claim of part (6).

To prove part (5) we observe that for $x \in (\xi_{4s+4}, \xi_{4s+2})$, one has $\theta(q, x) \leq 0$ and according to (3.1), $\theta(q, qx) = \theta(q, x)/(qx) - 1/(qx) < 0$ (because $qx > 0$). Hence $(q\xi_{4s+2}, q\xi_{4s+4}) \subset (\xi_{4s+1}, \xi_{4s+3})$, see [13, Fig. 3] (in [13] the latter inclusion is proved only for $q \in [-0.108, 0)$; for any $q \in (-1, 0)$, provided that the zeros ξ_{4s+1}, ξ_{4s+2}, ξ_{4s+3} and ξ_{4s+4} are real, it follows by continuity). Thus

$$Z_{(0,e^{\pi/2}]}(q) = Z_{[-e^{\pi/2}/|q|,0)}(q) + B = Z_{[-e^{\pi/2},0)}(q) + Z_{[-e^{\pi/2}/|q|,-e^{\pi/2})}(q) + B,$$

where $B = -1$, 0 or 1 indicates that the count might not concern the leftmost zero in $[-e^{\pi/2}/|q|, 0)$ and/or the rightmost zero in $(0, e^{\pi/2}]$. By parts (3) and (4) of the present theorem each of the summands $Z_{[-e^{\pi/2},0)}(q)$ and $Z_{[-e^{\pi/2}/|q|,-e^{\pi/2})}(q)$ is $o(1/(1 + q))$ which proves part (5). In the same way one proves the second claim of part (6) as well:

$$Z_{[e^{\pi/2},a]}(q) = Z_{[-a/|q|,-e^{\pi/2}/|q|]}(q) + B = Z_{[-a/|q|,-e^{\pi/2}]}(q) - Z_{(-e^{\pi/2}/|q|,-e^{\pi/2}]}(q) + B,$$

where $Z_{(-e^{\pi/2}/|q|,-e^{\pi/2}]}(q) = o(1/(1 + q))$ and

$$Z_{[-a/|q|,-e^{\pi/2}]}(q) = (\ln((a/|q|)/e^{\pi/2})/2)/(1+q) = (\ln(a/e^{\pi/2})/2)/(1+q) + o(1/(1+q)).$$

The theorem is proved. □

References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part II, Springer, New York, 2009.

[2] B. C. Berndt and B. Kim, Asymptotic expansions of certain partial theta functions, Proc. Amer. Math. Soc. 139(11) (2011) 3779–3788.

[3] K. Bringmann, A. Folsom and A. Milas, Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters, J. Math. Phys. 58(1) (2017) 011702, 19 pp.

[4] K. Bringmann, A. Folsom and R. C. Rhoades, Partial theta functions and mock modular forms as q-hypergeometric series, Ramanujan J. 29(1–3) (2012) 295–310.

[5] T. Creutzig, A. Milas and S. Wood, On regularised quantum dimensions of the singlet vertex operator algebra and false theta functions, Int. Math. Res. Not. 5 (2017) 1390–1432.

[6] G. H. Hardy, On the zeros of a class of integral functions, Messenger of Mathematics 34 (1904) 97–101.

[7] J. I. Hutchinson, On a remarkable class of entire functions, Trans. Amer. Math. Soc. 25 (1923) 325–332.

[8] O. M. Katkova, T. Lobova and A. M. Vishnyakova, On power series having sections with only real zeros, Comput. Methods Funct. Theory 3(2) (2003) 425–441.
[9] V. P. Kostov, On the zeros of a partial theta function, Bull. Sci. Math. 137(8) (2013) 1018–1030.
[10] V. P. Kostov, Asymptotics of the spectrum of partial theta function, Revista Mat. Complut. 27(2) (2014) 677–684.
[11] V. P. Kostov, A property of a partial theta function, C. R. Acad. Bulg. Sci. 67(10) (2014) 1319–1326.
[12] V. P. Kostov, Asymptotic expansions of zeros of a partial theta function, C. R. Acad. Bulg. Sci. 68(4) (2015) 419–426.
[13] V. P. Kostov, On a partial theta function and its spectrum, Proc. Royal Soc. Edinb. A 146(3) (2016) 609–623.
[14] V. P. Kostov, On the double zeros of a partial theta function, Bull. Sci. Math. 140(4) (2016) 98–111.
[15] V. P. Kostov, A separation in modulus property of the zeros of a partial theta function, Analysis Mathematica 44(4) (2018) 501–519.
[16] V. P. Kostov, On the zero set of the partial theta function, Serdica Math. J. 45 (2019) 225–258.
[17] V. P. Kostov and B. Shapiro, Hardy-Petrovitch-Hutchinson's problem and partial theta function, Duke Math. J. 162(5) (2013) 825–861.
[18] I. V. Ostrovskii, On zero distribution of sections and tails of power series, Israel Math. Conf. Proceedings 15 (2001) 297–310.
[19] M. Petrovitch, Une classe remarquable de séries entières, Atti del IV Congresso Internationale dei Matematici, Roma, Ser. 1, 2 (1908) 36–43.
[20] A. Sokal, The leading root of the partial theta function, Adv. Math. 229(5) (2012) 2603–2621.
[21] S. O. Warnaar, Partial theta functions. I. Beyond the lost notebook, Proc. London Math. Soc. 87(2) (2003) 363–395.

Received on February 3, 2025
Revised on June 16, 2025
Accepted on June 26, 2025

Vladimir Petrov Kostov

Université Côte d'Azur
CNRS, LJAD
FRANCE

E-mail: `vladimir.kostov@unice.fr`

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

A NEW REDUCIBILITY RESULT FOR MINIHYPERS IN FINITE PROJECTIVE GEOMETRIES

IVAN LANDJEV, ASSIA ROUSSEVA and KONSTANTIN VOROBEV

In this paper, we prove a new reducibility result for minihypers in projective geometries over finite fields. It is further used to characterize the minihypers with parameters $(70, 22)$ in $\mathrm{PG}(4, 3)$. The latter can be used to attack the existence problem for some hypothetical ternary Griesmer codes of dimension 6.

Keywords: linear codes, minihypers, reducibility, Griesmer bound

2020 Mathematics Subject Classification: 51E21, 51E22, 94B25, 94B65

1. Introduction

In this paper, we present a reducibility theorem for minihypers in the projective geometries $\mathrm{PG}(r, q)$. It can be used to characterize the minihypers with parameters $(70, 22)$ in $\mathrm{PG}(4, 3)$. These are a tool for solving the problem of the existence/nonexistence of several ternary Griesmer codes of dimension 6 [8–11, 13]. We do not impose a restriction on the maximal point multiplicity although for the 6-dimensional codes we need minihypers with a maximal point multiplicity of 2. The paper is structured as follows. In Section 2, we give some definitions and basic facts on arcs and minihypers in finite projective geometries. Section 3 contains our general reducibility result for minihypers. This theorem is then used in Section 4 to give a characterization of the $(70, 22)$-minihypers in $\mathrm{PG}(4, 3)$.

2. Preliminaries

In this section we introduce some basic notions and results on multisets of points in $\mathrm{PG}(r, q)$.

DOI: 10.60063/GSU.FMI.112.93-103

A *multiset* in $\mathrm{PG}(r,q)$ is a mapping $\mathcal{K}\colon \mathcal{P} \to \mathbb{N}_0$, from the pointset $\mathcal{P}$ of $\mathrm{PG}(r,q)$ to the non-negative integers. For a subset $\mathcal{Q}$ of $\mathcal{P}$, we define $\mathcal{K}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} \mathcal{K}(P)$. The integer $\mathcal{K}(\mathcal{Q})$ is called the multiplicity of the subset $\mathcal{Q}$. A point of multiplicity i is called an i-point. Similarly, i-lines, i-planes, i-solids are lines, planes, 3-dimensional subspaces of multiplicity i. The integer $\mathcal{K}(\mathcal{P})$ is called the cardinality of the multiset $\mathcal{K}$. Given a set of points $\mathcal{Q} \subseteq \mathcal{P}$ we define

$$\chi_{\mathcal{Q}}(P) = \begin{cases} 1 & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

A multiset is called *projective* if the multiplicity of each point takes on a value in $\{0,1\}$.

A multiset $\mathcal{K}$ in $\mathrm{PG}(r,q)$ is called an (n,w)-*arc*, if: (a) $\mathcal{K}(\mathcal{P}) = n$; (b) $\mathcal{K}(H) \le w$ for each hyperplane H in $\mathrm{PG}(r,q)$, and (c) there is a hyperplane H_0 with $\mathcal{K}(H_0) = w$. In a similar way, we define an (n,w)-*minihyper* (or (n,w)-*blocking set*) as a multiset $\mathcal{K}$ in $\mathrm{PG}(r,q)$ satisfying: (d) $\mathcal{K}(\mathcal{P}) = n$; (e) $\mathcal{K}(H) \ge w$ for each hyperplane H in $\mathrm{PG}(r,q)$, and (f) there is a hyperplane H_0 with $\mathcal{K}(H_0) = w$. Minihypers were introduced by Hamada [1]. Using this notion we indicate the presence of multiple points.

For a multiset $\mathcal{K}$ in $\mathrm{PG}(r,q)$, we denote by a_i the number of hyperplanes H with $\mathcal{K}(H) = i$, $i \ge 0$. By Λ_j we denote the number of points P from $\mathcal{P}$ with $\mathcal{K}(P) = j$. The sequence $a_0, a_1, a_2, \ldots$ is called *the spectrum* of $\mathcal{K}$. Sometimes when we want to stress the fact that a certain spectrum relates to the multiset $\mathcal{K}$, we write $a_i(\mathcal{K})$, resp. $\Lambda_j(\mathcal{K})$.

The existence of an $[n,k,d]_q$-code C of full length (no coordinate identically zero) is equivalent to that of a $(n, n-d)$-arc in $\mathrm{PG}(k-1,q)$. From any generator matrix G of C one can define a multiset $\mathcal{K}$ with points (with the corresponding multiplicities) the columns of G. This correspondence between $[n,k,d]_q$ codes and $(n,n-d)$-arcs maps isomorphic codes to projectively equivalent arcs and vice versa. If $\mathcal{K}$ is an (n,w)-arc in $\mathrm{PG}(k-1,q)$ with maximal point multiplicity s, then $\mathcal{F} = s - \mathcal{K}$ is an $(sv_k - n, sv_{k-1} - w)$-minihyper. Here as usual $v_k = (q^k-1)/(q-1)$.

Given an (n,w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$, we denote by $\gamma_i(\mathcal{K})$ the maximal multiplicity of an i-dimensional flat in $\mathrm{PG}(k-1,q)$, i.e., $\gamma_i(\mathcal{K}) = \max_\delta \mathcal{K}(\delta)$, $i = 0, \ldots, k-1$. If $\mathcal{K}$ is clear from the context we shall write just γ_i. It is well known that if $\mathcal{K}$ is an $(n, n-d)$-arc in $\mathrm{PG}(k-1,q)$ with $n = t + g_q(k,d)$, then

$$\gamma_j(\mathcal{K}) \le t + \sum_{i=k-1-j}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil .$$

In particular, for Griesmer arcs the maximal point multiplicity is at most $\lceil d/q^{k-1} \rceil$.

In terms of minihypers we have the following lower bounds on the multiplicity of subspaces of different dimensions. Both results are obtained by simple counting arguments.

Lemma 2.1. *Let $\mathcal{F}$ be an (n,w)-minihyper in* $\mathrm{PG}(r,q)$. *Then for any s-dimensional subspace S, it holds*

$$\mathcal{F}(S) \geq \left\lceil \frac{v_{r-s}w - v_{r-s-1}n}{q^{r-s-1}} \right\rceil. \tag{2.1}$$

Lemma 2.2. *Let $\mathcal{F}$ be an (n,w)-minihyper in* $\mathrm{PG}(r,q)$. *Let H be a hyperplane and let T be a subspace of codimension* 2 *contained in H. Then*

$$\mathcal{F}(T) \geq w - \frac{n - \mathcal{F}(H)}{q}. \tag{2.2}$$

The following argument will be used repeatedly throughout the paper. Let $\mathcal{K}$ be a multiset in $\mathrm{PG}(r,q)$. Fix an i-dimensional flat δ in $\mathrm{PG}(r,q)$, with $\mathcal{K}(\delta) = t$. Let further π be a j-dimensional flat in $\mathrm{PG}(r,q)$ of complementary dimension, i.e., $i + j = r - 1$ and $\delta \cap \pi = \varnothing$. Define the projection $\varphi = \varphi_{\delta,\pi}$ from δ onto π by

$$\varphi\colon \begin{cases} \mathcal{P} \setminus \delta & \to \quad \pi \\ Q & \to \quad \pi \cap \langle \delta, Q \rangle. \end{cases} \tag{2.3}$$

In other words, every point Q of $\mathrm{PG}(r,q)$, which is not in δ, is mapped in the point which is the intersection of π and the subspace generated by δ and Q. As before, $\mathcal{P}$ denotes the set of points of $\mathrm{PG}(r,q)$. Note that φ maps $(i+s)$-flats containing δ into $(s-1)$-flats in π. Given a set of points $F \subset \pi$, define the induced multiset $\mathcal{K}^{\varphi}$ by

$$\mathcal{K}^{\varphi}(F) = \sum_{\varphi_{\delta,\pi}(P) \in F} \mathcal{K}(P).$$

We shall exploit the obvious fact that if S is a flat in $\mathrm{PG}(k-1,q)$ through δ, then $\mathcal{K}^{\varphi}(\varphi(S)) = \mathcal{K}(S) - t$. In the next sections the subspace π will be a plane. A line in π which is incident with the points $P_0, \ldots, P_q$ is called a line of type $(\mathcal{K}^{\varphi}(P_0), \ldots, \mathcal{K}^{\varphi}(P_q))$.

The next few results have been proved for linear codes, but can be easily reformulated for arcs and blocking sets in finite projective geometries. This is done in the next theorems.

Theorem 2.3 ([12]). *Let $\mathcal{K}$ be an (n,w)-arc (resp. (n,w)-minihyper) in* $\mathrm{PG}(r,p)$, *where p is a prime. Let further $w \equiv n \pmod{p^e}$ for some $e \geq 1$. Then for every hyperplane H it holds that $\mathcal{K}(H) \equiv n \pmod{p^e}$.*

An (n,w)-arc in $\mathrm{PG}(r,q)$ is called *t-extendable* if the multiplicities of some of the points can be increased by a total of t, so that the obtained arc has parameters $(n+t,w)$. Similarly, an (n,w)-minihyper is called *t-reducible* if the multiplicities of some of the points can be reduced by a total of t, so that the obtained multiset is an $(n-t,w)$-minihyper. The following result by R. Hill and P. Lizak was proved initially for linear codes.

Theorem 2.4 ([2,3]). *Let $\mathcal{K}$ be an (n, w)-arc (resp. (n, w)-minihyper) associated with a Griesmer code in* $\mathrm{PG}(r, q)$ *with $(n-w, q) = 1$, such that the multiplicities of all hyperplanes are n or w modulo q. Then $\mathcal{K}$ is extendable to an $(n + 1, w)$-arc (resp. reducible to an $(n - 1, w)$-minihyper). Moreover the point of extension (resp. the point of reduction) is uniquely determined.*

The next theorem is a more sophisticated extension result by Hitoshi Kanda [4] which applies only to arcs (minihyper) in a geometry over $\mathbb{F}_3$.

Theorem 2.5 ([4]). *Let $\mathcal{K}$ be an (n, w)-arc (resp. (n, w)-blocking set) in* $\mathrm{PG}(r, 3)$. *Assume further that the multiplicity of every hyperplane H is congruent to n, $n + 1$, or $n + 2$ modulo 9 (resp. $n - 2$, $n - 1$, or n modulo 9). Then $\mathcal{K}$ is extendable to an $(n + 2, w)$-arc (resp. reducible to an $(n - 2, w)$-minihyper).*

3. A reducibility theorem for minihypers

Theorem 3.1. *Let p be a prime, with $w \equiv n - p \pmod{p^2}$, and let $\mathcal{F}$ be an (n, w)-minihyper in* $\mathrm{PG}(r, p)$, *that has the following properties:*

(1) *for every hyperplane H in* $\mathrm{PG}(r, p)$ *it holds $\mathcal{F}(H) \equiv n - p$ or $n \pmod{p^2}$;*

(2) *for every hyperplane H with $\mathcal{F}(H) \equiv n - p \pmod{p^2}$, the restriction $\mathcal{F}|_H$ is reducible to a divisible minihyper with divisor p;*

(3) *for every hyperplane H with $\mathcal{F}(H) \equiv n \pmod{p^2}$, the restriction $\mathcal{F}|_H$ is a divisible minihyper with divisor p.*

Then $\mathcal{F} = \mathcal{F}' + \chi_L$, where $\mathcal{F}'$ is a $(n - v_2, w - v_1)$-minihyper and L is a line. Moreover, the line L is uniquely determined.

Proof. By (1) the multiplicities of the hyperplanes are $w+ip^2$ and $w+ip^2+p$, where $i = 0, 1, 2, \ldots$. The hyperplanes H of multiplicity $w + ip^2$ are reducible to divisible minihypers by (2). Hence $\mathcal{F}|_H$ has parameters $(w + ip^2, u_i)$, where $u_i \equiv w + ip^2 - 1 \pmod{p}$. Moreover, all hyperlines in H have multiplicity $\equiv w - 1$ or $w \pmod{p}$, or equivalently, $n - 1$ or $n \pmod{p}$. The point of reduction is contained only in hyperplanes of multiplicity $\equiv n \pmod{p}$. Every hyperline that does not contain the point of reduction is of multiplicity $\equiv n - 1 \pmod{p}$.

In the hyperplanes of multiplicity $w + p + ip^2$ all hyperlines have multiplicity $\equiv n \pmod{p}$. So, for all hyperplanes H through a hyperline of multiplicity $\equiv n - 1 \pmod{p}$, one has $\mathcal{F}(H) \equiv n - p \pmod{p^2}$.

Consider a hyperline T of multiplicity $\mathcal{F}(T) \equiv n \pmod{p}$. Denote by x (resp. y) the number of hyperplanes of multiplicity $\equiv n-p \pmod{p^2}$ (resp. $\equiv n \pmod{p^2}$) through p. Obviously $x + y = p + 1$.

Denote by H_i, $i = 0, \dots, p$, the hyperplanes through T and set $\mathcal{F}(T) \equiv n + \alpha p \pmod{p^2}$. Now

$$\begin{aligned} n &= \sum_i \mathcal{F}(H_i) - p\mathcal{F}(T) \\ &\equiv x(n-p) + yn - p(n + \alpha p) \pmod{p^2} \\ &\equiv n(x+y) - px - np - \alpha p^2 \pmod{p^2}. \end{aligned}$$

This implies $px \equiv 0 \pmod{p^2}$, whence $x \equiv 0 \pmod{p}$ and $y \equiv 1 \pmod{p}$ (i.e., $y = 1$ or $p + 1$).

Define an arc $\widetilde{\mathcal{F}}$ in the dual geometry by

$$\widetilde{\mathcal{F}}(H) = \begin{cases} 1 & \text{if } \mathcal{F}(H) \equiv n \pmod{p^2}, \\ 0 & \text{if } \mathcal{F}(H) \equiv n - p \pmod{p^2}. \end{cases}$$

By the fact proved above, if a line contains two 1-points with respect to $\mathcal{F}$, then the whole line incident with them consists of 1-points. This means that all hyperplanes of multiplicity $n \pmod{p^2}$ form a subspace in the dual geometry.

Consider a minimal hyperplane H_0, i.e., a hyperplane of multiplicity w. All hyperlines through the point of reduction are contained in a unique hyperplane of multiplicity $n \pmod{p^2}$. This implies that the number of the hyperplanes of multiplicity $n \pmod{p^2}$ is equal to the number of the hyperlines in H_0 through a fixed point. This number is v_{r-1}. This implies that the hyperplanes of multiplicity $n \pmod{p^2}$ are all hyperplanes through a fixed line L.

It remains to show that all points on L have multiplicity at least 1 with respect to $\mathcal{F}$. Fix a minimal hyperplane H_0 and a hyperline T in H_0 of multiplicity $n - 1 \pmod{p}$. As noted above, all hyperplanes H_i, $i = 0, \dots, p$, through T are also of multiplicity $n - p \pmod{p^2}$. Denote by P_i the unique point of reducibility of the minihyper $\mathcal{F}|_{H_i}$. All points P_i are outside of the hyperline T. In addition, they are collinear since they form a blocking set with respect to the hyperplanes. Denote the line containing the points P_i by L'. Assume there is a hyperplane H of multiplicity $n \pmod{p^2}$ that meets L' in a single point, since then it meets $H_1, \dots, H_p$ in hyperlines of multiplicity $n - 1 \pmod{p}$, which is impossible. Hence every hyperplane of multiplicity $n \pmod{p^2}$ contains L' and hence $L \equiv L'$. □

4. Classification of $(70, 22)$-minihypers in $\mathrm{PG}(4, 3)$

As an application of Theorem 3.1 we shall characterize the $(70, 22)$-minihypers in $\mathrm{PG}(4, 3)$. This characterization is crucial for attacking the nonexistence of some ternary 6-dimensional codes whose existence is in doubt (cf [8]).

First, we claim without proof several characterization results for minihypers in $\mathrm{PG}(3, 3)$. The proofs can be found in [6].

Lemma 4.1. *A $(21, 6)$-minihyper in $\mathrm{PG}(3, 3)$ is one of the following:*

(α) *the sum of a plane and two lines;*

(β) *a minihyper with one double point with* $a_{12} = 2$*;*

(γ) *a projective minihyper and* $a_{12} = 1$.

Lemma 4.2. *Every* $(22, 6)$*-minihyper in* $\mathrm{PG}(3, 3)$ *with maximal point multiplicity* 2 *is either reducible to one of the* $(21, 6)$*-minihypers, or else the sum of a projective plane of order* 3 *and a non-canonical planar* $(9, 2)$*-minihyper* (*the complement of an oval in* $\mathrm{PG}(2, 3)$).

The next lemma follows from the classification of the linear codes with parameters $[50, 4, 33]_3$ and $[49, 4, 32]_3$ given in [5].

Lemma 4.3. *A* $(30, 9)$*-minihyper in* $\mathrm{PG}(3, 3)$ *is one of the following*:

(a) *the sum of two planes and a line (a canonical minihyper);*

(b) *the union of two planes plus two skew lines meeting the two planes in their common line;*

(c) *the complement of a* 10*-cap in* $\mathrm{PG}(3, 3)$ *(a projective minihyper).*

Every $(31, 9)$*-minihyper in* $\mathrm{PG}(3, 3)$ *is reducible to a* $(30, 9)$*-minihyper.*

Now we state the theorem which is the main result in this section. It describes the structure of the $(70, 22)$-minihypers in $\mathrm{PG}(4, 3)$.

Theorem 4.4. *Let* $\mathcal{F}$ *be a* $(70, 22)$*-minihyper in* $\mathrm{PG}(4, 3)$. *Then* $\mathcal{F}$ *is one of the following*:

(A) *the sum of a solid and a* $(30, 9)$*-minihyper in* $\mathrm{PG}(4, 3)$*;*

(B) *the sum of a* $(66, 21)$*-minihyper in* $\mathrm{PG}(4, 3)$ *and a line.*

Remark 4.5. The characterization of the $(66, 21)$-minihypers in $\mathrm{PG}(4, 3)$ is given in [7].

The proof of this theorem is split into several lemmas. Until the end of the section, we shall assume that $\mathcal{F}$ is a $(70, 22)$-minihyper in $\mathrm{PG}(4, 3)$.

Lemma 4.6. *Let* $\mathcal{F}$ *be a* $(70, 22)$*-minihyper in* $\mathrm{PG}(4, 3)$. *Then for every solid* S *in* $\mathrm{PG}(4, 3)$ *it holds* $\mathcal{F}(S) \equiv 1 \pmod 3$ (*i.e.,* $\mathcal{F}$ *is a divisible minihyper*).

Proof. Assume the maximal point multiplicity of such minihyper is s. Then $s - \mathcal{F}$ is a $(121s - 70, 40s - 22)$-arc in $\mathrm{PG}(4, 3)$, which is associated with a $[121s - 70, 5, 81s - 48]_3$-code which is readily checked to be a Griesmer code. By Ward's Theorem this code is divisible and hence, in turn, $\mathcal{F}$ is also divisible. Thus for each solid S in $\mathrm{PG}(4, 3)$ $\mathcal{F}(S) \equiv 1 \pmod 3$. $\square$

Lemma 4.7. *Let* $\mathcal{F}$ *be a* $(70, 22)$*-minihyper in* $\mathrm{PG}(4, 3)$. *If there exists a solid* S *with* $\mathcal{F}(S) \geq 49$*, then* $\mathcal{F}$ *is the sum of a solid and a* $(30, 9)$ *minihyper in* $\mathrm{PG}(4, 3)$.

Proof. Let us first note that a solid of multiplicity at least 49 does not have 0-points. Otherwise, the cardinality of $\mathcal{F}$ is $|\mathcal{F}| \geq 49 + 27 = 76$ since every line through the 0-point in that solid has to be blocked at least once. Now obviously $\mathcal{F} - \chi_S$ is a $(30, 9)$-minihyper since the multiplicity of each solid different from S is reduced by 13. □

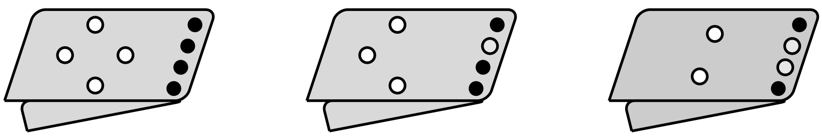
Lemma 4.8. *Let $\mathcal{F}$ be a $(70, 22)$-minihyper in* $\mathrm{PG}(4, 3)$ *in which every solid S is of multiplicity $\mathcal{F}(S) < 49$. Then there exist no solid S with*:

(a) $\mathcal{F}(S) = 28$;

(b) $\mathcal{F}(S) = 37$;

(c) $\mathcal{F}(S) = 46$.

Proof. (a) Let us start by noting that every $(28, 8)$-minihyper in $\mathrm{PG}(3, 3)$ is reducible to a $(26, 8)$-minihyper. Hence it is the sum of two planes and two points. The 22-solids through an 8-plane are just the irreducible $(22, 6)$-minihypers (the sum of a plane and the complement to an oval). There are three possibilities for such minihypers presented at the pictures below:



(i) (ii) (iii)

In the picture, the black points are 2-points, the white points are 0-points and the gray planes are planes of 1-points.

In case (i) the projection of the 22-plane from the 8-line is a line of type $(9, 5, 0, 0)$. The projection of a 28-plane from the 8-line is of type

$$(18, 0 + \varepsilon_1, 0 + \varepsilon_2, 0) \quad \text{or} \quad (9 + \varepsilon_1, 9 + \varepsilon_2, 0 + \varepsilon_3, 0)$$

with $\sum_i \varepsilon_i = 2$ or 3. Now consider a 22-solid S_0 of type (i) and denote by S_i, $i = 1, 2, 3$, the other three solids through the 8-plane π consisting of four 2-points. Now in the projection plane there exist three collinear 0 or $0 + \varepsilon$ points. The line incident with them is either of type $(18, 0, 0, 0)$ or of type $(9 + \varepsilon', 0 + \varepsilon'', 0, 0)$ which forces a solid of multiplicity at most 19, a contradiction.

In case (ii) the proof is similar. We consider a projection from a 5-line in an 8-plane. Now the image of a 22-solid has one of the types

$$(11, 3, 2, 1), \quad (10, 3, 3, 1) \quad \text{or} \quad (10, 3, 2, 2).$$

The image of a 28-solid is

$$(12 + \varepsilon_1, 3 + \varepsilon_2, 3 + \varepsilon_3, 3),$$

with $\sum_i \varepsilon_i = 2$. Now if the points of multiplicity at least 10 are not collinear then there is a line in the projection plane of multiplicity at most 14, which forces a solid of multiplicity at most 19, a contradiction. Otherwise the projection plane has a line of multiplicity at least 42. This gives a solid with at least 47 points. This case was completed in Lemma 4.7.

(b) An 11-plane is forced to have a 2-line whence there are no $(37, 11)$-minihypers.

(c) By Lemma 4.2 a $(22, 6)$-minihyper does not have 14-planes. Fix a 14-plane π in S and denote by $S_0 = S, S_1, S_2, S_3$ the solids through π. Clearly $\mathcal{F}(S_0) = 46$, and $\mathcal{F}(S_i) \geq 25$ for $i = 1, 2, 3$. Then

$$|\mathcal{F}| = \sum_i \mathcal{F}(S_i) - 3\mathcal{F}(\pi) \geq 46 + 3 \cdot 25 - 3 \cdot 14 = 79,$$

a contradiction. □

We have proved so far that if $\mathcal{F}$ is a $(70, 22)$-minihyper in PG$(4, 3)$ with maximal hyperplane of multiplicity at most 46, then the possible multiplicities lie in the set $\{22, 25, 31, 34, 40, 43\}$.

Lemma 4.9. *Let $\mathcal{F}$ be a $(70, 22)$-minihyper in* PG$(4, 3)$ *and let S solid of multiplicity* 43*. Then $\mathcal{F}|_S$ is a divisible minihyper with parameters* $(43, 13)$.

Proof. It is clear that there are no planes in S of multiplicity 14 since $(22, 6)$-minihypers do not have 14-planes. Furthermore, 15-planes in S are also impossible. This in turn implies that there are no planes of multiplicity $\equiv -1, 0 \pmod 3$. □

Lemma 4.10. *Let $\mathcal{F}$ be a $(70, 22)$-minihyper and let S be a 22-solid. Then $\mathcal{F}|_S$ is a reducible $(22, 6)$-minihyper.*

Proof. Denote by (a_i) the spectrum of $\mathcal{F}$. Using simple counting argument, we get the standard identities:

$$a_{22} + a_{25} + a_{31} + a_{34} + a_{40} + a_{43} = 121$$
$$22a_{22} + 25a_{25} + 31a_{31} + 34a_{34} + 40a_{40} + 43a_{43} = 2800$$
$$231a_{22} + 300a_{25} + 465a_{31} + 561a_{34} + 780a_{40} + 903a_{43} = 35 \cdot 69 \cdot 13 + 27 \sum_{i=2}^{4} \binom{i}{2} \Lambda_i,$$

where Λ_i is the number of i-points. (Note that every point is on a minimal hyperplane and the maximal point multiplicity on a minimal hyperplane is 4.) This implies

$$a_{31} + 2a_{34} + 5a_{40} + 7a_{43} = 10 + \Lambda_2 + 3\Lambda_6 + 6\Lambda_4. \quad (4.1)$$

The spectra (b_i) of the irreducible $(22, 6)$-minihypers are the following:

(a) $b_6 = 18$, $b_7 = 12$, $b_8 = 9$, $b_{22} = 1$;

(b) $b_6 = 18$, $b_7 = 12$, $b_8 = 8$, $b_{13} = 1$, $b_{17} = 1$;

(c) $b_6 = 18$, $b_7 = 11$, $b_8 = 9$, $b_{13} = 1$, $b_{16} = 1$;

(d) $b_6 = 17$, $b_7 = 12$, $b_8 = 9$, $b_{13} = 1$, $b_{15} = 1$.

We are going to rule out each of the possibilities (a)–(d). The argument is similar in all four cases. We fix a 22-solid S_0 and for each plane δ in S_0 we consider the maximal contribution of the other three solids S_1, S_2, S_3, to the left-hand side of (4.1). Table 1 gives the maximal contributions for planes δ in S_0 of different multiplicity.

Table 1

$\mathcal{F}(\delta)$	$\mathcal{F}(S_1)$	$\mathcal{F}(S_2)$	$\mathcal{F}(S_3)$	Contribution
6	22	22	22	0
7	22	22	25	0
8	22	25	25	0
13	22	25	40	5
15	22	31	40	6
16	22	34	40	7
17	25	34	40	7
22	31	40	43	13

(a) The left-hand side is bounded from above by 13, since 6-, 7-, and 8-planes give contribution of 0. So, we have $13 \geq 10 + \Lambda_2$. But in this case we have obviously $\Lambda_2 \geq 9$ (since the irreducible 22-plane alone has nine 2-points), which gives a contradiction.

(b) The left-hand side is at most $1 \cdot 5 + 1 \cdot 7 = 12 \geq 10 + \Lambda_2$. In this case $\Lambda_2 \geq 4$, a contradiction.

(c) We have again $1 \cdot 5 + 1 \cdot 7 \geq 10 + \Lambda_2$, and $\Lambda_2 \geq 3$, a contradiction.

(d) The left-hand side is at most $1 \cdot 5 + 1 \cdot 6 = 11$. Hence $11 \geq 10 + \Lambda_2$, but $\Lambda_2 \geq 2$ again a contradiction. □

Lemma 4.11. *Let $\mathcal{F}$ be a $(70, 22)$-minihyper in* PG(4, 3)*, and let S solid of multiplicity* 25*, or* 34*. Then $\mathcal{F}|_S$ is a $(25, 7)$-divisible minihyper in* PG(3, 3)*, respectively a divisible $(34, 10)$-minihyper in* PG(3, 3)*.*

Proof. Assume $\mathcal{F}|_S$ is a $(25, 7)$-minihyper and assume that there exists an 8-plane in S. A minimal solid can have 8-planes only if it is irreducible. But such solids were ruled out by Lemma 4.10. Hence counting the multiplicities of the solids through π we get $|\mathcal{F}| \geq 4 \cdot 25 - 3 \cdot 8 = 76$, a contradiction. If we assume that there exist a 9-plane, then a 2-line in this 9-plane is forced to be contained in an 8-plane, which was already ruled out. In the same way we can rule out the existence of planes of multiplicity $-1, 0 \pmod 3$.

Next assume that $\mathcal{F}|_S$ is a $(34, 10)$ minihyper. It is immediate that there exist no 11-planes in S since there exist no $(11, 3)$-minihypers in PG(2, 3). From this point on the proof is completed in the case of 25-solids. □

Lemma 4.12. *Let $\mathcal{F}$ be a $(70, 22)$-minihyper in* $\mathrm{PG}(4, 3)$, *and let S be a solid of multiplicity* 40. *Then $\mathcal{F}|_S$ is a* $(40, 12)$*-minihyper in* $\mathrm{PG}(3, 3)$, *that is reducible to a* $(39, 12)$*-minihyper.*

Proof. We have to rule out the existence of planes of multiplicity $\equiv -1 \pmod 3$. By the previous results such a plane can be contained only in 40-solids. If we denote its multiplicity by x, we have $x \equiv -1 \pmod 3$ and $4 \cdot 40 - 3x = 70$, But the equation implies $x \equiv 0 \pmod 3$, a contradiction. □

Now it is easily checked that a $(70, 22)$-minihyper in $\mathrm{PG}(4, 3)$ with hyperplanes of multiplicity at most 43 satisfies the conditions of Theorem 3.1:

- Condition (1) follows from Lemmas 4.6 and 4.8;
- Condition (2) follows from Lemmas 4.2, 4.3, 4.10, and 4.12;
- Condition (3) follows from Lemmas 4.9 and 4.11.

Acknowledgements

The first author was supported by the Bulgarian National Science Fund under Contract KP-06-N72/6-2023. The second author was supported by the Research Fund of Sofia University under Contract 80-10-164/18.04.2024. The research of the third author was supported by the NSP P. Beron Project CP-MACT.

References

[1] N. Hamada, A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry, Discrete Math. 116 (1993) 229–268.
[2] R. Hill, An extension theorem for linear codes, Des. Codes Cryptogr. 17 (1999) 151–157.
[3] R. Hill and P. Lizak, Extensions of linear codes, in: Proc. IEEE Int. Symposium on Inf. Theory, Whistler, Canada, 1995, p. 345.
[4] H. Kanda, A new extension theorem for ternary linear codes and its application, Finite Fields and Appl. 67 (2020) 1017111, https://doi.org/10.1016/j.ffa.2020.101711.
[5] I. Landjev, The non-existence of some optimal ternary codes of dimension five, Designs, Codes and Cryptography 15 (1998) 245–258.
[6] I. Landjev, A. Rousseva and E. Rogachev, On a class of minihypers in the geometries $\mathrm{PG}(r, q)$, in: Proc. 18th EAI Int. Conf. on CSECS, ed. by T. Zlateva and R. Goleva, LNICST 450, Springer, Cham, 2022, 142–153, https://doi.org/10.1007/978-3-031-17292-2_12
[7] I. Landjev, A. Rousseva and E. Rogachev, Characterization of some minihypers in $\mathrm{PG}(4, 3)$, Ann. Sofia Univ. Fac. Math. and Inf. 109 (2022) 91–98, https://doi.org/10.60063/GSU.FMI.109.91-98.
[8] T. Maruta, Griesmer bound for linear codes over finite fields, https://mars39.lomo.jp/opu/griesmer.htm.
[9] T. Maruta and Y. Oya, On optimal ternary linear codes of dimension 6, Adv. Math. Comm. 5(3) (2011) 505–520.

[10] T. Sawashima and T. Maruta, Nonexistence of some ternary linear codes with minimum weight −2 modulo 9, Adv. Math. Comm. 17(6) (2023) 1338–1357, https://doi.org/10.3934/amc.2021052.

[11] M. Takenaka, K. Okamoto and T. Maruta, On optimalnon-projective termary linear codes, Discrete Math. 308 (2008) 842–854.

[12] H. N. Ward, Divisibility of codes meeting the Griesmer bound, J. Comb. Theory Ser. A 83(1) (1998) 79–93.

[13] Y. Yoshida and T. Maruta, Ternary linear codes and quadrics, Electronic J. Combin. 16 (2009) #R9.

Ivan Landjev and Konstantin Vorobev

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia
BULGARIA

E-mail: ivan@math.bas.bg
konstantin.vorobev@gmail.com

Assia Rousseva

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: assia@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

ON THE ERROR BOUNDS OF THE GAUSS-TYPE QUADRATURE FORMULAE ASSOCIATED WITH SPACES OF PARABOLIC AND CUBIC SPLINE FUNCTIONS WITH DOUBLE EQUIDISTANT KNOTS

GENO NIKOLOV and PETAR B. NIKOLOV

In two papers from 1995 P. Köhler and G. Nikolov showed that Gauss-type quadrature formulae associated with spaces of spline functions with equidistant knots are asymptotically optimal in certain Sobolev classes of functions. In particular, Gauss-type quadratures associated with the spaces of spline functions of degree $r-1$ with double equispaced knots are asymptotically optimal definite quadrature formulae of order r when r is even, and it is conjectured that the asymptotical optimality property persists also in the case of odd r. For $r = 3$, 4, these quadrature formulae have been constructed by G. Nikolov, who also proved estimates for their error constants. The aim of this note is to refine the estimates for the error constant in the case $r = 3$, and to point out to some error estimates in both cases $r = 3$ and $r = 4$, which are easier to evaluate and could be sharper than those which involve the uniform norm of the r-th derivative of the integrand.

Keywords: spline functions, monosplines, Peano representation of linear functionals, definite quadrature formulae, error estimation of quadratures, Bernoulli polynomials

2020 Mathematics Subject Classification: 41A55, 65D30, 65D32

1. Introduction and statement of the results

A standard way to evaluate approximately the definite integral

$$I[f] := \int_0^1 f(x)\,dx$$

DOI: 10.60063/GSU.FMI.112.105-121

is to use quadrature formulae, which are linear functionals of the form

$$Q[f]=\sum_{i=1}^{n}a_i f(\tau_i),\quad 0\le\tau_1<\cdots<\tau_n\le 1. \tag{1.1}$$

We start with introducing some notation and definitions. Throughout this paper, π_m stands for the set of algebraic polynomials of degree not exceeding m. A quadrature formula Q is said to have algebraic degree of precision m (in short, $ADP(Q)=m$) if m is the largest non-negative integer such that its remainder functional

$$R[Q;f]:=I[f]-Q[f]$$

vanishes on π_m.

The Sobolev classes of functions $W_p^r[0,1]$, ($r\in\mathbb{N}$, $p\ge 1$), are defined by

$$W_p^r[0,1]:=\{f\in C^{r-1}[0,1]\colon f^{(r-1)}\text{ loc. abs. cont.},\ \int_0^1|f^{(r)}(t)|^p\,dt<\infty\}$$

(note that $C^r[0,1]\subset W_p^r[0,1]$ for every $p\ge 1$). Henceforth, $\|\cdot\|$ designates the supremum norm in $[0,1]$, and the usual $L_p[0,1]$-norm is shortly denoted by $\|\cdot\|_p$,

$$\|f\|_p=\begin{cases}\left(\int_0^1|f(t)|^p\,dt\right)^{1/p}, & \text{if } 1\le p<\infty,\\ \operatorname{vrai\,sup}_{t\in[0,1]}|f(t)|, & \text{if } p=\infty.\end{cases}$$

If $ADP(Q)=m\ge r-1$ and $f\in W_1^r[0,1]$, then by Peano representation theorem for linear functionals (cf. [14]), the remainder $R[Q;f]$ can be written in the form

$$R[Q;f]=\int_0^1 K_r(Q;t)f^{(r)}(t)\,dt, \tag{1.2}$$

where $K_r(Q;t)$ is referred to as the r-th Peano kernel of Q and is given by

$$K_r(Q;t)=\frac{1}{(r-1)!}R\big[Q;(\cdot-t)_+^{r-1}\big], \tag{1.3}$$

where $(x)_+^{r-1}=\max\{x,0\}^{r-1}$ is the truncated power function. In literature, $K_r(Q;t)$ is also termed as monospline of degree r. For quadrature formula Q in (1.1) the explicit form of $K_r(Q;t)$, $t\in[0,1]$, is

$$K_r(Q;t)=\frac{(1-t)^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(\tau_i-t)_+^{r-1} \tag{1.4}$$

$$=(-1)^r\left\{\frac{t^r}{r!}-\frac{1}{(r-1)!}\sum_{i=1}^{n}a_i(t-\tau_i)_+^{r-1}\right\}. \tag{1.5}$$

If $f\in W_p^r[0,1]$, then application of Hölder's inequality to (1.2) implies the unimprovable error estimate

$$\big|R[Q;f]\big|\le c_{r,p}(Q)\|f^{(r)}\|_p,\quad\text{where}\quad c_{r,p}(Q)=\|K_r(Q;\cdot)\|_q,\quad \frac{1}{p}+\frac{1}{q}=1.$$

Usually, $c_{r,p}(Q)$ is called the error constant of Q in the Sobolev class $W_p^r[0,1]$. In what follows, the subscript "n" in Q_n is used to emphasize that Q_n is an n-point quadrature formula, i.e., a quadrature formula which has n nodes. Quadrature formulae Q_n with the smallest possible error constant $c_{r,p}(Q_n)$ are called optimal quadrature formulae in $W_p^r[0,1]$. Without going into details, let us mention that the existence and uniqueness of optimal quadrature formulae in Sobolev classes of functions have been established by Bojanov [1–3] and Zhensykbaev [16, 17].

In the present paper we study certain definite quadrature formulae. A quadrature formula Q_n is said to be definite of order r ($r \geq 1$), if there exists a constant $c_r(Q_n) \neq 0$ such that

$$R[Q_n; f] = c_r(Q_n) f^{(r)}(\xi)$$

for every $f \in C^r[0,1]$ with some $\xi \in [0,1]$ depending on the integrand f. More precisely, Q_n is called positive, resp. negative, definite quadrature formula of order r if $c_r(Q_n) > 0$, resp. $c_r(Q_n) < 0$. Since $c_r(Q_n) = c_{r,\infty}(Q_n)$ if Q_n is positive definite and $c_r(Q_n) = -c_{r,\infty}(Q_n)$ if Q_n is negative definite, $c_r(Q_n)$ will also be referred to as the error constant of Q_n. The importance of definite quadrature formulae of order r stems from the fact that they provide one-sided approximation to $I[f]$ when $f^{(r)}$ has a permanent sign in $(0,1)$. The midpoint and the trapezium quadrature formulae are best-known examples of positive, resp. negative, definite quadrature formulae of order two.

Definite n-point quadrature formulae of order r with the smallest positive or the largest negative c_r are called optimal definite quadrature formulae. It is known that optimal definite quadrature formulae exist and are unique, cf. [6, 11, 15] and [5, Chapter VII.8]. We denote by $c_{n,r}^+$ and $c_{n,r}^-$ the error constants of the optimal n-point definite quadrature formulae of order r:

$$c_{n,r}^+ := \inf\{c_r(Q_n) \colon Q_n \text{ is positive definite of order } r\},$$
$$c_{n,r}^- := \sup\{c_r(Q_n) \colon Q_n \text{ is negative definite of order } r\}.$$

In [9] estimates have been established for the error constants of the Gauss-type quadrature formulae associated with the spaces of spline functions with double equidistant knots. These estimates in turn provide bounds for the error constants of the optimal definite quadrature formulae. Below we restate the main result from [9], denoting by B_r the Bernoulli polynomial of order r with leading coefficient $1/r!$.

Theorem A ([9, Theorem 1.1])**.** (a) *For even r with $2 \leq r \leq 2n$, there holds*

$$c_{n,r}^+ \leq -\frac{B_r(j/2)}{(n+1-r/2)^r}, \quad \textit{if} \quad r = 4m + 2j, \ j = 0, 1.$$

(b) *For even r with $2 \leq r \leq 2n-2$, there holds*

$$c_{n,r}^- \geq -\frac{B_r(j/2)}{(n-r/2)^r}, \quad \textit{if} \quad r = 4m + 2 - 2j, \ j = 0, 1.$$

(c) *For odd r with $1 \leq r \leq 2n-1$, there holds*

$$c_{n,r}^+ \leq \frac{\|B_r\|}{(n-(r-1)/2)^r} \quad \textit{and} \quad c_{n,r}^- \geq -\frac{\|B_r\|}{(n-(r-1)/2)^r}.$$

Comparison with results of Lange [10] shows that Gauss-type quadrature formulae associated with the spaces of spline functions of degree $r-1$ with double equidistant knots are asymptotically optimal definite quadrature formulae of order r when r is even, and it is conjectured that the asymptotical optimality property persists also in the case of odd r. Two particular cases of Theorem A relevant to the object of this paper are

$$c^{+}_{n+1,3} \le \frac{\sqrt{3}}{216n^3}, \quad c^{-}_{n+1,3} \ge -\frac{\sqrt{3}}{216n^3}, \tag{1.6}$$

$$c^{+}_{n+1,4} \le \frac{1}{720n^4}. \tag{1.7}$$

The right-hand sides of the inequalities in (1.6) and (1.7) are in fact bounds for the error constants of Gauss-type quadrature formulae associated with the linear spaces of spline functions $S_{n,3}$ and $S_{n,4}$, respectively, where for $r \ge 3$ and $n \ge 2$,

$$\begin{aligned} &S_{n,r} = \{f\colon f \in C^{r-3}[0,1],\ f_{|(x_k,x_{k+1})} \in \pi_{r-1},\ k=0,\dots,n-1\},\\ &x_k = x_{k,n} := \frac{k}{n}, \quad k=0,\dots,n. \end{aligned} \tag{1.8}$$

The functions $\{1, x, x^2, (x-x_1)_+, (x-x_1)_+^2, \dots, (x-x_{n-1})_+, (x-x_{n-1})_+^2\}$ form a basis for $S_{n,3}$, therefore $\dim S_{n,3} = 2n+1$ and the Gauss-type quadratures associated with $S_{n,3}$ are left and right $(n+1)$-point Radau quadrature formulae. These quadrature formulae were found, among others, in [12].

Theorem B ([12, Theorem 2]). *The right Radau quadrature formula associated with the space of parabolic splines $S_{n,3}$ is*

$$Q^{R,r}_{n+1}[f] = \sum_{k=0}^{n-1} a_{k,n} f\Big(\frac{k+\theta_k}{n}\Big) + a_{n,n} f(1) \approx I[f],$$

where $\theta_0 = 1/3$,

$$\begin{aligned} \theta_k &= \frac{1-\theta_{k-1}}{5-6\theta_{k-1}}, \quad k=1,\dots,n-1,\\ a_{k,n} &= \frac{1}{6n\theta_k(1-\theta_k)}, \quad k=0,\dots,n-1, \end{aligned}$$

and

$$a_{n,n} = \frac{2-3\theta_{n-1}}{6n(1-\theta_{n-1})}.$$

$Q^{R,r}_{n+1}$ is negative definite quadrature formula of order three with error constant

$$c_3(Q^{R,r}_{n+1}) = -\frac{\sqrt{3}}{216n^3} + O(n^{-4}).$$

Remark 1.1. The left Radau quadrature formula $Q^{R,l}_{n+1}$ associated with $S_{n,3}$ is obtained from $Q^{R,r}_{n+1}$ by reflection, i.e., $Q^{R,l}_{n+1}[f(\cdot)] = Q^{R,r}_{n+1}[f(1-\cdot)]$. Clearly, $Q^{R,l}_{n+1}$ is positive definite of order three and $c_3\big(Q^{R,l}_{n+1}\big) = -c_3\big(Q^{R,r}_{n+1}\big)$.

Since $\dim S_{n,4} = 2n+2$, associated with $S_{n,4}$ are the $(n+1)$-point Gauss quadrature formula Q^G_{n+1} and the $(n+2)$-point Lobatto quadrature formula Q^{Lo}_{n+2}. These quadrature formulae were investigated in [13]. The following theorem gives the construction and summarizes some of the properties of Q^G_{n+1} (cf. [13, Section 2]).

Theorem C. *Let*

$$Q^G_{n+1}[f] = \sum_{i=1}^{n+1} a^G_{i,n+1} f(\tau^G_{i,n+1}), \quad 0 < \tau^G_{1,n+1} < \cdots < \tau^G_{n+1,n+1} < 1,$$

be the Gauss quadrature formula associated with $S_{n,4}$, i.e., determined uniquely by the property $I[f] = Q^G_{n+1}[f]$ for every $f \in S_{n,4}$. Then:

(a) *Q^G_{n+1} is symmetrical: $a^G_{k,n+1} = a^G_{n+2-k,n+1}$ and $\tau^G_{k,n+1} = 1 - \tau^G_{n+2-k,n+1}$ for $k = 1,\dots,n+1$.*

(b) *Let $a^G_{i,n+1} = \delta_i/n$, $\tau^G_{i,n+1} = (i-\theta_i)/n$ for $i = 1,\dots,[n/2]+1$. Then the sequences $\{\delta_i\}$ and $\{\theta_i\}$ are determined by $\delta_1 = 16/27$, $\theta_1 = 3/4$ and, for $i = 1,\dots,[n/2]-1$, by the recurrence relations*

$$\theta_{i+1} = \frac{1-\delta_i(1-\theta_i)^2(5\theta_i+1)}{1-\delta_i(1-\theta_i)^2(4\theta_i+1)}, \quad \delta_{i+1} = \frac{1-\delta_i(1-\theta_i)^2(4\theta_i+1)}{\theta^2_{i+1}}.$$

If n is even ($n = 2m$), then $\theta_{m+1} = 1$ and $\delta_{m+1} = 1 - 2\delta_m(1-\theta_m)^2(2\theta_m+1)$; if n is odd ($n = 2m-1$), then $\delta_m = 1 - \delta_{m-1}(1-\theta_{m-1})^2(2\theta_{m-1}+1)$ and θ_m is the greater root of the equation

$$\theta_m(1-\theta_m) = \frac{\delta_{m-1}\theta_{m-1}(1-\theta_{m-1})^2}{1-\delta_{m-1}(1-\theta_{m-1})^2(2\theta_{m-1}+1)}.$$

(c) *Q^G_{n+1} is positive definite quadrature formula of order four and its error constant $c_4(Q^G_{n+1})$ obeys the representation*

$$c_4(Q^G_{n+1}) = \frac{1}{720n^4} - \frac{1}{12}\sum_{i=1}^{[(n+1)/2]} a^G_{i,n+1}(x_{i-1}-\tau^G_{i,n+1})^2(x_i-\tau^G_{i,n+1})^2.$$

For all $n \ge 4$ there holds

$$\frac{1}{720n^4} - \frac{1}{551.9775n^5} \le c_4(Q^G_{n+1}) \le \frac{1}{720n^4} - \frac{1}{552n^5}. \tag{1.9}$$

(d) *Let $f \in C^4[0,1]$. Then $R[Q^G_{n+1};f] = o(n^{-4})$ if and only if $f'''(0) = f'''(1)$. Moreover, if $\operatorname{sign}\{f'''(1) - f'''(0)\} = \epsilon \ne 0$, then there exists $n_0 \in \mathbb{N}$ such that $\epsilon R[Q^G_{n+1};f] \ge 0$ for all $n \ge n_0$.*

(e) *If $f \in W^4_1[0,1]$ and $f^{(4)} \ge 0$ a.e. in $[0,1]$, then for all $n \ge 2$,*

$$0 \le R\big[Q^G_{2n+1};f\big] \le R\big[Q^G_{n+1};f\big].$$

Remark 1.2. In [13] recurrence formulae have been proposed also for computation of the weights and nodes of the Lobatto quadrature formula Q_{n+2}^{Lo} associated with $S_{n,4}$, which is negative definite of order four. However, unlike the case with Q_{n+1}^{G}, this procedure is of numerical nature, as it requires determination of an initial parameter, cf. [13, Theorem 2.5].

Our first goal in this paper is to prove properties of the Radau quadrature formulae associated with $S_{n,3}$, which are the analogues of those of Q_{n+1}^{G}, presented in parts (c), (d) and (e) of Theorem C.

Theorem 1.1. *Let $Q_{n+1}^{R,l}$ and $Q_{n+1}^{R,r}$ be the $(n+1)$-point left and right Radau quadrature formulae associated with $S_{n,3}$, i.e., determined uniquely by the property $R[Q_{n+1}^{R,l};f]=R[Q_{n+1}^{R,r};f]=0$ for every $f\in S_{n,3}$. Then:*

(a) *The error constants of $Q_{n+1}^{R,l}$ and $Q_{n+1}^{R,r}$ are given by*

$$c_3\big(Q_{n+1}^{R,l}\big)=-c_3\big(Q_{n+1}^{R,r}\big)=\frac{\sqrt{3}}{216n^3}-\frac{\sqrt{3}}{108n^4}\sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1}. \tag{1.10}$$

With $*$ standing for both r and l, the following inequalities hold true for all $n\geq 4$:

$$\frac{\sqrt{3}}{216n^3}-\frac{1}{269.13n^4}<\big|c_3\big(Q_{n+1}^{R,*}\big)\big|<\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}. \tag{1.11}$$

(b) *Let $f\in C^3[0,1]$. With $*$ standing for both r and l, $R[Q_{n+1}^{R,*};f]=o(n^{-3})$ as $n\to\infty$ if and only if $f''(0)=f''(1)$. Moreover, if $\operatorname{sign}\{f''(1)-f''(0)\}=\epsilon\neq 0$, then there exists $n_0\in\mathbb{N}$ such that $\epsilon\, R[Q_{n+1}^{R,r};f]\leq 0$ and $\epsilon\, R[Q_{n+1}^{R,l};f]\geq 0$ for all $n\geq n_0$.*

(c) *If $f\in W_1^3[0,1]$ and $f'''\geq 0$ a.e. in $[0,1]$, then for all $n\geq 2$,*

$$0\geq R\big[Q_{2n+1}^{R,r};f\big]\geq R\big[Q_{n+1}^{R,r};f\big]\quad \text{and}\quad 0\leq R\big[Q_{2n+1}^{R,l};f\big]\leq R\big[Q_{n+1}^{R,l};f\big].$$

Remark 1.3. Theorem C and Theorem 1.1 provide the following improvement of the estimates (1.6) and (1.7) for the error constants of the optimal positive definite quadrature formulae of orders two and three with $n+1$ nodes, $n\geq 4$:

$$c_{n+1,3}^{+}\leq c_3\big(Q_{n+1}^{R,l}\big)<\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4},$$
$$c_{n+1,4}^{+}\leq c_4\big(Q_{n+1}^{G}\big)<\frac{1}{720n^4}-\frac{1}{552n^5}.$$

As a consequence we have the following

Corollary 1.1. (a) *If $f\in C^3[0,1]$ and $f'''\geq 0$ in $[0,1]$, then for all $n\geq 4$,*

$$\begin{aligned}0\leq R\big[Q_{n+1}^{R,l};f\big]&\leq\Big(\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}\Big)\|f'''\|,\\ 0\geq R\big[Q_{n+1}^{R,r};f\big]&\geq-\Big(\frac{\sqrt{3}}{216n^3}-\frac{1}{269.14n^4}\Big)\|f'''\|.\end{aligned} \tag{1.12}$$

(b) *If* $f \in C^4[0,1]$ *and* $f^{(4)} \geq 0$ *in* $[0,1]$, *then for all* $n \geq 4$

$$0 \leq R\big[Q^G_{n+1}; f\big] \leq \Big(\frac{1}{720n^4} - \frac{1}{552n^5}\Big)\|f^{(4)}\|. \tag{1.13}$$

Alternative error estimates are provided by the following theorem.

Theorem 1.2. (a) *If* $f \in C^3[0,1]$ *and* $f''' \geq 0$ *in* $[0,1]$, *then for all* $n \geq 2$,

$$\begin{aligned} 0 &\leq R\big[Q^{R,l}_{n+1}; f\big] \leq \frac{\sqrt{3}}{108n^3}\Big(f''(1) - f''(0)\Big), \\ 0 &\geq R\big[Q^{R,r}_{n+1}; f\big] \geq \frac{\sqrt{3}}{108n^3}\Big(f''(0) - f''(1)\Big). \end{aligned} \tag{1.14}$$

(b) *If* $f \in C^4[0,1]$ *and* $f^{(4)} \geq 0$ *in* $[0,1]$, *then for all* $n \geq 2$,

$$0 \leq R\big[Q^G_{n+1}; f\big] \leq \frac{1}{384n^4}\Big(f'''(1) - f'''(0)\Big). \tag{1.15}$$

Since the supremum norms of f''' and $f^{(4)}$ may be not accessible or difficult to evaluate, evidently the error bounds in Theorem 1.2 are easier to apply than those in Corollary 1.1. Even in the cases when $\|f'''\|$ or $\|f^{(4)}\|$ is known, it can still happen that the estimates in Theorem 1.2 are superior to those from Corollary 1.1.

Before concluding this section, we find appropriate to briefly mention a few more facts about Peano kernel representation of the remainders of quadrature formulae, for more details the reader is referred to [5].

It follows from (1.3) that the requirement $K_r(Q; u) = 0$ for some $u \in (0,1)$ is equivalent to $I[f_u] = Q[f_u]$, where $f_u(x) = (x-u)_+^{r-1}$. Hence, in order that Q evaluates to the exact value definite integrals of functions from a linear space of splines of degree $r-1$ with maximal dimension, it is necessary that the monospline $K_r(Q;\cdot)$ has the maximal possible number of zeros in $(0,1)$. The problem of the existence and uniqueness of monosplines satisfying boundary conditions and having maximal number of prescribed zeros in $(0,1)$ (the fundamental theorem of algebra for monosplines) has been resolved by Karlin and Micchelli [7]. Quadrature formulae corresponding to monosplines of the form (1.4)–(1.5) with maximal number of preassigned zeros in $(0,1)$ are called Gauss-type quadratures associated with the space of spline functions of degree $r-1$ having knots at these zeros. The results in [7] assert that Gauss-type quadratures for spaces of splines exist and are unique, and as in the case of classical Gauss-type quadratures associated with spaces of algebraic polynomials, all their weights are positive.

We finally point out that, in view of (1.2), a quadrature formula Q is definite of order r if and only if $ADP(Q) = r-1$ and $K_r(Q;t)$ does not change its sign in $(0,1)$. Therefore, all zeros of $K_r(Q;\cdot)$ in $(0,1)$ must have even multiplicities.

Theorem 1.1 is proved in the next section, and in Section 3 we present the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In view of Remark 1.1, it suffices to prove only the claims of Theorem 1.1 concerning $Q^{R,r}_{n+1}$. We denote the right Radau quadrature formula associated with $S_{n,3}$ by

$$Q^{R,r}_{n+1}[f] = \sum_{k=0}^{n-1} a_k\, f(\tau_k) + a_n\, f(1), \quad 0 < \tau_0 < \cdots < \tau_{n-1} < 1,$$

where, for the sake of simplicity, we skip the second indices in the weights and nodes (we also write $x_k = k/n$, $k = 0, \ldots, n$, see (1.8)).

According to Theorem B, we have

$$\tau_k = \frac{k + \theta_k}{n}, \quad k = 0, \ldots, n-1, \tag{2.1}$$

with

$$\theta_0 = \frac{1}{3}, \quad \theta_k = \frac{1 - \theta_{k-1}}{5 - 6\theta_{k-1}}, \quad k = 1, \ldots, n-1, \tag{2.2}$$

$$a_k = \frac{1}{6n\theta_k(1 - \theta_k)}, \quad 0 \le k \le n-1, \tag{2.3}$$

and

$$a_n = \frac{2 - 3\theta_{n-1}}{6n(1 - \theta_{n-1})}. \tag{2.4}$$

Lemma 2.1. *The sequence $\{\theta_k\}$ in (2.2) has the explicit representation*

$$\theta_k = \frac{s_k}{s_k + s_{k+1}}, \quad s_k = (2 + \sqrt{3})^k + (2 - \sqrt{3})^k, \quad k \in \mathbb{N}.$$

Proof. We apply induction with respect to k. The statement is true for $k = 0$, since $s_0 = 2$ and $s_1 = 4$. Assuming $\theta_{k-1} = \dfrac{s_{k-1}}{s_{k-1} + s_k}$ for some $k \in \mathbb{N}$, then

$$\theta_k = \frac{1 - \theta_{k-1}}{5 - 6\theta_{k-1}} = \frac{s_k}{5s_k - s_{k-1}} \stackrel{?}{=} \frac{s_k}{s_k + s_{k+1}}.$$

The last equality follows from the identity $s_{k-1} + s_{k+1} = 4s_k$, which is verified using $(2 \pm \sqrt{3})^2 + 1 = 4(2 \pm \sqrt{3})$. This accomplishes the induction step and thereby the proof of Lemma 2.1. The proposed method of proof does not give a clue about the way the explicit form of the solution of this recurrence equation was deduced. Equations like (2.2) are called Riccati difference equations, see e.g. [4] for a general approach to their solutions. □

For $f \in C^3[0, 1]$ the remainder of $Q^{R,r}_{n+1}$ admits the representation

$$R[Q^{R,r}_{n+1}; f] = \int_0^1 K_3(Q^{R,r}_{n+1}; t) f'''(t)\, dt,$$

where, according to (1.5),

$$K_3(Q^{R,r}_{n+1};t) = -\frac{t^3}{6} + \frac{1}{2}\sum_{i=0}^{n-1} a_i(t-\tau_i)_+^2 \le 0, \quad t \in (0,1).$$

The zeros of $K_3(Q^{R,r}_{n+1};\cdot)$ in $(0,1)$ are $\{x_k\}_{k=1}^{n-1}$, and each of them is double. The error constant of $Q^{R,r}_{n+1}$ is given by

$$c_3(Q^{R,r}_{n+1}) = \int_0^1 K_3(Q^{R,r}_{n+1};t)\,dt = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} K_3(Q^{R,r}_{n+1};t)\,dt =: \sum_{k=0}^{n-1} I_k. \tag{2.5}$$

Clearly,

$$I_k = \frac{1}{24}\big(x_k^4 - x_{k+1}^4\big) + \frac{1}{6}\,J_k, \tag{2.6}$$

where

$$J_k = \sum_{i=0}^{k} a_i(x_{k+1}-\tau_i)^3 - \sum_{i=0}^{k-1} a_i(x_k-\tau_i)^3.$$

By using $x_{k+1} - x_k = 1/n$, we obtain

$$\begin{aligned}
J_k &= \frac{1}{n}\sum_{i=0}^{k} a_i\Big[(x_{k+1}-\tau_i)^2 + (x_{k+1}-\tau_i)(x_k-\tau_i) + (x_k-\tau_i)^2\Big] - a_k(\tau_k - x_k)^3\\
&= \frac{1}{n}\sum_{i=0}^{k} a_i\Big[2(x_{k+1}-\tau_i)^2 - \frac{1}{n}(x_{k+1}-\tau_i) + (x_k-\tau_i)^2\Big] - a_k(\tau_k - x_k)^3\\
&= \frac{2}{n}Q^{R,r}_{n+1}\big[(x_{k+1}-\cdot)_+^2\big] - \frac{1}{n^2}Q^{R,r}_{n+1}\big[(x_{k+1}-\cdot)_+\big] + \frac{1}{n}Q^{R,r}_{n+1}\big[(x_k-\cdot)_+^2\big]\\
&\quad + a_k(\tau_k - x_k)^2(x_{k+1}-\tau_k).
\end{aligned}$$

Since $Q^{R,r}_{n+1}[f] = I[f]$ for every $f \in S_{n,3}$, we have

$$\begin{aligned}
J_k &= \frac{2}{n}I\big[(x_{k+1}-\cdot)_+^2\big] - \frac{1}{n^2}I\big[(x_{k+1}-\cdot)_+\big] + \frac{1}{n}I\big[(x_k-\cdot)_+^2\big]\\
&\quad + a_k(\tau_k - x_k)^2(x_{k+1}-\tau_k)\\
&= \frac{2}{3n}x_{k+1}^3 - \frac{1}{2n^2}x_{k+1}^2 + \frac{1}{3n}x_k^3 + a_k(\tau_k - x_k)^2(x_{k+1}-\tau_k).
\end{aligned}$$

Substituting this expression for J_k in (2.6) and replacing x_k, x_{k+1}, a_k and τ_k using (1.8), (2.1) and (2.3), we obtain

$$I_k = \frac{2\theta_k - 1}{72n^4}. \tag{2.7}$$

By using Lemma 2.1, we find

$$
\begin{aligned}
1-2\theta_k &= \frac{s_{k+1}-s_k}{s_{k+1}+s_k} = \frac{(\sqrt{3}+1)(2+\sqrt{3})^k-(\sqrt{3}-1)(2-\sqrt{3})^k}{(3+\sqrt{3})(2+\sqrt{3})^k+(3-\sqrt{3})(2-\sqrt{3})^k} \\
&= \frac{\sqrt{3}}{3}\,\frac{(2+\sqrt{3})^k-\frac{\sqrt{3}-1}{\sqrt{3}+1}(2-\sqrt{3})^k}{(2+\sqrt{3})^k+\frac{\sqrt{3}-1}{\sqrt{3}+1}(2-\sqrt{3})^k} = \frac{\sqrt{3}}{3}\,\frac{(2+\sqrt{3})^k-(2-\sqrt{3})^{k+1}}{(2+\sqrt{3})^k+(2-\sqrt{3})^{k+1}} \\
&= \frac{\sqrt{3}}{3}\Big(1-\frac{2(2-\sqrt{3})^{k+1}}{(2+\sqrt{3})^k+(2-\sqrt{3})^{k+1}}\Big) \\
&= \frac{\sqrt{3}}{3}\Big(1-\frac{2}{(2+\sqrt{3})^{2k+1}+1}\Big).
\end{aligned}
$$

By plugging this expression in (2.7), we arrive at

$$
I_k = -\frac{\sqrt{3}}{216n^4}\Big(1-\frac{2}{(2+\sqrt{3})^{2k+1}+1}\Big), \qquad k=0,\dots,n-1. \tag{2.8}
$$

The representation (1.10) of $c_3(Q_{n+1}^{R,r})$ in Theorem 1.1(a) now follows from (2.5) and (2.8). As was already mentioned, $c_3(Q_{n+1}^{R,l}) = -c_3(Q_{n+1}^{R,r})$. The two-sided estimates (1.11) are derived using the inequalities

$$
\sum_{k=0}^{3}\frac{1}{(2+\sqrt{3})^{2k+1}+1} \le \sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1} \le \sum_{k=0}^{3}\frac{1}{(2+\sqrt{3})^{2k+1}+1} + \sum_{k=4}^{\infty}\frac{1}{(2+\sqrt{3})^{2k+1}}.
$$

With this Theorem 1.1(a) is proved, and we proceed with the proof of part (b). If $f\in C^3[0,1]$, then by the mean value theorem,

$$
R[Q_{n+1}^{R,r};f] = \sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} K_3(Q_{n+1}^{R,r};t)f'''(t)\,dt = \sum_{k=0}^{n-1} I_k\,f'''(\xi_k)
$$

with $\xi_k\in(x_k,x_{k+1})$, $k=0,\dots,n-1$. We split the last sum into two parts:

$$
R[Q_{n+1}^{R,r};f] = -\frac{\sqrt{3}}{216n^3}\sum_{k=0}^{n-1}\frac{1}{n}\,f'''(\xi_k) + \sum_{k=0}^{n-1}\Big[I_k+\frac{\sqrt{3}}{216n^4}\Big]\,f'''(\xi_k) =: A+B.
$$

The sum in A is a Riemann sum for the continuous (hence integrable) function f''' on $[0,1]$, therefore

$$
A = -\frac{\sqrt{3}}{216n^3}\,\big(f''(1)-f''(0)\big) + o(n^{-3}).
$$

For B we have, in view of (2.8),

$$
B = \frac{\sqrt{3}}{108n^4}\sum_{k=0}^{n-1}\frac{1}{(2+\sqrt{3})^{2k+1}+1}f'''(\xi_k) = O(n^{-4}).
$$

Hence,

$$R[Q_{n+1}^{R,r};f] = A + B = -\frac{\sqrt{3}}{216n^3}\big(f''(1) - f''(0)\big) + o(n^{-3}),$$

which proves Theorem 1.1(b) for the remainders of $Q_{n+1}^{R,r}$.

For the proof of Theorem 1.1(c) we need the estimate for the number of zeros of a spline function in a given interval (a,b), provided by the Budan-Fourier theorem for splines. For a real-valued function f defined on the finite interval $[a,b]$, $Z_f(a,b)$ stands for the total number of the zeros of f in (a,b) counted with their multiplicities. By $S^-(a_1,a_2,\dots,a_m)$ and $S^+(a_1,a_2,\dots,a_m)$ we denote the number of strong and weak sign changes, respectively, in the finite sequence of real numbers $a_1,a_2,\dots,a_m$.

Lemma 2.2 ([8], Theorem 2.1). *If f is a polynomial spline function of exact degree r on (a,b) (i.e., of degree r with $f^{(r)}(t) \neq 0$ for some $t \in (a,b)$ with finitely many (active) knots in (a,b), all simple), then*

$$\begin{aligned} Z_f(a,b) \le Z_{f^{(r)}}(a,b) + S^-\big(f(a), f'(a), \dots, f^{(r-1)}(a), f^{(r)}(\sigma+)\big) \\ - S^+\big(f(b), f'(b), \dots, f^{(r-1)}(b), f^{(r)}(\tau-)\big), \end{aligned}$$

where $[\sigma,\tau] \subset [a,b]$ is the largest interval such that $f^{(r)}(\sigma+) \neq 0$ and $f^{(r)}(\tau-) \neq 0$.

The difference $s(t) = K_3(Q_{n+1}^{R,r};t) - K_3(Q_{2n+1}^{R,r};t)$ of the third Peano kernels of the right Radau quadrature formulae associated with $S_{n,3}$ and $S_{2n,3}$ is a spline function of degree two with $3n$ knots in $(0,1)$, which has double zeros at the points $x_k = k/n$, $k = 1,\dots,n-1$. In view of (1.4) and (1.5), s can be represented in two alternative ways,

$$s(t) = \frac{1}{2}\Big(\sum_{k=0}^{n-1} a_{k,n}(t-\tau_{k,n})_+^2 - \sum_{k=0}^{2n-1} a_{k,2n}(t-\tau_{k,2n})_+^2\Big), \tag{2.9}$$

$$s(t) = \frac{1}{2}\Big(\sum_{k=0}^{2n} a_{k,2n}(\tau_{k,2n}-t)_+^2 - \sum_{k=0}^{n} a_{k,n}(\tau_{k,n}-t)_+^2\Big). \tag{2.10}$$

Recall that all weights $a_{k,n}$ and $a_{k,2n}$ of Radau quadrature formulae are positive, therefore $s(t)$ is a spline function of exact degree two. Indeed,

$$s''(t) = \sum_{k=0}^{n-1} a_{k,n}(t-\tau_{k,n})_+^0 - \sum_{k=0}^{2n-1} a_{k,2n}(t-\tau_{k,2n})_+^0$$

is a piecewise constant function whose n positive jumps cannot be canceled out by the $2n$ negative jumps. This observation implies also that the number of sign changes of s'' in $(0,1)$ does not exceed $2n$, i.e.,

$$Z_{s''}(0,1) \le 2n. \tag{2.11}$$

By Theorem B, $\tau_{0,2n} = \dfrac{1}{6n} < \tau_{0,n} = \dfrac{1}{3n}$, therefore $s''(\tau_{0,2n}+) = -a_{0,2n} < 0$ while $s(t) \equiv 0$ for $t \in [0, \tau_{0,2n})$. From $\tau_{n,n} = \tau_{2n,2n} = 1$ and (2.10) we obtain $s''(1-) = a_{2n,2n} - a_{n,n}$. We shall show that $a_{2n,2n} - a_{n,n} \neq 0$, in fact,

$$a_{2n,2n} - a_{n,n} < 0. \tag{2.12}$$

From (2.4) and Lemma 2.1 we find

$$a_{n,n} = \frac{2s_n - s_{n-1}}{6n\, s_n},$$

hence (2.12) is equivalent to inequality

$$2\,\frac{s_{n-1}}{s_n} - \frac{s_{2n-1}}{s_{2n}} < 2,$$

which obviously is true since $0 < s_{k-1} < s_k$, $k \in \mathbb{N}$.

Lemma 2.2 applied with $r = 2$, $f = s$, $[a, b] = [0, 1]$ and $[\sigma, \tau] = [\tau_{0,2n}, 1]$ yields

$$\begin{aligned} Z_s(0,1) &\le Z_{s''}(0,1) + S^-\big(s(0), s'(0), s''(\tau_{0,2n}+)\big) - S^+\big(s(1), s'(1), s''(1-)\big) \\ &\le 2n + S^-\big(0, 0, -a_{0,2n}\big) - S^+\big(0, 0, a_{2n,2n} - a_{n,n}\big) \\ &\le 2n - 2. \end{aligned}$$

Recalling that s has double zeros at the points k/n, $k = 1, \ldots, n-1$, we conclude that s has no other zeros in $(0, 1)$. Since $s(1) = s'(1) = 0$ and $s''(1-) < 0$, it follows that $s(t) \le 0$, $t \in (0, 1)$, i.e.,

$$K_3(Q_{n+1}^{R,r}; t) \le K_3(Q_{2n+1}^{R,r}; t) \le 0, \quad t \in (0, 1).$$

If $f \in W_1^3[0, 1]$ and $f'''(t) \ge 0$ a.e. in $[0, 1]$, then

$$R\big[Q_{n+1}^{R,r}; f\big] - R\big[Q_{2n+1}^{R,r}; f\big] = \int_0^1 s(t) f'''(t)\, dt \le 0,$$

and consequently

$$R\big[Q_{n+1}^{R,r}; f\big] \le R\big[Q_{2n+1}^{R,r}; f\big] \le 0.$$

With this Theorem 1.1(c) is proved. □

Figure 1 illustrates the situation when $n = 4$. Its left part depicts the graphs of the third Peano kernels of the 5-point and 9-point right Radau quadrature formulae. We observe that the difference of the two Peano kernels, depicted on the right, vanishes on the interval $[0, \tau_{0,9}]$.

3. Proof of Theorem 1.2

Proof of Theorem 1.2(a). In view of Remark 1.1, it suffices to prove the estimates (1.14) only for $R\big[Q_{n+1}^{R,r}; f\big]$. We apply the argument from [9] to the proof of (1.6), comparing $K_3(Q_{n+1}^{R,r}; t)$ with the adjusted one-periodic Bernoulli monospline

$$g(t) = \frac{1}{n^3}\Big(B_3(\theta) - B_3\big(\{nt + \theta\}\big)\Big), \quad \theta = \frac{3 + \sqrt{3}}{6}. \tag{3.1}$$

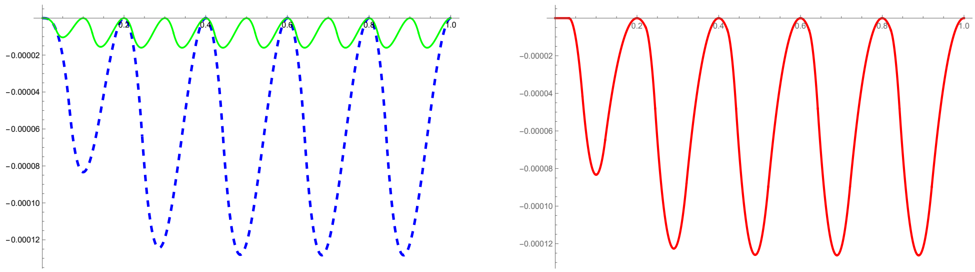


Figure 1. Left: graphs of $K_3(Q_5^{R,r};t)$ (dashed) and $K_3(Q_9^{R,r};t)$ (solid);
Right: graph of $K_3(Q_5^{R,r};t) - K_3(Q_9^{R,r};t)$.

Here, $\{\cdot\}$ is the fractional part function, B_3 is the third Bernoulli polynomial with leading coefficient $1/6$,

$$B_3(t) = \frac{1}{6}\Big(t^3 - \frac{3}{2}t^2 + \frac{1}{2}t\Big),$$

and

$$B_3(\theta) = -\|B_3\| = -\frac{\sqrt{3}}{216}.$$

We need the following properties of $g(t)$, defined in (3.1):

(i) The zeros of $g(t)$ in $(0,1)$ are $x_k = \dfrac{k}{n}$, $k = 1,\dots,n-1$, each of them double;

(ii) $g(t)$ satisfies the inequalities

$$-\frac{\sqrt{3}}{108n^3} \le g(t) \le 0, \quad t \in [0,1];$$

(iii) $g(t)$ has n simple knots in $(0,1)$ located at the points $\dfrac{k-\theta}{n}$, $k = 1,\dots,n$;

(iv) $g(0) = g'(0) = g(1) = g'(1) = 0$ and

$$g''(0+) = g''(1-) = \frac{1-2\theta}{2n} < 0.$$

The set of zeros of $K_3(Q_{n+1}^{R,r};t)$ in $(0,1)$ coincides with that of the zeros of $g(t)$, namely, the double zeros at x_k, $k = 1,\dots,n-1$. Furthermore,

$$\begin{aligned} &K_3(Q_{n+1}^{R,r};0) = K_3'(Q_{n+1}^{R,r};0) = K_3(Q_{n+1}^{R,r};1) = K_3'(Q_{n+1}^{R,r};1) = 0,\\ &K_3''(Q_{n+1}^{R,r};0+) = 0, \quad K_3''(Q_{n+1}^{R,r};1-) = a_{n,n} > 0. \end{aligned} \tag{3.2}$$

Then $s(t) = g(t) - K_3(Q_{n+1}^{R,r};t)$ is a spline function of degree two with $2n$ knots in $(0,1)$. We apply Lemma 2.2 to s and obtain

$$\begin{aligned} 2n-2 \le Z_s(0,1) &\le Z_{s''}(0,1) + S^-\big(s(0),s'(0),s''(0+)\big) - S^+\big(s(1),s'(1),s''(1-)\big)\\ &\le 2n + S^-\Big(0,0,\frac{1-2\theta}{2n}\Big) - S^+\Big(0,0,\frac{1-2\theta}{2n} - a_{n,n}\Big)\\ &\le 2n-2. \end{aligned}$$

Hence, $s(t)$ has no other zeros in $(0,1)$ except the double ones at x_k, $k=1,\dots,n-1$, therefore $s(t)$ does not change its sign in $(0,1)$. From (iv) and (3.2) it follows that $s(t)\le 0$ on $[0,1]$, which together with (ii) implies

$$-\frac{\sqrt{3}}{108n^3}\le g(t)\le K_3(Q_{n+1}^{R,r};t)\le 0,\quad t\in[0,1]. \tag{3.3}$$

If $f\in C^3[0,1]$ and $f'''(t)\ge 0$, $t\in[0,1]$, then (3.3) implies

$$\begin{aligned}0\ge R[Q_{n+1}^{R,r};f]&=\int_0^1 K_3(Q_{n+1}^{R,r};t)f'''(t)\,dt\ge\min_{t\in[0,1]}K_3(Q_{n+1}^{R,r};t)\int_0^1 f'''(t)\,dt\\ &\ge\min_{t\in[0,1]}g(t)\int_0^1 f'''(t)\,dt=-\frac{\sqrt{3}}{108n^3}\big(f''(1)-f''(0)\big).\end{aligned}$$

The proof of Theorem 1.2(a) is complete. □

Figure 2 shows how close to each other are the graphs of the third Peano kernel of a right Radau quadrature formula and the associated adjusted Bernoulli monospline $g(t)$ defined in (3.1) in the case $n=4$. For larger n, except for a small neighborhood of the left end-point of the interval, the two graphs are practically undistinguishable.

Proof of Theorem 1.2(b). The argument is similar to that in the proof of part (a). The fourth Peano kernel of the $(n+1)$-point Gaussian quadrature formula

$$Q_{n+1}^G[f]=\sum_{i=1}^{n+1}a_{i,n+1}^G f(\tau_{i,n+1}^G),\quad 0<\tau_{1,n+1}<\cdots<\tau_{n+1,n+1}<1,$$

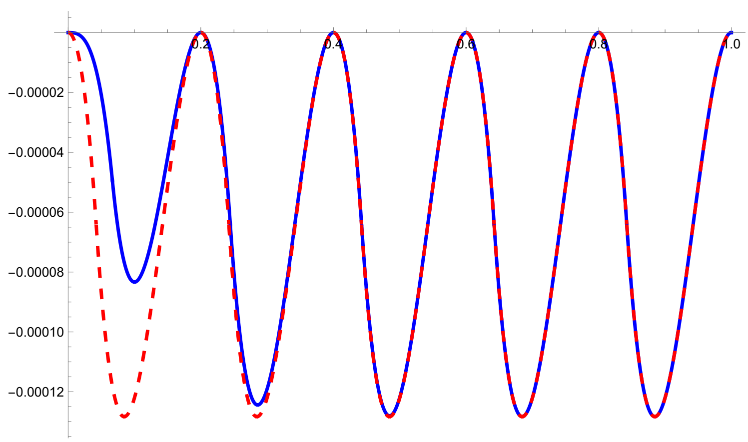


Figure 2. Graphs of Peano kernel $K_3(Q_{n+1}^{R,r};t)$ (solid) and of the associated adjusted Bernoulli monospline $g(t)$ defined in (3.4) (dashed), $n=4$.

associated with $S_{n,4}$, is compared with the adjusted Bernoulli monospline

$$g(t) = \frac{1}{n^4}\Big(B_4(\{nt\}) - B_4(0)\Big) = \frac{1}{n^4}\Big(B_4(\{nt\}) + \frac{1}{720}\Big), \tag{3.4}$$

where $B_4(t)$ is the fourth Bernoulli polynomial

$$B_4(t) = \frac{1}{24}\Big(t^4 - 2t^3 + t^2 - \frac{1}{30}\Big).$$

Now $g(t)$ is a monospline of degree four which has $n-1$ simple knots in $(0,1)$ located at the points $x_k = \dfrac{k}{n}$, $k = 1, \dots, n-1$. It follows from

$$-B_4(0) = -B_4(1) = \frac{1}{720} = \|B_4\|$$

that $g(t) \ge 0$, $t \in (0,1)$, and g has double zeros in $(0,1)$ at x_k, $k = 1, \dots, n-1$. Moreover,

$$\|g\| = \frac{1}{n^4}\Big(\max_{t\in[0,1]} B_4(t) + \frac{1}{720}\Big) = \frac{1}{384n^4}. \tag{3.5}$$

The difference $s(t) = g(t) - K_4(Q^G_{n+1}; t)$ is a spline function of degree three with $2n$ simple knots in $(0,1)$, namely, $\{x_k\}_{k=1}^{n-1} \cup \{\tau^G_{k,n+1}\}_{k=1}^{n+1}$. We have

$$\begin{aligned} &s(0) = s'(0) = s(1) = s'(1) = 0,\\ &s''(0) = s''(1) = \frac{1}{12n^2},\\ &s'''(0+) = -\frac{1}{2n}, \quad s'''(1-) = \frac{1}{2n}. \end{aligned}$$

The explicit form of $s'''(t)$ for $t \in (0,1)$ is

$$s'''(t) = -\frac{1}{2n} - \frac{1}{n}\sum_{k=0}^{n-1} (t - x_k)^0_+ + \sum_{k=1}^{n+1} a^G_{k,n+1}\big(t - \tau^G_{k,n+1}\big)^0_+.$$

Taking into account that all Gaussian weights $a^G_{k,n+1}$ are positive, we conclude that

$$Z_{s'''}(0,1) \le 2n - 1.$$

By applying Lemma 2.2 we obtain

$$\begin{aligned} 2n - 2 \le Z_s(0,1) &\le Z_{s'''}(0,1) + S^-\big(s(0), s'(0), s''(0), s'''(0+)\big)\\ &\qquad - S^+\big(s(1), s'(1), s''(1), s'''(1-)\big)\\ &\le 2n - 1 + S^-\Big(0, 0, \frac{1}{12n^2}, -\frac{1}{2n}\Big) - S^+\Big(0, 0, \frac{1}{12n^2}, \frac{1}{2n}\Big)\\ &= 2n - 2. \end{aligned}$$

Hence, the only zeros of $s(t)$ in $(0,1)$ are the double zeros at x_k, $k=1,\dots,n-1$, and $s(t)$ does not change its sign in $(0,1)$. Since $s(0)=s'(0)=0$ and $s''(0)>0$, it follows that $s(t)\geq 0$ on $[0,1]$, hence

$$g(t)\geq K_4(Q_{n+1}^G;t)\geq 0,\quad t\in[0,1]. \tag{3.6}$$

If $f\in C^4[0,1]$ and $f^{(4)}(t)\geq 0$ on $[0,1]$, then (3.4) and (3.6) imply

$$\begin{aligned}0\leq R\big[Q_{n+1}^G;f\big]&=\int_0^1 K_4(Q_{n+1}^G;t)f^{(4)}(t)\,dt\leq \max_{t\in[0,1]} g(t)\int_0^1 f^{(4)}(t)\,dt\\&=\frac{1}{384n^4}\Big(f'''(1)-f'''(0)\Big).\end{aligned}$$

The proof of Theorem 1.2(b) is complete. □

Remark 3.1. Using Lemma 2.2, error estimates analogous to those in Theorem 1.2 can be proved for all Gauss-type quadrature formulae associated with the spaces $S_{n,r}$, $r>4$, defined in (1.8). However, since the Gauss-type quadrature formulae are not known for $r>4$, these estimates are not of practical importance.

Acknowledgements

This research is supported by the Bulgarian National Science Fund under Contract KP-06-N62/4.

References

[1] B. D. Bojanov, Uniqueness of the monosplines of least deviation, in: Numerische Integration, ed. by G. Hämmerlin, ISNM 45, Birkhäuser, Basel, 1979, 67–97.
[2] B. D. Bojanov, Existence and characterization of monosplines of least L_p deviation, in: Constructive Function Theory'77, ed. by Bl. Sendov and D. Vačov, Sofia, BAN, 1980, 249–268.
[3] B. D. Bojanov, Uniqueness of the optimal nodes of quadrature formulae, Math. Comput. 36 (1981) 525–546.
[4] L. Brand, A sequence defined by a difference equation, Amer. Math. Monthly 62 (1955) 489–492.
[5] H. Brass, Quadraturverfahren, Vandenhoeck & Ruprecht, Göttingen, 1977.
[6] K. Jetter, Optimale Quadraturformeln mit semidefiniten Kernen, Numer. Math. 25 (1976) 239–249.
[7] S. Karlin and C. Micchelli, The fundamental theorem of algebra for monosplines satisfying boundary conditions, Isr. J. Math. Soc. 11 (1972) 405–451.
[8] P. Köhler and G. Nikolov, Error bounds for Gauss type quadrature formulae related to spaces of splines with equidistant knots, J. Approx. Theory 81 (1995) 368–388.
[9] P. Köhler and G. Nikolov, Error bounds for optimal definite quadrature formulae, J. Approx. Theory 81 (1995) 397–405.
[10] G. Lange, Beste und optimale definite Quadraturformeln, PhD Thesis, Technical University Clausthal, 1977.

[11] G. Lange, Optimale definite Quadraturformeln, in: Numerische Integration, ed. by G. Hämmerlin, Birkhäuser Verlag, Basel, 1979, 187–197.

[12] G. Nikolov, Gaussian quadrature formulae for splines, in: Numerische Integration, IV, ed. by G. Hämmerlin and H. Brass, ISNM 112, Birkhäuser Verlag, Basel, 1993, 267–281.

[13] G. Nikolov, On certain definite quadrature formulae, J. Comp. Appl. Math. 75 (1996) 329–343.

[14] G. Peano, Resto nelle formule di quadratura espresso con un integrale definito, Atti della Reale Accademia dei Lincei: Rendiconti Ser. 5 22 (1913) 562–569.

[15] G. Schmeisser, Optimale Quadraturformeln mit semidefiniten Kernen, Numer. Math. 20 (1972) 32–53.

[16] A. Zhensykbaev, Best quadrature formulae for some classes of periodic differentiable functions, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977) (in Russian); English translation in: Math. USSR Izv. 11 (1977) 1055–1071.

[17] A. Zhensykbaev, Monosplines and optimal quadrature formulae for certain classes of non-periodic functions, Anal. Math. 5 (1979) 301–331 (in Russian).

Geno Nikolov

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: geno@fmi.uni-sofia.bg

Petar B. Nikolov

Faculty of Pharmacy
Medical University of Sofia
2, Dunav Str.
1000 Sofia
BULGARIA

E-mail: p.nikolov@pharmfac.mu-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

BUILDING MENTAL HEALTH RESOURCES: THE HUMAN-CENTERED INTERACTION DESIGN IMPLEMENTED IN me_HeLi-D PLATFORM

ELITSA PELTEKOVA, DAFINKA MITEVA and IOANNIS PATIAS

This paper analyzes user feedback on the Moodle-based digital mental health literacy platform developed under the Erasmus+ project “me_HeLi-D: Mental Health Literacy and Diversity. Enhancing Mental Health and Resilience through digital Resources for Youth” (me_HeLi-D). The platform aimed to provide interactive mental health literacy content, and supported by user feedback contribute for engagement and improve learning outcomes. A mixed-methods approach, combining structured surveys and open-ended comments, was used to collect data on user experiences. Feedback was systematically categorized into general comments, module-specific critiques, and technical issues. Analysis helped identify recurring issues related to navigation, content accessibility, and technical difficulties with multimedia components, particularly with the multimedia functionality. Inconsistencies like the lack of availability for a “back to course” option confused users, and duplicated titles in some modules created additional navigational obstacles. Technical issues, such as the multimedia functionality, made content engagement difficult. These findings underscore the critical importance of user-centered design principles in developing effective digital learning environments. The study highlights the need for rigorous testing across diverse devices and operating systems to ensure accessibility and functionality. Furthermore, the feedback emphasizes the demand for more interactive learning experiences, suggesting that incorporating active learning strategies can enhance engagement and knowledge retention. Recommendations for platform revisions include addressing navigation inconsistencies, improving multimedia functionality, and incorporating interactive elements to promote active learning. This study contributes to the growing body of knowledge on best practices for designing and evaluating digital mental health literacy interventions and emphasizes the iterative nature of development based on continuous user feedback.

Keywords: mental health literacy (MHL), digital mental health literacy (DMHL), information technology (IT), interactive digital resources, human-centered design, H5P, Moodle

CCS Concepts:

• Human-centered computing∼Interaction design∼Empirical studies in interaction design (High Relevance)

DOI: 10.60063/GSU.FMI.112.123-135

1. Introduction

The Erasmus+ project "me_HeLi-D: Mental Health Literacy and Diversity. Enhancing Mental Health and Resilience through digital Resources for Youth" (me_HeLi-D) [7] recognized the potential of digital tools to enhance 'mental health literacy' [4] among young people. The me_HeLi-D project aimed to develop and implement a digital mental health literacy platform using Moodle, a widely used open-source learning management system [2]. Moodle provides a flexible framework for creating interactive learning environments, allowing for the integration of various multimedia elements and facilitating communication between learners and educators. The me_HeLi-D platform was designed to provide young people with access to engaging and informative content related to mental health, covering topics such as common mental health conditions, recognizing symptoms, seeking help, and promoting resilience. The platform aimed to not only increase knowledge about mental health but also to foster positive attitudes towards mental health and reduce stigma associated with mental illness.

The rapid advancement of digital technologies has revolutionized various aspects of modern life, including healthcare and education. This so-called digital transformation opened up new horizons for addressing critical public health challenges, such as mental health literacy, which has an additional importance particularly among youth. Mental health literacy, defined as the knowledge and beliefs about mental disorders which aid in their recognition, management, or prevention [1], is a crucial factor in promoting help-seeking behavior, reducing stigma, and improving overall mental well-being [10]. Young people, facing unique developmental challenges and often experiencing the onset of mental health conditions during adolescence or young adulthood [6], represent a key target group for mental health literacy interventions. However, traditional approaches to mental health education may not always reach or effectively engage this population. Digital platforms, with their accessibility, flexibility, and potential for interactive learning experiences, offer a promising alternative and supplement to traditional methods.

The me_HeLi-D platform is designed to support digital learning through interactive modules. It is organized into thematic units, each containing multimedia content, self-assessment tools, and discussion forums. The interaction design follows principles of intuitive navigation, responsive feedback, and accessibility compliance. Users engage with curated content including instructional videos, annotated readings, and peer-reviewed assignments.

The development of the me_HeLi-D platform was grounded in the principles of human-centered computing and interaction design principles [9]. The user-centered design emphasizes on importance of understanding the needs, preferences, and experiences of the target users throughout the design and development process. This approach recognizes that technology is most effective when it is tailored to the specific needs and characteristics of the individuals who will be using it. In the context of digital mental health literacy interventions, user-centered design is particularly critical, as it can influence the acceptability, usability, and ultimately, the effectiveness of the intervention [3]. By involving young people in the development process, the me_HeLi-D project sought to create a platform that was relevant, engaging, and accessible to its target audience.

While the me_HeLi-D platform held great promise for enhancing mental health literacy, the project team recognized the importance of evaluating the platform's effectiveness and identifying areas for improvement [11]. User feedback plays a crucial role in formative evaluation, providing valuable insights into user experiences and informing iterative design processes [8]. Collecting and analyzing user feedback allows developers to identify potential usability issues, technical problems, and areas where the content or design can be enhanced to better meet the needs of the target audience [5]. This iterative approach to development, where feedback is continuously gathered and incorporated into platform revisions, is essential for creating effective and sustainable digital health interventions [7].

This paper presents an analysis of user feedback received on the me_HeLi-D digital mental health literacy platform. The study employed a mixed-methods approach, combining structured surveys and open-ended comments to gather rich data on user experiences. The feedback was systematically categorized and analyzed to identify recurring themes and specific suggestions for improvement.

This study is guided by the following research questions:

- Which design, and technical aspects hinder or facilitate engagement and learning in me_HeLi-D platform?

  and

- How can the me_HeLi-D platform be improved to enhance learning effectiveness?

The findings of this study will provide valuable insights for the me_HeLi-D project team and other developers working in the field of digital mental health literacy. Furthermore, this research underscores the importance of user-centered design and continuous feedback in the development of effective digital learning environments. The ultimate goal is to create digital resources that empower young people to improve their mental health literacy and promote their overall well-being [7].

2. Methodology

The study employed a mixed-methods approach to investigate user feedback on the me_HeLi-D digital mental health literacy platform. Mixed-methods research, combining both qualitative and quantitative data collection and analysis techniques, allows a wider understanding of complex user expectations than either approach alone. In this context, qualitative data provided rich insights into the nuances of user experiences, while quantitative data helped to identify patterns and trends in user feedback. This combined approach facilitated a deeper understanding of the platform's strengths and weaknesses and informed actionable recommendations for improvement.

2.1. Participants

Most of the participants had solid prior experience with mental health resources, but limited tech comprehensibility. They were formed by the work groups engaged in preparing the pilot deployment. Precisely, the participants were 22 adults aged 21–35, with a gender distribution of 18 females and 4 males. Ethical considerations were paramount throughout the study. All participants provided informed consent before participating in the study. Participants were assured of anonymity and confidentiality, and all data were stored securely.

2.2. Data collection

As mentioned, data were collected using a combination of structured surveys and open-ended feedback prompts. This approach allowed for the collection of both quantifiable data and rich qualitative insights into user experiences.

Structured surveys. The structured survey consisted of the specific aspects of the platform it assesses, namely, navigation, content, design, technical functionality. All modules, and all activities were on the survey. For the responses a simple scale was used providing multiple choice. The survey included user satisfaction with the platform's navigation, content relevance, and technical performance. Items were rated on a 3-point scale ranging from 'Disagree', 'Neutral' to 'Agree'. Examples of survey questions included: 'The platform was easy to navigate' and 'The content was relevant to my needs'. The survey was administered online using direct access to a spreadsheet.

Open-ended feedback. In addition to the structured survey, participants were provided with open-ended feedback prompts to elicit more detailed information about their experiences with the platform. Participants were asked to provide open-ended feedback on all aspects of the platform. These open-ended responses allowed participants to express their thoughts and feelings about the platform in their own words, providing valuable contextual information.

2.3. Data analysis

The data collected through the structured surveys were analyzed using descriptive statistics. This analysis provided an overview of user ratings and identified areas of general satisfaction or dissatisfaction. The qualitative data from the open-ended feedback prompts were analyzed by subject. This analysis was used for identifying, organizing, and interpreting patterns of meaning within qualitative data. Finally, both the qualitative and quantitative findings were integrated to provide a comprehensive understanding of user experiences with the me_HeLi-D platform.

3. Results

This section presents the findings from the analysis of user feedback on the me_HeLi-D digital mental health literacy platform. The results are organized according to the main themes that emerged from the thematic analysis of the open-ended feedback and are supplemented by quantitative data from the structured surveys where relevant. The themes reflect the key areas of user experience with the platform, highlighting both strengths and weaknesses.

3.1. Navigation and user interface

A significant portion of user feedback focused on the platform's navigation and user interface. While some participants found the platform easy to navigate, a substantial percentage reported difficulties in finding information and moving between different sections (see Table 1).

Table 1

Navigation and user interface-related feedback

Navigation challenge	Percentage of users reporting issue	Example user comment	Suggested improvement
Inconsistent "Back" button functionality	82%	"Sometimes the 'back' button took me to the previous page, other times it disappeared."	Implement a consistent 'back' button across all modules.
Difficulty finding specific modules	68%	"I couldn't easily find the module on stress management. The menu was confusing."	Improve the organization and labeling of the main menu.
Confusing module titles	45%	"The titles of some modules were unclear and didn't tell me what the content was about."	Use more descriptive and user-friendly module titles.
Lack of visual tips	23%	"I felt lost navigating within some modules. There weren't clear visual tips to guide me."	Add visual tips, such as progress bars to show users their location.
Mobile navigation issues (e.g., menu not displaying correctly)	9%	"The menu was difficult to use on my phone. It wouldn't expand properly."	Optimize the mobile navigation for different screen sizes.

Inconsistent navigation. A recurring issue was the inconsistency in navigation controls across different modules. For instance, the "back to course" button (see Figure 1, lower left corner) was not available on all pages, leading to user frustration and confusion. One participant stated, 'I got lost several times because the back button disappeared, and I had to click through multiple pages to find my way back.' This inconsistency disrupted the user flow and made it difficult for participants to orient themselves within the platform.

Messy layout. Some users found the layout of certain modules to be messy. Several participants commented on the excessive amount of text and visual elements on some pages, making it difficult to focus on the key information. This visual overload negatively impacted the user experience and potentially hindered information processing.

Table 1 and Figure 2 summarize the most commonly reported navigation challenges. The majority of users (82%) experienced inconsistent behavior of the 'Back'

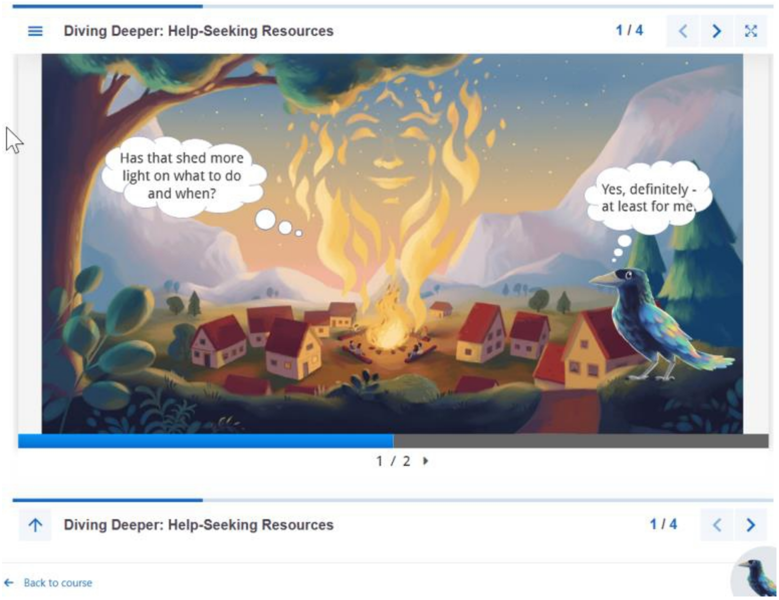


Figure 1. H5P Interactive Book Activity “H4 | Diving deeper: Help-seeking Resources”

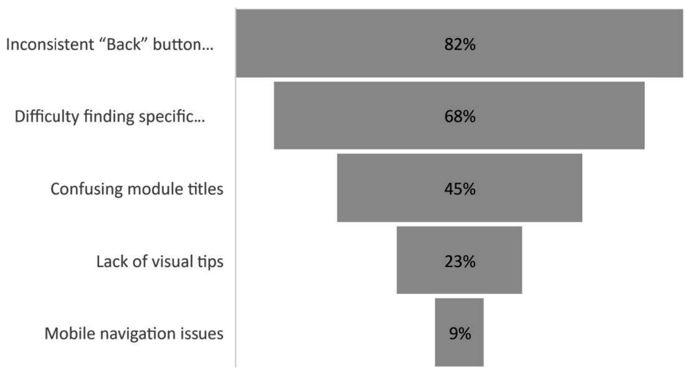


Figure 2. Navigation challenge

button, which disrupted their workflow. Additionally, 68% found it difficult to locate specific modules, citing unclear menu organization. These findings highlight the need for improved interface consistency and clearer content labeling.

3.2. Content and features

Feedback regarding the platform’s content and features was varied. While many participants appreciated the informative nature of the content, some suggestions were made for improvement (see Table 2).

Table 2

Content and features-related feedback

Content theme	Percentage of positive comments	Percentage of negative comments	Example quote (Positive)	Example quote (Negative)
Information on anxiety	68%	23%	"The information on anxiety was very helpful and clear."	"I felt the section on anxiety could have been more detailed."
Coping strategies	82%	5%	"I appreciated the practical coping strategies provided."	"Some of the coping strategies were not relevant to me."
Recognizing mental health issues	91%	9%	"The platform helped me better understand mental health."	"I already knew most of the information presented."

Content relevance. The majority of participants found the content to be relevant and informative. For example: Many participants referred to the platform for providing valuable information on mental health topics that were absolutely relevant.

Interactive elements. While the platform included some interactive elements, several participants suggested incorporating more interactive activities to enhance engagement. Participants suggested including quizzes, interactive scenarios, and videos to make the learning experience more engaging.

Table 2 presents user sentiment toward key content themes. Feedback was overwhelmingly positive (Figure 3), particularly regarding coping strategies and mental health awareness. However, 23% of users felt the anxiety section lacked depth, suggesting an opportunity for content expansion. Also, the majority of respondents expressed positive feedback across all content themes, with "Recognizing Mental Health Issues" receiving the highest approval (91%). Negative feedback was minimal, though some users desired more depth in the 'Information on Anxiety' section.

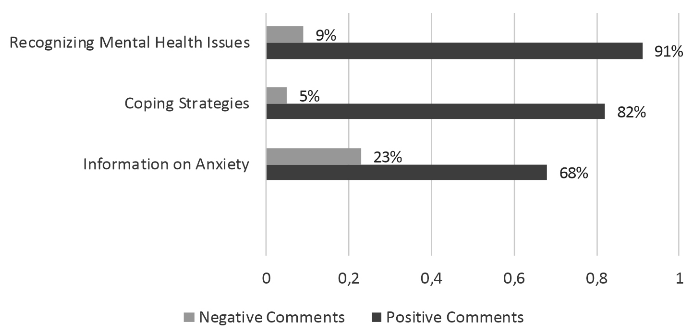


Figure 3. Content and features-related feedback

3.3. Technical issues

Multimedia functionality. Technical issues with multimedia components, particularly video playback, were frequently reported (68%). A significant percentage of participants reported problems with video playback, with some experiencing audio-only playback or buffering issues.

Technical issues, beyond those related to multimedia, also emerged as a theme in the user feedback (see Figure 4).

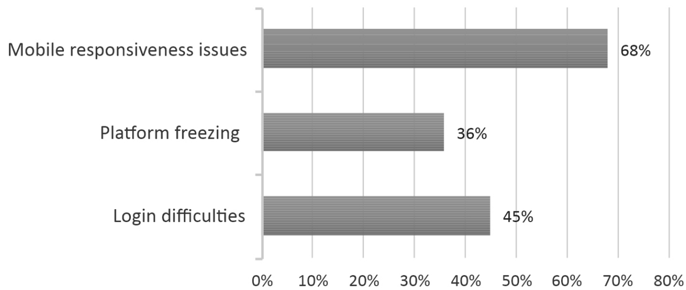


Figure 4. Technical issues-related feedback

Platform stability. Some participants reported experiencing occasional platform instability. Participants mentioned experiencing error messages or being logged out unexpectedly. Also reported "The platform froze a couple of times, and I lost my progress." These technical issues disrupted the learning process and negatively impacted the overall user satisfaction.

Login issues. Difficulties with the login process were also reported by some users. Some participants reported difficulties creating accounts or logging in to the platform. The smooth and issues-free login process is essential for ensuring user access to the platform.

Figure 4 summarizes the technical issues reported by users. Video playback and mobile responsiveness were the most common problems, each cited by 68% of respondents. These findings suggest that multimedia delivery and mobile compatibility are critical areas for platform improvement.

3.4. Overall user experience

Despite the challenges mentioned above, many participants expressed a positive overall impression of the me_HeLi-D platform. Many participants acknowledged the platform's potential as a valuable resource for mental health information. One participant summarized the overall sentiment by stating, "Despite some technical issues, I think this platform has the potential to be very helpful for young people." However, the feedback clearly highlighted areas where improvements are needed to enhance user experience and maximize the platform's effectiveness (see Table 3).

Table 3 and Figure 5 present user-suggested improvements to the HeLi-D platform. The vast majority (91%) requested more interactive activities, such as quizzes

Table 3

Overall user experience-related feedback

Suggested improvement	Percentage of suggestions	Example user suggestion
More interactive activities	91%	"Include more quizzes and interactive exercises."
Improved navigation	68%	"Make the navigation more consistent across all modules."
Enhanced multimedia content	45%	"Add more videos and multimedia examples."
Mobile optimization	36%	"Improve the platform's responsiveness on mobile devices."

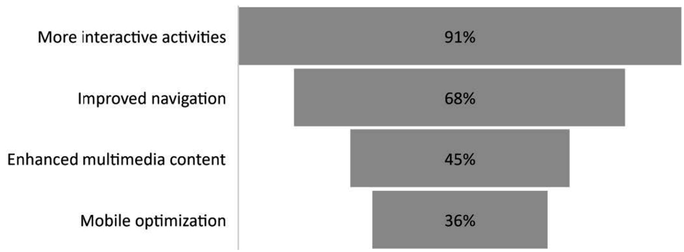


Figure 5. Suggested improvement

and exercises. Navigation enhancements (68%) and multimedia content expansion (45%) were also common themes. These insights highlight key areas for future development to improve user engagement and satisfaction.

4. Discussion

The results of this study offer valuable insights into user experiences with the me_HeLi-D digital mental health literacy platform, highlighting both its strengths and areas requiring improvement. The recurring theme of navigation difficulties underscores the critical importance of user-centered design specially in the specific area of educational platforms. Inconsistent navigation, as evidenced by the issue related to the availability of the "back to course" button and confusing module labeling, disrupts user flow and creates dissatisfaction. When users struggle to navigate the platform, their cognitive resources are diverted from learning the content to simply trying to find their way around, undervaluing the effect of knowledge acquisition.

The technical issues reported, particularly with video playback and mobile responsiveness, highlight the challenges related to ensuring accessibility and functionality across diverse devices that can support various users. In an educational platform, the users may have varying levels of access to technology and different device access and/or preference. Thus, it is of crucial importance to test and optimize the platform's performance across a range of devices and operating systems. The difficulties encountered with video playback not only limit access to multimedia content but also raise concerns about equity and inclusivity. If some users are unable to fully engage with the platform's content due to technical limitations, it can exacerbate existing disparities in access to mental health resources.

The feedback regarding content and features suggests that while users appreciated the relevance and informativeness of the content, they clearly state that they desired a more interactive learning experience. Passive consumption of information, such as simply reading text, is less effective than interactive activities that encourage learners to actively process and apply the material. Incorporating more interactive elements, such as quizzes, scenarios, and gamified activities, could significantly enhance user engagement and learning outcomes.

The study also revealed the importance of addressing technical issues beyond navigation and multimedia. Reports of platform instability and login difficulties point to the need for robust technical infrastructure and ongoing maintenance. A seamless and reliable user experience is essential for maintaining user motivation and encouraging continued engagement with the platform. If users encounter frequent technical problems, they are likely to become discouraged and abandon the platform altogether.

This study has some limitations. The sample size, 22 participants so far, may limit the generalizability of the findings to a broader population of young people. The developed environment is intended for young people (15–24 years old), but the testing was carried out with adults (21–35 years old) who had solid knowledge on the field, so this affects the final results of the study. Furthermore, the study relied on self-reported feedback, which may be subject to biases. Thus, the research is ongoing and further more results could incorporate more objective measures of user behavior and provide a more comprehensive understanding of user engagement. Despite these limitations, this study provides valuable insights into user experiences with the me_HeLi-D platform and offers practical recommendations for improvement.

5. Recommendations

Based on the findings of this study, several key recommendations can be made to enhance the me_HeLi-D digital mental health literacy platform and improve user engagement and learning outcomes.

1. *Enhance navigation and user interface.*

    - Implement consistent navigation. Ensure that navigation controls, such as the "back to course" button, are consistently available across all modules. This will prevent user confusion and improve overall platform usability.
    - Improve menu organization. Reorganize the main menu to make it easier for users to find specific modules. Consider using clear and descriptive labels and grouping related modules together.

- Provide visual tips. Incorporate visual tips, such as progress bars or breadcrumb trails, to help users track their progress and orient themselves within the platform.

2. *Improve technical functionality.*

- Thorough testing. Conduct rigorous testing of the platform across a range of devices and operating systems to identify and resolve any technical issues, particularly those related to video playback and mobile responsiveness. Multimedia optimization. Optimize multimedia content for different devices and internet connections to ensure smooth playback and accessibility for all users.
- Platform stability. Address any reported issues with platform instability and login difficulties to ensure a seamless and reliable user experience.

3. *Enhance content and features.*

- Incorporate interactive elements. Integrate more interactive activities, such as quizzes, scenarios, and gamified challenges, to promote active learning and enhance user engagement.
- Diversify content formats. Explore the use of different content formats, such as videos, infographics, and podcasts, to cater to diverse learning styles and preferences.
- Regularly update content. Ensure that the platform's content is regularly updated to reflect the latest research and best practices in mental health.

By implementing these recommendations, the me_HeLi-D platform can be significantly enhanced to provide a more engaging, effective, and accessible learning experience for young people seeking to improve their mental health literacy. The iterative nature of digital platform development necessitates a commitment to continuous evaluation and improvement, grounded in the principles of user-centered design, to maximize the platform's impact and reach.

6. Conclusions

This study provides valuable insights into user experiences interaction design with a digital mental health literacy platform, highlighting the crucial role of user feedback in the development of effective educational technology. The findings underscore the importance of prioritizing human-centered computing and interaction design principles to ensure platform usability, accessibility, and engagement. The identified navigation inconsistencies, technical issues with multimedia, and requests for more interactive content point to specific areas requiring attention. The proper addressing of those issues will be critical for maximizing the platform's potential to improve mental health literacy among youth. The study reinforces the iterative nature of digital platform development, emphasizing the need for continuous user feedback and ongoing evaluation. Future iterations of the me_HeLi-D platform should prioritize the recommendations outlined, including streamlining navigation, thoroughly testing multimedia functionality across various devices, and incorporating interactive elements to promote active learning. Further research could explore the long-term impact of the platform on mental health knowledge and behavior, as well as investigate the effectiveness of different interactive learning strategies in this context. By embracing user-centered design and prioritizing continuous improvement, digital mental health platforms can become powerful tools for promoting well-being and empowering individuals to manage their mental health.

Acknowledgements

The research of these authors was supported by me_HeLi-D: Mental Health Literacy and Diversity. Enhancing Mental Health and Resilience through Digital Resources for Youth, 2022-1-AT01-KA220-SCH-000087294, funded by the Erasmus+ Programme, Key Action 2: Partnerships for Cooperation.

References

[1] P. Boonpichachan, S. Panuthai and D. Tamdee, Mental health literacy and psychological well-being among dependent elders, Nurs. J. CMU 49(1) (2022) 186–199, https://he02.tci-thaijo.org/index.php/cmunursing/article/view/254229.

[2] S. H. P. W. Gamage, J. R. Ayres and M. B. Behrend, A systematic review on trends in using Moodle for teaching and learning, Int. J. STEM Educ. 9(9) (2022) 22 pp, https://doi.org/10.1186/s40594-021-00323-x.

[3] A. Gulliver et al., Internet-based interventions to promote mental health help-seeking in elite athletes: An exploratory randomized controlled trial, J. Med. Internet Res. 14(3) (2012) e69, https://doi.org/10.2196/jmir.1864.

[4] A. F. Jorm, A. E. Korten, P. A. Jacomb, H. Christensen, B. Rodgers and P. Pollitt, Mental health literacy: a survey of the public's ability to recognise mental disorders and their beliefs about the effectiveness of treatment, Med. J. Aust. 166(4) (1997) 182–186, https://doi.org/10.5694/j.1326-5377.1997.tb140071.x.

[5] K. Kaloyanova, N. Leventi and E. Kaloyanova, Evaluating computing students' digital skills and health literacy: A case from Bulgaria, Front. Public Health 10 (2023) 1085842, https://doi.org/10.3389/fpubh.2022.1085842.

[6] R. C. Kessler, P. Berglund, O. Demler, R. Jin, K. R. Merikangas and E. E. Walters, Lifetime prevalence and age-of-onset distributions of DSM-IV disorders in the National Comorbidity Survey Replication, Arch. Gen. Psychiatry 62(6) (2005) 593–602, https://doi.org/10.1001/archpsyc.62.6.593.

[7] me_HeLi-D, About, University of Graz, 2025, https://project.meheli-d.uni-graz.at/about/.

[8] I. Patias and K. Kaloyanova, Digital health literacy and well-being of students at Sofia University – Bulgaria, Eur. J. Public Health 32(3) (2022) https://doi.org/10.1093/eurpub/ckac129.712.

[9] B. Still and K. Crane, Fundamentals of User-Centered Design: A Practical Approach, First edition, CRC Press, 2017, https://doi.org/10.4324/9781315200927.

[10] H. Stuart, Reducing the stigma of mental illness, Glob. Ment. Health 3 (2016) e17, https://doi.org/10.1017/gmh.2016.11.

[11] M. Wright et al., Enhancing help-seeking efficacy in a sustainable and adaptive digital environment, Eur. J. Public Health 33(2) (2023) 238–239, https://doi.org/10.1093/eurpub/ckad160.606.

Received on April 14, 2025
Revised on September 10, 2025
Accepted on September 15, 2025

Elitsa Peltekova, Dafinka Miteva and Ioannis Patias

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: epeltekova@fmi.uni-sofia.bg
dafinca@fmi.uni-sofia.bg
patias@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

ALMOST ZERO AND HIGH ω-TURING DEGREES

ANDREY C. SARIEV

We show that the class of the high degrees is definable in the terms of the almost zero degrees in the local substructure of the ω-Turing degrees. Namely, we prove that a degree is high if and only if it bounds all almost zero degrees.

Keywords: Turing reducibility, degree structures, definability

2020 Mathematics Subject Classification: 03D28, 03D30

1. Introduction

Soskov [15] introduces the ω-enumeration reducibility between sequences of sets of natural numbers. As a preorder, this reducibility induces a degree structure – the upper semi-lattice $\mathcal{D}_\omega$ of the ω-enumeration degrees. Soskov also defines a jump operation, whose properties are later explored in [16]. Its local theory (i.e., the theory of the degrees below the first jump of the least ω-enumeration degree) is studied in [15, 16].

The ω-Turing reducibility $\leq_{T,\omega}$ is introduced by Sariev and Ganchev in [13] as an analogue of the ω-enumeration reducibility, which is based on the Turing reducibility. In this computational framework the informational content of a sequence is uniquely determined by the set of the Turing degrees of the sets that *code* the sequence. We say that a set codes a sequence if and only if *uniformly* in k, it can compute the k-th element of the considered sequence in its k-th Turing jump:

$$X \subseteq \omega \text{ codes } \{A_k\}_{k<\omega} \iff A_k \leq_T X^{(k)} \text{ uniformly in } k.$$

Having this, we shall say that the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ if and only if each set that codes $\mathcal{B}$ also codes $\mathcal{A}$:

$$\mathcal{A} \leq_{T,\omega} \mathcal{B} \iff (\forall X \subseteq \omega)[X \text{ codes } \mathcal{B} \Rightarrow X \text{ codes } \mathcal{A}].$$

DOI: 10.60063/GSU.FMI.112.137-152

The relation $\leq_{T,\omega}$ is a preorder on the set of the sequences of sets of natural numbers and in the standard way it induces a degree structure – the upper semi-lattice $\mathcal{D}_{T,\omega}$ of the ω-Turing degrees.

The least ω-Turing degree $\mathbf{0}$ is that one containing all sequences coded by $\varnothing$. The requirement for uniformity in the definition of coding allows us to define a new notion of "lowness" or "closeness" to $\mathbf{0}$, which has no equivalent in the Turing degrees. Indeed, note that if $\{A_k\}_{k<\omega}$ is a sequence such that for each $k < \omega$, $A_k \leq_T \varnothing^{(k)}$, then its degree is not necessarily $\mathbf{0}$, because of the lack of uniformity of the reduction. We shall refer to such kind of sequences and to the degrees containing them as *almost zero*. There are continuum many almost zero degrees so they are not bounded by any degree. Also, the substructure of all almost zero degrees is sufficiently rich to embed each countable partial order [12].

In [13] a jump operation on sequences is defined. Namely the jump $\mathcal{A}'$ of the sequence $\mathcal{A}$ is defined in such a way that for each $X \subseteq \omega$

$$X \text{ codes } \mathcal{A}' \iff (\exists Y)[X \equiv_T Y' \mathbin{\&} Y \text{ codes } \mathcal{A}].$$

Being a degree invariant, the jump on sequences induces in the standard way a jump operation in the degree structure.

The structure of the ω-Turing degrees extends in a natural way the structure of the Turing degrees. The mapping

$$\deg_T(A) \mapsto \deg_{T,\omega}(A, \varnothing, \dots, \varnothing, \dots)$$

defines an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$ which preserves the least element, the order, the l.u.b. and jump operations. Also, the both structures have an isomorphic groups of automorphisms [13].

The jump operator gives rise to the *local structure* $\mathcal{G}_{T,\omega}$ of the ω-Turing degrees – namely this is the substructure consisting of the degrees below the first jump $\mathbf{0}'$ of the least ω-Turing degree. As usual a *high/low hierarchy* can be introduced. This hierarchy partitions the local substructure on layers, based on the informational content of the jumps of the degrees. Intuitively, a degree is high$_n$ or low$_n$ according as its n-th jump takes its greatest or least value. The degrees which are high$_n$ for some n we shall refer as *high*; similarly, a degree is *low* if it is low$_n$ for some n.

The structures of ω-Turing and ω-enumeration degrees are closely related. There is a natural embedding of $\mathcal{D}_{T,\omega}$ into $\mathcal{D}_\omega$ (it is based on the natural embedding of the Turing degrees into the enumeration degrees), which preserves the least element, the l.u.b. operation and the jump, [13]. Although the both structures share a lot of common properties, they are not elementary equivalent. For example, there are minimal ω-Turing degrees [13], while the ω-enumeration degrees are dense [15].

Since the underlying structures of the Turing and the enumeration degrees are quite different, in many cases the results from $\mathcal{D}_\omega$ can not be transferred directly to $\mathcal{D}_{T,\omega}$. For example, the so called Kalimullin pairs which are extensively used in a sequence of definability results concerning both enumeration and ω-enumeration degrees, do not even exist in the Turing and the ω-Turing degrees.

This paper explores definability issues concerning the high and almost zero degrees. In [13] Sariev and Ganchev show that each of the classes of the high$_n$ and the low$_n$ degrees are first order definable in the local structure. In [5] they prove that the classes of the high and the low degrees are definable in $\mathcal{G}_{T,\omega}$. As a consequence, the first order definability of the almost zero degrees in the local structure is shown. Here we continue the investigation by showing a tighter connection between the definability of the high and the almost zero degrees. Namely, we show that in the local structure a degree is high if and only if it bounds all almost zero degrees.

A similar result holds in the structure of the ω-enumeration degrees, [16]. Namely Soskov and Ganchev show that for each non-high degree there is an almost zero degree which is not below it. Just like us, they use a modification of the Sacks agreement method [9, 11]. In order to make their construction below the first jump of the least element, they use the so-called *good approximation* of a sequence of sets of natural numbers. Since such approximations are not usable in the case of the ω-Turing degrees, we consider Δ^0_2-approximations for sequences – a uniform version of the Δ^0_2-approximations for sets. First, we characterize the degrees in the local structure exactly as the degrees which contain a sequence with a Δ^0_2-approximation. This allows us to build degrees below $\mathbf{0}'$. Since the Δ^0_2-approximations have a different nature form the good approximations used in [16] and the Σ^0_1-approximations used in the original Sacks' construction, specific modifications in the construction are made. For example, in order to make the constructed approximation to converge, we consider *maximum length of agreement* function, instead of the standard length of agreement function used in both Sacks' and Soskov and Ganchev constructions.

2. Preliminaries

2.1. Basic notions

We shall denote the set of natural numbers by ω. We also shall consider each natural number n as the set of natural numbers strictly less than n: $n = \{m \in \omega \mid m < n\}$. In this way when we write $D \subseteq n$, we mean that D is a finite set of natural numbers and all of its members are less than n.

If not stated otherwise, a, b, c, ... shall stand for natural numbers, A, B, C, ... for sets of natural numbers, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ... for degrees and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... for sequences of sets of natural numbers. We shall further follow the following convention: whenever a sequence is denoted by a calligraphic Latin letter, then we shall use the roman style of the same Latin letter, indexed with a natural number, say k, to denote the k-th element of the sequence (we always start counting from 0). Thus, if not stated otherwise, $\mathcal{A} = \{A_k\}_{k<\omega}$, $\mathcal{B} = \{B_k\}_{k<\omega}$, $\mathcal{C} = \{C_k\}_{k<\omega}$, etc. We shall denote the set of all sequences (*of length* ω) of sets of natural numbers by $\mathcal{S}_\omega$.

As usual $A \oplus B$ shall stand for the set $\{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\}$. By A^+ we shall denote the set $A \oplus (\omega \setminus A)$.

2.2. The Turing reducibility and degrees

We assume that the reader is familiar with the notion of Turing reducibility $\leq_T$, and with the structure of the Turing degrees $\mathcal{D}_T$ (for a survey of basic results on the Turing degree structure we refer the reader to [8–10, 14]).

Let $\{\widehat{P}_e\}_{e<\omega}$ be a fixed effective numbering of all Turing machine oracle programs. Note that $\widehat{P}_e$ is independent of the oracle A. We shall write $\Phi_e^A(x) = y$ if oracle program $\widehat{P}_e$ with A on its oracle tape and input x halts and yields output y. In this case we say that $\Phi_e^A(x)$ *converges* ($\Phi_e^A(x)\downarrow$); otherwise, $\Phi_e^A(x)$ diverges ($\Phi_e^A(x)\uparrow$). We refer to Φ_e^A as a e-th *Turing functional*. We shall write $\Phi_{e,s}^A(x) = y$ if $x, y, e < s$, and y is the output from $\Phi_e^A(x)$ in less that s steps of the oracle Turing program $\widehat{P}_e$. Note that always $\operatorname{dom}(\Phi_{e,s}^A) \subseteq s$.

We shall write W_e^A and $W_{e,s}^A$ for $\operatorname{dom}(\Phi_e^A)$ and $\operatorname{dom}(\Phi_{e,s}^A)$ respectively. By the Post's Theorem, the computably enumerable in A sets are exactly the sets $\{W_e^A\}_{e<\omega}$. Note that for each e,

$$W_{e,0}^A \subseteq W_{e,1}^A \subseteq \cdots \subseteq W_{e,s}^A \subseteq \cdots$$

and $W_e^A = \bigcup_s W_{e,s}^A$. Thus $x \in W_e^A$ if and only if $(\exists s)[x \in W_{e,s}^A]$ for all $x < \omega$. Also, each one of the sets $W_{e,s}^A$ is finite and computable in A. Later, we shall call such a sequence a Σ_1^A-approximation of W_e^A. By the Post's Theorem, a set is computably enumerable in A if and only if it has a Σ_1^A-approximation.

In order to simplify the constructions, we need the following corollary of the Recursion Theorem.

Corollary 2.1. *Let f be a computable function. Then there exists $e < \omega$, such that for all $A \subseteq \omega$,*

$$\Phi_{f(e)}^A = \Phi_e^A.$$

Proof. By the uniform version of the relativized Recursion Theorem [14], there is a computable function e such that for all sets $A \subseteq \omega$ and all $n \in \omega$, if Φ_n^A is a total function, then

$$\Phi_{\Phi_n^A(e(n))}^A = \Phi_{e(n)}^A.$$

Note that if f is computable function, then there is a natural number a such that for all A, $f = \Phi_a^A$. Indeed, just let $\widehat{P}_a$ be an oracle program which computes f that does not use the oracle tape. Now by the Recursion Theorem

$$\Phi_{f(e(a))}^A = \Phi_{\Phi_a^A(e(a))}^A = \Phi_{e(a)}^A.$$

Since the index a is universal for all sets A, we can take $e = e(a)$. □

We shall say that A is *Turing reducible* to (or *computable* in) B (written $A \leq_T B$) if there is a Turing functional Φ_e such that $A = \Phi_e^B$. The relation $\leq_T$ is a preorder on the powerset 2^ω of the natural numbers and induces a nontrivial equivalence relation $\equiv_T$. The equivalence classes under $\equiv_T$ are called Turing degrees. The Turing degree which contains the set A is denoted by $\deg_T(A)$. The set of all Turing degrees is denoted by $\mathbf{D}_T$. The Turing reducibility between sets induces a partial order $\leq_T$ on $\mathbf{D}_T$ by

$$\deg_T(A) \leq_T \deg_T(B) \iff A \leq_T B.$$

We denote by $\mathcal{D}_T$ the partially ordered set $\langle \mathbf{D}_T, \leq_T \rangle$. The least element of $\mathcal{D}_T$ is the Turing degree $\mathbf{0}_T$ of $\varnothing$. Also, the degree of $A \oplus B$ is the least upper bound of the degrees of A and B. Therefore $\mathcal{D}_T$ is an upper semi-lattice with least element.

The (Turing) jump A' of $A \subseteq \omega$ is defined as the halting problem for machines with an oracle A,

$$A' = \{e \mid \Phi_e^A(e)\downarrow\}.$$

The jump operation preserves the Turing reducibility, so we can define $\deg_T(A)' = \deg_T(A')$. Since $A <_T A'$, then we have $\mathbf{a} <_T \mathbf{a}'$ for every Turing degree $\mathbf{a}$. The jump operator is uniform, i.e., there exists a computable function j such that for every sets A and B, if $A \leq_T B$ via the Turing operator with index e, then $A' \leq_T B'$ via the operator with index $j(e)$.

The jump A' of A is computable enumerable in A uniformly in A. That is, there exists z such that for all sets A, $A' = W_z^A$. Thus $\{W_{z,s}^A\}_{s<\omega}$ is a Σ_1^A-approximation of A'.

2.3. The ω-Turing degrees

The ω-Turing reducibility and the corresponding degree structure $\mathcal{D}_{T,\omega}$ are introduced by Sariev and Ganchev in [13]. An equivalent, but more approachable definition in the terms of the uniform Turing reducibility is derived again in the same paper. Here we shall present only on the latter. It is based on the notion of *jump sequence* – it was introduced by Soskov [15] in order to describe the ω-enumeration reducibility.

For each $\mathcal{X} = \{X_k\}_{k<\omega}$, its corresponding *jump sequence* $\mathcal{P}(\mathcal{X})$ is defined as the sequence $\{P_k(\mathcal{X})\}_{k<\omega}$ such that:

- $P_0(\mathcal{X}) = X_0$
- $P_{k+1}(\mathcal{X}) = (P_k(\mathcal{X}))' \oplus X_{k+1}$, for each $k < \omega$.

Now, the sequence $\mathcal{A}$ is ω-Turing reducible to the sequence $\mathcal{B}$ (written $\mathcal{A} \leq_{T,\omega} \mathcal{B}$), if and only if

$$A_n \leq_T P_n(\mathcal{B}) \text{ uniformly in } n.$$

Clearly $\leq_{T,\omega}$ is a reflexive and transitive relation, and the relation $\equiv_{T,\omega}$ defined by

$$\mathcal{A} \equiv_{T,\omega} \mathcal{B} \iff \mathcal{A} \leq_{T,\omega} \mathcal{B} \text{ and } \mathcal{B} \leq_{T,\omega} \mathcal{A}$$

is an equivalence relation. The equivalence classes under this relation are called ω-Turing degrees. In particular the equivalence class $\deg_{T,\omega}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{A} \equiv_{T,\omega} \mathcal{B}\}$ is called the ω-Turing degree of $\mathcal{A}$. The relation $\leq$ defined by

$$\mathbf{a} \leq \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b} (\mathcal{A} \leq_{T,\omega} \mathcal{B})$$

is a partial order on the set of all ω-Turing degrees $\mathbf{D}_{T,\omega}$. By $\mathcal{D}_{T,\omega}$ we shall denote the structure $\langle \mathbf{D}_{T,\omega}, \leq \rangle$. The ω-Turing degree $\mathbf{0}$ of the sequence $\varnothing_\omega = \{\varnothing\}_{k<\omega}$ is the least element in $\mathcal{D}_{T,\omega}$. Further, the ω-Turing degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_{T,\omega}(\mathcal{A})$ and $\mathbf{b} = \deg_{T,\omega}(\mathcal{B})$. Thus $\mathcal{D}_{T,\omega}$ is an upper semi-lattice with least element.

It is not difficult to notice that each sequence and its jump sequence belong to the same ω-Turing degree, i.e., for all $\mathcal{A} \in \mathcal{S}_\omega$,

$$\mathcal{A} \equiv_{T,\omega} \mathcal{P}(\mathcal{A}). \tag{2.1}$$

In this way, $\mathcal{P}(\mathcal{A})$ is equivalent to an $\mathcal{A}$ sequence, whose members are monotone with respect to $\leq_T$ and each of its member decides the halting problems of the previous members.

Given a set $A \subseteq \omega$, denote by $A \uparrow \omega$ the sequence $(A, \varnothing, \varnothing, \dots, \varnothing, \dots)$. The definition of $\leq_{T,\omega}$ and the uniformity of the jump operation imply that for all sets of natural numbers A and B,

$$A \uparrow \omega \leq_{T,\omega} B \uparrow \omega \iff A \leq_T B. \tag{2.2}$$

The last equivalence means, that the mapping $\kappa \colon \mathbf{D}_T \to \mathbf{D}_{T,\omega}$, defined by

$$\kappa(\deg_T(X)) = \deg_{T,\omega}(X \uparrow \omega),$$

is an embedding of $\mathcal{D}_T$ into $\mathcal{D}_{T,\omega}$. Further, the so defined embedding κ preserves the order, the least element and the binary least upper bound operation.

We shall refer to κ as *the natural embedding* of the Turing degrees into the ω-Turing degrees. The range of κ shall be denoted by $\mathbf{D}_1$ and shall be called *the natural copy* of the Turing degrees.

2.4. The jump operator and jump inversion

In Computability Theory, often in addition to the considered reducibility between objects (sets, sequences of sets or functions, countable structures, etc.) a *jump operation* $'$ is introduced – for example the Turing jump ([6]) and the enumeration jump ([7]) over subsets of ω, or the jump of a structure ([17]). Usually the jump operator is monotone ($\alpha \leq \beta$ implies $\alpha' \leq \beta'$) and strictly expansive ($\alpha < \alpha'$). Its monotonicity allows to transfer the jump on the degrees (usually the degrees are the equivalence classes with respect to the relation $\equiv$ defined as: $\alpha \equiv \beta \iff \alpha \leq \beta \;\&\; \beta \leq \alpha$).

The notion of ω-Turing jump of a sequence of sets of natural numbers is defined by Sariev and Ganchev [13]. All the properties of the ω-Turing jump mentioned in this section are proved in the same paper. Following their lines, the ω-Turing jump $\mathcal{A}'$ of $\mathcal{A} = \{A_k\}_{k<\omega}$ is defined as the sequence

$$\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \dots, A_k, \dots). \tag{2.3}$$

Note that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_{T,\omega} \{P_{k+1}(\mathcal{A})\}_{k<\omega}$. The jump operator is strictly expansive and monotone with respect to the ω-Turing reducibility. This allows us to define a jump operation on the ω-Turing degrees by setting

$$\deg_{T,\omega}(\mathcal{A})' = \deg_{T,\omega}(\mathcal{A}').$$

Clearly for all $\mathbf{a}, \mathbf{b} \in \mathbf{D}_{T,\omega}$, $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$.

Also the jump operation on ω-Turing degrees agrees with the jump operation on the Turing degrees, i.e., we have

$$\kappa(\mathbf{x}') = \kappa(\mathbf{x})', \text{ for all } \mathbf{x} \in \mathbf{D}_T.$$

We shall denote by $\mathcal{A}^{(n)}$ the n-the iteration of the jump operator on $\mathcal{A}$. Let us note that

$$\mathcal{A}^{(n)} = (P_n(\mathcal{A}), A_{n+1}, A_{n+2}, \dots) \equiv_{T,\omega} \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{2.4}$$

It is clear that if $\mathcal{A} \in \mathbf{a}$, then $\mathcal{A}^{(n)} \in \mathbf{a}^{(n)}$, where $\mathbf{a}^{(n)}$ denotes the n-th iteration of the jump operation on the degree $\mathbf{a}$.

Having a jump operation, a natural question that arises is what is the range of the operation? The Friedberg Jump Inversion Theorem [4] gives us that the range of the Turing jump over the upper cone of the degree $\mathbf{b}$ is exactly the upper cone of $\mathbf{b}'$ and the range of the enumeration jump consists exactly of the total degrees above $\mathbf{b}'$. Similar inversion result holds for the operation jump of structure, [17]. So, as one may expect, the range of the ω-Turing jump over the upper cone of $\mathbf{b}$ is exactly the upper cone over $\mathbf{b}'$, [13]. Further, stronger jump inversion properties can be asked, by specifying additional characteristics for the preimage of the jump.

For example in the case of the Turing degrees we have the Cooper Jump Inversion Theorem [2], by which each Turing degree above $\mathbf{0}'_T$ is a jump of a minimal degree (a degree is minimal if and only if it bounds only the least element $\mathbf{0}_T$). A similar result [7] by McEvoy in the enumeration degrees reveals that each total degree above $\mathbf{0}'_e$ is a jump of a quasi-minimal degree (an enumeration degree is quasi-minimal if and only if the only total degree below it is $\mathbf{0}_e$). As it is shown in [13], a ω-Turing degree $\mathbf{m}$ is minimal (i.e., bounds only the least element) if and only if there is a natural number n and a set M such that the Turing degree of M is a low over $\mathbf{0}_T^{(n)}$ (i.e., $\varnothing^{(n)} <_T M <_T M' \equiv_T \varnothing^{(n+1)}$) minimal cover of $\mathbf{0}_T^{(n)}$ (i.e., there are no A such that $\varnothing^{(n)} <_T A <_T M$) and

$$(\underbrace{\varnothing, \dots, \varnothing}_{n}, M, \varnothing, \dots) \in \mathbf{m}.$$

Therefore all minimal ω-Turing degrees are low – the jump of each of them is equal to $\mathbf{0}'$; so such a jump inversion theorem does not hold in $\mathcal{D}_{T,\omega}$. There is a sequel of jump inversion theorems in the Computable Structure Theory as well ([1, 17, 18]).

Using the Cooper Jump Inversion Theorem, one can show that there is no least jump inversion – there are countably many minimal degrees with jump equal to $\mathbf{a} >_T \mathbf{0}'_T$ and they are mutually incomparable; so there is no least degree with jump equal to $\mathbf{a}$. In the enumeration degrees the jump cannot be inverted to least degree, too. In a difference to $\mathcal{D}_T$ and $\mathcal{D}_e$, the structure of the ω-Turing degrees possesses a least jump inversion. More precisely, for each natural number n if $\mathbf{b}$ is above $\mathbf{a}^{(n)}$, then the set

$$\{\mathbf{x} \mid \mathbf{a} \leq \mathbf{x} \ \& \ \mathbf{x}^{(n)} = \mathbf{b}\}$$

has a least element. We shall denote this degree by $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$. An explicit representative of $\mathbf{I}^n_{\mathbf{a}}(\mathbf{b})$ can be given by setting

$$I^n_{\mathcal{A}}(\mathcal{B}) = (A_0, A_1, \dots, A_{n-1}, B_0, B_1, \dots, B_k, \dots), \tag{2.5}$$

where each $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$ are arbitrary.

In the case when $\mathbf{a} = \mathbf{0}$, for the sake of simplicity, we shall use the notation $\mathbf{I}^n$ instead of $\mathbf{I}^n_{\mathbf{0}}$. The operation $\mathbf{I}^n$ is monotone:

$$\mathbf{0}^{(n)} \leq \mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{I}^n(\mathbf{x}) \leq \mathbf{I}^n(\mathbf{y}).$$

3. The local structure and the a.z. degrees

The degree substructure lying beneath the first jump of the least element is usually referred to as the local structure of the degree structure. In the case of the ω-Turing degrees, we shall denote this structure by $\mathcal{G}_{T,\omega}$,

$$\mathcal{G}_{T,\omega} = \langle \{\mathbf{x} \in \mathbf{D}_{T,\omega} \mid \mathbf{x} \leq \mathbf{0}'\}, \leq \rangle ,$$

where $\leq$ is the inherited from $\mathcal{D}_{T,\omega}$ order. Proofs of all stated properties in this section can be found in [12, 13].

As usual, a degree in the local structure is said to be *high$_n$* if and only if its n-th jump is as high as possible. A degree is *high* if and only if it is high$_n$ for some n. We shall denote the set of all high degrees in $\mathcal{G}_{T,\omega}$ by

$$\mathbf{H} = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \ \& \ (\exists n)[\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}]\}.$$

We need the following sufficient condition when a degree is in $\mathbf{H}$.

Proposition 3.1. *Let $\mathbf{a} \leq \mathbf{0}'$ and $\mathcal{A} \in \mathbf{a}$. Then $\mathbf{a} \in \mathbf{H}$, if there is a k such that $\varnothing^{(k+1)} \equiv_T P_k(\mathcal{A})$.*

Proof. Let $\mathcal{A} \leq_{T,\omega} \varnothing'_\omega$ and k be such that $\varnothing^{(k+1)} \equiv_T P_k(\mathcal{A})$. Then for each $r \geq 0$, $\varnothing^{(k+r+1)} \leq_T P_{k+r}(\mathcal{A})$ uniformly in r. Hence $\varnothing^{(k+1)}_\omega \leq_{T,\omega} \{P_{k+r}(\mathcal{A})\}_{r<\omega} \equiv_{T,\omega} \mathcal{A}^{(k)}$. Since the ω-Turing jump is monotone, then $\mathbf{a}$ is high$_k$ and so in $\mathbf{H}$. □

Similarly a degree in the local structure is said to be *low$_n$* if and only if its n-th jump is as low as possible. A degree is *low* if and only if it is low$_n$ for some n. We shall denote the set of all low degrees in $\mathcal{G}_{T,\omega}$ by

$$\mathbf{L} = \{\mathbf{a} \mid \mathbf{a} \leq \mathbf{0}' \ \& \ (\exists n)[\mathbf{a}^{(n)} = \mathbf{0}^{(n)}]\}.$$

The degrees that are neither high nor low, shall be referred to as *intermediate* degrees. The corresponding degree class is denoted by $\mathbf{I}$.

Further, for each $n < \omega$ define $\mathbf{o}_n$ as the least degree, satisfying the equation $\mathbf{x}^{(n)} = \mathbf{0}^{(n+1)}$:

$$\mathbf{o}_n = \mathbf{I}^n(\mathbf{0}^{n+1}).$$

In other words, $\mathbf{o}_n$ is the least high$_n$ degree. Thus, a degree $\mathbf{a} \in \mathcal{G}_{T,\omega}$ is high$_n$ if and only if $\mathbf{o}_n \leq \mathbf{a}$. Note also that $\mathbf{o}_0 = \mathbf{0}'$. Since each high$_n$ degree is also high$_{n+1}$ degree but not all high$_{n+1}$ degrees are high$_n$ we have that

$$\mathbf{0}' = \mathbf{o}_0 > \mathbf{o}_1 > \cdots > \mathbf{o}_n > \cdots$$

By (2.3) and (2.5), for each $n < \omega$ the degree $\mathbf{o}_n$ contains the sequence $\mathcal{O}_n = \{O_n^k\}_{k<\omega}$, where $O_n^k = \varnothing$ for $k < n$, and $O_n^k = \varnothing^{(k+1)}$ for $k \geq n$.

Finally, we introduce the notion of *almost zero* (a.z.) degrees.

Definition 3.2. We shall say that the ω-Turing degree $\mathbf{a}$ is almost zero if and only if there is a sequence $\mathcal{A} \in \mathbf{a}$ such that

$$(\forall k)[P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}].$$

Note that the class of a.z. degrees is downward closed; also it is closed under the operation $\vee$ for l.u.b. There are continuum many a.z. degrees, hence not all of them are in $\mathcal{G}_{T,\omega}$. In fact, the almost zero degrees less than $\mathbf{0}'$ are fully characterized by the $\mathbf{o}_n$ degrees.

Proposition 3.3. *Let* $\mathbf{a} \leq \mathbf{0}'$. *Then* $\mathbf{a}$ *is a.z. if and only if for each* n, $\mathbf{a} < \mathbf{o}_n$.

Proof. Let $\mathbf{a}$ be an a.z. degree below $\mathbf{0}'$. Clearly $\mathbf{a} < \mathbf{o}_0 = \mathbf{0}'$. Now, fix a natural number $n \geq 1$. Fix a sequence $\mathcal{A}$ in it such that for each k, $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}$. Since $\mathbf{a} \leq \mathbf{0}'$, then uniformly in $k \geq n$:

$$P_k(\mathcal{A}) \leq_T \varnothing^{(k+1)} \equiv_T P_k(\mathcal{O}_n).$$

But for $k < n$, $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)} \equiv_T P_k(\mathcal{O}_n)$. Hence $P_k(\mathcal{A}) \leq_T P_k(\mathcal{O}_n)$ uniformly in k and thus $\mathcal{A} \leq_{T,\omega} \mathcal{O}_n$.

For the converse direction, let $\mathbf{a} \leq \mathbf{0}'$ be such that $\mathbf{a} < \mathbf{o}_n$ for each n. Let $\mathcal{A}$ be a sequence in $\mathbf{a}$. Then its jump sequence $\mathcal{P}(\mathcal{A})$ has the same degree. Since for each n, $\mathcal{P}(\mathcal{A}) \leq_{T,\omega} \mathcal{O}_{n+1}$, then for each n

$$P_n(\mathcal{A}) \leq_T P_n(\mathcal{O}_{n+1}) \equiv_T \varnothing^{(n)}.$$

Therefore $\mathbf{a}$ is an a.z. degree. □

There are no nonzero almost zero degrees neither in the high degrees, nor in the low degrees.

Proposition 3.4. *Let* $\mathbf{a}$ *be a nonzero a.z. degree. Then* $\mathbf{a} \in \mathbf{I}$.

Proof. Let $\mathbf{a}$ be a nonzero a.z. degree. Then for each n, $\mathbf{a} < \mathbf{o}_n$ and hence $\mathbf{a}$ is a non-high$_n$. So $\mathbf{a} \notin \mathbf{H}$.

Let $\mathcal{A}$ be a sequence in $\mathbf{a}$ such that $P_k(\mathcal{A}) \equiv_T \varnothing^{(k)}$ for each k. Then $\mathcal{A}^{(n)} \equiv_{T,\omega} (P_n(\mathcal{A}), P_{n+1}(\mathcal{A}), \dots)$. Thus, $I^n_{\varnothing_\omega}(\mathcal{A}^{(n)})=(\varnothing, \dots, \varnothing^{(n-1)}, P_n(\mathcal{A}), P_{n+1}(\mathcal{A}), \dots) \equiv_{T,\omega} \mathcal{P}(\mathcal{A}) \equiv_{T,\omega} \mathcal{A}$. Hence $\mathbf{I}^n(\mathbf{a}^{(n)}) = \mathbf{a}$.

Now, suppose that $\mathbf{a}$ is low$_n$ for some n. Then $\mathbf{a} = \mathbf{I}^n(\mathbf{a}^{(n)}) = \mathbf{I}^n(\mathbf{0}^{(n)}) = \mathbf{0}$. A contradiction. Therefore $\mathbf{a}$ is a non-low. Thus $\mathbf{a}$ is intermediate. □

4. Δ^0_2-APPROXIMATIONS

The aim of this section is to provide a machinery for constructing degrees below $\mathbf{0}'$. The idea is to use uniform relativization of the Δ^0_2-approximations used in the Turing degrees. We start with some basic notions and properties.

Definition 4.1. We say a computable sequence $\{A^s\}_{s<\omega}$ of finite sets is a Δ^0_2-approximation for A if for all x, $\lim_{s\to\infty} A^s(x)$ exists and it is equal to $A(x)$.

If A has Δ^0_2-approximation $\{A^s\}_{s<\omega}$, then we shall call a natural number $e < \omega$ an Δ^0_2-*index* for A corresponding to the approximation $\{A^s\}_{s<\omega}$ if and only if for all $s, x < \omega$, $A^s(x) = \Phi^{\varnothing}_e(s, x)$.

The sets possessing Δ^0_2-approximations are exactly those which are Turing reducible to the Halting problem (see, for example [2]).

Proposition 4.2 (The Limit Lemma)**.** *$A \leq_T \varnothing'$ if and only if A has a Δ^0_2-approximation.*

Moreover, we can pass effectively between a Δ^0_2-index for A and index for the reduction of A to $\varnothing'$ (i.e., such e that $A = \Phi^{\varnothing'}_e$), [14, Ch. 4.2].

The definition of Δ^0_2-approximation can be easily relativized to oracle B (see [14, Ch. 4.4]).

Definition 4.3. We say a computable in B sequence $\{A^s\}_{s<\omega}$ of finite sets is a Δ^B_2-approximation for A if for all x, $\lim_{s\to\infty} A^s(x)$ exists and it is equal to $A(x)$.

If $\{A^s\}_{s<\omega}$ is a Δ^B_2-approximation for A and $e < \omega$ is such that for all s, x, $A^s(x) = \Phi^B_e(s, x)$, then we shall call e an Δ^B_2-index for A, corresponding to the approximation $\{A^s\}_{s<\omega}$.

We have the relativized version of the Limit Lemma.

Proposition 4.4 (The Relativized Limit Lemma)**.** *$A \leq_T B'$ if and only if A has a Δ^B_2-approximation.*

Again, we can pass effectively between Δ^B_2-indices for A and indices for the reduction of A to B'.

Using a Δ^B_2-approximation of A we can build a Δ^B_2-approximation of each computable in A set.

Proposition 4.5. *Let $\{A^s\}_{s<\omega}$ be a Δ_2^B-approximation of A and e be such that Φ_e^A is a total characteristic function. Then $\{\Phi_{e,s}^{A^s}\}_{s<\omega}$ is a Δ_2^B-approximation of Φ_e^A.*

Proof. Let $x < \omega$. Then $\Phi_e^A(x)\downarrow$. Fix a $n < \omega$ greater than the maximal use of the oracle A in the computation $\Phi_e^A(x)$. Then

$$\Phi_e^{A\upharpoonright n}(x)\downarrow = \Phi_e^A(x).$$

Let s_0 and s_1 be such that

$$(\forall t \geq s_0)[A^t \upharpoonright n = A \upharpoonright n]$$

and

$$(\forall t \geq s_1)[\Phi_{e,t}^{A\upharpoonright n}(x)\downarrow = \Phi_e^{A\upharpoonright n}(x)].$$

Hence, for each $t \geq \max\{s_0, s_1\}$ we have that

$$\Phi_{e,t}^{A^t}(x)\downarrow = \Phi_{e,t}^{A^t\upharpoonright n}(x) = \Phi_{e,t}^{A\upharpoonright n}(x) = \Phi_e^{A\upharpoonright n}(x) = \Phi_e^A(x).$$

Thus for each $x < \omega$, $\lim_s \Phi_{e,s}^{A^s}(x) = \Phi_e^A(x)$. Since $\mathrm{dom}(\Phi_{e,s}^{A^s}) \subseteq s$, then $\Phi_{e,s}^{A^s}$ is finite for each s. Finally, it is not difficult to notice that the sequence $\{\Phi_{e,s}^{A^s}\}_{s<\omega}$ is computable in B uniformly in s. □

Finally, we define the notion of Δ_2^0-approximation for a sequence.

Definition 4.6. We say a sequence $\{A_n^s\}_{n,s<\omega}$ is a Δ_2^0-approximation for the sequence $\{A_n\}_{n<\omega}$ if $\{A_n^s\}_{s<\omega}$ is $\Delta_2^{\varnothing^{(n)}}$-approximation for A_n uniformly in n.

Again, the sequences with Δ_2^0-approximations are exactly those below $\varnothing'_\omega$.

Proposition 4.7. *$\mathcal{A} \leq_{T,\omega} \varnothing'_\omega$ if and only if $\mathcal{A}$ has a Δ_2^0-approximation.*

Proof. First, let $\mathcal{A} \leq_{T,\omega} \varnothing'_\omega$. Let f be a total computable function such that for each n, $A_n = \Phi_{f(n)}^{\varnothing^{(n+1)}}$. Let z be such that for all n, $\varnothing^{(n+1)} = W_z^{\varnothing^{(n)}}$. Denote $\varnothing_s^{(n+1)} = W_{z,s}^{\varnothing^{(n)}}$ for each $s, n < \omega$. Thus uniformly in n, $\{\varnothing_s^{(n+1)}\}_{s<\omega}$ is a $\Sigma_1^{\varnothing^{(n)}}$-approximation for $\varnothing^{(n+1)}$. Let for each n and s, $A_n^s = \Phi_{f(n),s}^{\varnothing_s^{(n+1)}}$.

Then for each n and s, A_n^s is finite, because $A_n^s \subseteq s$. Using the computability of f, it is easy to notice that uniformly in n the sequence $\{A_n^s\}_{s<\omega}$ is computable in $\varnothing^{(n)}$. Let $x < \omega$ and u be greater than the maximum use of the oracle $\varnothing^{(n+1)}$ in the computation of $A_n(x)$ as $\Phi_{f(n)}^{\varnothing^{(n+1)}}$. Let s_0 and s_1 be such that

$$(\forall t \geq s_0)[\varnothing^{(n+1)} \upharpoonright u = \varnothing_t^{(n+1)} \upharpoonright u]$$

and

$$(\forall t \geq s_1)[\Phi_{f(n),t}^{\varnothing^{(n+1)}}(x)\downarrow = \Phi_{f(n)}^{\varnothing^{(n+1)}}(x)].$$

Now, for each $t \geq \max\{s_0, s_1\}$, we have that

$$A_n^s(x) = \Phi_{f(n),t}^{\varnothing_t^{(n+1)}}(x) = \Phi_{f(n),t}^{\varnothing^{(n+1)}\upharpoonright u}(x) = \Phi_{f(n),t}^{\varnothing^{(n+1)}}(x) = \Phi_{f(n)}^{\varnothing^{(n+1)}}(x) = A_n(x).$$

Hence, $\{A_n^s\}_{n,s<\omega}$ is a Δ_2^0-approximation for $\mathcal{A}$.

Conversely, let $\{A_n^s\}_{n,s<\omega}$ be a Δ_2^0-approximation for the sequence $\mathcal{A}$. Since for each n, $\{A_n^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(n)}}$-approximation for A_n, then $A_n \leq_T \varnothing^{(n+1)}$. It remains to show that this reduction is uniform in n. We shall use the standard relativizations of the Enumeration Theorem and the s-m-n Theorem, [14]. Fix a natural number z and a computable function q such that for all sets $A \subseteq \omega$ and all $x, y, z_1, z_2 < \omega$: $\Phi_z^A(x, y) = \Phi_x^A(y)$ and $\Phi_{q(x,y)}^A(z_1, z_2) = \Phi_x^A(y, z_1, z_2)$.

Since $\{A_n^s\}_{s<\omega}$ is computable in $\varnothing^{(n)}$ uniformly in n, then there is a computable function f such that for all $n, s < \omega$

$$A_n^s = \Phi_{f(n,s)}^{\varnothing^{(n)}}.$$

Then for all $x, n, s < \omega$, $\Phi_{f(n,s)}^{\varnothing^{(n)}}(x) = \Phi_z^{\varnothing^{(n)}}(f(n, s), x)$. Since f is computable, then there is z' (which is uniformly computable by z and an index for f as a partial computable function) such that $\Phi_z^{\varnothing^{(n)}}(f(n, s), x) = \Phi_{z'}^{\varnothing^{(n)}}(n, s, x)$. But $\Phi_{z'}^{\varnothing^{(n)}}(n, s, x) = \Phi_{q(z',n)}^{\varnothing^{(n)}}(s, x)$ and hence there is a computable function $r(n) = q(z', n)$ such that for all $x, s, n < \omega$, $A_n^s(x) = \Phi_{r(n)}^{\varnothing^{(n)}}(s, x)$.

In this way we have a computable function which gives a $\Delta_2^{\varnothing^{(n)}}$-index for A_n corresponding to the approximation $\{A_n^s\}_{s<\omega}$ and hence there is a computable function which gives an index for the reduction of A_n to $\varnothing^{(n+1)}$. □

5. Defining the high degrees

In this section we shall prove our main result – namely that in the local substructure of the ω-Turing degrees, the set of all high degrees is definable in the terms of the a.z. degrees and the structure order $\leq$. First, we need the following result, revealing that there is no intermediate degree which bounds all a.z. degrees. The proof uses Δ_2^0-approximations of sequences and the construction is based on that one used in the Sacks Splitting Theorem [3, 11, 14].

Lemma 5.1. *Let* $\mathbf{a} \in \mathbf{I}$. *Then there exists an a.z. degree* $\mathbf{d}$ *such that* $\mathbf{d} \leq \mathbf{0}'$ *and* $\mathbf{d} \not\leq \mathbf{a}$.

Proof. Let $\mathbf{a} \in \mathbf{I}$ and $\mathcal{A} \in \mathbf{a}$. Fix Δ_2^0-approximations $\{P_n^s\}_{n,s<\omega}$ and $\{P_n^s(\mathcal{A})\}_{n,s<\omega}$ for $\mathcal{P}(\varnothing_\omega{}')$ and $\mathcal{P}(\mathcal{A})$ respectively. We shall construct a sequence $\{D_k^s\}_{k,s<\omega}$ such that:

1. for each k and each x, $\lim_{s\to\infty} D_k^s(x)$ exists and $D_k = \lim_{s\to\infty} D_k^s$ is finite;
2. uniformly in k, $\{D_k^s\}_{s<\omega}$ is computable in $\varnothing^{(k)}$;

3. for each k, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$.

The satisfaction of 1. and 2. immediately gives us that $\{D_k\}_{k<\omega}$ has an a.z. degree. As we will see, 3. implies that $\mathcal{D} \not\leq_{T,\omega} \mathcal{A}$.

We shall need the following notations. For each k and s, denote by $\Psi_k^s = \Phi_{k,s}^{P_k^s(\mathcal{A})}$. By Proposition 4.5, $\{\Psi_k^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(k)}}$-approximation of $\Phi_k^{P_k(\mathcal{A})}$. Given two sets X and Y of natural numbers and $s < \omega$, denote by $l^s(X, Y)$ the *length of agreement function* up to s:

$$l^s(X, Y) = \max\{n \leq s \mid X \upharpoonright (n + 1) = Y \upharpoonright (n + 1)\}.$$

Finally, for each $k, s < \omega$ let

$$l_k^s = l^s(D_k^s, \Psi_k^s)$$

and

$$m_k^s = \max\{l_k^t \mid t \leq s\}$$

stay respectively for the length of agreement and the maximum length of agreement between D_k^s and Ψ_k^s.

Construction. For each $k < \omega$, set $D_k^0 = \varnothing$. Suppose that for each $k < \omega$, D_k^s is constructed. Then for each k set

$$D_k^{s+1} = \{x \mid x \leq m_k^s \ \&\ x \in P_k^s\}.$$

End of the construction.

Note that $\{D_k^s\}_{s<\omega}$ is computable in $\varnothing^{(k)}$ uniformly in k. The next claim shows that $\{D_k^s\}_{k,s<\omega}$ is a Δ_2^0-approximation.

Claim 1. *For each k and x, $\lim_{s\to\infty} D_k^s(x)$ exists.*

Proof. Consider arbitrary k and x. Let s be such that the approximations $\{P_k^t\}_{t<\omega}$ and $\{\Psi_k^t\}_{t<\omega}$ are stabilized up to x after stage s, i.e., for all $t \geq s$,

$$P_k^s \upharpoonright (x + 1) = P_k(\varnothing_\omega{}') \upharpoonright (x + 1),$$

and

$$\Psi_k^t \upharpoonright (x + 1) = \Phi_k^{P_k(\mathcal{A})} \upharpoonright (x + 1).$$

If $x \notin P_k(\varnothing_\omega{}')$, then for each $t \geq s$, $x \notin P_k^t$ and so $D_k^t(x) = 0$. Now let $x \in P_k(\varnothing_\omega{}')$. Then for each $t \geq s$, $P_k^t(x) = 1$. On stage $s + 1$ of the construction we have two cases:

- $m_k^s < x$. Then $l_k^s < x$ and hence $D_k^{s+1}(x) = 0 \neq 1 = \Psi_k^{s+1}(x)$. Therefore $l_k^{s+1} < x$ and thus $m_k^{s+1} < x$. Now using an induction on $t \geq s$, one can easily show that for all $t \geq s$, $D_k^t(x) = 0$.
- $x \leq m_k^s$. Since for each $t \geq s$, then $m_k^t \geq m_k^s \geq x$. But for all $t \geq s$, $x \in P_k^t$ and thus $D_k^t(x) = 1$.

This proves the claim. □

Denote by D_k for each $k < \omega$, the limit of $\{D_k^s\}_{s<\omega}$. Thus $\{D_k^s\}_{s<\omega}$ is a $\Delta_2^{\varnothing^{(k)}}$-approximation of the set D_k. So $\{D_k\}_{k<\omega} \leq_{T,\omega} \varnothing_\omega{}'$. The next claim is needed in order to show that the degree of $\{D_k\}_{k<\omega}$ is an a.z. and not below **a**.

Claim 2. *For each $k < \omega$, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$.*

Proof. Suppose that k is such that $D_k = \Phi_k^{P_k(\mathcal{A})}$. Then $D_k = P_k(\varnothing_\omega{}')$.

Indeed, let $x \in D_k$. Let s be such that for all $t \geq s$, $D_k^t(x) = 1$. By the construction, for each $t \geq s$, $x \leq m_k^t$ and $x \in P_k^t$. Hence $x \in P_k(\varnothing_\omega{}')$.

Now, let $x \in P_k(\varnothing_\omega{}')$. Let s_0 be such that after stage s_0 the approximation $\{P_k^s\}_{s<\omega}$ is stabilized on x:

$$(\forall t \geq s_0)[P_k^t(x) = P_k(\mathcal{A})(x) = 1].$$

Let s_1 be such that after stage s_1, the approximations $\{D_k^s\}_{s<\omega}$ and $\{\Psi_k^s\}_{s<\omega}$ are stabilized on each $y \leq x$:

$$(\forall t \geq s_1)(\forall y \leq x)[D_k^t(y) = D_k(y) \ \&\ \Psi_k^t(y) = \Phi_k^{P_k(\mathcal{A})}(y)].$$

Thus for each $t \geq s_1$ and for all $y \leq x$, $D_k^t(y) = D_k(y) = \Phi_k^{P_k(\mathcal{A})} = \Psi_k^t(y)$. Thus for each $t \geq \max\{s_0, s_1\}$, $x \leq l_k^t \leq m_k^t$ and since $x \in P_k^t$, then $x \in D_k^{t+1}$. Therefore $x \in D_k$.

Having that $D_k = P_k(\varnothing_\omega{}')$ and $D_k = \Phi_k^{P_k(\mathcal{A})}$, then $\varnothing^{(k+1)} \equiv_T P_k(\varnothing_\omega{}') \leq_T P_k(\mathcal{A})$. By Proposition 3.1, **a** must be a high degree, which contradicts with the fact that **a** is an intermediate. □

The following claim shows that the sequence $\{D_k\}_{k<\omega}$ has an a.z. degree.

Claim 3. *For each $k < \omega$, the set D_k is finite.*

Proof. Fix k. Then, by the previous claim, $D_k \neq \Phi_k^{P_k(\mathcal{A})}$. Let x be the least natural number such that $D_k(x) \neq \Phi_k^{P_k(\mathcal{A})}(x)$. Fix s such that after it the approximations $\{D_k^t\}_{t<\omega}$ and $\{\Psi_k^t\}_{t<\omega}$ are stabilized on each $y \leq x$. Then for all $t \geq s$:

- for all $y < x$, $D_k^t(y) = \Psi_k^t(y)$,
- $D_k^t(x) = D_k^s(x) \neq \Psi_k^s(x) = \Psi_k^t(y)$.

Thus for each $t \geq s$, $l_k^t = x - 1$ and hence $m_k^t = \max\{m_k^s, x - 1\}$. Therefore, $D_k^t \subseteq m_k^{s+1} + 1$, which implies that $D_k \subseteq m_k^{s+1} + 1$; so D_k is finite. □

Finally, we use the Corollary 2.1 of the Recursion Theorem in order to prove that $\{D_k\}_{k<\omega}$ is not below $\mathcal{A}$.

Claim 4. $\{D_k\}_{k<\omega} \not\leq_{T,\omega} \mathcal{A}$.

Proof. Suppose that $\{D_k\}_{k<\omega} \leq_{T,\omega} \mathcal{A}$ and let f be a computable function such that for each k, $D_k = \Phi_{f(k)}^{P_k(\mathcal{A})}$. By Corollary 2.1, let $e < \omega$ be such that for all $A \subseteq \omega$, $\Phi_e^A = \Phi_{f(e)}^A$. Then

$$D_e = \Phi_{f(e)}^{P_e(\mathcal{A})} = \Phi_e^{P_e(\mathcal{A})},$$

which contradicts Claim 2. □

This proves the lemma. □

Finally in our main result, we characterize the high degrees exactly as the degrees bounding all almost zero degrees.

Theorem 5.2. *Let* $\mathbf{a} \leq \mathbf{0}'$. *Then* $\mathbf{a} \in \mathbf{H}$ *if and only if for each a.z. degree* $\mathbf{d} \leq \mathbf{0}'$, $\mathbf{d} \leq \mathbf{a}$.

Proof. Let $\mathbf{a}$ be a high degree and $\mathbf{d} \leq \mathbf{0}'$ be an a.z. degree. Let n be such that $\mathbf{a}$ is a high$_n$. Hence $\mathbf{d} \leq \mathbf{o}_n \leq \mathbf{a}$.

Now let $\mathbf{a} \leq \mathbf{0}'$ be a degree that bounds all a.z. degrees below $\mathbf{0}'$. By the previous lemma, $\mathbf{a} \notin \mathbf{I}$. Let $\mathbf{d} \leq \mathbf{0}'$ be an a.z. degree. Since $\mathbf{d} \notin \mathbf{L}$, then for each n, $\mathbf{0}^{(n)} < \mathbf{d}^{(n)} \leq \mathbf{a}^{(n)}$. Hence $\mathbf{a}$ is not low$_n$ for any n. So $\mathbf{a} \notin \mathbf{L}$ and thus $\mathbf{a} \in \mathbf{H}$. □

The similar problem of the definability of the low degrees in the terms of the almost zero degrees still remains an open question.

References

[1] W. Calvert, A. Frolov, V. Harizanov, J. Knight, C. McCoy, A. Soskova and S. Vatev, Strong jump inversion, Journal of Logic and Computation 28(7) (2018) 1499–1522.
[2] S. B. Cooper, Minimal degrees and the jump operator, Journal of Symbolic Logic 38 (1973) 249–71.
[3] S. B. Cooper, Computability Theory, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D.C., 2004.
[4] R. M. Friedberg, A criterion for completeness of degrees of unsolvability, Journal of Symbolic Logic 22 (1957) 159–60.
[5] H. Ganchev and A. C. Sariev, Definability in the local structure of the ω-Turing degrees, Mathematical Structures in Computer Science 29(7) (2019) 927–937.
[6] S. L. Kleene and E. L. Post, The uppersemilattice of degrees of recursive unsolvability, Annals of Mathematics 59 (1954) 379–407.
[7] K. McEvoy, Jumps of quasi-minimal enumeration degrees, Journal of Symbolic Logic 50 (1985) 839–848.
[8] P. G. Odifreddi, Classical recursion theory, Volume I, Studies in logic and the foundations of mathematics, ed. by R. A. Soare, A. S. Troelstra, S. Abramsky and S. Artemov, vol. 125, Elsevier, 1989.
[9] P. G. Odifreddi, Classical recursion theory, Volume II, Studies in logic and the foundations of mathematics, ed. by R. A. Soare, A. S. Troelstra, S. Abramsky and S. Artemov, vol. 143, Elsevier, 1999.
[10] H. Rogers Jr., Theory of recursive functions and effective computability, McGraw-Hill, New York, 1967.
[11] G. E. Sacks, On the degrees less than $\mathbf{0}'$,Annals of Mathematics 2(77) (1963) 211–231.

[12] A. C. Sariev, Definability in degree structures, PhD dissertation, Sofia University, 2015 (in Bulgarian).
[13] A. C. Sariev and H. Ganchev, The ω-Turing degrees, Annals of Pure and Applied Logic 165(9) (2014) 1512–1532.
[14] R. Soare, Turing computability: Theory and applications of computability, Springer-Verlag, Berlin, Heidelberg, 2016.
[15] I. N. Soskov, The ω-enumeration degrees, Journal of Logic and Computation 6 (2007) 1193–1214.
[16] I. N. Soskov and H. Ganchev, The jump operator on the ω-enumeration degrees, Annals of Pure and Applied Logic 160(30) (2009) 289–301.
[17] A. A. Soskova and I. N. Soskov, A jump inversion theorem for the degree spectra, Journal of Logic and Computation 19 (2009) 199–215.
[18] S. Vatev, Conservative extensions of abstract structures, in: Models of Computation in Context, CiE 2011, ed. by B. Löwe, D. Normann, I. N. Soskov and A. A. Soskova, LNCS, vol 6735, Springer, Berlin, Heidelberg, 2011, 300–309.

Received on March 19, 2025
Revised on July 21, 2025
Accepted on July 22, 2025

Andrey C. Sariev

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: andreys@fmi.uni-sofia.bg

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

AN APPROACH FOR ECOLOGICAL WATER BODIES' ENVIRONMENT MODELING

SNEZANA SAVOSKA, ANDRIJANA BOCEVSKA,
NATASHA BLAZESKA-TABAKOVSKA,
BLAGOJ RISTEVSKI AND ILIJA JOLEVSKI

The study presents an approach for modeling ecological water bodies' protection in order to support the transformation of agricultural land surrounding water springs into woodlands. Google Earth Pro was chosen for its enhanced capability to import Geographic Information System (GIS) data. It enables the measurement of ground area, radius, and circumference, along with the creation of high-resolution screenshots and offline videos for sharing. SketchUp, a 3D modeling tool, offers various drawing applications and features an extensive online library of free model assemblies. The combined use of these tools allows the creation of file formats that can be accessed in Google Earth Pro for virtual presentations. The efficacy of the proposed approach was illustrated using data from two springs located in the region of North Macedonia, which are utilized for bottled water.

Keywords: Google Earth Pro, SketchUp, EU Water Framework Directive (WFD)

2020 Mathematics Subject Classification: 68U07

CCS Concepts:

• Computing methodologies~Modeling and simulation~Model development and analysis~ Modeling methodologies

1. Introduction

The EU Water Framework Directive (WFD) aims to support the restoration of water bodies in EU countries as well as across Europe. The aim is to gain water with a "good quality status" by 2027, which is already achieved in many EU countries. Many activities were undertaken in and outside the EU countries in this goal. The objective is to create mechanisms and practices for sustainably restoring water

DOI: 10.60063/GSU.FMI.112.153-164

bodies without utilizing costly chemical technologies and purification methods. The purification methods raise the cost of water and result in low-quality drinking water for the population in Europe. Planning the water bodies' protection is the best way of achieving this goal. It is one of the most effective sustainable solutions to the issue of providing a reliable, high-quality water supply. The protection of water bodies involves the use of effective technologies for accurate planning and safeguarding of water springs, river flows and other water reservoirs. According to COST Action PASFOR-W recommendations, the gradual transformation of agricultural land into woodland and planned planting and management of certain types of trees are one of the most sustainable ways of water bodies' protection [24].

This paper proposes an approach for modeling ecological water bodies' environment according to recommendations of EU WFD directives. It is used for creating scenarios for transforming agricultural land around water springs into woodlands. Additionally, it offers benefits to society, including the contribution of companies that utilize springs for bottling water. For this purpose, Google Earth Pro and SketchUp are utilized as user-friendly tools for planning and modeling the land surrounding water bodies.

For nearly two decades, the Republic of North Macedonia has been without integrated policies to address water resource management [12,14]. Consequently, the water quality and quantity of drinking water for the population constantly decreased. The separation of authorizations for water resources protection affects the occurrence of many problems with water resources [25]. The responsibilities are fragmented among water management companies, sanitary inspection services, the Ministry of Ecology and the Ministry of Economy. Some uncertainties in the legal regulations that refer to water management also influence the current conditions. Another worrying factor is the evasive cutting of forests throughout the territory and their usage for heating. Bad policies with the forests that have been carried out over the past two decades have produced damage to the forest fund and a growing absence of cutting control. This arises due to poverty of the population, insufficient control of the forest fund, deliberate forest fires and the lack of effective state mechanisms for protection, unpunished executors and non-application of law. The influence of illegal landfills and low ecological awareness of the population is also high [3]. All these consequences have a negative impact on the water supply and quality of drinking water. In order to meet the needs for drinking water, water wells are included, but they cause a decrease in water quality. It is therefore not surprising that according to the latest statistics [22], the percentage of people with diseases is tending to a steady increase. Citizens are facing eco-problems every day and there is no integrated solution.

The depletion of the forest reserve in recent years has resulted in a consistently high level of air pollution. It additionally affects the quality of water through polluted atmospheric influences (precipitation). The contaminated soils with heavy metals and pollutants also have a significant impact on water quality. All these pollutants are the reasons for the external threat to human life, as exposome [20]. The inadequate policies of the previous government for massive tree planting have resulted in only 11% success [24] despite the huge resources spent on afforestation.

Guided by these problems, we propose an approach for modeling and planning eco-design of water springs and water bodies in accordance with the EU Water Framework Directive using Google Earth Pro and SketchUp. This software provides a highly detailed, interactive 3D map of the world, using satellite imagery, aerial photography, and other geospatial data, while SketchUp is a 3D modeling software that allows users to design and create detailed 3D models. The combination of Google Earth Pro and SketchUp offers a powerful toolset for creating and visualizing 3D models in a geospatial context. The tool also allows for measuring the area, radius, and circumference of the ground. This approach provides printing high-resolution screenshots and makes compelling offline movies to share. SketchUp is a 3D modeling software for a wide range of drawing applications. It can work with ready-made objects or models that can be embedded in the proposed approach. It supports importing objects created with many other 3D modeling tools such as Autodesk 3DS Max [23] or others. Furthermore, an online library of free model assemblies and 3D Warehouse can be used. This library has a possibility for contributing models by adding some layout functionality.

The paper is organized as follows. After the introduction and related works, we present the aims of the proposed approach according to EU WFD policy. The next section describes the general framework for creating an approach based on the GIS system. The created approach for selected locations around the water springs is explained in the fourth section. It is created as scenarios for selected springs for two locations. Concluding remarks highlight the benefits of applying the proposed approach and some possible scenarios and plans for future work.

2. Related works

Many efforts were made by EU countries to provide an ecological framework for good water status. The EU WFD aims to assist in the restoration of water bodies in Europe to achieve "good ecological status" by 2027. In this context, the catchments of EU rivers must be protected from agricultural pollutants as well as other forms of pollution. Accumulating evidence shows that recent improvements to agricultural practices can be very effective in water quality improvement, but not sufficient. The COST action PASFOR-W proposed a framework for obtaining growing support for land use change. The purpose is to help bridge the gap, with a particular focus on tree planting to intercept and reduce the delivery of diffuse pollutants to water [24].

The PESFOR-W recommendations consolidate and expand knowledge on the potential reach and scope of F4W PES schemes. In addition, they provide PES environmental effectiveness [24], PES cost-effectiveness and agreed standards and guidance. The platform has to link research findings to frontline users in water [24], forestry, agriculture, and environmental finance sectors. It considers supporting tools such as GIS, Decision support systems (DSS), CAD, and tools such as SketchUp. SketchUp is one of the user-friendly tools for creating 3D objects in the GIS environment.

GIS systems have a wide range of applications, especially when ecosystems have to be analyzed, protected and planned [9]. Taking into consideration GIS systems, some researchers [7] interpret GIS as GISy – Systems, GISc – Science and GISt – Studies. GISy focuses mainly on technology for the acquisition and management of spatial information. GISc is connected and focused on conceptual issues of representing spatial data, achieving a deeper understanding of the meanings and creative analysis. GISt is focused on social, legal and ethical issues taking into consideration their complexity. The researchers figured out increasing of the market of cheap software, and raster systems on very basic hardware. This fact provided GIS education with great tutorials and video materials. They also empowered broader usage of GIS systems [7] due to various factors that facilitate the use of GIS in marketing data. User-friendly software with distributed data and wider computer adoption was provided. Ever since, when environmental issues are considered, GIS has been widely utilized as a strategic tool. GIS is suitable for planning the agriculture economy with exploratory spatial analysis [8], human capital knowledge analysis [6], health care analysis [17], government issues, e-commerce and e-government support [1, 6, 7, 19].

Many projects bring symbiosis of GIS with other technologies such as remote sensing activities, providing information for ecosystems for wider communities [13]. Some academic researchers are focused on creating local and hybrid models [13], providing cloud options and accessibility for a wider population. Some of them are moved into the cloud with a semantic specification for spatial data infrastructure. In this way, interoperability and framework for web applications based on open geospatial standards were provided [5]. All activities and steps already provide a frame for geospatial web services, cloud GIS and semantic web for GIS, allowing data to be shared and reused. These steps provided interoperability and spatial data infrastructure (SDI) with Geography Markup Language (GML) or Extensible Markup Language (XML) [10].

Nowadays, many Earth Observation (EO) data are available on the web. They are provided by government agencies that support additional measures and results as a benefit of increasing technological capacities [2, 5]. They support the next steps of GIS system development as a connection to creating standards as OGC Web Services (OWS). In this way, Cloud services quality requirements set by INSPIRE directives (EU directives) are met [4]. These activities aim to provide compatible Geospatial Open Source Software standards and semantic interoperability with tracking changes possibilities [28]. All these factors provide support for numerous projects, standards and various GIS software tools [5].

GIS filtering approaches are used within Woodland for Water (W4W) projects to determine the possibilities for woodland creation. It is used in the first phase of understanding catchment issues and opportunities [15]. Using GIS technology, the W4W measure can be targeted strategically at particular sites. This is important because GIS can assist in regional woodland creation schemes by providing spatial information on the existing woodland occurrence. It can assist environmental managers who wish to connect existing sites or create new woodlands in largely treeless landscapes [16]. By implementation of GIS, a method of identifying spatial components for regional conservation planning is developed [18]. An ecological conceptual model of a landscape can be combined with assessment methods to evaluate ecosystem service functions. They can also establish a spatial analysis model for urban minimum ecological land using GIS [27]. GIS-based landscape planning and cost-benefit analysis can be applied in a methodology for designing and implementing PES systems [11]. A suitable method for identifying key eco-spaces using GIS is proposed in [26].

All these uses of the GIS system have prerequisites for high skills in GIS modeling and usage of specific complex tools for planning landscapes. They can be used only by planning experts with high IT skills. For this reason, we proposed an easy-to-use approach. This approach considers using a simple combination of GIS and 3D tools, in a way that non-IT experts can use it with a short course. It will empower planners and foresters in forestry and water supply. This simulation tool can be used to plan the transformation of land into woodland or forest.

3. Aims of the proposed approach for planning woodland modeling

The knowledge gained from the research conducted in COST Action PESFOR-W [24] can enhance the design and environmental effectiveness of planning activities for the protection of ecological water bodies. This paper focuses on the use of a combination of the GIS system Google Earth Pro and SketchUp, which enhances planning efficiency in addressing the issue of diffuse water pollution [24]. This is in accordance with the report on sustainable forest management in Europe: to "develop appropriate policies and strategies for managing forests and water resources sustainably to adapt to climate change and contribute to its mitigation" [21]. We consider the four principal research objectives [24]:

1. Characterize and critically evaluate the governance models and design structure of EU WFD schemes;

2. Evaluate the environmental effectiveness of targeted woodland planting in reducing a range of agricultural diffuse pollutants;

3. Develop a European repository of Case Studies that investigate lessons from existing EU WFD schemes;

   and

4. Develop User Guidance on the suitability of pollutant, ecosystem service and catchment scale models to quantify the effectiveness of tree planting to reduce diffuse pollution.

The focus was on the second principle, proposing an approach for planning the afforestation of landscapes around the springs. The combination of these software tools is chosen due to the synergy they offer in modeling the environment of ecological water bodies. The user-friendly combination can bring numerous benefits to the water protection sectors.

4. Proposed ecological water bodies' protection approach for modeling

Considering the aims of the approach for modeling ecological water bodies' environments for planning woodland around water resources, we use the powerful and easy-to-use combination of tools – Google Earth Pro and SketchUp.

Firstly, we consider the water springs in the Republic of North Macedonia and choose two of them to examine the real situation. We identified and analyzed their GIS coordinates along with the requirements of the four main research objectives [24].

The main steps that we differentiate in our approach are given in the following.

- Choosing the object that should be explored. For our purpose, we choose agricultural land surrounding water springs.
- Setting the location and directional orientation by importing an image from Google Earth. The SketchUp model is set to the same latitude and longitude as the chosen location. The imported terrain image is aligned with the axes in SketchUp.
- Modeling using SketchUp tools (selecting the tree type and layout, determining the spacing between trees, and defining the area size).
- Uploading the model into Google Earth.

This synergy enables users around the globe to populate the virtual Earth with 3D models, and the living Google Earth is a testimony to this vision.

In practice, the forest engineers have to follow the steps of the scenario proposed in Figure 1 to gain the best results.

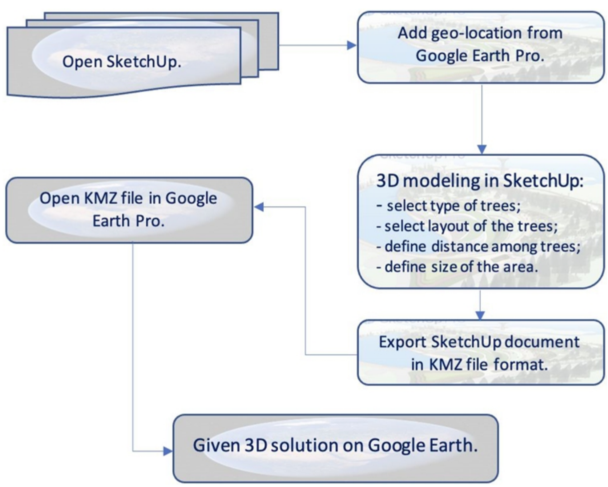


Figure 1. Proposed PES F4W approach for woodland planning

The proposed approach can be illustrated with the following use case scenarios. Their aim is to support the transformation of agricultural land surrounding water springs "Dobra voda" (Figure 2) and "Medzitlija" (Figure 3) into woodlands.

Figure 2. Google Earth Pro Map of the spring "Dobra voda"

Figure 3. Google Earth Pro Map of the spring "Medzitlija"

- To geolocate a model in SketchUp, it is necessary to import terrain using the Add Location tool. We import terrain that enables the creation of the model based on the site where it is planned to be built (or just display it on Google Earth).
- In SketchUp, we select the type of trees as a 3D model (appropriate to the area and height of the location, a decision made by the forest engineer), define the distance among trees, and the size of the area to be reforested or protected. There are a number of websites available on the internet that offer ready-to-use SketchUp models. The layout of the trees can be different. In the first case, we used a rectangular and in the second use case, we used a circular layout. These parameters must be previously known and defined by forest engineers.
- When placing a model created in SketchUp, it is essential to ensure the model's dimensions are exact so that its scale aligns proportionally with the surrounding landscape or buildings in Google Earth. Previewing a SketchUp model in Google Earth is a great way to see how the created model looks in the context of its surroundings. When the model is ready for a preview, it is needed to export it as a Google Earth KMZ file. The KMZ format is a zipped form of Google Earth's proprietary Keyhole Markup Language (KML). A KMZ can contain location data (latitude and longitude) along with other information, such as SketchUp geometry. A KMZ file allows you to view a model in Google Earth without needing SketchUp installed. Furthermore, it allows sharing of the model for viewing in Google Earth in SketchUp's 3D Warehouse.
- When a 3D model is imported into Google Earth, it can be translated, rotated, and scaled to fit into the Earth coordinate system. Within Google Earth, Google Earth's navigation tools can be used to move around the area and see how your model looks on its intended site. The created use case in Google Earth Pro of the spring "Dobra voda" after opening the KMZ file gained by SketchUp can be seen in Figure 4. The same approach is applied in the next case study for the spring "Medzitlija", shown in Figure 5.

It is important to note that the use of this approach for modeling the environmental protection of ecological water bodies is not solely limited to afforestation. It can be used with different Google Earth Pro locations as well as different ready-to-use 3D objects from SketchUp. Because of their user-friendly usage and flexibility, they can be used for modeling GIS-connected plans from a different scope of interest. Planning of river protection with woods, barriers, bridges and other objects, planning the protection of springs, wells and watercourses, as well as urban planning suitable to the concept of smart cities can also be performed.

4.1. Benefits from the PES F4W approach for the society

In general, several stakeholder groups can profit from this user-friendly approach. First, the Ministry of Agriculture, Forestry and Water Management can use

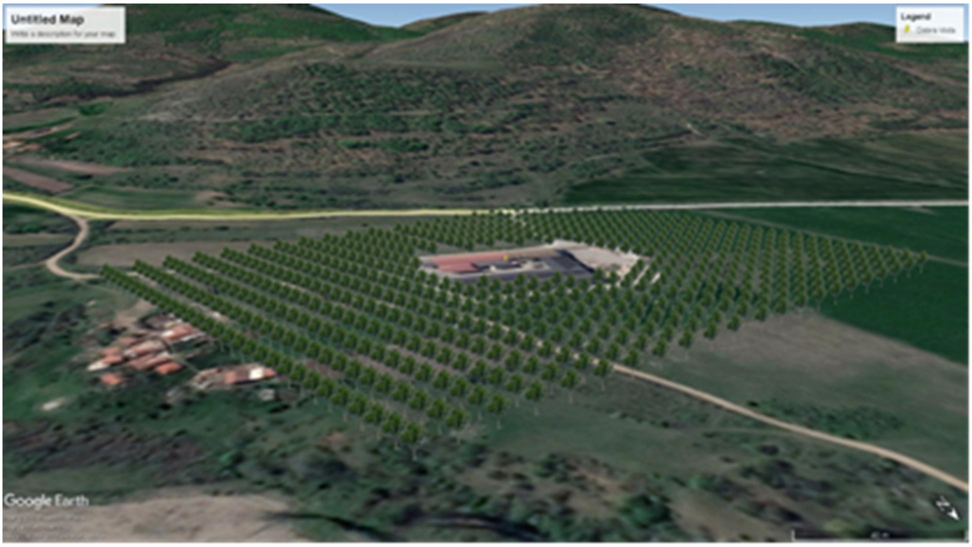


Figure 4. Google Earth Pro Map of the spring "Dobra voda" after opening the KMZ file gained by SketchUp

Figure 5. Google Earth Pro Map of the spring "Medzitlija" after opening the file gained by SketchUp

it for planning and modeling the spring and the water body's protection. In general, several stakeholder groups can utilize this user-friendly approach. Firstly, the Ministry of Agriculture, Forestry, and Water Management can use it for planning and modeling the protection of springs and water bodies. In addition, it can be used for planting suitable trees or forest around the springs, basins, and rivers. It is important because it has to provide clean water and protect the citizens, according to the EU demands and the EU WFD. The WFD aims to ensure that all surface water and groundwater are in good quality status by 2027. They will benefit from this approach in short and long-term planning. They can prepare the needed seedlings, appropriate for the properties of the land and solve the land ownership.

The Ministry of Ecology can benefit from the usage of this approach to protect the environment around the water springs, supporting the EU WFD. The Ministry of Economy, as the entity responsible for granting water concessions to companies, will also benefit. The companies that bottle and sell drinking water, food, and beverages will benefit from meeting EU standards for clean water by using this approach.

The Ministry of Healthcare can benefit from the implementation due to a reduction in diseases linked to issues caused by polluted water sources. This is related to the use of artificial fertilizers, other fertilizers, insecticides, and pesticides around springs, sources, and rivers. This approach can be used to protect all water sources utilized for food production and agricultural irrigation.

Citizens will benefit from the approach implementation with improved living conditions by having access to "water with a good quality", without pollutants from land (agriculture or insecticide). In the end, this is the main goal of the EU WFD.

5. Conclusions

In this paper, we propose an approach for modeling the environmental protection of ecological water bodies based on the goals of the EU Water Framework Directive. The main aim is to gain water with a "good quality status" by 2027. This framework intends to provide activities in planning the protection of water bodies as recommended. The most effective sustainable way to protect water bodies is effective and precise planning of their protection in areas around water springs, river flows, and other water accumulations. As advised in [24], the gradual transformation of agricultural land into woodland is another sustainable way of water bodies' protection.

The combination of the GIS system Google Earth Pro and the 3D modeling SketchUp tool is used for this approach for modeling plans and creating scenarios for transforming agricultural land around water springs into woodlands. This approach demonstrates the virtual afforestation implemented in the selected area surrounding the springs. It is demonstrated using two selected springs, "Dobra voda" and "Medzitlija," which serve as case studies for bottled water.

It is worth mentioning that the approach is applicable to a wide range of GIS planning and modeling activities using different 3D objects. Applications include but are not limited to planning river protection with woods, barriers, bridges, springs, wells, and watercourses. It is also suitable for urban planning, essential for implementing the concept of smart cities.

As future work, we plan to develop a framework for calculating Payments for Ecosystem Services based on the recommendations of PASFOR-W [24]. These schemas are connected with gaining easy-to-use calculation expert system intended for forester and water experts in the process of decision-making in woodland and water bodies protection. Specifically, a tool is needed for planning the type of trees, the spacing between them, and the number of tree seedlings. This can be determined based on factors such as the land's geology, sea level, weather conditions, and other relevant factors.

References

[1] Y. Du, C. Yu and J. Liu, A Study of GIS Development Based on KML and Google Earth, Fifth International Joint Conference, Seoul (25–27 Aug. 2009) 1581–1585, https://doi.org/10.1109/NCM.2009.17.

[2] EOSDIS, Earth Observing System Data and Information System, National Aeronautics and Space Administration, https://www.earthdata.nasa.gov/about/esdis/eosdis.

[3] Environmental statistics, State Statistical Office – Republic of North Macedonia, https://www.stat.gov.mk/PrikaziPoslednaPublikacija.aspx?id=20.

[4] European Commission, European Commission Directive 2007/2/EC of the European Parliament and of the Council of 14 March 2007 establishing an Infrastructure for Spatial Information in the European Community (INSPIRE), Off. J. Eur. Union 50 (2007) 1–14.

[5] K. Evandelidis et al., Geospatial services in the Cloud, Computers & Geoscience 63 (February 2014) 116–122, https://doi.org/10.1016/j.cageo.2013.10.007.

[6] A. Faggian and P. McCann, Human capital flows and regional knowledge assets: a simultaneous equation approach, Oxford Economic Papers 58(3) (2006) 475–500, https://doi.org/10.1093/oep/gpl010.

[7] P. Forer and D. Unwin, Enabling progression GIS and education, http://www.geos.ed.ac.uk/~gisteac/gis_book_abridged/files/ch54.pdf.

[8] S. J. Goetz and A. Rupasingha, High-Tech Firm Clustering: Implications for Rural Areas, American Journal of Agricultural Economics 84(5) (Dec. 2002) 1229–1236.

[9] J. R. Haslett, Geographic information systems. A new approach to habitat definition and the study of distributions, Science Direct, Trends in Ecology & Evolution 5(7) (July 1990) 214–218, https://doi.org/10.1016/0169-5347(90)90134-Y.

[10] T. Johansson and P. Pellikka, GISAS – Geographical Information systems applications for Schools, In: AGILE 2006, 9-th AGILE Conference on Geographic Information Science, Shaping the future on Geographic Information Science in Europe, 317–318, ALTO Press, Hungary, 2006.

[11] P. Martínez de Anguita, S. Rivera, J. M. Beneitez, F. Cruz and F. M. Espinal, A GIS Cost-Benefit Analysis-Based Methodology to Establish a Payment for Environmental Services System in Watersheds: Application to the Calan River in Honduras, Journal of Sustainable Forestry 30(1–2) (2011), https://doi.org/10.1080/10549811003742225.

[12] Menadziranje so vodeni resursi vo R.S.M, http://www.mes.org.mk/PDFs/MAKNEN/EN&Watermanagement.pdf.

[13] H. J. Miller, Potential Contributions of Spatial Analysis to Geographic Information Systems for Transportation (GIS-T), Geographical Analysis 31(4) (03 September 2010), https://doi.org/10.1111/j.1538-4632.1999.tb00991.x.

[14] Ministry of Environment and Physical Planning, https://www.moepp.gov.mk/en-GB.

[15] T. Nisbet and S. Broadmeadow, Woodlands for Water opportunity mapping – national, https://www.forestresearch.gov.uk/publications/opportunity-mapping-for-trees-and-floods.
[16] P. M. Perrin et al., National Survey of Native Woodlands 2003–2008: Volume I: Main report, Report submitted to National Parks & Wildlife Service, Dublin, 2008.
[17] D. B. Richardson et al., Spatial Turn in Health Research, Science 339(6126) (22 March 2013) 1390–1392, https://doi.org/10.1126/science.1232257.
[18] M. Rouget, R. M. Cowling, R. L. Pressey and D. M. Richardson, Identifying spatial components of ecological and evolutionary processes for regional conservation planning in the Cape Floristic Region, South Africa, Divers. Distrib. 9 (2003) 191–210, https://doi.org/10.1046/j.1472-4642.2003.00025.x.
[19] S. Savoska, V. Muaremi, A. Boceska, B. Ristevski and Z. Kotevski, Modelling of GIS Based Visual System for Local Educational Institutions' Stakeholders, ISGT 2016, Sofia, 24–35.
[20] S. Savoska, B. Ristevski, N. Blazheska-Tabakovska and I. Jolevski, Towards integration exposome data and personal health records in the age of IoT, in: 11th ICT Innovations Conference 2019, 17–19 October, Ohrid, Republic of Macedonia, 2019, 237–246.
[21] State of Europe's Forests 2007, The MCPFE Report on Sustainable Forest Management in Europe, https://unece.org/DAM/timber/publications/State_of_europes_forests_2007.pdf.
[22] State Statistical Office – Republic of North Macedonia, Mortality, https://www.stat.gov.mk/PrikaziSoopstenie.aspx?rbrtxt=7.
[23] I. Stojanov, B. Ristevski, Z. Kotevski and S. Savoska, Application of 3ds Max for 3D Modelling and Rendering, in: International Conference on Applied Internet and Information Technologies, 2016, 133–144.
[24] G. Valatin et al., PESFOR-W: Improving the design and environmental effectiveness of woodlands for water Payments for Ecosystem Services, Research Ideas and Outcomes 3 (2017), https://doi.org/10.3897/rio.3.e13828.
[25] Water law (2022), https://faolex.fao.org/docs/pdf/mac202054.pdf.
[26] H. Xie, G. Yao and P. Wang, Identifying Regional Key Eco-Space to Maintain Ecological Security Using GIS, International Journal of Environmental Research and Public Health 11(3) (Mar 2014) 2550–2568, https://doi.org/10.3390/ijerph110302550.
[27] Y. Zhang, Q. Wang and B. J. Li, Study on forecasting ecological land demand with carbon-oxygen balance method, China Land Sci. 27 (2007) 23–28.
[28] P. Zhao et al., The Geoprocessing Web, Computers & Geosciences 47 (October 2012) 3–12, https://doi.org/10.1016/j.cageo.2012.04.021.

Received on December 18, 2024
Accepted on March 14, 2025

Snezana Savoska, Andrijana Bocevska, Natasha Blazeska-Tabakovska,
Blagoj Ristevski and Ilija Jolevski

University St. Kliment Ohridski – Bitola
Faculty of Information and Communication Technologies – Bitola
Prvi maj bb, 7000 Bitola
REPUBLIC OF NORTH MACEDONIA

E-mails: snezana.savoska@uklo.edu.mk
andrijana.bocevska@uklo.edu.mk
natasa.tabakovska@uklo.edu.mk
blagoj.ristevski@uklo.edu.mk
ilija.jolevski@uklo.edu.mk

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 112, 2025

ANNUAL OF SOFIA UNIVERSITY "ST. KLIMENT OHRIDSKI"
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 112, 2025

LARGE LANGUAGE MODELS: A TOOL FOR SOLVING MATHEMATICAL PROBLEMS IN HIGH SCHOOL

Ralitza Stamenkova

Mathematics learning resources have evolved from static textbooks to collaborative online forums and conversational artificial intelligence (AI) tools. This evolution reflects students' ongoing demand for clarity, adaptability, and accessible support. While traditional textbooks offered privacy, they limited flexibility. Online forums, such as Yahoo! Answers, Math Stack Exchange, and Mathematika.bg, enabled collaborative problem-solving, but they required public participation. Large language models (LLMs), including ChatGPT, now offer private, adaptive "comfort mode" interactions, combining the autonomy of self-study with the responsiveness of a personal tutor.

The potential of LLMs to support mathematics education in Bulgaria is examined through a dual approach in this study: (1) eleven models solving problems from the Bulgarian National External Assessment (NEA) for 10th grade are empirically evaluated, and (2) practicing mathematics teachers enrolled in a Master program are qualitatively observed. Model performance was evaluated based on accuracy, methodological alignment with the national curriculum, and linguistic appropriateness.

The findings indicate that, although several models, such as Mistral (22B) and DeepSeek R1, achieved perfect accuracy, they often used solution strategies that deviated from national standards. Locally fine-tuned models (e.g., BgGPT) demonstrated stronger curriculum alignment and the use of precise Bulgarian mathematical terminology. Teacher feedback revealed recognition of AI's potential for personalized student support, as well as caution toward integration, reflecting a preference to retain creative and methodological control. The study concludes that three conditions are necessary for successful AI integration in mathematics education: mathematical accuracy, adherence to curriculum-specific methods, and linguistically precise explanations. Large language models (LLMs) can complement, but not replace, the teacher's role when deliberately embedded in the continuum of educational resources, from books to forums to conversational AI.

Keywords: mathematics education, large language models, ChatGPT, AI in education, curriculum alignment, teacher perceptions, personalised learning

2020 Mathematics Subject Classification: 97D40

DOI: 10.60063/GSU.FMI.112.165-183

1. Introduction

The evolution of digital technologies and the emergence of AI have profoundly transformed the landscape of mathematics education, leading to "a paradigm shift in how education is delivered ... from static content to dynamic, personalized learning environments" [3,9]. From the limited print-based resources of the late 20th century to the vast digital ecosystem of today, students and educators have gained access to a wide array of tools, ranging from educational videos and interactive platforms to AI-powered assistants. However, this transformation has also introduced challenges, including inconsistencies in content quality, unequal access, and shifting expectations regarding the roles of teachers and technology in the learning process. As noted by Ali et al. [2], while ChatGPT offers real-time, adaptive support, its use also raises concerns about overreliance, academic dishonesty, and uneven student autonomy.

According to Dey [7], AI is "reshaping educational experiences, enhancing learning outcomes, and addressing long-standing challenges in the education sector", offering significant potential for personalization and innovation across disciplines. In this context, the present study explores the intersection of AI and mathematics education, with a particular focus on the perceptions and behaviours of practicing teachers enrolled in the master's program "Innovation and Multidisciplinarity in Teaching Mathematics, Computer Modelling, and Information Technologies" during the winter semester of 2024/2025. The cohort under observation is currently enrolled in secondary education programs and is being observed as part of two graduate-level courses: "Educational Methods and Tools and Innovative Approaches" and "Project-Based Learning." The objective of this study is to investigate how these experienced educators interact with emerging technologies, particularly LLM, in the design of learning tasks and pedagogical strategies.

The study also considers the evolving needs of school-aged learners, who increasingly seek diverse, personalized, and on-demand forms of academic support. According to Ayeni et al. [3], AI enables "individualized learning trajectories that dynamically adapt to student needs," making it a powerful tool for inclusivity and motivation. The way students seek assistance in mathematics has evolved over time. Initially, students predominantly relied on textbook-based study methods. Subsequently, there has been a shift toward online peer collaboration, which has emerged as an increasingly prevalent approach. More recently, there has been a notable increase in the use of AI-powered platforms as a tool for seeking academic assistance. This progression is exemplified by the transition from early web forums, such as Yahoo! Answers, to structured question-and-answer communities, including Math Stack Exchange and `Matematika.bg`. The current era is characterized by the emergence of platforms such as ChatGPT, Le Chat (Mistral AI), and DeepSeek, which offer immediate solutions and, in some cases, curriculum-aligned explanations. These developments have diversified the forms of academic support available to learners, offering both new opportunities and new challenges for integration into formal education.

In light of these considerations, the present study aims to address three interconnected objectives:

(1) To examine how practicing teachers engage with AI tools in their own pedagogical design.
(2) To explore how students' historical and present-day needs shape their interactions with online mathematical resources.
(3) To analyse the potential and limitations of chatbot-based AI systems for delivering curriculum-aligned mathematical explanations and solutions.

To achieve these aims, the present study evaluates the capacity of contemporary language models to solve mathematics problems from the Bulgarian National External Assessment (NEA) for 10th grade. The evaluation focuses on three factors: accuracy, methodological alignment with the national curriculum, and linguistic appropriateness. The empirical results are complemented by qualitative insights from practicing teachers, offering a nuanced understanding of both the opportunities and constraints of integrating AI into mathematics teaching and learning. The central question guiding this research is how these technologies can be meaningfully integrated into the curriculum to meet authentic educational needs while preserving pedagogical integrity.

2. Observations

2.1. Master of education student

The observation focuses on students from the Master's program "Innovation and multidisciplinarity in teaching Mathematics, Computer Modelling and Information Technologies" in the winter semester of 2024/2025. A review of their work on ongoing assignments in the courses "Educational methods and tools and innovative approaches" and "Project-based learning" has been conducted. It is noteworthy that the participants in the study were master's-level students who also practice as mathematics teachers in well-regarded secondary schools, primarily based in the capital. Most of these individuals possess extensive experience in teaching and demonstrate a robust comprehension of the subject matter.

One of the assignments given in their coursework posed the following question: "Is artificial intelligence considered an educational technology?" A total of nine written responses were submitted, with most of these responses limited to a single printed page. One student explicitly stated that they initiated the process by consulting ChatGPT's response to the prompt. However, it is challenging to ascertain the extent to which the other submissions were independently authored or supported by AI tools.

The prevailing sentiment among the student cohort was that AI has the potential to function as a substantial support instrument for educators, particularly in the domains of data processing, test creation, and the automation of administrative tasks. The participants underscored the merits of intelligent educational platforms, which possess the capacity to adapt content in accordance with students' proficiency levels. However, it was acknowledged that such platforms are not widely used in the Bulgarian educational context, rendering many of the arguments more theoretical than experiential. Many students have expressed ethical and social concerns regarding the implementation of artificial intelligence in educational settings. There was unanimous consensus that AI cannot substitute for the teacher as the primary figure in the educational process. However, AI can enhance and support the teacher's work. This mirrors findings suggesting that while educators see potential in AI, they remain sceptical and cautious in adopting it widely due to unclear guidelines and insufficient training [7]. This is further supported by findings from Pepin et al. [19], who emphasize that while teachers acknowledge the exploratory potential of tools like ChatGPT, they often remain hesitant to integrate them meaningfully into their pedagogy without clearer didactic frameworks or institutional guidance.

In the course entitled "Educational Methods and Tools and Innovative Approaches," students were assigned the following task: "Formulate a mathematics task that could be used as the basis for a mathematical essay, presentation, group activity, or practice-based individual project. In the case of employing a large language model (LLM) during task development, it is necessary to specify the model's name and version, the original prompt, and the generated result. The objective is to ingeniously modify the AI-generated content to create a final version of the task." The course was conducted online, enabling continuous feedback and iterative improvement of student work, which fostered both consultation and reflection. A study of the submitted assignments revealed that only one of the ten was explicitly inspired or co-authored with the help of AI. The remaining submissions appear to be authentic and original works, as evidenced by their style and depth. In subsequent discussions, it became clear that the students viewed the creation of assignments independently as a matter of professional pride. This attitude is in partial conflict with the course's underlying objective, which is to cultivate proficiency in the utilization of digital technologies and their pedagogical applications. In a similar vein, the course titled "Project-Based Learning" incorporated two assignments that advocated for the utilization of AI. However, the same cohort of students predominantly demonstrated a reluctance to engage with AI tools.

From a research perspective, this offers an insightful glimpse into teachers' attitudes toward creative collaboration with AI. Despite the modest sample size, the findings offer a promising line of inquiry for future studies delving into teachers' perceptions and inclination to incorporate AI in educational practice.

In another assignment, participants were obliged to select a problem from their own teaching practice:

"Choose a problem from your practice and write it down.

Look at the solution, using one or more of the different tools, like PhotoMath, ChatGPT, WolframAlpha, or feedback in a forum about a specific topic (these are just some examples, and they are not mandatory).

Pay attention to the steps of the generated solutions and answer the following questions:

(1) How similar is the generated solution to the one you would show your students, and in what ways are they different? In your analysis, clearly indicate which tool was used to generate the solution you are discussing.

(2) How did you choose the tool? Was it based on experience, skills, the specifics of the task, the students' knowledge, or other factors?
(3) What ideas does the solution give you for changing the task or making a new one?"

The responses indicated that the university students were aware of AI tools commonly used by school learners. It was observed that PhotoMath, a tool that was once prevalent among students, appears to be losing its relevance. Conversely, WolframAlpha is regarded as a favoured instrument among advanced students, university learners, and educators. Despite that the students demonstrated familiarity with the listed tools, their contributions were characterized by a lack of initiative and innovation in expanding the list of technologies. This finding suggests a potential area for curricular enhancement. According to Ravšelj et al. [23], teachers recognize the growing role of tools like ChatGPT in education but emphasize the need for clear institutional support and ethical guidelines for integration, noting a lack of consistent policy and training as barriers to meaningful adoption. This is true not only for educators themselves but also because educators serve as role models and conduits through which these competencies are transmitted to students. As emphasized by Pepin et al. [19], educators have a significant impact on classroom practices, shaping not only the pedagogical approach but also modelling digital engagement and the judicious use of tools such as ChatGPT. This role, as highlighted by the researchers, is crucial in determining the acceptance or restriction of these technological tools within educational settings.

2.2. The online materials vs. school students needs

The expansion of digital technologies and the ubiquity of internet access has led to a substantial augmentation in the array of mathematics resources accessible to students. In the contemporary educational landscape, learners have access to a variety of educational materials, including instructional videos created by individual enthusiasts and educational organizations, as well as collections of solved examination problems offered by private tutoring institutions and online portals. While this abundance provides unprecedented opportunities for independent learning, it also poses a significant challenge: the quality, clarity, and pedagogical consistency of these materials vary considerably.

In order to accurately assess the impact of AI in education, it is essential to understand the specific needs of students in the context of evolving educational resources.

In the late 20th century, students had limited access to resources for independent study. Mathematics textbooks were frequently minimalistic in nature, with answers to exercises being included only sporadically. Problem books, which are designed to address the most challenging aspects of mathematics education, seldom offer solutions to these problems. However, these solutions are often accompanied by expressions that may be perceived as discouraging, such as "it follows obviously from here," which may not provide adequate support to learners who are still grappling with fundamental concepts. In that context, students relied heavily on peer collaboration and the goodwill of their teachers as the primary sources of additional guidance. The phenomenon of self-directed learning was largely constrained by a lack of accessible, comprehensive, and pedagogically appropriate materials.

The rise of the Internet, however, has had a profound impact on the educational landscape. Online forums have emerged as valuable spaces where students can seek clarification, engage in discourse, and disseminate solutions. These forums evolved into informal learning communities that reflected the diversity of students' learning styles. As Raban and Harper observe, "in general, question asking communities are formed thanks to two complementary human instincts: the need to ask questions and the inclination of answer people to contribute their knowledge to others" [22]. For some students, a fully written solution was sufficient; for others, the opportunity to pose follow-up questions and receive clarifying explanations was essential. This shift signalled a global trend toward more interactive and learner-centred support environments.

During this period, platforms such as Yahoo! Answers (active from 2005 to 2021) gained international popularity as public knowledge-sharing hubs. Yahoo! Answers, launched in 2005 [11], was among the first large-scale community-driven platforms that enabled users to ask and answer questions across a wide array of domains. The mathematics section of the website rapidly became a popular resource for students, educators, and enthusiasts seeking assistance with problem-solving, ranging from basic arithmetic to advanced calculus. Yahoo! Answers was a prominent example of an online knowledge-sharing community, characterized by its substantial size and diversity. This platform facilitated a wide range of interactions, encompassing both technical and social domains. Notably, it featured a robust Mathematics section that was specifically designed to cater to high school learners [1]. Yahoo! Answers and similar platforms have served not only as information sources but as informal learning communities, where the interplay of social cognition and perceived knowledge value drives both asking and answering behaviours [22].

Despite its initial success, Yahoo! Answers experienced a steady decline after 2010 due to diminished content quality, reduced moderation, and increased spam. Consequently, its educational value has diminished. In April 2021, the platform was officially deactivated [4]. Despite the absence of a systematic curation of its contents, the impact of this collection remains substantial. Subsequently, a considerable number of users and contributors migrated to more specialized platforms such as Math Stack Exchange, Reddit, and Quora. These platforms continue to address the evolving needs of students in the digital age.

The inception of Math Stack Exchange (MSE) in 2010 coincided with the broader expansion of the Stack Exchange network. The establishment of MSE was driven by an increasing demand for specialized, peer-reviewed mathematical support. In contrast to earlier platforms such as Yahoo! Answers, MSE implemented a reputation system, stringent moderation policies, and LaTeX integration, thereby facilitating engagement with mathematical content across diverse levels of proficiency, ranging from secondary education to advanced research. As Mansouri et al. observe, "the presence of Community Question Answering sites such as Math Stack

Exchange and Math Overflow suggests that there is a great public interest in finding answers to mathematical questions posed in natural language, using both text and mathematical notation" [14].

It is evident that, over time, Math Stack Exchange evolved into one of the most active and trusted online mathematics communities. The platform's emphasis on systematic topic categorization, the collaborative oversight of content by its members, and the establishment of trust based on reputation have collectively fostered a remarkably coherent and sustainable environment for knowledge sharing. Preliminary research on Stack Exchange communities indicates that "active communities have higher local cohesiveness and develop stable and more strongly connected cores." Social trust, as determined by reputation, plays a pivotal role in the long-term sustainability of these communities [27].

Concurrently, `Matematika.bg` was established by Yordan Petrov and remains under his maintenance [20]. Since its inception in 2005, the platform has been instrumental in fostering the development of online mathematics resources, with a particular emphasis on interactive content such as national assessment tests (e.g., for Bulgarian students in Grades 4 and 7). Notably, the platform has been led by Petrov in its development and authorship. The site offers a distinctive integration of several components. The materials have been meticulously aligned with the curriculum and include the official NEA tests for various grades. Petrov's engagement in the robust forum system is characterized by a wide range of participation in discussions, encompassing a diverse array of subjects, from high school-level topics to advanced mathematics, including differential equations. Under Petrov's leadership, `Matematika.bg` has evolved into a distinctive Bulgarian educational platform that integrates teacher-led resource development, learner interaction, and an ongoing vision for adaptive, personalized learning through exercises and structured guidance.

These platforms have addressed a significant gap in educational resources by providing students with mechanisms to seek clarification and receive explanations tailored to their cognitive needs and mathematical maturity levels. Rather than merely serving as answer banks, these communities foster social learning environments that reinforce mathematical learning through peer dialogue and explanation. According to Padayachee and Campbell [18], online mathematics forums can extend the benefits of face-to-face discussions to digital spaces. These digital spaces help students build a "community of inquiry" that promotes deep, interactive engagement with complex mathematical ideas.

The rapid dissemination of learning resources accelerated further due to the COVID-19 pandemic [26]. Notably, the number of educational videos created by teachers to methodically guide students through problem-solving processes increased. These videos promote conceptual understanding and demonstrate mathematical thinking in action. Meanwhile, private tutoring and after-school academies remained popular alternatives for customized educational support. Instructional videos, especially those with heuristic worked examples and interactive features, have proven to be valuable resources for fostering metacognitive engagement and supporting diverse learning needs. As Wirth and Greefrath [30] observe, such videos allow learners to adapt the content to their own pace and provide step-by-step visual and verbal guidance, which students consider essential for grasping complex mathematical modelling processes.

2.3. From foundations to frontiers

The theoretical underpinnings of AI were established in the mid-20th century with foundational work in symbolic reasoning, machine learning, and computational models of cognition [24]. However, a substantial paradigm shift occurred at the end of the century with the emergence of deep learning, a subset of machine learning that uses multi-layered neural networks to model complex patterns in large datasets [8, 12]. The implementation of deep learning techniques has made training large-scale models, including LLMs, possible using extensive textual data sets [5, 25]. These techniques have enabled the generation of human-like language and the successful execution of a wide range of tasks with increasing precision [6].

A significant turning point in the evolution of LLMs and their public perception occurred in 2022 with the release of ChatGPT by OpenAI. Initially based on the GPT-3.5 architecture, ChatGPT later transitioned to the GPT-4 architecture [6, 17]. This release substantially increased access to conversational AI, showcasing the potential of LLMs in various domains, such as education, programming, creative writing, and research assistance [13, 17]. In 2025, the landscape evolved further with the introduction of DeepSeek [15], a next-generation model that expanded the capabilities of multilingual reasoning and domain specialization.

All these marked a pivotal shift in the ecosystem's evolution, introducing a new dimension. The reactions ranged from enthusiasm to scepticism. However, the rapid adoption of LLMs by students, particularly for mathematical tasks, underscores the ongoing demand for personalized, context-sensitive academic support [31].

It has become evident that students have consistently sought out learning platforms that align with their preferred learning styles [2], whether these styles are visual, verbal, exploratory, or procedural. The emergence of forums, videos, and now AI-powered assistants does not signify a fundamental shift in the needs of students. Rather, it indicates a shift in the methods through which these needs can be effectively and flexibly addressed. While some students continue to engage with conventional methods, such as individual teacher consultations or peer interaction within online forums, chatbot-based AI systems are increasingly being adopted as a widely accepted and accessible alternative. This is further supported by recent findings: "ChatGPT has a large positive impact on improving learning performance ($g = 0.867$) and a moderately positive impact on enhancing learning perception ($g = 0.456$) and fostering higher-order thinking ($g = 0.457$)" [28]. This evolution necessitates a critical reassessment of the design, integration, and evaluation of educational technologies. This re-evaluation should not be limited to the novelty of the technologies but should also consider their responsiveness to longstanding and authentic learner demands [28].

A comprehensive understanding of the expectations that contemporary students hold toward tools like ChatGPT necessitates a multifaceted examination of several factors, including their motivational profiles, learning styles, and prior experiences with academic support systems. While some students continue to exhibit high levels of engagement through conventional methods such as individual teacher consultations or peer interaction within online forums, there has been an increasing adoption of chatbot-based AI systems as a widely accepted and accessible alternative [23].

What kind of support do students typically receive from AI Chatbots? Most of the time, it is a thorough, step-by-step plan with the necessary reasons and explanations [6, 29]. Often, these solutions follow a chain-of-thought format that breaks down complex problems into smaller, logically connected steps [23]. This type of reasoning has been shown to significantly improve the interpretability and effectiveness of AI responses in educational contexts. A valuable feature of this interaction is that students can ask clarifying questions about any stage of the solution process, creating a dynamic, adaptive learning experience.

One of the key advantages of using AI tools like ChatGPT is the psychologically safe environment they create. Students can engage with academic content without facing judgment or embarrassment, which are common in traditional classroom settings. The AI does not evaluate performance, assign grades, or offer opinions about knowledge. Instead, it functions as an impartial, always-available assistant whose sole purpose is to provide support and clarification. This neutrality makes ChatGPT especially appealing to learners who lack confidence or hesitate to participate actively in class. Research also shows that students value ChatGPT for its nonjudgmental tone and accessibility, especially when they are struggling with subjects such as mathematics [23].

3. Empirical evaluation

3.1. Dataset

The empirical study employed eight mathematics problems from the Bulgarian NEA for 10th grade, administered at the conclusion of the 2023/2024 academic year. The selected items were identified as Problems 1, 2, 4, 6, 7, 8, 9, and 10 [16].

The original examination contained multiple-choice items, with four options and one correct answer. For the purposes of this study, these multiple-choice items were adapted into a short-answer format. This transformation ensured that evaluation would account for the accuracy and format of the final answer in the context of the given problem. According to the official scoring rubric, each of these tasks is worth 4 points. The preparation of the problem involved the transcription of all mathematical formulae, expressions, and symbols into LaTeX. This was done to ensure precise automated processing.

3.2. Methodology

An automated evaluation was conducted using eleven popular large language models (LLMs) representing different architectures and parameter scales, all current as of March 2025, and obtained from Ollama, a public repository of pre-trained generative models.

The following models were analyzed:

- BigGPT (9B and 27B)
- LLaMA 3 (8B and 70B; labeled `llama_31_8b` and `llama_33_70b`)
- DeepSeek R1 (7B)
- Mathstral (7B)
- Mixtral (8×22B and 8×7B)
- Mistral (7B, 22B, and 123B).

In addition to the automated batch evaluation, a manual assessment was performed on Le Chat (Mistral), DeepSeek, and ChatGPT. In these sessions, the same problems were posed interactively. In certain instances, supplementary constraints, guidelines, or requirements are delineated. The constraints are predominantly contingent upon the implementation of a particular mathematical approach, with the objective of adhering to the outlined curriculum.

4. Synopsis of the results

4.1. Automated evaluation

As illustrated in Figure 1, the success rate (i.e., the percentage of correctly solved problems) varies across models. The validation of a solution was contingent upon its capacity to yield an exact match with the anticipated outcome, presented in the prescribed format. In addition to raw success rates, the evaluation incorporated four qualitative scoring dimensions derived from the rubric.

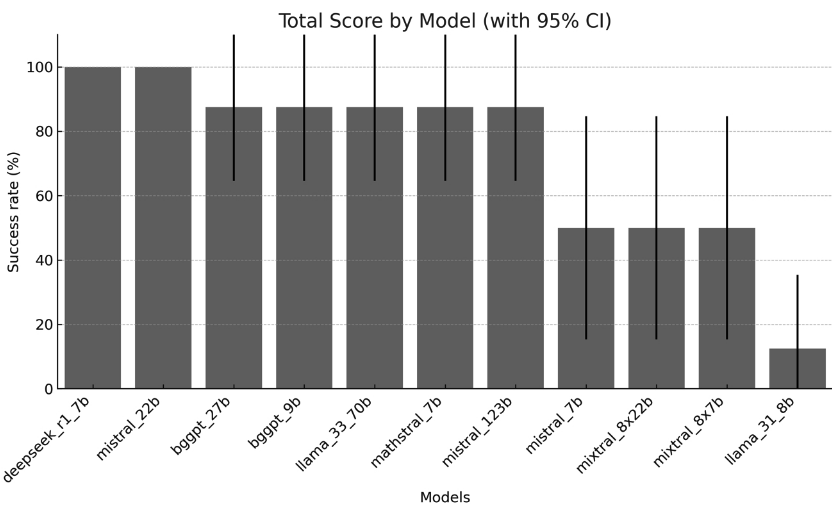


Figure 1. Total score by model

- The *Structured solution* (0–2) is the extent to which the model's response is characterized by a coherent, stepwise derivation leading to the conclusion;
- *Pedagogical quality* (0–2) – the alignment of the reasoning and methods with the official Bulgarian educational plan;
- *Critical error*, otherwise referred to as a penalty, is characterized by deductive or conceptual flaws, ranging from minor miscalculations that result in a score of -1 to incorrect underlying concepts that yield a score of -2;
- *Language clarity* (0–2) – the degree to which the language used is correct and appropriate, including grammar, orthography, and domain-specific terminology.

As illustrated in Table 1, there is a strong correlation between performance in raw accuracy and structured presentation and pedagogical quality, though the relationship is not perfect. For instance:

Table 1

Models by rubrics

Model	**Success rate (%)**	**Mean quality (%)**	**Structured solution (0–2)**	**Pedagogical quality (0–2)**	**Critical error (pen.)**	**Language clarity (0–2)**
DeepSeek R1 (7B)	100.0	54.17	1.50	1.50	0.00	0.25
Mistral (22B)	100.0	81.25	1.75	1.75	0.00	1.38
BgGPT (27B)	87.50	79.17	1.75	1.50	−0.38	1.88
BgGPT (9B)	87.50	81.25	1.75	1.50	−0.25	1.88
LLaMA 3.3 (70B)	87.50	56.25	1.38	1.00	−0.13	1.13
Mathstral (7B)	87.50	62.50	1.88	1.63	0.00	0.25
Mistral (123B)	87.50	85.42	1.88	1.88	−0.25	1.63
Mistral (7B)	50.00	41.67	1.13	0.75	−1.00	1.63
Mixtral (8×22B)	50.00	56.25	1.43	1.29	−0.50	1.50
Mixtral (8×7B)	50.00	33.33	1.00	0.88	−0.75	0.88
LLaMA 3.1 (8B)	12.50	12.50	0.75	0.38	−1.13	0.75

The DeepSeek R1 (7B) and Mistral (22B) models both demonstrated 100% success rates; however, a divergence in their mean quality scores was observed (54.17% vs. 81.25%). DeepSeek's lower score is attributable to more frequent minor deviations from the anticipated Bulgarian curriculum style, despite the presence of mathematically correct answers.

Mathstral (7B) attained the highest structured solution score (1.88/2), yet exhibited a lower mean quality (62.5%), frequently attributable to minor methodological discrepancies with the Bulgarian syllabus, such as the utilization of internationally prevalent solution techniques in conjunction with those dictated by the national curriculum.

Statistical significance

A one-way analysis of variance (ANOVA) was conducted on the success rates across the eleven models to determine whether the differences were statistically significant. The results of the study indicate the following:

The F-statistic $(10, 77)$ was determined to be 15.42, and the p-value was found to be less than 0.001. This indicates that there is a statistically significant difference in the success rates across the models.

Subsequent post-hoc Tukey HSD comparisons revealed that:

- The performance of DeepSeek R1 (7B) and Mistral (22B) was found to be significantly superior to that of LLaMA 3.1 (8B) ($p < 0.001$).
- The BgGPT variants (9B & 27B) and Mistral (123B) formed a statistically indistinguishable high-performance cluster, with success rates ranging from 87.5% to 100%.
- The lowest-performing group comprised of Mixtral (8×7B) and LLaMA 3.1 (8B).

Error analysis

A qualitative review of the findings indicated that linguistic and methodological issues predominated over purely mathematical errors.

The linguistic issues encompassed solutions that were entirely in English despite Bulgarian inputs, or grammatically awkward phrasing. In the majority of these cases, mathematical correctness remained unaltered, suggesting a sufficient understanding of the Bulgarian problem statement. Terminology frequently reflected literal translation rather than the locally accepted mathematical vocabulary.

The methodological deviations that were identified included correct but inefficient approaches. For example, solving for the individual roots and then multiplying, instead of directly applying Viète's formulas; or tackling higher-degree inequalities by case analysis rather than interval methods prescribed in the Bulgarian curriculum. These deviations appear to reflect global training bias rather than failure of reasoning.

Structured solution

It has been observed that models in the top performance group – namely, Mistral (123B), Mathstral, and BgGPT variants – typically exhibited clear multistep solutions accompanied by formula citations prior to utilization. In contrast, LLaMA 3.1 (8B) exhibited the lowest structured score (0.75/2), frequently producing only minimal reasoning before yielding a definitive answer.

Pedagogical quality

The highest pedagogical alignment was observed in Mistral (123B) and Mistral (22B) (1.88 and 1.75/2, respectively). These models demonstrated a strong adherence to the Bulgarian curriculum methods, as evidenced by their utilization of the interval method for inequalities, which superseded the more ad hoc and case-specific approaches. Lower-scoring models frequently resorted to alternative yet valid methods derived from global training data.

Critical errors

The most proficient group exhibited a low incidence of critical errors, with DeepSeek R1 (7B), Mathstral (7B), and Mistral (22B) demonstrating no penalties. In contrast, LLaMA 3.1 (8B) accumulated the largest average penalty (-1.13), indicative of multiple conceptual misunderstandings rather than isolated arithmetic slips.

Language clarity

The BgGPT variants demonstrated a high level of linguistic appropriateness, with an average score of 1.88 out of 2. These variants exhibited a capacity to produce grammatically correct Bulgarian, incorporating locally relevant terminology, thereby exhibiting a strong alignment with the subject's linguistic nuances. DeepSeek R1 (7B) demonstrated near-flawless accuracy, yet its clarity score (0.25/2) was notably lower due to frequent mixing of Bulgarian with Russian or English, a combination that may present challenges in classroom adoption.

Observed patterns and implications

A thorough examination of rubric data reveals a discernible trade-off between accuracy and educational alignment in select models. Global models such as DeepSeek R1 and LLaMA 3.3 (70B) have been shown to achieve high levels of accuracy; however, these models have been observed to occasionally overlook region-specific pedagogical norms. Locally trained or fine-tuned models (BgGPT) demonstrate enhanced alignment with the national curriculum and superior language utilization, despite their inability to attain absolute precision. From an instructional perspective, structured reasoning and adherence to curricular conventions are as important as raw correctness, particularly for formative assessment and classroom demonstration. Consequently, while models such as DeepSeek R1 can function as problem-solving backends, their outputs may necessitate post-processing or fine-tuning to align with the pedagogical and linguistic standards for Bulgarian education.

The findings indicate that, despite noteworthy advancements in mathematical reasoning, prevailing LLMs lack the reliability necessary for autonomous tool use in this category of mathematics assessment problems, particularly in scenarios where pedagogical alignment and linguistic precision are paramount.

4.2. Manual assessment

In order to evaluate the applicability of contemporary AI chatbots in supporting Bulgarian high school mathematics education, three widely discussed models, DeepSeek, Le Chat, and ChatGPT, were examined with respect to their problem-solving capabilities, alignment with national curricular standards, and suitability for both teacher and student use.

DeepSeek

A notable constraint of the DeepSeek chatbot is its restricted capacity for image recognition. Consequently, the necessity arose to assess the efficacy of the LaTeX version of the mathematical problems. This option is not realistic for the average high school student. As with automated testing scenarios, the model produced accurate solutions. Notably, the production of grammatically correct and contextually appropriate Bulgarian in the formulation of solutions was identified as a significant advantage. Once more, discrepancies in problem-solving methodologies became evident. For instance, when determining the maximum of a quadratic function, the model's initial solution incorporated mathematical analysis. After the implementation of constraints that confined the methods to those pertinent to the curriculum, the vertex coordinates were obtained through the utilization of the method of completing the square. When prompted to use a standard formula, the model produced the desired solution. However, iterative refinement and methodological adjustment of this sort would exceed the capabilities of a typical tenth-grade student seeking assistance. Consequently, it is not appropriate to regard DeepSeek as a substitute for direct consultation with a teacher or participation in a peer discussion forum.

Le Chat (by Mistral)

Conversely, the Le Chat model has been shown to effectively implement image recognition, thereby enabling the direct processing of problem statements presented in image format. In contrast to the automated testing scenarios involving other Mistral-based models, all issues encountered in the interactive chat sessions were resolved correctly. However, disparities in the implemented solution strategies – relative to those with which Bulgarian students are more familiar – became evident, primarily due to the specificity of national educational standards. While Le Chat may serve as a convenient tool for teachers, enabling efficient recording and formatting of solutions and the generation of similar problems, for students, these methodological discrepancies limit its value as an autonomous learning aid.

ChatGPT

ChatGPT is historically the first model to overcome the language barrier, enabling seamless communication in Bulgarian. It continues to be the most prevalent model among students and educators in Bulgaria. Furthermore, product integrations have been developed; for example, SmarTest incorporates ChatGPT, thereby enabling automatically generated test questions to be copied and used in test creation [21]. However, the discrepancy between ChatGPT's problem-solving methodologies and the Bulgarian educational curriculum gives rise to compatibility issues. While many solutions are mathematically correct, they may not align with the methods or presentation formats expected within the national educational system, rendering them inapplicable in certain classroom or examination contexts.

The comparative analysis of DeepSeek, Le Chat, and ChatGPT in the context of mathematical problem-solving for Bulgarian high school students reveals distinct strengths and limitations.

5. Discussion and conclusions

The trajectory of mathematics learning resources is indicative of an ongoing process of adaptation to the evolving demands of students, who increasingly seek clarity in the material, accessibility in the form of support services, and a degree of personalization in their academic experiences. For many years, Bulgarian students, like their counterparts around the world, primarily relied on printed textbooks and problem books. These texts frequently provided only rudimentary explanations and omitted comprehensive solutions, impeding independent study. The additional guidance provided was largely derived from teacher consultations or peer interactions, thereby constraining the flexibility and breadth of self-directed learning.

The development of the Internet signified a substantial paradigm shift. The advent of online forums such as Yahoo! Answers, Math Stack Exchange, and the Bulgarian platform `Matematika.bg` has engendered novel opportunities for learners to pose questions, access explanations, and collaborate asynchronously. These communities proffered a plethora of problem-solving perspectives and social engagement, yet also introduced challenges, including inconsistent quality, varying depth, and delayed responses. While forums fostered a sense of community and allowed students to learn from multiple perspectives, they also required public participation, which some learners found uncomfortable.

Conversational AI tools, in contrast, facilitate the creation of a private, adaptive space where students can explore questions without peer visibility, receive immediate feedback, and adjust the depth or pace of explanations to suit their individual preferences. This development signifies a transition from socially mediated, frequently asynchronous learning models to an on-demand, personalized interaction paradigm. This paradigm emulates the psychological comfort of working one-on-one with a tutor. LLMs, such as ChatGPT, exemplify the latest stage in this progression, as they represent a shift from delayed, public exchanges to instant, private "comfort mode" interactions. It has been demonstrated that students could receive step-by-step solutions on demand, to inquire further without the concern of being judged, and to adapt conversations to their own pace and style. This evolution mirrors broader educational trends towards customized, adaptive systems that can cater to individual learning preferences [3, 28]. The psychological safety and flexibility of AI-powered chats have been shown to be especially appealing for learners who are hesitant to participate actively in class or in public online forums.

However, the empirical findings demonstrate that while LLMs frequently attain high levels of accuracy – at times reaching 100% on NEA mathematics problems – their methodologies do not invariably align with the Bulgarian curriculum. While these divergences are mathematically valid, they can pose challenges in high-stakes assessments, where strict adherence to established methods is mandatory [19, 23]. This observation aligns with a previously identified forum-era reality: the correctness of a resource does not inherently guarantee its alignment with the needs of its intended audience.

Linguistic clarity remains a further challenge. Locally fine-tuned models such as BgGPT score highly for using precise Bulgarian mathematical terminology, whereas some global models mix Bulgarian with English or Russian terms. In mathematics learning, as in forum exchanges, language precision and familiarity strongly influence comprehension and student confidence [30].

Insights from the parallel qualitative study reinforce these findings. Master level mathematics teachers recognized AI's potential for providing tailored student support but were cautious about integrating it into their own practice. Many preferred to create materials independently, viewing this as part of their professional identity. This cautious stance echoes earlier hesitations in adopting forums for formal teaching, reflecting a desire to maintain control over content and methodology. Yet, as students increasingly gravitate toward comfort-mode learning environments, the absence of guided integration risks widening the gap between formal instruction and self-directed study [2, 10, 13].

From static books to peer-driven forums, to interactive AI chats, each stage in this evolution reflects a shift not only in technology but also in the social dynamics of learning. Books offered privacy but little adaptability; forums introduced collaboration and multiple perspectives but required public participation; conversational AI combines the privacy of independent study with the adaptability and responsiveness of a personal tutor. Today's LLMs are uniquely positioned to meet these evolving needs by offering private, adaptive, and judgment-free interactions, while retaining the ability to guide students step-by-step.

However, for successful integration into formal education to occur, three specific criteria must be met.

- Mathematical accuracy is critical for maintaining the reliability that fosters student trust.
- Methodological alignment signifies the process of ensuring that solutions are congruent with local curricular standards, particularly within the context of assessment.
- Linguistic appropriateness is defined as the employment of precise, culturally and pedagogically relevant terminology.

The findings of this study suggest that in the absence of teacher mediation, the outputs of mathematically correct AI systems may potentially mislead students if they employ solution methods or language with which the students are unfamiliar. This evidence aligns with broader research indicating that effective AI adoption is contingent upon technical accuracy, institutional support, teacher training, and ethical safeguards [2, 3, 10, 13, 28]. In the domain of mathematics education, these considerations are particularly salient, as the presentation of solutions is as important as the correctness of the solutions themselves [19, 23, 30].

Moving forward, educational institutions should:

- Provide clear guidelines for aligning AI-generated content with curricular standards.
- Offer professional development enabling teachers to adapt AI outputs for local contexts.

- Ensure equitable access and address ethical considerations in AI-supported learning.

When incorporated deliberately into this continuum, ranging from books to forums to comfort-mode AI, LLMs have the potential to augment rather than substitute for the role of the teacher. These resources have the potential to combine the richness of peer-based learning with the safety and adaptability of private study, thereby ensuring that mathematics education remains both rigorous and responsive to students' changing expectations.

Acknowledgements

The research and preparation of this article were funded and supported by the National Program "Young Scientists and Postdoctoral Fellows – 2," administered by the Ministry of Education and Science of the Republic of Bulgaria.

References

[1] L. A. Adamic, J. Zhang, E. Bakshy and M. S. Ackerman, Knowledge sharing and Yahoo Answers: everyone knows something, in: Proc. 17th Int. Conf. on World Wide Web, WWW, Beijing, 2008.

[2] D. Ali, Y. Fatemi, E. Boskabadi, M. Nikfar, J. Ugwuoke and H. Ali, ChatGPT in teaching and learning: a systematic review, Education Sciences 14(6) (2024) 643.

[3] O. O. Ayeni, N. M. A. Hamad, O. N. Chisom, B. Osawaru and O. E. Adewusi, AI in education: a review of personalized learning and educational technology, GSC Advanced Research and Reviews 18(2) (2024) 261–271.

[4] I. Bonifacic, Yahoo!News, Yahoo, 06 04 2021. Available at https://www.yahoo.com/news/yahoo-answers-shutdown-may-4th-210240460.html. Accessed August 2025.

[5] T. Brown, B. Mann, N. Ryder et al., Language models are few-shot learners, in: 34th Conference on Neural Information Processing Systems, Vancouver, 2020.

[6] S. Bubeck, V. Chandrasekaran, R. Eldan et al., Sparks of artificial general intelligence: early experiments with GPT-4, arXiv preprint arXiv:2303.12712, 2023.

[7] D. N. C. Dey, Enhancing educational tools through artificial intelligence in perspective of need of AI, SSRN, 2024.

[8] I. Goodfellow, Y. Bengio and A. Courville, Deep learning, MIT Press, 2016.

[9] P. Halachev, Integration of ChatGPT in e-Learning systems: comprehensive review, Periodicals of Engineering and Natural Sciences 12(1) (2024) 169–182.

[10] G.-J. Hwang and Y.-F. Tu, Roles and research trends of artificial intelligence in mathematics education: a bibliometric mapping analysis and systematic review, Mathematics 9 (2021) 584.

[11] U. P. International, Yahoo! launches Web answering site, 08.12.2005. Available at https://phys.org/news/2005-12-yahoo-web-site.html. Accessed August 2025.

[12] M. I. Jordan and T. M. Mitchell, Machine learning: trends, perspectives, and prospects, Science 349(6245) (2015) 255–260.

[13] E. Kasneci, K. Sessler, S. Küchemann et al., ChatGPT for good? On opportunities and challenges of large language models for education, Learning and Individual Differences 103 (2023) 102274.

[14] B. Mansouri, A. Agarwal, D. Oard and R. Zanibbi, Finding old answers to new math questions: the ARQMath lab at CLEF 2020, in: Advances in Information Retrieval, 42nd European Conference on IR Research, Lisbon, 2020.
[15] A. Mathew, Deep seek vs. ChatGPT: a deep dive into AI language mastery, International Journal for Multidisciplinary Research 7(1) (2025) 5.
[16] MES, NEA, 06/2024. Available at https://www.mon.bg/nfs/2024/06/nvo_10kl_math_12062024.pdf. Accessed August 2025.
[17] OpenAI, GPT-4 Technical Report, 2023. Available at https://cdn.openai.com/papers/gpt-4.pdf. Accessed August 2025.
[18] P. Padayachee and A. L. Campbell, Supporting a mathematics community of inquiry through online discussion forums: towards design principles, Int. J. Math. Educ. Sci. Technol. 53(1) (2021) 35–63.
[19] B. Pepin, N. Buchholtz and U. Salinas-Hernández, Mathematics education in the era of ChatGPT: investigating its meaning and use for school and university education, Digital Experiences in Mathematics Education 11 (2025) 1–8.
[20] Y. Petrov, `Matematika.bg`, Available at https://www.matematika.bg/. Accessed August 2025.
[21] A. Popov, Artificial intelligence to assist teachers. Available at https://departments.unwe.bg/Uploads/Department/padmin_5c4a3_UNWE-Teacher-Lecture.pdf. Accessed August 2025 (in Bulgarian).
[22] D. R. Raban and F. M. Harper, Motivations for answering questions online, Burda Center Publishing, 2008.
[23] D. Ravšelj, D. Keržič, N. Tomaževič et al. Higher education students' perceptions of ChatGPT: A global study of early reactions, PLoS ONE 20(2) (2025) e0315011, https://doi.org/10.1371/journal.pone.0315011.
[24] S. Russell and P. Norvig, Artificial intelligence: a modern approach, Pearson, 2020.
[25] A. Vaswani, N. Shazeer, N. Parmar et al., Attention is all you need, in: 31st Conference on Neural Information Processing Systems, Long Beach, CA, 2017.
[26] R. Videla, S. Rossel, C. Muñoz and C. Aguayo, Online mathematics education during the COVID-19 pandemic: didactic strategies, educational resources, and educational contexts, Education Sciences 12(7) (2022) 492.
[27] A. Vranić, A. Tomašević, A. Alorić and M. M. Dankulov, Sustainability of Stack Exchange Q&A communities: the role of trust, EPJ Data Science 12 (2023) 4.
[28] J. Wang and W. Fan, The effect of ChatGPT on students' learning performance, learning perception, and higher-order thinking: insights from a meta-analysis, Humanities and Social Sciences Communications 12 (2025) 621.
[29] J. Wei, X. Wang, D. Schuurmans et al., Chain of thought prompting elicits reasoning in Large Language Models. Available at https://doi.org/10.48550/arXiv.2201.11903. Accessed August 2025.
[30] L. Wirth and G. Greefrath, Working with an instructional video on mathematical modeling: uppersecondary students' perceived advantages and challenges, ZDM – Mathematics Education 56 (2024) 573–587.
[31] K. Žáková, D. Urbano, R. Cruz-Correia et al., Exploring student and teacher perspectives on ChatGPT's impact in higher education, Education and Information Technologies 30 (2024) 649–692.

Received on August 14, 2025
Accepted on November 1, 2025

Ralitza Stamenkova

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mail: r.stamenkova@fmi.uni-sofia.bg

Submission of manuscripts. The *Annual* is published in December each year. A `pdf` file of the manuscript should be submitted on the journal website https://annual.uni-sofia.bg/index.php/fmi. The manuscript will be subject to an initial check. If the manuscript does not pass, it will be declined; otherwise, it will proceed to a rapid but thorough peer review process. If accepted, it will be immediately scheduled for the nearest forthcoming issue. No page charges are applied.

The submission of a paper implies that it has not been published and is not under consideration for publication elsewhere. If accepted, it also implies that the author(s) transfer copyright to the Faculty of Mathematics and Informatics at Sofia University "St. Kliment Ohridski", including the right to adapt the article for use in conjunction with computer systems and programs, as well as reproduction or publication in machine-readable form and incorporation in retrieval systems. A printable pdf file of the published paper will be available on the journal website.

Instructions for authors. Preference will be given to papers not longer than 25 pages, written in English and typeset using LaTeX or MS Word. MS Word files will be converted to LaTeX. Templates are available on the journal website.

Manuscripts should be submitted for editorial consideration in `pdf` format on the journal website. Upon acceptance, authors should submit the paper in `tex` or `docx` format, as well as the graphic files of the figures (if any) in separate files.

The first page of the manuscript must contain a title, name(s) of the author(s), a short abstract, a list of keywords, and appropriate codes according to Mathematics Subject Classification 2020 (MSC2020), see https://msc2020.org/, or the 2012 ACM Computing Classification System (CCS), see https://dl.acm.org/ccs. The affiliation(s), including the electronic address, should be given at the end of the manuscript.

Figures must be inserted in the text near their first reference. Graphic files in any of the following formats: `jpg`, `png`, `pdf`, or `eps` should be submitted together with the final version of the manuscript after acceptance. All illustrations must be in grayscale or black-and-white; otherwise, appropriate conversion will be performed.

Tables should be inserted in the text as close to the point of reference as possible. Some space should be left above and below the table.

Footnotes must be numbered consecutively and should be brief.

References must be cited in the text in square brackets, such as [3], [5, 7], [11, p. 123], or [16, Ch. 2.12]. They must be numbered alphabetically by the family name of the first author. Examples (please note the order, style, and punctuation of the bibliographical description of each entry):

For books: R. R. Phelps, Convex functions, monotone operators and differentiability, Second Edition, Lecture Notes in Mathematics, 1364 Springer-Verlag, Berlin, 1993.

For journal articles: J. M. Borwein and D. Preiss, A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions, Trans. Amer. Math. Soc. 303(2) (1987) 517–527.

For articles in edited volumes or proceedings: R. Deville and N. Ghoussoub, Perturbed minimization principles and applications, in: Handbook of the geometry of Banach spaces, ed. by W. B. Johnson and J. Lindenstrauss, vol. 1, North-Holland, Amsterdam, 2001, 393–435.